Limits and Continuity Practical - 1

NE

1. $\lim_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

Soln

$$\lim_{x \to a} \frac{a+2\sqrt{x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}}.$$

$$\lim_{x \to a} \frac{(a+2x-3x)\left(\sqrt{3a+x} + 2\sqrt{x}\right)}{(3a-3x)\left(\sqrt{a+2x} + \sqrt{3x}\right)}$$

$$\lim_{x \to a} \frac{(a-x)\left(\sqrt{3a+x} + 2\sqrt{x}\right)}{(3a-3x)\left(\sqrt{a+2x} + \sqrt{3x}\right)}$$

$$\frac{1}{3} \quad \frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{2\sqrt{a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\overset{2}{\cancel{4}}\, \cancel{\sqrt{a}}}{\cancel{2}\sqrt{3}\cancel{\sqrt{a}}}$$

$$= \frac{2}{3\sqrt{3}}$$

2. $$\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$$

Soln

$$\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right]$$

$$\lim_{y \to 0} \frac{a+y-a}{y\sqrt{a+y}\left(\sqrt{a+y} + \sqrt{a}\right)}$$

$$\lim_{y \to 0} \frac{\cancel{y}}{\cancel{y}\sqrt{a+y}\left(\sqrt{a+y} + \sqrt{a}\right)}$$

$$\frac{1}{\sqrt{a+0}\left(\sqrt{a+0} + \sqrt{a}\right)}$$

$$\frac{1}{\sqrt{a}\left(\sqrt{a} + \sqrt{a}\right)}$$

$$\frac{1}{\sqrt{a}\left(2\sqrt{a}\right)} = \frac{1}{2a}$$

3. $$\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$$

Soln

By Substituting $x - \frac{\pi}{6} = h.$

$$x = h + \frac{\pi}{6}$$

where $h \to 0.$

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{6}\right) - \sqrt{3}\sin\left(h+\frac{\pi}{6}\right)}{\pi - 6\left(h+\frac{\pi}{6}\right)}$$

Using

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\lim_{h\to 0} \frac{\cosh\cdot\cos\frac{\pi}{6} - \sin h\sin\frac{\pi}{6} - \sqrt{3}\sin h\cos\frac{\pi}{6} + \cos h\sin\frac{\pi}{6}}{\pi - \cancel{6}\left(\frac{6h+\pi}{\cancel{6}}\right)}$$

$\cos\frac{\pi}{6} = \cos 30^\circ$

$$\lim_{h\to 0} \frac{\cosh\cdot\frac{\sqrt{3}}{2} - \sin h\frac{1}{2} - \sqrt{3}\left(\sin h\frac{\sqrt{3}}{2} + \cosh\frac{1}{2}\right)}{\cancel{\pi} - 6h + \cancel{\pi}}$$

$\sin\frac{\pi}{6} = \sin 30^\circ$
$= \frac{1}{\sqrt{2}}$

$$\lim_{h\to 0} \frac{\cancel{\cos\frac{\sqrt{3}h}{2}} - \frac{\sin h}{2} - \sin\frac{3h}{2} + \cancel{\cos\frac{\sqrt{3}h}{2}}}{-6h}$$

$$\lim_{h\to 0} \frac{\cancel{-}\sin\frac{4h}{2}}{\cancel{-}6h}$$

$$\lim_{h\to 0} \frac{\cancel{\sin 4h}}{\cancel{3\cdot2}h}$$

$$\frac{1}{3}\lim_{h\to 0}\frac{\sin h}{h} = \frac{1}{3}\times 1 = \frac{1}{3}$$

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4. $\lim_{x\to\infty}\left[\dfrac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$

Soln:

By rationalizing Numerator & Denominator both

$$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\times\frac{\sqrt{x^2+5}+\sqrt{x^2-3}}{\sqrt{x^2+5}+\sqrt{x^2-3}}\times\frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2+1}}\right]$$

$$\lim_{x\to\infty}\left[\frac{(x^2+5-x^2+3)\left(\sqrt{x^2+3}+\sqrt{x^2+1}\right)}{(x^2+3-x^2-1)\left(\sqrt{x^2+5}+\sqrt{x^2-3}\right)}\right]$$

$$\lim_{x\to\infty}\frac{\overset{4}{\cancel{8}}\left(\sqrt{x^2+3}+\sqrt{x^2+1}\right)}{\cancel{2}\left(\sqrt{x^2+5}+\sqrt{x^2-3}\right)}$$

$$4\lim_{x\to\infty}\frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)}+\sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After applying limit we get,

$= 4.$

5. i] $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x \le \pi/2$

$= \dfrac{\cos x}{\pi - 2x}$, for $\dfrac{\pi}{2} < x < \pi$

at $x = \dfrac{\pi}{2}$

Soln:

(a) $f\left(\frac{\pi}{2}\right) = \dfrac{\sin 2\left(\frac{\pi}{2}\right)}{1-\cos 2\left(\frac{\pi}{2}\right)}$ $\therefore f\left(\frac{\pi}{2}\right) = 0$

f at $x = \dfrac{\pi}{2}$ define.

(b) $\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos x}{\pi - 2x}$

By Substituting Method $x - \frac{\pi}{2} = h$

$x = h + \pi/2$

where $h \to 0$.

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - 2\left(h+\frac{\pi}{2}\right)}$$

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - \cancel{2}\left(\frac{2h+\pi}{\cancel{2}}\right)}$$

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{-2h}$$ using $\cos(A+B) = \cos A\cos B - \sin A \sin B$.

$$\lim_{h\to 0} \frac{\cos h\cdot\cos\frac{\pi}{2} - \sin h\cdot\sin\frac{\pi}{2}}{-2h}$$

$$\lim_{h\to 0} \frac{\cos h\cdot 0 - \sin h}{-2h}$$

$$\lim_{h\to 0} \frac{-\sin h}{-2h}$$

$$\frac{1}{2}\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \frac{1}{2}.$$

(c) $\lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2^-} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$ Using $\sin 2x = 2\sin x \cdot \cos x$

$\lim_{x \to \pi/2^-} \frac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$

$\lim_{x \to \pi/2^-} \frac{2\cancel{\sin x} \cdot \cos x}{\sqrt{2}\cdot\cancel{\sin x}}$

$\lim_{x \to \pi/2^-} \frac{2\cos x}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cos x.$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ f is not continous at $x = \pi/2$.

ii) $f(x) = \frac{x^2-9}{x-3}$ $\quad 0 < x < 3$

$= x+3$ $\quad 3 \le x \le 6$

$= \frac{x^2-9}{x+3}$ $\quad 6 \le x < 9$

at $x=3$ & $x=6$

Soln:- $f(3) = \frac{x^2-9}{x-3} = 0$

f at $x=3$ define.

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+3$

a) $f(3) = x+3 = 3+3 = 6$

f is define at $x=3$.

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x+3) = 6$.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

$\therefore$ L.H.L = R.H.L

f is continous at $x=3$.

for $x=6$.

$f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{\cancel{27}^{3}}{\cancel{9}} = 3$

b) $\lim_{x\to 6^+} \frac{x^2-9}{x+3}$

$\lim_{x\to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$

$\lim_{x \to 6^+} (x-3) = 6-3 = 3$

$\lim_{x \to 6^-} x+3 = 3+6 = 9$

$\therefore$ L.H.L $\neq$ R.H.L

function is not continuous

6. i) $f(x) = \frac{1-\cos 4x}{x^2}$ $x < 0$
$= k$ $x = 0$ } at $x = 0$

Soln:

f is continuous at $x=0$

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = k$

$\lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = k$

$2\lim_{x \to 0} \frac{\sin^2 2x}{x^2} = k$

$2\lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^2 = k$

$2(2)^2 = k$

$\therefore$ $k = 8$

ii) $f(x) = (\sec^2 x)^{\cot^2 x}$ $x \neq 0$
$= k$ $x = 0$ } at $x = 0$

Soln:-

$f(x) = (\sec^2 x)^{\cot^2 x}$

Using

$\tan^2 x - \sec^2 x = 1$

$\therefore \sec^2 x = 1 + \tan^2 x$

&

$\cot^2 x = \dfrac{1}{\tan^2 x}$.

$$\therefore \lim_{x \to 0} (\sec^2 x)^{\cot^2 x}.$$

$$\lim_{x \to 0} \left(1 + \tan^2 x\right)^{\frac{1}{\tan^2 x}}.$$

We know that

$$\lim_{x \to 0} (1 + px)^{\frac{1}{px}} = e.$$

$\therefore = e$

$\therefore k = e.$

iii) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$ $\quad x \neq \frac{\pi}{3}$

$x = \pi/3$

at $x = \pi/2$

$x - \frac{\pi}{3} = h$

$x = h + \frac{\pi}{3}$

where $h \to 0$

$$f(\pi/3 + h) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\frac{\pi}{3} + h)}$$

$$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

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Using

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\lim_{h\to 0} \frac{\sqrt{3} - \dfrac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3}\tan h}}{\cancel{\pi} - \cancel{\pi} - 3h}$$

$$\lim_{h\to 0} \frac{\dfrac{\sqrt{3}\left(1 - \tan\frac{\pi}{3}\cdot\tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3}\cdot\tan h}}{-3h}$$

Using $\tan$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3}\times\sqrt{3}\cdot\tan h) - (\sqrt{3} + \tan h)}{1 - \tan\pi/3\cdot\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3\tan h) - (\sqrt{3} + \tan h)}{1 - \sqrt{3}\cdot\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\cancel{\sqrt{3}} - 3\tan h - \cancel{\sqrt{3}} - \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\cancel{-}4\tan h}{\cancel{-}3h\,(1 - \sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{4-\tan h}{3h\,(1-\sqrt{3}\tan h)}$$

$$\frac{4}{3}\lim_{h\to 0}\frac{\tan h}{h}\lim_{h\to 0}\frac{1}{(1-\sqrt{3}\tan h)}$$

$$\frac{\tan h}{h}=1$$

$$=\frac{4}{3}\cdot\frac{1}{1-\sqrt{3}(0)}$$

$$=\frac{4}{3}\left(\frac{1}{1}\right) = \frac{4}{3}.$$

7.

i. $f(x) = \dfrac{1-\cos 3x}{x\tan x}$ $\quad x \neq 0$ $\Bigg\}$ at $x=0$

$\quad = a \quad x=0$

$$f(x) = \frac{1-\cos 3x}{x\tan x}$$

$$\lim_{x\to 0} = \frac{2\sin^2 \frac{3}{2}x}{x\tan x}$$

$$\lim_{x\to 0} = \frac{\cancel{2}\,\frac{\sin^2\frac{3x}{2}}{x^2}\times x^2}{x\cdot\frac{\tan x}{x^2}\times x^2}$$

$$2\lim_{x\to 0}\frac{\left(\frac{3}{2}\right)^2}{1} = \frac{\cancel{2}\times 9}{\cancel{4}_2} = \frac{9}{2}$$

$$\lim_{x\to 0} f(x) = \frac{9}{2} \qquad 9 = f(0)$$

$\therefore$ f is not continous at $x=0$

Redefine function

$$f(x) = \begin{cases} \dfrac{1-\cos 3x}{x\tan x} & x \neq 0 \\ \dfrac{9}{2} & x = 0 \end{cases}$$

Now $\lim\limits_{x\to 0} f(x) = f(0)$

f has removable discontinuity at $x=0$

ii) $$f(x) = \left.\begin{aligned} &\frac{(e^{3x}-1)\sin x^{\circ}}{x^2} && x \neq 0 \\ &= \frac{\pi}{6} && x = 0 \end{aligned}\right\} \text{at } x=0$$

$$\lim_{x\to 0} \frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$$

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} \quad \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$\lim_{x\to 0} 3 \cdot \frac{e^{3x}-1}{3x} \quad \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$3\lim_{x\to 0} \frac{e^{3x}-1}{3x} \quad \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$f(i)$

$\cancel{3}\log_e \frac{\pi}{\cancel{180}_{60}} = \frac{\pi}{60} = f(0)$

f is continuous at $x=0$

8. $f(x) = \frac{e^{x^2} - \cos x}{x^2} \quad x=0$

is continuous at $x=0$

Soln:

$\therefore$ Given

f is continuous at $x=0$

$\lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$

$\lim_{x\to 0} \frac{(e^{x^2} - 1) + (1 - \cos x)}{x^2}$

$\lim_{x\to 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x\to 0} \frac{1 - \cos x}{x^2}$

$\log e + \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2}$

$\log e + 2 \lim_{x\to 0} \left(\frac{\sin x/2}{x}\right)^2$

Multiply with 2 on Numerator & Denominator

$= 1 + \cancel{2} \times \frac{1}{\cancel{4}\,2} = \frac{3}{2} = f(0)$

9. $f(x)$ $\dfrac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}$, $x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$$\lim_{x\to\pi/2} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$\lim_{x\to\pi/2} \frac{2-1+\sin x}{\cos^2 x\left(\sqrt{2}+\sqrt{1+\sin}\right)}$$

$$= \lim_{x\to\pi/2} \frac{1+\sin x}{1-\sin^2 x\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to\pi/2} \frac{\cancel{1+\sin x}}{(1-\sin x)\cancel{(1+\sin x)}\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to\pi/2} \frac{1}{(1-\sin x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$= \frac{1}{2\left(\sqrt{2}+\sqrt{2}\right)}$$

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Practical 2

Derivative

Q1 Show that the following function defined from IR to IR are differentiable.

i) cotx

$f(x) = \cot x$

$$f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x-a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x-a}$$

$$= \lim_{x \to 0} \frac{\tan a - \tan x}{(x-a)\tan x \tan a}$$

put $x-a=h$

$x = a+h$.

as $x \to a$ $h \to 0$.

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(\cancel{a}+h-\cancel{a})\tan(a+h)\tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h)\tan a}$$

formula $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$

$\tan A - \tan B = \tan(A-B)(1+\tan A \tan B)$.

$$= \lim_{h \to 0} \frac{(a-a-h) - (1+\tan a \tan(a+h))}{h \times \tan(a+h)\tan a}$$

$$\lim_{h\to 0} \frac{-\tan h}{h} \times \frac{1+\tan a \tan(a+h)}{\tan(a+h)\tan a}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a}$$

$$= \frac{-\sec^2 a}{\tan^2 a}$$

$$= \frac{-1}{\cancel{\cos^2} a} \times \frac{\cancel{\cos^2} a}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a.$$

$\therefore$ $Of(a) = -\cos^2 a.$

$\therefore$ f is differentiable $\forall a \in R$

ii) cosecx.

$f(x) = \operatorname{cosec} x$

$$Of(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x-a}$$

put $x-a=h$

$x = a+h$

as $x\to a$, $h\to 0$.

$$Df(h) = \lim_{h\to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \sin(a+h)}$$

Formula:

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

SN

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \sin(a+h)}$$

$$= \lim_{h\to 0} \frac{-\sin\frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \sin(a+h)}$$

$$= -\frac{1}{\cancel{2}} \times \frac{\cancel{2}\cos\left(\frac{2a+c}{2}\right)}{\sin(a+c)}$$

$$= \frac{-\cos a}{\sin^2 a} = -\cot a \ \operatorname{cosec} a$$

iii) $\sec x$.

$f(x) = \sec x$

$$Df(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x-a}$$

$$= \lim_{x\to a} \frac{\cos a - \cos x}{(x-a)\cos a \cos x}$$

put $x-a=h$

$x = a+h$

as $x\to a$, $h\to 0$

$$Df(h) = \lim_{x\to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cos(a+h)}$$

Formula: $-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

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$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{\cancel{a}-\cancel{a}-h}{2}\right)}{h \times \cos a \cos(a+h)}$$

$$\lim_{h \to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin\frac{-h}{2}}{\cos a \cos(a+h) \times \frac{-h}{2}} \times \frac{-1}{2}$$

$$= \frac{-1}{2} \times \frac{-2\sin\left(\frac{2a+c}{2}\right)}{\cos a \cos(a+c)}$$

$$= -\frac{1}{\cancel{2}} \times \cancel{2}\frac{\sin a}{\cos a \times \cos a}$$

$$= \tan a \sec a$$

Q2 If $f(x) = 4x+1$, $x \le 2$
$= x^2+5$, $x > 0$ at $x=2$, then find function is differentiable or not.

Soln :-

LHD :

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4\times 2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x+2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{(x-2)} = 4$$

$Df(2^-) = 4$

RHD :

$$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 2+2 = 4.$$

$Df(2^+) = 4$

RHD = LHD.

f. is differentiable at $x=2$.

Q3 If $f(x) = 4x+7$ $\quad x<3$.

$= x^2+3x+1$, $x \geq 3$ at $x=3$ then find f is differentiable or not.

Soln :-

R.H.D :

$$Df(+3) = \lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+3\times 3+1)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$Df(3^+) = 9$

L.H.D $= Df(3^-)$

$$= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x + 7 - 19}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x - 12}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$$

$Df(3^+) = 4.$

R.H.D $\neq$ L.H.D.

f is not differentiable at $x = 3$.

Q4 If $f(x) = 8x - 5$, $x \leq 2$.

$= 3x^2 - 4x + 7$; $x > 2$ at $x = 2$, Then

find f is differentiable or not.

Soln :-

$f(2) = 8 \times 2 - 5 = 16 - 5 = 11$

R.H.D:

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 3 \times 2 + 2 = 8$

$Df(2^+) = 8$

L.H.D

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 8$$

$Df(2^-) = 8$

L.H.D = R.H.D.

f is differentiable at $x = 3$.

20/12/19

Practical 3 — Application of Derivative

Q1

i) $f(x) = x^3 - 5x - 11$

$f'(x) = 3x^2 - 5$

f is increasing iff $f'(x) > 0$.

$3x^2 - 5 > 0$

$3x^2 > 5$

$x^2 > \frac{5}{3}$.

$x > \pm\sqrt{\frac{5}{3}}$

$x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

f is decreasing iff $f'(x) < 0$

$3x^2 - 5 < 0$

$3x^2 < 5$

$x^2 < \frac{5}{3}$

$x < \pm\sqrt{\frac{5}{3}}$

$+ \quad - \quad +$

$-\sqrt{\frac{5}{3}} \qquad \sqrt{\frac{5}{3}}$

$x \in \left(-\frac{\sqrt{5}}{\sqrt{3}}, \sqrt{\frac{3}{3}}\right)$

ii) $f(x) = x^2 - 4x$

$f(x) = 2x - 4$

f is increasing iff $f(x) > 0$

$\therefore \quad 2x - 4 > 0$

$2(x-2) > 0$

$x - 2 > 0$

$x > 2$

$\therefore \; x \in (2, \infty)$

$\therefore$ f is decreasing iff $f'(x) < 0$

$2x - 4 < 0$

$2(x-2) < 0$

$x - 2 < 0$

$x < 2$

$\therefore \; x \in (-\infty, 2)$.

iii) $f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

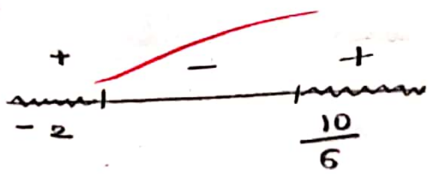
$\therefore$ f is increasing iff $f'(x) > 0$

$3x^2 + 2x - 20 > 0$

$6x^2 + 12x - 10x - 20 > 0$

$6x(x+2) - 10(x+2) > 0$

$(6x - 10)(x+2) > 0$.



$\therefore \; x \in (-\infty, -2) \cup \left(\frac{10}{6}, \infty\right)$

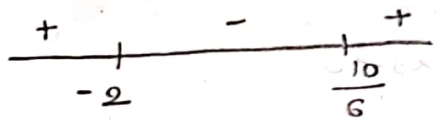
f is decreasing iff $f'(x) < 0$

$6x^2 + 2x - 20 < 0$

$6x^2 + 12x - 10x - 20 < 0$

$6x(x+2) - 10(x-2) < 0$

$(6x-10)(x+2) < 0$.



$\therefore x \in \left(-2, \frac{10}{6}\right)$

iv) $f(x) = x^3 - 27x + 5$

$= 3x^2 - 27$

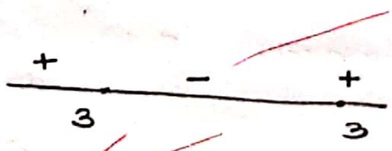
$= 3(x^2 - 9)$

f is increasing iff $f'(x) > 0$

$\therefore 3(x^2 - 9) > 0$

$x^2 - 9 > 0$

$(x-3)(x+3) > 0$



$x \in (-\infty, -3) \cup (3, \infty)$

f is decreasing iff $f'(x) < 0$

$3(x^2 - 9) < 0$

$x^2 - 9 < 0$

$(x-3)(x+3) < 0$

$+ \quad - \quad +$

$\quad -3 \qquad 3$

$\therefore \; x \in (-3, 3)$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = -24 - 18x - 24$

i.e $6x^2 - 18x - 24$

$6(x^2 - 3x - 4)$

f is increasing iff $f'(x) > 0$

$\therefore \; 6(x^2 - 3x - 4) > 0$

$x^2 - 3x - 4 > 0$

$x^2 - 4x + x - 4 > 0$

$x(x-4) + 1(x-4) > 0$

$(x+1)(x-4) > 0$

$+ \quad - \quad +$

$\quad -1 \qquad 4$

$\therefore \; x \in (-\infty, -1) \cup (4, \infty)$

f is decreasing ~~iff~~ $f'(x) < 0$.

$f \; 6(x^2 - 3x - 4) < 0$

$x^2 - 3x - 4 < 0$

$x^2 - 4x + x - 4 < 0$.

$x(x-4) + 1(x-4) < 0$

$(x+1)(x-4) < 0$

$+ \quad - \quad +$

$\quad 1 \qquad 4$

$x \in (1, 4)$

Q2

i) $y = 3x^2 - 2x^3$

Let,

$f(x) = y = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$f''(x) = 6 - 12x$

$= 6(1-2x)$

$f''(x)$ is concave upward iff,

$f''(x) > 0$

$6(1-2x) > 0.$

$1 - 2x > 0$

$-2x > -1$

$2x < 1$

$x < \frac{1}{2}$

$\therefore x \in \left(-\infty, \frac{1}{2}\right)$

$f''(x)$ is concave downwards iff

$f''(x) < 0$

$6(1-2x) < 0$

$1-2x < 0$

$-2x < -1$

$2x > 1$

$x > \frac{1}{2}$

$\therefore x \in \left(\frac{1}{2}, \infty\right)$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

Let

$f(x) = y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$\therefore f'(x) = 4x^3 - 18x^2 + 24x + 5$

$\therefore f''(x) = 12x^2 - 36x + 24$

$= 12(x^2 - 3x + 2)$

$f''(x)$ is concave upwords iff

$f''(x) > 0$

$12(x^2 - 3x + 2) > 0$

$x^2 - 3x + 2 > 0$

$x^2 - x - 2x + 2 > 0$

$x(x-1) - 2(x-1) > 0$

$(x-2)(x-1) > 0$

$\therefore x \in (-\infty, 1) \cup (2, \infty)$

$f''(x)$ is concave downward iff

$f''(x) < 0$

$12(x^2 - 3x + 2) < 0$

$x^2 - 3x + 2 < 0$

$x^2 - x - 2x + 2 < 0$

$x(x-1) - 2(x-1) < 0$

$(x-2)(x-1) < 0$

$\therefore x \in (1, 2)$

iii) $y = x^3 - 27x + 5$

Let,

$f(x) = y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$.

$f''(x)$ is concave upwards iff.

$f''(x) > 0$

$6x > 0$

$x > 0$

$\therefore x \in (0, \infty)$

$f''(x)$ is concave downward iff,

$f''(x) < 0$

$6x < 0$

$x < 0$

$\therefore x \in (-\infty, 0)$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

Let,

$f(x) = y = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = -24x - 18x + 6x^2$

$f''(x) = -18 + 12x.$

$f''(x)$ is concave upward iff,

$f''(x) > 0$

$-18 + 12x > 0$

$12x > 18$

$x > \frac{18}{12}$

$\therefore x \in \left(\frac{3}{2}, \infty\right)$

$f''(x)$ is concave downwards iff

$f''(x) < 0$

$-18 + 12x < 0$

$12x < 18$

$x < \frac{18}{12}$

$x \in \left(-\infty, \frac{3}{2}\right)$

4) $y = 2x^3 + x^2 - 20x + 4$

Let,

$f(x) = y = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

$= 2(6x+1)$

$f''(x)$ is concave upwards iff,

$f''(x) > 0.$

$2(6x+1) > 0$

$6x + 1 > 0$

$6x > 1$

$x > \frac{-1}{6}$

$x \in \left(\frac{1}{6}, \infty\right)$

$f''(x)$ is concave downward iff,

$f''(x) < 0$

$2(6x+1) < 0$

$6x+1 < 0$

$6x < -1$

$x < \frac{-1}{6}$

$\therefore x \in \left(-\infty, \frac{-1}{6}\right)$

28/12/19

Scanned with CamScanner

Practical 4

Q1

i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now Consider.

$f'(x) = 0$

$2x - \frac{32}{x^3} = 0.$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6.$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$.

$\therefore\ f(2) = 2^2 + \frac{16}{2^2}$

$= 4 + \frac{16}{4}$

$= 4 + 4$

$= 8.$

$f''(-2) = 2 + \frac{96}{-2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = -2$

$\therefore$ Function reaches minimum values at $x = 2$ & $x = -2$.

$f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

Consider,

$f'(x) = 0$

$-16x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$\therefore\ f''(x) = -30x + 60x^3.$

$f(1) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minimum value at $x = 1$.

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$f''(-1) = 30(-1) + 60(-1)^3$

$= 30 - 60$

$= 1$

$f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0.$

$\therefore$ f has maximum value at $x = 1$.

$f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 5$

$\therefore$ f has the maximum value 5 at $x = -1$ and has the minimum value 1 at $x = 1$.

iii) $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

Consider

$f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$3x(x - 2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$\therefore x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore$ f has maximum value at $x = 0$.

$\therefore f(0) = (0)^3 - 3(0)^2 + 1$

$= 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore$ f has minimum value at $x = 2$.

$f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore$ f has maximum value 1 at $x = 0$ and

f has minimum value -3 at $x = 2$.

iv)

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

Consider,

$f'(x) = 0$

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0.$

$\therefore x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6.$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0.$

$\therefore f$ has minimum value at $x = 2.$

$\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19.$

$f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore f$ has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1$

$= 8.$

$\therefore$ f has maximum value 8 at $x = -1$ &

f has minimum value -19 at $x = 2$.

Q2

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = 0 + \frac{9.5}{55}$$

$\therefore x_1 = 0.1727.$

$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= -0.0829.$

$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0895 - 1.0362 - 55$

$= -55.9467.$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{0.0829}{55.9467}$$

$$= 0.1712$$

$$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$$

$$= 0.0050 - 0.0879 - 9.416 + 9.5$$

$$= 0.0011$$

$$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$$

$$= 0.0879 - 1.0272 - 55$$

$$= -55.9393$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.1712 + \frac{0.0011}{55.9393}$$

$$= 0.1712$$

$\therefore$ The root of the equation is 0.1712.

ii) $f(x) = x^3 - 4x - 9$

$f'(x) = 3x^2 - 4$

$f(2) = 2^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9$

$f(3) = 3^3 - 4(3) - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$ be the initial approximation

$\therefore$ By Newton's Method.

$x_{n+1} = x_n - f(x_n)/f(x_n)$

$x_1 = x_0 - f(x_0)/f'(x_0)$

$= 3 - 6/23.$

$= 2.7392.$

$f(x_1) = (2.7392)^3 - 4(2.7392) - 9.$

$= 20.5528 - 10.9568 - 9$

$= 0.596.$

$f'(x_1) = 3(2.7392)^2 - 4.$

$= 22.5096 - 4$

$= 18.5096.$

$x_2 = x_1 - f(x_1)/f'(x_1)$

$= 2.7392 - 0.596 / 18.5096$

$= 2.7071.$

$f(x_2) = (2.7071)^3 - 4(2.7071)$

$= 19.8386 - 10.8284$

$= 0.0102$

$f'(x_2) = 3(2.7071)^2 - 4$

$= 21.9851 - 4$

$= 17.9851$

$= 2.7071 - \dfrac{0.0102}{17.9851}$

$= 2.7071 - 0.0056 = \underline{\underline{2.7015}}$

$f(x_3) = (2.7015)^3 - 4(2.7015) - 9.$

$= 19.7158 - 10.806 - 9 = -0.0901$

$f'(3) = 3(2.7015)^2 - 4 = 21.8943 - 4 = 17.8943$

$x_4 = 2.7015 + 0.0901 / 17.8943 = 2.7015 + 0.0050.$

$= \cancel{20}\ 2.7065.$

3) $f(x) = x^3 - 1.8x^2 - 10x + 17 \quad [1, 2]$

$f'(x) = 3x^2 - 3.6x - 10.$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$

$= -1.8 - 10 + 17$

$= 6.2.$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17.$

$= 8 - 7.2 - 20 + 17 = -2.2.$

Let $x_0 = 2$ be initial approximation By Newton's Method.

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$x_1 = x_0 - f(x_0) / f'(x_0)$

$= 2 - 2.2 / 5.2.$

$= 2 - 0.4230 = 1.577.$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - 4.4764 - 15.77 + 17$

$= 0.6755.$

$$f'(x) = 3(1.577)^2 - 3.6(1.577) - 10.$$
$$= 7.4608 - 5.6772 - 10.$$
$$= -8.2164.$$

$$x_2 = x_1 - f(x_1)/f'(x_1).$$
$$= 1.577 + 0.6755/8.2164$$
$$= 1.577 + 0.0822$$
$$= 1.6592$$

$$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$$
$$= 4.5677 - 4.9553 - 16.592 + 17.$$
$$= 0.0204$$

$$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$$
$$= 8.2588 - 5.97312 - 10.$$
$$= -7.7143$$

$$x_3 = x_2 - f(x_2)/f'(x_2)$$
$$= 1.6592 + 0.0204/7.7143$$
$$= 1.6592 + 0.0026$$
$$= 1.6618$$

$$f(x_3) = (1.66(8)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$$
$$= 4.5892 - 4.9708 - 16.618 + 17$$
$$= 0.0004$$

$$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$$
$$= 8.2847 - 5.9824 - 10$$
$$= -7.6977$$

$$x_4 = x_3 - f(x_3)/f'(x_3)$$
$$= 1.6618 + \frac{0.0004}{7.6977}$$
$$= 1.6618$$

Practical 5

Integration

Solve the following integration.

i) $$\frac{dx}{\sqrt{x^2+2x-3}}$$

$$= \int \frac{1}{x^2+2x-3} \, dx.$$

$$= \int \frac{1}{\sqrt{x^2+2x-3}} \, dx$$

$$= \int \frac{1}{\sqrt{x^2+2x+1-4}}$$

$a^2 + 2ab + b^2 = (a+b)^2$

$$= \int \frac{1}{\sqrt{(x+1)^2-4}} \, dx.$$

Substitute

put $x+1 = t$

$dx = \frac{1}{t} \times dt$ where $t=1$ $t = x+1$

$$\int \frac{1}{\sqrt{t^2-4}} \, dt$$

using

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(|x+\sqrt{x^2-a^2}|\right)$$

$$= \ln\left(|t+\sqrt{t^2-4}|\right)$$

$t = x+1$

$$= \ln\left(|x+1+\sqrt{(x+1)^2-4}|\right)$$

$$= \ln\left(|x+1+\sqrt{x^2+2x-3}|\right).$$

$$= \ln\left(|x+1+\sqrt{x^2+2x-3}|\right)+c$$

2) $\int (4e^{3x}+1)\,dx$

$$\int 4e^{3x}\,dx + \int 1\,dx$$

$$= 4\int e^{3x}\,dx + \int 1\,dx \quad \# \int e^{dx}dx = \frac{1}{d} \times e^{dx}$$

$$= \frac{4e^{3x}}{3} + x.$$

$$= \frac{4e^{3x}}{3} + x + c$$

3. $\int 2x^2 - 3\sin(x) + 5\sqrt{x}\,dx$

$$= \int 2x^2 - 3\sin(x) + 5x^{1/2}\,dx$$

$$= \int 2x^2\,dx - \int 3\sin(x)\,dx + \int 5x^{1/2}\,dx$$

$$\frac{2x^3}{3} + 3\cos x + \frac{10x\sqrt{x}}{3} + c.$$

$$= \frac{2x^3+10x\sqrt{x}}{3} + 3\cos x + c.$$

4. $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx.$

$I = \int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$= \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{4}{\sqrt{x}}\right) dx$

$= \int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\right) dx$

$= \int \left(x^3 \cdot x^{-1/2} + 3x \cdot x^{-1/2} + 4x^{-1/2}\right) dx$

$= \frac{2}{7} x^{7/2} + 3 \cdot \frac{2}{3} x^{3/2} + 4x^{1/2} \cdot 2 + c.$

$= \frac{2}{7} x^{7/2} + 2x^{3/2} + 8\sqrt{x} + c.$

v) $\int t^7 \sin(2t^4)\,dt$

$I = \int t^7 \sin(2t^4)\,dt$

Let $t^4 = x$,

$4t^3 dt = dx.$

$I = \frac{1}{4}\int 4t^3 . t^4 \sin(2t^4)\,dt.$

$= \frac{1}{4}\int x. \sin(2x)\,dx.$

$= \frac{1}{4}\left[x\cancel{\int \sin 2x} - \int\left[\int \sin 2x \cdot \frac{d}{dx}(x)\right]\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x \cdot 1\right]$

$$= \frac{1}{4}\left[\frac{-x\cos 2x}{2} + \frac{1}{4}\sin 2x\right] + c$$

$$= -\frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x + C.$$

Resubstituting $x = t^4$.

$$\therefore I = -\frac{1}{8}t^4\cos(2t^4) + \frac{1}{6}\sin(2t^4) + c$$

6) $\int \sqrt{x}\,(x^2-1)\,dx$

$$I = \int \sqrt{x}\,(x^2-1)\,dx$$

$$= \int \left(\sqrt{x}.x^2 - \sqrt{x}\right)dx$$

$$= \int x^{5/2}dx - \int \sqrt{x}\,dx$$

$$= \frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2} + c$$

7) $\frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

$$I = \int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$$

Let,

$$\frac{1}{x^2} = t$$

$$\therefore x^{-2} = t$$

$$\therefore \frac{-2}{x^3}dx = dt$$

$$I = \frac{1}{-2}\int \frac{-2}{x^3}\sin\left(\frac{1}{x^2}\right)dx$$

$$= \frac{-1}{2}\int \sin t\, dt.$$

$$= \frac{-1}{2}\left[-\cos t\right] + c$$

$$= \frac{1}{2}\cos t + c$$

Resubstituting $t = \frac{1}{x^2}$

$$I = \frac{1}{2}\cos\left(\frac{1}{x^2}\right) + C$$

8 $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx$

$$I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx$$

let $\sin x = t$

$\cos x\, dx = dt$

$$I = \int \frac{dt}{\sqrt[3]{t^2}}$$

$$I = \int \frac{dt}{t^{2/3}}$$

$$= \int t^{-2/3}\, dt$$

$$= 3t^{1/3} + c.$$

$$= 3(\sin x)^{1/3} + c.$$

$$= 3\sqrt[3]{\sin x} + c$$

9 (x) $\int e^{\cos^2 x} \sin 2x\, dx$

$$I = \int e^{\cos^2 x} \sin 2x\, dx$$

Let $\cos^2 x = t$

$$-2\cos x \sin x\, dx = dt$$

$$-\sin 2x\, dx = dt.$$

$$I = -\int -\sin 2x\; e^{\cos^2 x}\, dx.$$

$$= -\int e^{t}\, dt.$$

$$= -e^{t} + c.$$

Resubstituting $t = \cos^2 x$.

$$I = -e^{\cos^2 x} + c.$$

19. $\int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$

$$I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$$

Let

$x^3 - 3x^2 + 1 = t$

$(3x^2 - 6x)\,dx = dt$

$3(x^2 - 2x)\,dx = dt$

$(x^2 - 2x)\,dx = \frac{dt}{3}$

$\therefore I = \int \frac{1}{t} \frac{dt}{3}$

$\frac{1}{3} \int \frac{dt}{t}$

$\frac{1}{3} \log t + c.$

Resubstituting $t = x^3 - 3x^2 + 1$,

$I = \frac{1}{3} \log (x^3 - 3x^2 + 1) + c$

AK
03/01/2020

Practical 6 — Application of integration & Numerical integration.

Q1. Find the length of the following curve.

1. $x = t\sin t$, $y = 1-\cos t$, $t \in [0, 2\pi]$

Soln:-

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = t - \sin t$ $\quad \therefore \frac{dx}{dt} = 1-\cos t.$

$y = 1-\cos t$ $\quad \therefore \frac{dy}{dt} = 0 - (-\sin t) = \sin t.$

$$L = \int_0^{2\pi} \sqrt{(1-\cos t + \sin t)^2 + (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1-2\cos t + 1}\, dt$$

$$= \int_0^{2\pi} \sqrt{2-2\cos t}\, dt = \int_0^{2\pi} \sqrt{2(1-\cos t)} \quad \int_0^{2\pi} \sqrt{2 \times 2\sin^2 t/2}$$

$$= \sqrt{4\sin^2 t/2}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{t}{2}\right| dt \quad \therefore \sin^2\frac{t}{2} = \frac{1-\cos t}{2}.$$

$$= \int_0^{2\pi} 2\sin\frac{t}{2}\, dt.$$

$$= \left(-4\cos\left(\frac{t}{2}\right)\right)_0^{2\pi} - (-4\cos\pi) - (-4\cos 0)$$

$= 4+4$

$= 8$

2. $y = \sqrt{4 - x^2} \qquad x \in [-2, 2]$

Soln :-

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$y = \sqrt{4 - x^2} \quad \therefore \frac{dy}{dx} = 2\int_0^2 1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2 dx.$$

$$= 2\int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

$$= 4\int_0^2 \frac{1}{\sqrt{4 - x^2}}\, dx.$$

$$= 4\left(\sin^{-1} x/2\right)_0^2$$

$$= 2\pi.$$

3. $y = x^{3/2}$ in $[0, 4]$

Soln :-

$$f'(x) = \frac{3}{2} x^{1/2}$$

$$[f'(x)]^2 = \frac{9}{4} x.$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$\int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx$$

$$\int_0^4 \sqrt{\frac{4+9x}{4}}\, dx$$

$$= \frac{1}{2} \int_0^4 \sqrt{4+9x}\, dx$$

$$= \frac{1}{2} \left[\frac{(4+9x)^{1/2+1}}{1/2+1} \right]_0^4 dx$$

$$= \frac{1}{27} \left[(4+9x)^{3/2} \right]_0^4$$

$$= \frac{1}{27} \left[(4+0)^{3/2} - (4+36)^{3/2} \right]$$

$$= \frac{1}{27} \left[(4)^{3/2} - (40)^{3/2} \right]$$

4 ii) $x = 3\sin t \qquad y = 3\cos t$

$\frac{dx}{dt} = 3\cos t \qquad \frac{dy}{dt} = -3\sin t$

$$L = \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(1)}\, dt$$

$$= \int_0^{2\pi} 3\, dt$$

$$= 3 \int_0^{2\pi} dt$$

$$= 3 [x]_0^{2\pi}$$

$$= 3(2\pi - 0)$$

$$= 6\pi$$

5. $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1, 2]$.

Soln:

$$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} = \frac{y^4-1}{2y^2}$$

$$= \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y)^2}}\, dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$$

$$= \frac{1}{2}\int_1^2 y^2 dy + \frac{1}{2}\int_1^2 y^{-2} dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right] = \frac{17}{12}$$

Q2 Using simpson Rule Solve the following.

1. $\int_0^2 e^{x^2} dx$ with $n = 4$

Soln :-

$a = 0$, $b = 2$, $n = 4$

$$h = \frac{2-0}{4} = \frac{1}{2} = 0.5.$$

x	0	0.5	1	1.5	2
y	1	1.2840	2.7182	9.4877	54.5981
	y_0	y_1	y_2	y_3	y_4

By Simpson's Rule

$$\int_0^2 e^{x^2}\,dx = \frac{0.5}{3}\,(1+54.5981)+4\,(1.2840+9.4877) + 2\,(2.7182+54.5981)$$

$$= \frac{0.5}{3}\left[55.5981 + 43.0868 + 114.6326\right]$$

$$= 1.17779$$

2. $\int_0^4 x^2\,dx$

$$L = \frac{4-0}{4} = \cancel{0}\ 1$$

x	0	1	2	3	4
y	0	1	4	9	16

$$\int_0^4 x^2\,dx = \frac{1}{3}\left[16+4(10)+8\right]$$

$$= 64/3.$$

$$\int_0^4 x^2\,dx = 21.533$$

88

3. $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ with $n = 6$.

Soln :-

$$L = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$6\pi/18$	$8\pi/18$
y	0	0.4166	0.58	0.70	0.80687	0.8727	0.9908

$$\int_0^{\pi/3} \sqrt{\sin x} = \pi/54 \times 12.1163$$

$$\int_0^{\pi/3} \sqrt{\sin x}\, dx = 0.7049.$$

Practical 7

Differential Equation.

1. $$x\frac{dy}{dx}+y = e^x$$

$$\frac{dy}{dx}+\frac{1}{x}y = \frac{e^x}{x}$$

$$(x) = \frac{1}{x}$$

$$IF = e^{\int(x)}dx$$

$$= e^{\int 1/x}dx$$

$$= e^{\ln x}$$

$$IF = x$$

$$y(IF) = \int Q(x)\,(IF)\,dx+c$$

$$= \int\frac{e^x}{x}\,x\cdot dx+c$$

$$= \int e^x dx+c$$

$$xy = e^x+c.$$

2. $$e^x\frac{dy}{dx}+2e^x y = 1$$

$$\frac{dy}{dx}+\frac{2e^x}{e^x}y = \frac{1}{e^x} \quad (\div \text{ by } e^x)$$

$$\frac{dy}{dx}+2y = \frac{1}{e^x}$$

$$\frac{dy}{dx}+2y = e^x.$$

$P(x) = 2 \qquad Q(x) = e^{-x}$

$\int P(x)\,dx$

$IF = e^{\int 2dx}$

$= e^{2x}$

$y(IF) = \int Q(x)(IF)\,dx + c$

$y \cdot e^{2x} \int e^{x} + 2x\,dx + c.$

$= \int e^{x} dx + c$

$y \cdot e^{2x} = e^{x} + c.$

3. $x\frac{dy}{dx} = \frac{\cos x}{x} \cdot 2y.$

$x\frac{dy}{dx} = \frac{\cos x}{x} \cdot 2y$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}.$

$P(x) = 2(x) \qquad Q(x) = \frac{\cos x}{x^2}$

$IF = e^{\int P(x)\,dx}$

$\cancel{e^{\int 2/x\,dx}}.$

$= \ln x^2$

$y(IF) = \int Q(x)(IF)\,dx + c.$

$= \int \cos x + c$

$x^3 y = \sin x + c$

4. $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$

$\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{\sin x}{x^3}$

$P(x) = 3/x \quad Q(x) = \sin x / x^3$

$= e^{\int P(x)\,dx}$

$= e^{\int 3/x\,dx}$

$= e^{3/x\,dx}$

$= e^{\ln x^3}$

$IF = x^3$.

$y\,(IF) = \int Q(x)\,(IF)\,dx + c$.

$= \int \dfrac{\sin x}{x^3} \cdot x^3\,dx + c$

$= \int \sin x\,dx + c$

$x^3 y = -\cos x + c$

5. $e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = 2x$

$\dfrac{dy}{dx} + 2y = \dfrac{2x}{e^{2x}}$

$P(x) = 2 \quad Q(x) = 2x/e^{2x} = 2x^{-2x}$.

$IF = e^{\int P(x)\,dx}$

$= e^{\int 2\,dx} = e^{2x}$

99

$$y(IF) = \int Q(x)(IF)\,dx + c.$$

$$= \int 2x e^{-2x} e^{2x}\,dx + c$$

$$ye^{2x} = x^2 + c$$

6. $\sec^2x \cdot \tan y\,dx + \sec^2y\,\tan x\,dy = 0$

$$\sec^2x \cdot \tan y\,dx = -\sec^2y \cdot \tan x\,dy$$

$$\frac{\sec^2x}{\tan x}\,dx = \frac{-\sec^2y}{\tan y}\,dy$$

$$\int \frac{\sec^2x\,dx}{\tan x} = -\int \frac{\sec^2y}{\tan y}\,dy$$

$$\log|\tan x| = -\log|\tan y| + c$$

$$\log|\tan x \cdot \tan y| + c$$

$$\tan x \cdot \tan y = e^c$$

7. $\frac{dy}{dx} = \sin^2(x-y+1)$.

put $x-y+1 = u$

Differentiating BTS

$$x-y+1 = u$$

$$1 - \frac{dy}{dx} = \frac{du}{dx}$$

$$1 - \frac{du}{dx} = dy/dx$$

$$1 - \frac{du}{dx} = \sin^2 u$$

$$\frac{dv}{dx} = 1-\sin^2 v.$$

$$\frac{dv}{dx} = \cos^2 v$$

$$\frac{dv}{\cos^2 v} = dx$$

$$\int \sec^2 v \, dv = \int dx$$

$$\tan v = x+c$$

$$\tan(x+y-1) = x+c.$$

8. $$\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$$

Put $2x+3y = v$

$$2+3\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} \cdot 2\right)$$

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\frac{(v-1)}{(v+2)}$$

$$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$$

$$= \frac{2v+3}{v+2}$$

67

$$\frac{3(v+1)}{v+2}$$

$$\int\left(\frac{v+2}{v+1}\right)dv = 3dx.$$

$$\int \frac{v+1}{v}\,dx + \int \frac{1}{v+1}\,dv = 3x$$

$$v + \log|x| = 3x + c$$

$$2x + 3y + \log|2x+3y+1| = 3x + c$$

$$3y = x - \log|2x+3y+1| + c$$

10/01/2020

Practical No. 8

1. $\frac{dy}{dx} = y + e^x - 2$

$f(x, y) = y + e^x - 2$, $y_0 = 2$, $x_0 = 0$, $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.57435
2	1	3.5743	4.2925	5.3615.

$y_{n+1} = y_n + h f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
3	1.5	5.3615	7.8431	9.28305
4	2	9.2831		

$\therefore$ By Euler's formula,
$y(2) = 9.2831$.

2. $\frac{dy}{dx} = 1+y^2$

$f(x,y) = 1+y^2$, $y_0 = 0$, $x_0 = 0$, $h = 0.2$

Using Euler's iteration formula.

$y_{n+1} = y_n + h\, f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1665	0.6413
3	0.6	0.6413	1.4113	0.9236
4	0.8	0.9236	1.8630	1.2942
5	1	1.2942		

∴ By Euler's formula
$y(1) = 1.2942.$

3. $\frac{dy}{dx} \sqrt{\frac{x}{y}}$ $y(0) = 1$ $x_0 = 0$ ~~x~~ $h = 0.2$

Using Euler's iteration formula.

$$y_{n+1} = y_n + h f(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	0
1	0.2	0		
2	0.4			
3	0.6			
4	0.8			
5	1			

07

4. $\frac{dy}{dx} = 3x^2+1$ $y_0 = 2$, $x_0 = 1$, $h = 0$

for $h = 0.5$

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n,y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	49	28.5
2	2	28.5		

By Euler's Formula

$y(2) = 28.5$

For $h = 0.25$

n	x_n	y_n	$f(x_n,y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4219
2	1.5	4.4219	7.75	6.3594
3	1.75	6.3594	10.1815	8.9048.
4	2	8.9048		

By Euler's formula

$y(2) = 8.9048.$

5) $\frac{dy}{dx} = \sqrt{xy} + 2$ $y_0 = 1$, $x_0 = 1$, $h = 0.2$

Using Euler's iteration formula

$y_{n+1} = y_n + h f(x_n ; y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	1.6
1	1.2	1.6		

∴ By Euler's formula,

$y(1.2) = 1.6$

13/01/2020

Practical 9 Limit & Partial Order derivatives.

Q1

i) $$\lim_{(x,y)\to(-4,1)} \frac{x^3-3y+y^2-1}{xy+5}$$

At $(-4,-1)$ Denominator $\neq 0$

$\therefore$ By applying limit

$$= \frac{(-4)^3-3(-1)+(-1)^2-1}{-4(-1)+5}$$

$$= \frac{-64+3+1-1}{4+5}$$

$$= \frac{-61}{9}$$

ii) $$\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$$

At $(2,0)$, Denominator $\neq 0$

$\therefore$ By applying limit,

$$= \frac{(0+1)\left((2)^2+0-4(2)\right)}{2+0}$$

$$= \frac{1(4+0-8)}{2}$$

$$= \frac{-4}{2}$$

$$= -2$$

iii) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2 \cdot z^2}{x^3-x^2\, yz}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz} = \frac{(x-yz)(x+yz)}{x^2(x-yz)}$

Apply limit

$= \frac{(1)^2+(1)^2(1)}{\cancel{(1)^3}-(1)^2(1)\cancel{(1)}}$

$\cancel{\frac{1+1}{1-1}} = \cancel{0}\ 2$

$= \frac{1+1(1)}{(1)^2} = 2$

Q2 $f(x,y) = xy\,e^{x^2+y^2}$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(xy\,e^{x^2+y^2})$

$= ye^{x^2+y^2}(2x)$

$\therefore f_x = 2xy\,e^{x^2+y^2}$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(xy\,e^{x^2+y^2})$

$= xe^{x^2+y^2}(2y)$

$\therefore f_y = 2yxe^{x^2+y^2}$

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ii) $f_x = e^x \cos y$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(e^x \cos y)$

$f_y = -e^x \sin y$

iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(x^3y^2 - 3x^2y + y^3 + 1)$

$\therefore f_x = 3x^2y^2 - 6xy$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$\frac{\partial}{\partial y}(x^2y^2 - 3x^2y^2 + y^3 + 1)$

$2x^3y - 3x^2 + 3y^2$

Q3

i) $f(x,y) = \dfrac{2x}{1+y^2}$

$$f_x = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{2+2y^2-0}{(1+y^2)^2}$$

$$= \frac{2\cancel{(1+y^2)}}{\cancel{(1+y^2)}(1+y)^2}$$

$$= \frac{2}{1+y^2}$$

At $(0,0)$

$$= \frac{2}{1+0}$$

$$= 2$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{1+y^2(0) - 2x(2y)}{(1+y^2)^2}$$

$$= \frac{-4xy}{(1+y^2)^2}$$

At $(0,0)$

$$= \frac{-4(0)(0)}{(1+0)^2}$$

$$= 0$$

Q4

1. $f(x,y) = \frac{y^2-xy}{x^2}$

$$f_x = \frac{x^2 \frac{\partial}{\partial x} y^2-xy) \frac{\partial}{\partial x}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(-y) - (y^2-xy)(2x)}{x^4}$$

$$= \frac{x^2(+y) - 2x}{x^4}(y^2 - xy)$$

$$f_y \quad = \frac{2y - x}{x^2}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-x^2y - 2x(y^2 - xy)}{x^4}\right)$$

$$= \frac{x^4\left(\frac{\partial}{\partial x} -x^2y - 2xy^2 + 2x^2y\right) - -x^2y - 2xy + 2x^2y)\frac{\partial}{\partial x}(x^4)}{(x^4)^2}$$

$$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x^3(-x^2y - 2xy + 2x^2y)}{x^6} \quad — \quad 1$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{2-0}{x^2} = \frac{2}{x^2} \quad — \quad 2$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{-x^2y - 2xy^2 + 2x^2y}{x^4}\right) \quad 3$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{x^2\frac{d}{dx}(2y-x) - (2y-x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2 - 4xy + 2x^2}{x^4} \quad — \quad 4$$

From 3 & 4

$$f_{xy} = f_{xx}$$

ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$f_x = \frac{\partial}{\partial x}\left(x^3 + 3x^2y - \log(x^2+1)\right)$

$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$

$f_y = \frac{\partial}{\partial y}\left(x^2 + 3x^2y^2 - \log(x^2+1)\right]$

$= 0 + 6x^2y - 0$

$= 6x^2y$

$f_{xx} = 6x + 6y^2 - \left(\frac{x^2+1 \frac{\partial(2x)}{\partial x} - 2x \frac{\partial(x^2+1)}{\partial x}}{(x^2+1)^2}\right)$

$= 6x + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2}\right)$ — 1

$f_{yy} = \frac{\partial}{\partial y}(6x^2y)$

$= 6x^2$ — 2

$f_{xy} = \frac{\partial}{\partial y}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 0 + 12xy - 0$

$= 12xy$ — 3

$f_{yx} = \frac{\partial}{\partial x}(6x^2y)$

$= 12xy$ — 4

From 3 & 4

$f_{xy} = f_{yx}$

iii) $f(x,y) = \sin(xy) + e^{x+y}$

$f_x = y\cos(xy) + e^{x+y}(1)$

$= y\cos(xy) + e^{x+y}$

$f_y = x\cos(xy) + e^{x+y}(1)$

$= x\cos(xy) + e^{x+y}$

$f_{xx} = \frac{\partial}{\partial x} y\cos(xy)(y) + e^{x+y}\cancel{(1)}$ —— ①

$= -y\sin(xy)\cdot(y) + e^{x+y}(1)$

$= -y^2\sin(xy) + e^{x+y}$ —— ①

$f_{yy} = \frac{\partial}{\partial y}\left(x\cos(xy) + e^{x+y}\right)$

$= -x\sin(xy)(x) + e^{x+y}(1)$

$= x^2\sin(xy) + e^{x+y}$ —— ②

$f_{xy} = \frac{\partial}{\partial y}\left(y\cos(xy) + e^{x+y}\right)$

$= -y^2\sin(xy) + \cos(xy) + e^{x+y}$ —— ③

$f_{yx} = \frac{\partial}{\partial x}\left(x\cos(xy) + e^{x+y}\right)$

$= -x^2\sin(xy) + \cos(xy) + e^{x+y}$ —— ④

From ③ & ④

$f_{xy} \neq f_{yx}$

Q5

i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$f(1,1) = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$f_x = \dfrac{1}{2\sqrt{x^2+y^2}} \cdot (2x)$

$= \dfrac{x}{\sqrt{x^2+y^2}}$ $\quad f_x$ at $(1,1) = \dfrac{1}{\sqrt{2}}$

$f_y = \dfrac{1}{2\sqrt{x^2+y^2}} (2y)$

$= \dfrac{y}{\sqrt{x^2+y^2}}$ $\quad f_y$ at $(1,1) = \dfrac{1}{\sqrt{2}}$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \dfrac{1}{\sqrt{2}}(x-1) + \dfrac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \dfrac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \dfrac{1}{\sqrt{2}}(x+y-2)$

$= \cancel{\sqrt{2}} + \dfrac{1}{\sqrt{2}}x + \dfrac{1}{\sqrt{2}}y - \dfrac{2}{\cancel{\sqrt{2}}}$

$= \dfrac{x+y}{\sqrt{2}}$

ii) $f(x,y) = 1 - x + y\sin x$ at $\left(\frac{\pi}{2}, 0\right)$

$f\left(\frac{\pi}{2}, 0\right) = 1 - \frac{\pi}{2} + 0 = 1 - \frac{\pi}{2}$

$f_x = 0 - 1y\cos x$ $\quad$ $f_y = 0 - 0 + \sin x$

f_x at $\left(\frac{\pi}{2}, 0\right) = -1 + 0$ $\quad$ f_y at $\left(\frac{\pi}{2}; 0\right) = \sin\frac{\pi}{2} = 1$

$= -1$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 1 - \frac{\pi}{2} + (-1)\left(x - \frac{\pi}{2}\right) + 1(y-0)$

$= 1 - \cancel{\frac{\pi}{2}} - x + \cancel{\frac{\pi}{2}} + y$

$= 1 - x + y$

iii) $f(x,y) = \log x + \log y$ at $(1,1)$

$f(1,1) = \log(1) + \log(1) = 0$

$f_x = \frac{1}{x} + 0$ $\quad$ $f_y = 0 + \frac{1}{y}$

f_x at $(1,1) = 1$ $\quad$ f_y at $(1,1) = 1$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 0 + 1(x-1) + 1(y-1)$

$= x - 1 + y - 1$

$= x + y - 2$

24/01/2020

Practical 10

Q1
i. $f(x,y) = x + 2y - 3$

Here

$u = 3i - j$ is not a unit vector

$\bar{u} = 3\bar{i} - \bar{j}$

$|u| = \sqrt{10}$

$\therefore$ Unit vector along u is $\frac{\bar{u}}{|u|} = \frac{1}{\sqrt{10}}(3\bar{i} - \bar{j})$

$= \frac{1}{\sqrt{10}}(3, -1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

Now

$f(a+hu) = f\left((1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)\right)$

$= f\left(1 + \frac{3h}{\sqrt{10}}, -1 - \frac{h}{\sqrt{10}}\right)$

$= 1 + \frac{3h}{\sqrt{10}} + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 - 2 - 3 + \frac{3h}{\sqrt{10}} - \frac{2h}{\sqrt{10}}$

$= -4 + \frac{h}{\sqrt{10}}$

$$\therefore \quad f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-4} + \frac{h}{\sqrt{10}} - (\cancel{-4})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{10}}$$

$$= \frac{1}{\sqrt{10}}$$

ii) $f(x, y) = y^2 - 4x + 1$, $a = (3, 4)$, $u = i + 5j$

Here

$u = i + 5j$ is not a unit vector

$\bar{u} = i + 5j$

$|\bar{u}| = \sqrt{26}$

$\therefore$ Unit vector along u is $\frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{26}}(i + 5j)$

$$= \frac{1}{\sqrt{26}}(1, 5)$$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

Now,

$$f(a+hu) = f\left((3, 4) + h\left(\frac{1}{\sqrt{26}} + \frac{5}{\sqrt{26}}\right)\right)$$

$$= f\left(\frac{3+h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right)$$

Du

iii)

$$= \left(4+\frac{5h}{\sqrt{26}}\right)^2 - 4\left(3+\frac{h}{\sqrt{26}}\right)+1$$

$$= 16+\frac{25h^2}{26}+\frac{40h}{\sqrt{26}}-12-\frac{4h}{\sqrt{26}}+1$$

$$= \frac{26h^2}{26}+\frac{36h}{\sqrt{26}}+5$$

$$D_u f(a) = \lim_{h\to 0}\frac{f(a+hu)-f(a)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{25h^2}{26}+\frac{36}{\sqrt{26}}+5-5}{h}$$

$$= \lim_{h\to 0}\frac{25h^2}{26}+\frac{36h}{\sqrt{26}}$$

$$= \lim_{h\to 0}\frac{\cancel{h}\left(\frac{25h}{26}+\frac{36}{\sqrt{26}}\right)}{\cancel{h}}$$

$$= \frac{25(0)}{26}+\frac{36}{\sqrt{26}}$$

$$= \frac{36}{\sqrt{26}}$$

iii) $f(x,y) = 2x+3y$ $\quad a(1,2)$, $\cancel{4=3i+}$ $u = 3i+4j$.

Here,

$u = 3i+4j$ is not a unit vector

$\bar{u} = 3i+4j$

$|\bar{u}| = \sqrt{25} = 5$

$\therefore$ Unit vector along $u = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{5}(3\hat{i} + 4\hat{j})$

$$= \frac{1}{5}(3, 4)$$

$$= \left(\frac{3}{5}, \frac{4}{5}\right)$$

Now

$$f(a+hu) = f\left((1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right)$$

$$= f\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$$

$$= 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$$

$$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$$

$$= 8 + \frac{18h}{5}$$

$$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{8} + \frac{18h}{5} - \cancel{8}}{h}$$

$$= \lim_{h \to 0} \frac{18\cancel{h}}{5\cancel{h}}$$

$$= \frac{18}{5}$$

Q2

i) $f(x,y) = x^y + y^x$ $\qquad a = (1,1)$

$f_x = y(x^{y-1}) + y^x \log y$

$f_y = x(y^{x-1}) + x^y \log x$

$\nabla f(x,y) = (f_x, f_y)$

$= (y x^{y-1} + y^x \log y,\ x y^{x-1} + x^y \log x)$

$\nabla f(x,y)$ at $(1,1)$

$= (1\ (1)^0 + 1^1 \log(1),\ 1\,(1)^{(1-1)} + \log 1)$

$= (1,1)$

ii) $f(x,y) = (\tan^{-1}x) \cdot y^2$ $\qquad a = (1,-1)$

$f_x = y^2\left(\frac{1}{1+x^2}\right) = \frac{y^2}{1+x^2}$

$f_y = 2y \tan^{-1}x$

$\nabla f(x,y) = (f_x, f_y)$

$\left(\frac{y^2}{1+x^2},\ 2y\tan^{-1}x\right)$

$\nabla f(x,y)$ at $(1,-1)$

$= \left(\frac{(-1)^2}{1+(1)^2},\ 2(-1)\tan^{-1}(1)\right)$

$= \left(\frac{1}{2},\ \frac{-2\pi}{4}\right)$

$= \left(\frac{1}{2},\ \frac{-\pi}{2}\right)$

iii) $f(x,y,z) = xyz - e^{x+y+z}$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= \left(yz - e^{x+y+z}, \ xz - e^{x+y+z}, \ xy - e^{x+y+z}\right)$

$\nabla f(x,y,z)$ at $(1,-1,0)$

$= \left(-1(0) - e^{1-1+0}, \ 1(0) - e^{1-1+0}, \ 1(1) - e^{1-1+0}\right)$

$= (0-1, \ 0-1, \ -1-1)$

$= (-1, -1, -2)$

Q3

i. $x^2 \cos y + e^{xy} = 2$ at $(1, 0)$

$f(x,y) = x^2 \cos y + e^{xy} - 2$

$f_x = 2x \cos y + y e^{xy}$

$f_y = -x^2 \sin y + x e^{xy}$

$(x_0, y_0) = (1, 0)$

f_x at $(1,0) = 2(1)\cos 0 + 0$

$= 2$

f_y at $(1,0) = -(1)^2 \sin(0) + 1(e)^{1(0)}$

$= 1$

$f_x (x - x_0) + f_y (y - y_0) = 0$

$2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$2x + y - 2 = 0$ ——— Equation of Tanget

Now,

For equation of Normal;

$bx + ay + d = 0$

$x + 2y + d = 0$

$(1) + 2(0) + d = 0$ At $(1, 0)$

$1 + d = 0$

$d = -1$

$\therefore x + 2y - 1 = 0$ — Equation of Normal

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2, -2)$

$f(x, y) = x^2 + y^2 - 2x + 3y + 2$

$f_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$\therefore f_x$ at $(2, -2) = 2(2) - 2$

$= 2$

$f_y = 0 + 2y - 0 + 3 + 0$

$= 2y + 3$

f_y at $(2, -2) = 2(-2) + 3$

$= 1$

$\therefore$ Equation of tangent

$f_x (x - x_0) + f_y (y - y_0) = 0$

$2(x - 2) + (-1)(y + 2) = 0$

$2x - 4 - y - 2 = 0$

$2x - y - 6 = 0$ — Equation of Tangent.

For Equation of Normal

$bx + ay + d = 0$

$-x + 2y + d = 0$

$-(2) + 2(-2) + d = 0$ at $(2, -2)$

$-2 - 4 + d = 0$

$d = 6$

$\therefore -x + 2y + 6 = 0$ —— Equation of Normal.

Q4

i) $x^2 - 2yz + 3y + xz = 7$ at $(2, 1, 0)$.

$f(x, y, z) = x^2 - 2yz + 3y + xz - 7$

$f_x = 2x - 0 + 0 + z - 0$
$= 2x + z$

$\therefore f_x$ at $(2, 1, 0) = 2(2) + 0$
$= 4$

$f_y = -2z + 3 + 0 - 0$
$= -2z + 3$

f_y at $(2, 1, 0) = -2(0) + 3$
$= 3$

$f_z = 0 - 2y + 0 + x - 0$
$= -2y + x$

f_z at $(2, 1, 0) = -2(1) + 2$
$= 0$

Equation of tangent

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$4(x - 2) + 3(y - 1) + 0(z - 0) = 0$

$4x - 8 + 3y - 3 = 0$

$\therefore\ 4x + 3y - 11 = 0$ —— Equation of tangent.

Equation of normal

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z} \rightarrow \text{(Equation of normal)}$$

$$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z-0}{0}$$

ii) $3xyz - x - y + z = -4$ at $(1, -1, 2)$

$f(x, y, z) = 3xyz - x - y + z + 4$

$f_x = 3yz - 1 - 0 + 0 + 0$ $\quad$ f_x at $(1, -1, 2) = 3(1)(2) - 1$

$= 3yz - 1$ $\quad = -7$

$f_y = 3xz - 0 - 1 + 0 + 0$ $\quad$ f_y at $(1, -1, 2) = 3(1)(2)(-1)$

$= 3xz - 1$ $\quad = 5$

$f_z = 3xy - 0 - 0 + 1 - 0$ $\quad$ f_z at $(1, -1, 2) = 3(1)(-1) + 1$

$= 3xy + 1$ $\quad = -2$

Equation of tangent

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$-7(x - 1) + 5(y + 1) + (-2)(z - 2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$ —— Equation of tangent

Equation of normal:

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$ —— Equation of normal

Q5

i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$ — ①

$\therefore f_x = 6x + 0 - 3y + 6 - 0$
$= 6x - 3y + 6$

$f_y = 2y - 3x + 0 - 4$
$= 2y - 3x - 4$ — ②

$f_x = 0$
$6x - 3y + 6 = 0$
$3(2x - y + 2) = 0$
$2x - y + 2 = 0$
$2x - y = -2$ — ③

$f_y = 0$
$2y - 3x - 4 = 0$
$2y - 3x = 4$ — ④

Multiplying ③ by 2 & subtracting ④ from ③

$$\begin{array}{r} 4x - \cancel{2y} = -4 \\ -\ \cancel{2y} - 3x = 4 \\ \hline 7x \quad = 0 \end{array}$$

$x = 0$

Substituting value of x in ③

$2(0) - y = -2$
$-y = -2$
$y = 2$

$\therefore$ Critical points are $(0, 2)$

Now,

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

$rt - s^2 = 12 - 9$

$= 3 > 0$

Here, $r > 0$ and $rt - s^2 > 0$

f has minimum at $(0, 2)$

$f(0,2) = 3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$

$= 0 + 4 - 0 + 0 - 8$

$= -4$

ii) $f(x,y) = 2x^4 + 3x^2y - y^2$

$f_x = 8x^3 + 6xy - 0$

$= 8x^3 + 6xy$

$f_y = 0 + 3x^2 - 2y$

$= 3x^2 - 2y$

Now

$f_x = 0$

$8x^3 + 6xy = 0$

$2x(4x^2 + 6xy) = 0$

$4x^2 + 6y = 0$ — ①

$f_y = 0$

$3x^2 - 2y = 0$

$3x^2 - 2y = 0$

$3x^2 - 2y = 0$ — ②

Multiply in ① by 3 & ② by 4 & Subtracting ② from ①

$\cancel{12x^2} + 18y = 0$

$-\cancel{12x^2} - 8y = 0$

$24y = 0$

$y = 0$ — ③

Substituting ③ in ②

$x^2 - 2(0) = 0$

$3x^2 = 0$

$x^2 = 0$

$x = 0$ — ④

Critical points are $(0, 0)$

Now,

$r = f_{xx} = 24x^2 + 6y$

$t = f_{yy} = -2$

$s = f_{xy} = 6x$

$rt - s^2 = (24x^2 + 6y)(-2) - (6x)^2$

$= -48x^2 - 12y - 36x^2$

$= -84x^2 - 12y$

At $(0, 0)$

$r = 24(0)^2 + 6(0)$
$= 0$

$s = 6(0) = 0$

$rt - s^2 = -84(0)^2 - 12(0) = 0$

$r = 0$ & $rt - s^2 = 0$

$\therefore$ Nothing can be said.

iii) $f(x, y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x - 0 + 2 + 0 - 0$

$f_y = -2y + 0 + 8 - 0$
$= -2y + 8$

$f_x = 0$
$2x + 2 = 0$
$2(x+1) = 0$
$x + 1 = 0$
$x = -1$

$f_y = 0$
$-2y + 8 = 0$
$-2(y-4) = 0$
$y - 4 = 0$
$y = 4$

Critical points are $(-1, 4)$

$\therefore r = f_{xx} = 2$
$t = f_{yy} = -2$
$s = f_{xy} = 0$
$rt - s^2 = 2(-2) - 0^2$
$= -4 < 0$

Here $r > 0$ and $rt - s^2 < 0$
Nothing can be said.

AK
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