Question 1-1.8-5q

EE24BTECH11015 - Dhawal

1) If $\mathbf{A}$ and $\mathbf{B}$ be the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, find the equation of the set of points $\mathbf{P}$ such that $PA^2 + PB^2 = K^2$ where K is a constant.

Variable	Description	Values
A	Point given	$\begin{pmatrix} 3, 4, 5 \end{pmatrix}$
B	Point given	$\begin{pmatrix} -1, 3, -7 \end{pmatrix}$
P	Point satisfying the equation	To find
K	Constant	Unknown

TABLE 1: Variables Used

Solution:
Solving the equation,

$$PA^2 + PB^2 = K^2 \tag{1.1}$$

$$\left\|\mathbf{P} - \mathbf{A}\right\|^2 + \left\|\mathbf{P} - \mathbf{B}\right\|^2 = K^2 \tag{1.2}$$

$$(\mathbf{P} - \mathbf{A})^T (\mathbf{P} - \mathbf{A}) + (\mathbf{P} - \mathbf{B})^T (\mathbf{P} - \mathbf{B}) = K^2 \tag{1.3}$$

$$\left\|\mathbf{P}\right\|^2 - \mathbf{P}^T\mathbf{A} - \mathbf{A}^T\mathbf{P} + \left\|\mathbf{A}\right\|^2 + \left\|\mathbf{P}\right\|^2 - \mathbf{P}^T\mathbf{B} - \mathbf{B}^T\mathbf{P} + \left\|\mathbf{B}\right\|^2 = K^2 \tag{1.4}$$

For vectors,

$$\mathbf{P}^T\mathbf{A} = \mathbf{A}^T\mathbf{P} \text{ and } \mathbf{P}^T\mathbf{B} = \mathbf{B}^T\mathbf{P} \tag{1.5}$$

Then,

$$2\left\|\mathbf{P}\right\|^2 - 2\mathbf{A}^T\mathbf{P} + \left\|\mathbf{A}\right\|^2 - 2\mathbf{B}^T\mathbf{P} + \left\|\mathbf{B}\right\|^2 = K^2 \tag{1.6}$$

Equation without putting the values of $\mathbf{A}$ and $\mathbf{B}$,

$$2\left\|\mathbf{P}\right\|^2 - 2\mathbf{A}^T\mathbf{P} + \left\|\mathbf{A}\right\|^2 - 2\mathbf{B}^T\mathbf{P} + \left\|\mathbf{B}\right\|^2 - K^2 = 0 \tag{1.7}$$

Finding $\left\|\mathbf{A}\right\|^2$,

$$\left\|\mathbf{A}\right\|^2 = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 50 \tag{1.8}$$

Finding $\|\mathbf{B}\|^2$,

$$\|\mathbf{B}\|^2 = \mathbf{B}^T\mathbf{B} = \begin{pmatrix} -1 & 3 & -7 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} = 59 \tag{1.9}$$

Putting the values in equation,

$$2\|\mathbf{P}\|^2 - 2\begin{pmatrix} 3 & 4 & 5 \end{pmatrix}\mathbf{P} + 50 - 2\begin{pmatrix} -1 & 3 & -7 \end{pmatrix}\mathbf{P} + 59 - K^2 = 0 \tag{1.10}$$

$$2\|\mathbf{P}\|^2 - \begin{pmatrix} 6 & 8 & 10 \end{pmatrix}\mathbf{P} - \begin{pmatrix} -2 & 6 & -14 \end{pmatrix}\mathbf{P} + 109 - K^2 = 0 \tag{1.11}$$

Final equation:

$$2\|\mathbf{P}\|^2 - \begin{pmatrix} 4 & 14 & -4 \end{pmatrix}\mathbf{P} + 109 - K^2 = 0 \tag{1.12}$$

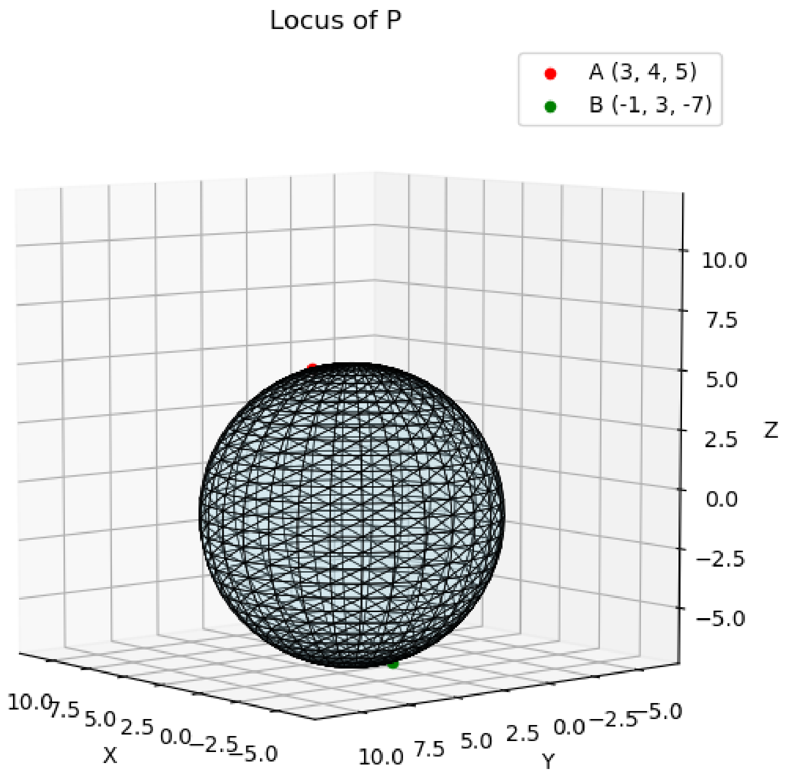

Fig. 1.1: Locus of P