

# Circuits and Transforms

## EE3900: Linear Systems and Signal Processing

### Indian Institute of Technology Hyderabad

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#### 1. DEFINITIONS

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

1.2 The Laplace transform of  $g(t)$  is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

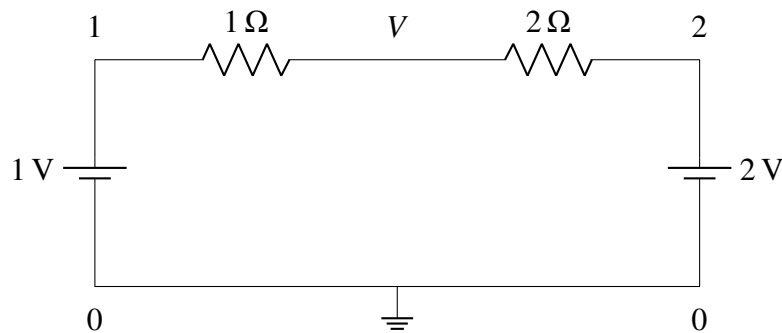


Fig. 2.3. Circuit diagram at steady state before flipping the switch

#### 2. LAPLACE TRANSFORM

2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes  $q_1 \mu\text{C}$ . Then S is switched to position Q. After a long time, the charge on the capacitor is  $q_2 \mu\text{C}$

2.2. Draw the circuit using latex-tikz

**Solution:**

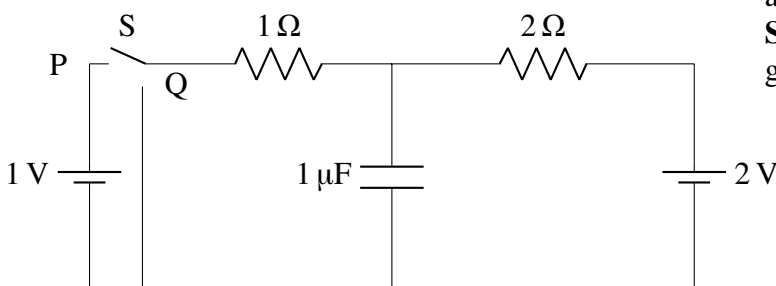


Fig. 2.2. Circuit diagram of the circuit in question

2.3. Find  $q_1$

**Solution:** After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

By Kirchoff's junction law, we get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \quad (2.1)$$

$$\Rightarrow V = \frac{4}{3} \text{ V} \quad (2.2)$$

$$\Rightarrow q_1 = CV = \frac{4}{3} \mu\text{C} \quad (2.3)$$

2.4. Show that the Laplace transform of  $u(t)$  is  $\frac{1}{s}$  and find the ROC

**Solution:** The Laplace transform of  $u(t)$  is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st} dt \quad (2.4)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.5)$$

$$= \lim_{R \rightarrow \infty} \frac{1 - e^{-sR}}{s} \quad (2.6)$$

This limit is finite only if  $\Re(s) > 0$ , which is going to be its ROC

Therefore

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \Re(s) > 0 \quad (2.7)$$

2.5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad a > 0 \quad (2.8)$$

and find the ROC

**Solution:** The Laplace transform of  $e^{-at}u(t)$  for  $a > 0$  is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} dt \quad (2.9)$$

$$= \int_0^{\infty} e^{-(s+a)t} dt \quad (2.10)$$

$$= \lim_{R \rightarrow \infty} \frac{1 - e^{-(s+a)R}}{s+a} \quad (2.11)$$

This limit is finite only if  $\Re(s+a) > 0$ , which is going to be its ROC

Therefore

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \Re(s) > -a \quad (2.12)$$

since  $a$  is real

2.6. Now consider the following resistive circuit transformed from Fig. 2.2

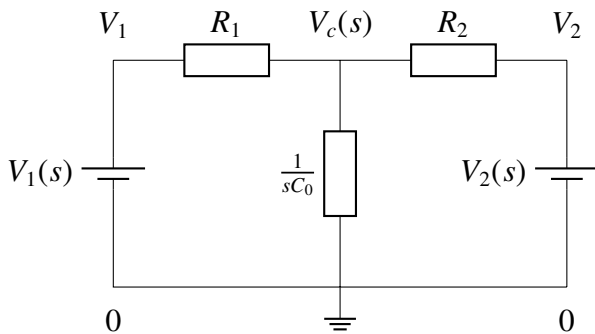


Fig. 2.6. Circuit diagram in  $s$ -domain before flipping the switch

where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \quad (2.13)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \quad (2.14)$$

Find the voltage across the capacitor  $V_c(s)$

**Solution:**

$$V_1(s) = \frac{1}{s} \quad \Re(s) > 0 \quad (2.15)$$

$$V_2(s) = \frac{2}{s} \quad \Re(s) > 0 \quad (2.16)$$

By Kirchoff's junction law, we get

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \quad (2.17)$$

$$\Rightarrow V_c \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \quad (2.18)$$

$$\Rightarrow V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (2.19)$$

$$= \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s \left( s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0} \right)} \quad (2.20)$$

2.7. Find  $v_c(t)$ . Plot using Python.

**Solution:** On performing partial fraction decomposition

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \left( \frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \right), \Re(s) > 0 \quad (2.21)$$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2} \left( u(t) - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) \right) \quad (2.22)$$

$$= \frac{2R_1 + R_2}{R_1 + R_2} \left( 1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} \right) u(t) \quad (2.23)$$

Substitute the values  $R_1 = 1 \Omega$ ,  $R_2 = 2 \Omega$ ,  $C_0 = 1 \mu\text{F}$

$$v_c(t) = \frac{4}{3} \left( 1 - e^{-\frac{3}{2} \times 10^6 t} \right) u(t) \text{ V} \quad (2.24)$$

2.8. Verify your result using ngspice

**Solution:** Download the following codes for simulation and plotting Fig. 2.8 respectively

```
wget https://github.com/Dhatrreddyy/circuits/
blob/main/codes/2.8.cir
wget https://github.com/Dhatrreddyy/circuits/
blob/main/codes/2.7.py
```

Run the codes by executing

```
ngspice 2.8.cir
python3 2.7.py
```

2.9. Obtain Fig. 2.6 using the equivalent differential equation

**Solution:** Using Kirchoff's junction law

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (2.25)$$

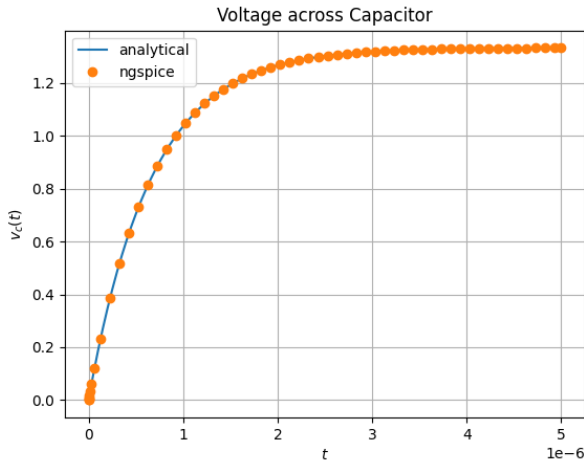


Fig. 2.8. Plot of  $v_c(t)$  before flipping the switch

where  $q(t)$  is the charge on the capacitor  
On taking the Laplace transform on both sides  
of this equation

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (2.26)$$

But  $q(0^-) = 0$  and

$$q(t) = C_0 v_c(t) \quad (2.27)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (2.28)$$

Thus

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) = 0 \quad (2.29)$$

$$\Rightarrow \frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - 0}{\frac{1}{sC_0}} = 0 \quad (2.30)$$

which is the same equation as the one we obtained from Fig. 2.6

### 3. INITIAL CONDITIONS

#### 3.1. Find $q_2$ in Fig. 2.2

**Solution:** After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

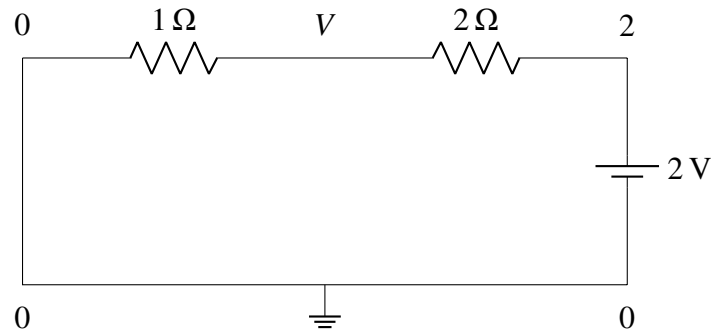


Fig. 3.1. Circuit diagram at steady state after flipping the switch

By Kirchoff's junction law, we get

$$\frac{V - 0}{1} + \frac{V - 2}{2} = 0 \quad (3.1)$$

$$\Rightarrow V = \frac{2}{3} \text{ V} \quad (3.2)$$

$$\Rightarrow q_2 = CV = \frac{2}{3} \mu\text{C} \quad (3.3)$$

#### 3.2. Draw the equivalent $s$ -domain resistive circuit when S is switched to position Q. Use variables $R_1, R_2, C_0$ for the passive elements. Use latex-tikz

**Solution:**

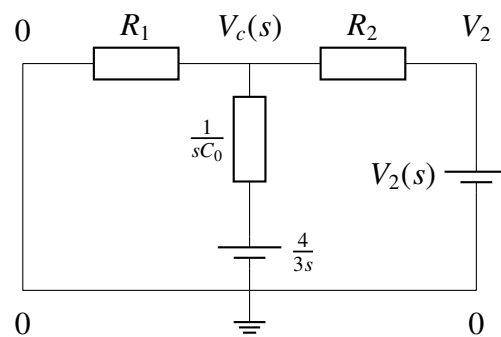


Fig. 3.2. Circuit diagram in  $s$ -domain after flipping the switch

The battery  $\frac{4}{3s}$  corresponds to the initial potential difference of  $\frac{4}{3}$  V across the capacitor just before switching it to Q

#### 3.3. Find $V_c(s)$

**Solution:** By Kirchoff's junction law, we get

$$\frac{V_c - 0}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.4)$$

$$\Rightarrow V_c \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_2}{R_2} + \frac{4}{3}C_0 \quad (3.5)$$

$$\Rightarrow V_c(s) = \frac{\frac{2}{sR_2} + \frac{4}{3}C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (3.6)$$

$$= \frac{\frac{2}{R_2C_0} + \frac{4}{3}s}{s \left( s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0} \right)} \quad (3.7)$$

3.4. Find  $v_c(t)$ . Plot using Python

**Solution:** On performing partial fraction decomposition

$$V_c(s) = \frac{4}{3} \left( \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \right) + \frac{\frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \left( \frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \right) \quad (3.8)$$

for  $\Re(s) > 0$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) + \frac{2R_1}{R_1 + R_2} \left( u(t) - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) \right) \quad (3.9)$$

Substitute the values  $R_1 = 1 \Omega$ ,  $R_2 = 2 \Omega$ ,  $C_0 = 1 \mu\text{F}$

$$v_c(t) = \frac{4}{3} e^{-\frac{3}{2} \times 10^6 t} u(t) + \frac{2}{3} \left( 1 - e^{-\frac{3}{2} \times 10^6 t} \right) u(t) \quad (3.10)$$

$$= \frac{2}{3} \left( 1 + e^{-\frac{3}{2} \times 10^6 t} \right) u(t) \text{ V} \quad (3.11)$$

3.5. Verify your result using ngspice

**Solution:** Download the following codes for simulation and plotting Fig. 3.5 respectively

```
wget https://github.com/Dhatrireddy/circuits/blob/main/codes/3.5.cir
wget https://github.com/Dhatrireddy/circuits/blob/main/codes/3.4.py
```

Run the codes by executing

```
ngspice 3.5.cir
python3 3.4.py
```

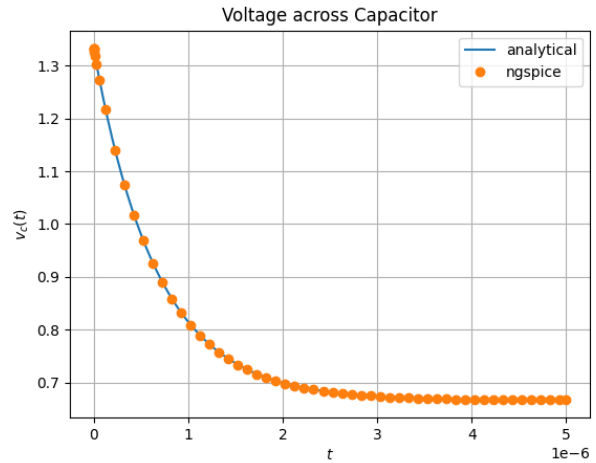


Fig. 3.5. Plot of  $v_c(t)$  after flipping the switch

3.6. Find  $v_c(0^-)$ ,  $v_c(0^+)$  and  $v_c(\infty)$

**Solution:** At  $t = 0^-$ , the switch still hasn't been switched to Q and the circuit is in steady state

$$v_c(0^-) = \frac{4}{3} \text{ V} \quad (3.12)$$

For  $t \geq 0$ , we can use the above formula

$$v_c(0^+) = \lim_{t \rightarrow 0^+} v_c(t) = \frac{4}{3} \text{ V} \quad (3.13)$$

$$v_c(\infty) = \lim_{t \rightarrow \infty} v_c(t) = \frac{2}{3} \text{ V} \quad (3.14)$$

3.7. Obtain Fig. 3.2 using the equivalent differential equation

**Solution:** Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (3.15)$$

where  $q(t)$  is the charge on the capacitor

On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (3.16)$$

But  $q(0^-) = \frac{4}{3}C_0$  and

$$q(t) = C_0 v_c(t) \quad (3.17)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (3.18)$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left( sC_0 V_c(s) - \frac{4}{3}C_0 \right) = 0 \quad (3.19)$$

$$\Rightarrow \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.20)$$

which is the same equation as the one we obtained from Fig. 3.2

#### 4. BILINEAR TRANSFORM

4.1. In Fig. 2.2, consider the case when  $S$  is switched to  $Q$  right in the beginning. Formulate the differential equation

**Solution:** The differential equation is the same as before

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (4.1)$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{dv_c}{dt} = 0 \quad (4.2)$$

but with a different initial condition

$$q(0^-) = q(0) = 0 \quad (4.3)$$

4.2. Find  $H(s)$  considering the output voltage at the capacitor

**Solution:** On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) - 0 = 0 \quad (4.4)$$

$$\Rightarrow V_c(s) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + sC_0 V_c(s) = \frac{V_2(s)}{R_2} \quad (4.5)$$

$$\Rightarrow \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (4.6)$$

The transfer function is thus

$$H(s) = \frac{\frac{1}{R_2 C_0}}{s + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \quad (4.7)$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \quad (4.8)$$

4.3. Plot  $H(s)$ . What kind of filter is it?

**Solution:** Download the following Python code that plots Fig. 4.7

```
wget https://github.com/Dhatireddy/circuits/blob/main/codes/4.3.py
```

Run the codes by executing

```
python 4.3.py
```

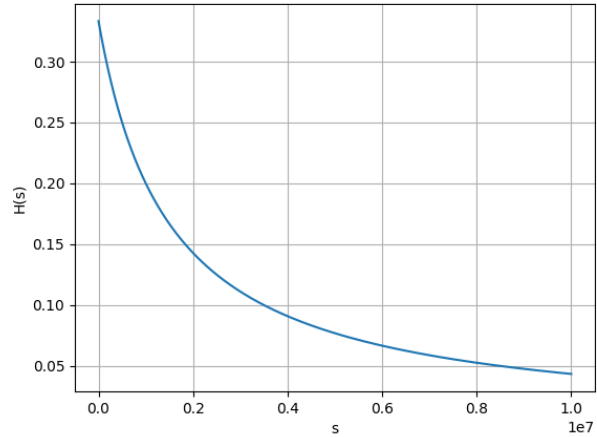


Fig. 4.7. Plot of  $H(s)$

Consider the frequency-domain transfer function by putting  $s = j\omega$

$$H(j\omega) = \frac{5 \times 10^5}{j\omega + 1.5 \times 10^6} \quad (4.9)$$

$$\Rightarrow |H(j\omega)| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \quad (4.10)$$

As  $\omega$  increases,  $|H(j\omega)|$  decreases

In other words, the amplitude of high-frequency signals gets diminished and they get filtered out

Therefore, this is a low-pass filter

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.11)$$

**Solution:**

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{dv_c}{dt} = 0 \quad (4.12)$$

$$\Rightarrow C_0 \frac{dv_c}{dt} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \quad (4.13)$$

$$\Rightarrow v_c(t)|_{t=n+1} = \int_n^{n+1} \left( \frac{2u(t) - v_c(t)}{R_2 C_0} - \frac{v_c(t)}{R_1 C_0} \right) dt \quad (4.14)$$

By the trapezoidal rule of integration

$$\int_a^b f(t)dt \approx \frac{b-a}{2}(f(a) + f(b)) \quad (4.15)$$

Consider  $y(t) = v_c(t)$

$$\begin{aligned} y(n+1) - y(n) &= \frac{1}{R_2 C_0} (u(n) + u(n+1)) \\ &- \frac{1}{2}(y(n+1) + y(n)) \left( \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0} \right) \end{aligned} \quad (4.16)$$

Thus, the difference equation is

$$\begin{aligned} y(n+1) &\left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= y(n) \left( 1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) \\ &+ \frac{1}{R_2 C_0} (u(n) + u(n+1)) \end{aligned} \quad (4.17)$$

4.5. Find  $H(z)$

**Solution:** Let  $\mathcal{Z}\{y(n)\} = Y(z)$

On taking the Z-transform on both sides of the difference equation

$$\begin{aligned} zY(z) &\left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= Y(z) \left( 1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) \\ &+ \frac{1}{R_2 C_0} \left( \frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}} \right) \end{aligned} \quad (4.18)$$

$$\begin{aligned} Y(z) &\left( z + \frac{z}{2R_1 C_0} + \frac{z}{2R_2 C_0} - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= \frac{1}{R_2 C_0} \frac{1+z}{1-z^{-1}} \end{aligned} \quad (4.19)$$

Also

$$v_2(t) = 2 \quad \forall t \geq 0 \quad (4.20)$$

$$\Rightarrow x(n) = 2u(n) \quad (4.21)$$

$$\Rightarrow X(z) = \frac{2}{1-z^{-1}} \quad |z| > 1 \quad (4.22)$$

Thus, the transfer function in  $z$ -domain is

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.23)$$

$$= \frac{\frac{1+z}{2R_2 C_0}}{z + \frac{z}{2R_1 C_0} + \frac{z}{2R_2 C_0} - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0}} \quad (4.24)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \quad (4.25)$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5 (1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.26)$$

with the ROC being

$$|z| > \max \left( 1, \left| \frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \right| \right) \quad (4.27)$$

$$\Rightarrow |z| > 1 \quad (4.28)$$

4.6. How can you obtain  $H(z)$  from  $H(s)$ ?

**Solution:** The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \quad (4.29)$$

where  $T$  is the step size of the trapezoidal rule (1 in our case)

This is known as the bilinear transform

Thus

$$H(z) = \frac{\frac{1}{R_2 C_0}}{2 \frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \quad (4.30)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 - z^{-1} + \left( \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) (1+z^{-1})} \quad (4.31)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \quad (4.32)$$

$$= \frac{2.5 \times 10^5 (1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.33)$$

which is the same as what we obtained earlier

4.7. Find  $y(n)$  from  $H(z)$  and verify whether  $y(n) = y(t)|_{t=n}$

**Solution:**

$$Y(z) = H(z)X(z) \quad (4.34)$$

$$= \frac{2.5 \times 10^5(1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \frac{2}{1 - z^{-1}} \quad (4.35)$$

$$= \frac{\frac{2}{3}}{1 - z^{-1}} - \frac{\frac{2}{3}}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.36)$$

On taking the inverse Z-transform by considering the ROC to be  $|z| > 1$ , we get

$$y(n) = \frac{2}{3}u(n) - \frac{2}{3} \left( -\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \right)^n u(n) \quad (4.37)$$

$$= \frac{2}{3} \left( 1 - \left( \frac{1 - 7.5 \times 10^5}{1 + 7.5 \times 10^5} \right)^n \right) u(n) \quad (4.38)$$

If we are sampling the signal at intervals of  $T \ll 10^{-5}$ , say  $10^{-7}$  s, i.e.,  $n = 10^{-7}, 2 \times 10^{-7}, \dots$

$$y(n) \approx \frac{2}{3} \left( 1 - \frac{1 - 7.5 \times 10^5 n}{1 + 7.5 \times 10^5 n} \right) u(n) \quad (4.39)$$

Now

$$Y(s) = H(s)X(s) \quad (4.40)$$

$$= \frac{5 \times 10^5}{s + 1.5 \times 10^6} \frac{2}{s} \quad (4.41)$$

$$= \frac{10^6}{1.5 \times 10^6} \left( \frac{1}{s} - \frac{1}{s + 1.5 \times 10^6} \right) \quad (4.42)$$

On taking the inverse Laplace transform by considering the ROC to be  $\Re(s) > 0$ , we get

$$y(t) = \frac{2}{3} \left( 1 - e^{-1.5 \times 10^6 t} \right) u(t) \quad (4.43)$$

But

$$e^{-1.5 \times 10^6 t} = \frac{e^{-0.75 \times 10^6 t}}{e^{0.75 \times 10^6 t}} \quad (4.44)$$

$$\approx \frac{1 - 7.5 \times 10^5 t}{1 + 7.5 \times 10^5 t} \quad \text{when } t \ll 10^{-6} \quad (4.45)$$

Therefore

$$y(t) \approx \frac{2}{3} \left( 1 - \frac{1 - 7.5 \times 10^5 t}{1 + 7.5 \times 10^5 t} \right) u(t) \quad (4.46)$$

$$\therefore y(n) = y(t)|_{t=n} \quad (4.47)$$

Download the following codes for simulation and plotting Fig. 4.7 respectively

```
wget https://github.com/Dhatrreddyy/
circuits/blob/main/codes/4.7.cir
wget https://github.com/Dhatrreddyy/
circuits/blob/main/codes/4.7.py
```

Run the codes by executing

```
ngspice 4.7.cir
python 4.7.py
```

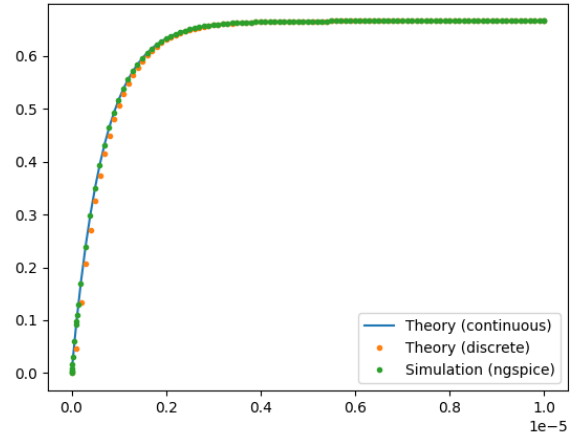


Fig. 4.7. Plots of  $y(t)$  and  $y(n)$

4.8. Find  $y(n)$  by solving the difference equation  
**Solution:**

$$y(n+1) \left( 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} \right) = y(n) \left( 1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0} \right) + \frac{1}{R_2C_0} (u(n) + u(n+1)) \quad (4.48)$$

For  $n \geq 0$ ,  $u(n) + u(n+1) = 2$

$$y(n+1) = y(n) \left( \frac{1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \right) + \frac{\frac{2}{R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \quad (4.49)$$

Let

$$a = \frac{1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} = -\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \quad (4.50)$$

$$b = \frac{\frac{2}{R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} = \frac{10^6}{7.5 \times 10^5 + 1} \quad (4.51)$$

Therefore, the difference equation is

$$y(n+1) = ay(n) + b \quad (4.52)$$

$$\implies y(n) = ay(n-1) + b \quad (4.53)$$

$$= a(ay(n-2) + b) + b \quad (4.54)$$

$$= a^2(ay(n-3) + b) + b(1+a) \quad (4.55)$$

On repeating this decomposition, we finally get

$$y(n) = a^n y(0) + b(1 + a + \dots + a^{n-1}) \quad (4.56)$$

$$= 0 + b \left( \frac{1 - a^n}{1 - a} \right) \quad (4.57)$$

$$y(n) = \frac{10^6}{7.5 \times 10^5 + 1} \times \frac{7.5 \times 10^5 + 1}{2 \times 7.5 \times 10^5} \times \left( 1 - \left( -\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \right)^n \right) \quad (4.58)$$

$$y(n) = \frac{10}{15} \left( 1 - \left( -\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \right)^n \right) \quad n \geq 0 \quad (4.59)$$

$$\therefore y(n) = \frac{2}{3} \left( 1 - \left( \frac{1 - 7.5 \times 10^5}{1 + 7.5 \times 10^5} \right)^n \right) u(n) \quad (4.60)$$

which is the same as what we obtained earlier