Ch8.Ex.6

EE24BTECH11015 - Dhawal

Question:
Find the area of the region founded by two parabolas $y = x^2$ and $y^2 = x$.

Solution:

Variable	**Description**	**values**
$\mathbf{V_1}$	Quadratic form of the matrix of $y = x^2$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathbf{u_1}$	Linear coefficient vector of $y = x^2$	$\begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}$
f_1	constant term of $y = x^2$	0
$\mathbf{V_2}$	Quadratic form of the matrix of $x = y^2$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{u_2}$	Linear coefficient vector of $x = y^2$	$\begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix}$
f_2	constant term of $x = y^2$	0

TABLE 0: Variables used

Theoritical Solution:

The intersection of two conics with parameters $V_i, u_i, f_i,\ i = 1, 2$ is defined as

$$x^T \left(V_1 + \mu V_2\right) x + 2 \left(u_1 + \mu u_2\right)^T x + \left(f_1 + \mu f_2\right) = 0 \tag{0.1}$$

Solving this, the points of intersection are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.2}$$

The area of the region founded by two parabolas $y = x^2$ and $y^2 = x$ is

$$= \int_0^1 \sqrt{x} dx - \int_0^1 x^2 dx \tag{0.3}$$

$$= \left(\frac{2x\sqrt{x}}{3} - \frac{x^3}{3}\right)_0^1 \tag{0.4}$$

$$= \frac{2}{3} - \frac{1}{3} \tag{0.5}$$

$$= 0.333333 \tag{0.6}$$

Computational Solution:

Taking trapezoid-shaped strips of a small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize the points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with the step size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) \tag{0.7}$$

$$= h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] \tag{0.8}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve $y\left(x\right)$ from $x = x_0$ to $x = x_n$, $\left(x_0, x_1, \ldots x_n\right)$ be equidistant points with step-size h.

$$A\left(x_n + h\right) = A\left(x_n\right) + \frac{1}{2}h\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{0.9}$$

We can repeat this till we get required area.
Discretizing the steps, making $A\left(x_n\right) = A_n, y\left(x_n\right) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{0.10}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left(\left(y_n + hy'_n\right) + y_n\right) \tag{0.11}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.12}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.13}$$

$$x_{n+1} = x_n + h \tag{0.14}$$

In the given question, $y_n = \sqrt{x_n} - {x_n}^2$ and $y'_n = \frac{2x_n\sqrt{x_n}}{3} - \frac{{x_n}^3}{3}$
The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.15}$$

$$A_{n+1} = A_n + h\left(\sqrt{x_n} - {x_n}^2\right) + \frac{1}{2}h^2\left(\frac{2x_n\sqrt{x_n}}{3} - \frac{{x_n}^3}{3}\right) \tag{0.16}$$

$$x_{n+1} = x_n + h \tag{0.17}$$

Iterating till we reach $x_n = 1$ will return required area.
Area obtained computationally: 0.333039 sq. units
Area obtained theoretically: 0.333333 sq.unis

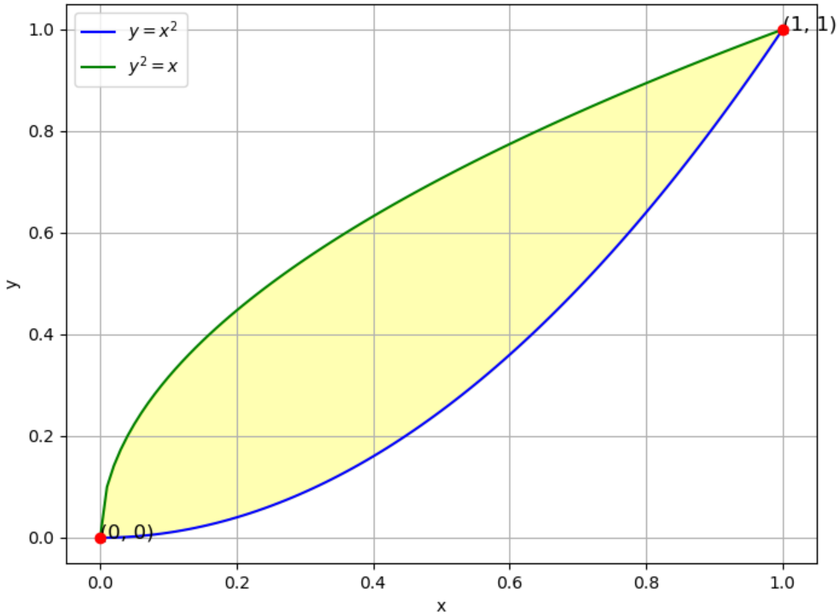

Fig. 0.1: Graph of the parabolas $y = x^2$ and $y^2 = x$ and the area enclosed between them