Introduction to POMDPs

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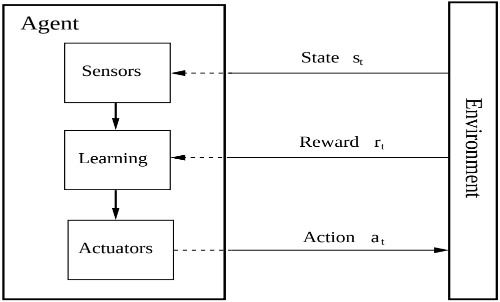
Partially Observable Markov Decision Processes

Uncertainty in Reinforcement Learning

Agent Architecture

Agent achieves goals by interacting with environment



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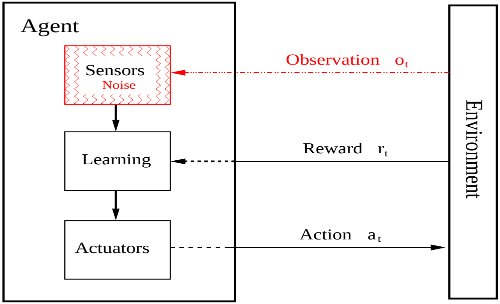

Partial Observability

- Uncertainty about state is induced through noisy sensory measurements

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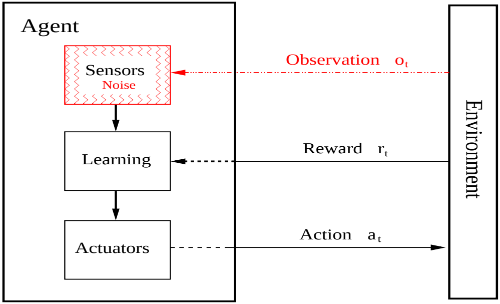

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- Uncertainty about state is induced through noisy sensory measurements
- Observations do not reveal complete state information

Problem Specification

Partially Observable Markov Decision Process

A POMDP is given by $M = (T, S, O, A, P_S, P_O, r)$

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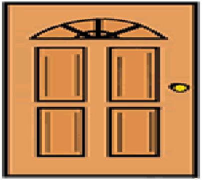
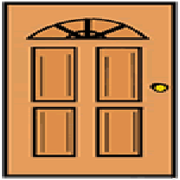
r: Reward Model Reward function $r : S \times A \to \mathbb{R}$

The Famous Tiger Problem

State Space



$S = \text{LEFT}$



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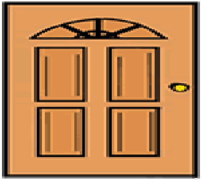

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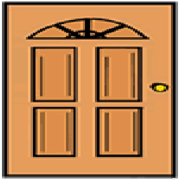

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Actions and Rewards

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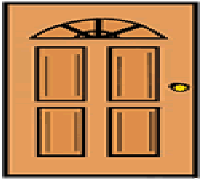

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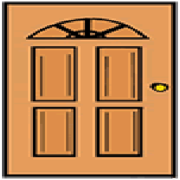

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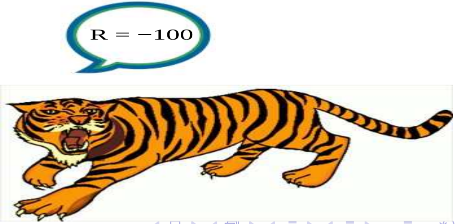

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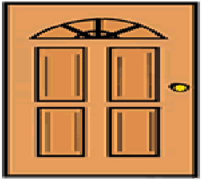

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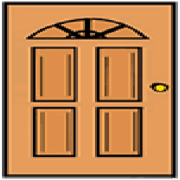

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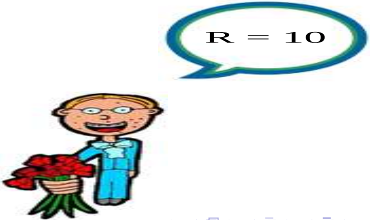

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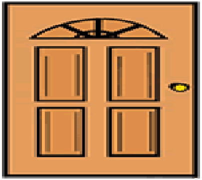

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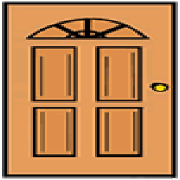

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Observations (Listening for $R = -1$)

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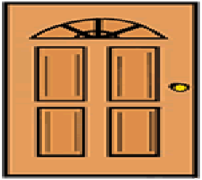

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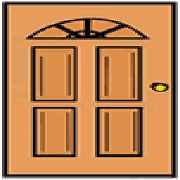

$\mathrm{S} = \mathrm{RIGHT}$

Observations (Listening for $R = -1$)

$$
\begin{aligned}
P(o = HL \mid s = LEFT) &= 0.85 \\
P(o = HR \mid s = LEFT) &= 0.15 \\
P(o = HL \mid s = RIGHT) &= 0.15 \\
P(o = HR \mid s = RIGHT) &= 0.85
\end{aligned}
$$

Finite Horizon Policies

Fully Observable Processes

An optimal policy is given by a sequence of mappings $\pi^* := (\pi^*_t)_{(0 \le t < T_F)}$ each constituting a rule $\pi_t : S \longrightarrow A$ for choosing actions

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$$\pi^* \in \arg\max_{\pi \in \Pi} E[\sum_{t=0}^{T_F - 1} r(s_t, \pi_t(s_t)) + r(s_{T_F}, \cdot)]$$

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Given that current state s_t is unknown, what information is available in order to choose optimal actions?

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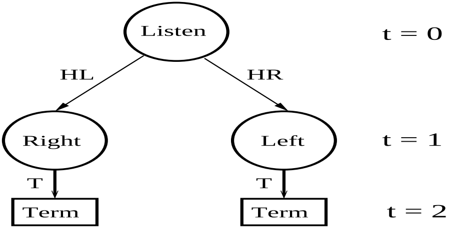
The complete sequence of past actions and observations

$\Rightarrow$ Optimal policy π^* depends on past actions and observations

Example Policy for $T_F = 2$

Policy $\equiv$ Tree



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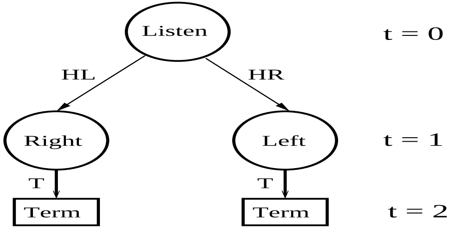

Policy $\equiv$ Mapping

Policy $\pi : (A \times O)^* \rightarrow A$

$$\pi([]) = Listen$$

$$\pi([Listen, HL]) = Right \quad \pi([Listen, HR]) = Left$$

Example Policy for $T_F = 3$

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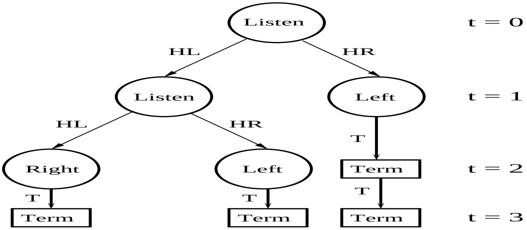

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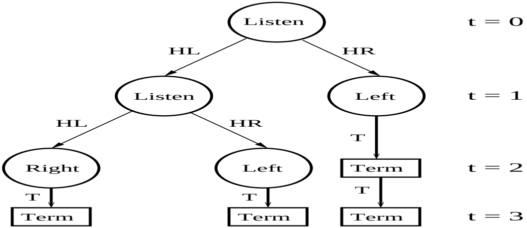

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Information State Space

How to solve POMDPs?

Question: How is it possible to compute an optimal policy tree?

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Definition (Information States)

The information state I_t is defined to be the complete sequence of actions and observations $[a_0, o_1, a_1, o_2, ..., a_{t-1}, o_t]$ until time t.
The information state space I contains all possible information states.

Information State MDP

Given a POMDP $M := (T, S, A, O, P_S, P_O, r)$, we compute an optimal policy for M by transforming M into an MDP called *Information State MDP*

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Definition (Information State MDP)

Discretized Time: Does not change

State Space: Given by information state space I

Action Space: Does not change

Transitions: $P_I(I_{t+1} = [a_0, o_1, ..., a_t, o_{t+1}] \mid a_t, I_t) = p(o_{t+1}|a_t, I_t)$

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Rewards $r_I(I_t, a_t) = \sum_{s \in S} p(s_t = s \mid I_t) r(s, a)$

Markov Property

Lemma

Information states constitute a markovian state space. It holds that $P_I(I_{t+1} \mid a_t, I_t) = P_I(I_{t+1} \mid a_t, I_t, I_{t-1}, \dots, I_0)$ *(memoryless process)*

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Proof.

$I_k = [a_0, o_1, ..., a_{k-1}, o_k] \subset [a_0, o_1, ..., a_{t-1}, o_t] = I_t \ (0 \leq k \leq t-1)$

Thus, it follows $P_I(I_{t+1} \mid a_t, I_t) = P_I(I_{t+1} \mid a_t, I_t, I_{t-1}, ..., I_0)$ □

Value Iteration on Information States

Value Functions on Information States

Compute sequence of value functions $(V_n)_{0 \leq n \leq T_F}$ defined on information states, $V_n : (A \times O)^* \longrightarrow \mathbb{R}$

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$$V_{T_F}(I_{T_F}) = \sum_{s \in S} p(s_{T_F} = s \mid I_{T_F}) r(s, \cdot)$$

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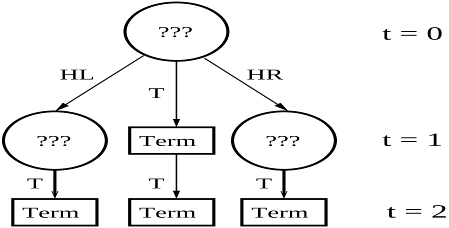
2. Bellman Equation

$$V_n^*(I_t) = \max_{a \in A} [\sum_{s \in S} p(s_t = s | I_t) r(s, a)$$
$$+ \beta \sum_{o \in O} p(o_{t+1} = o | I_t, a) V_{n+1}^*(I_{t+1} = \{a_0, .., a_t = a, o_{t+1} = o\})]$$

Value Iteration (Example)

Value Iteration



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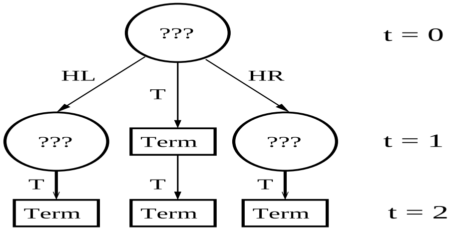

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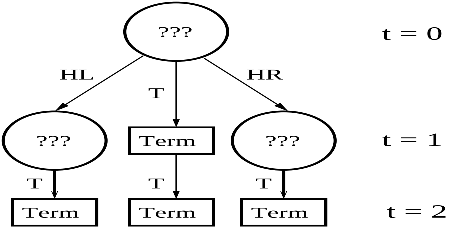

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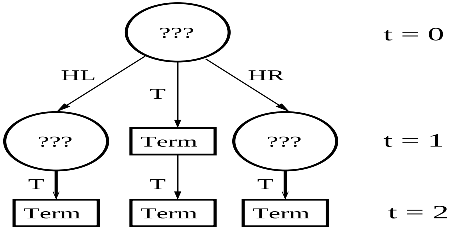

$$V_0([]) = \max\{r(Left) + V_1([Left, T]),\ r(Right) + V_1([Right, T]),\ r(Listen) + \sum_{o \in \{HL,HR\}} p(o \mid [Listen]) V_1([Listen, o])\}$$

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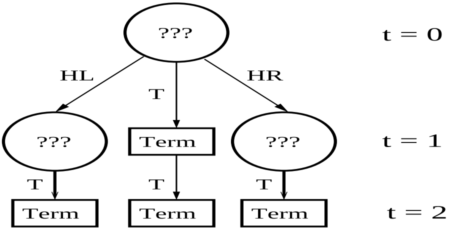

$$
\begin{aligned}
V_1([Left, T]) &= V_2([Left, T, \cdot, T]) \\
V_1([Right, T]) &= V_2([Right, T, \cdot, T])
\end{aligned}
$$

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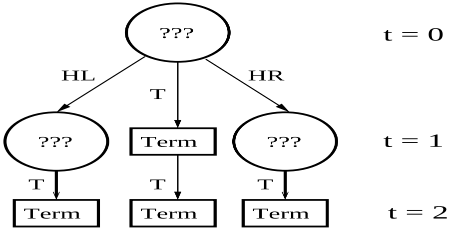

$$\begin{aligned} V_1([Listen, HL]) &= \max\{r(Left) + V_2([Listen, HL, Left, T]), \\ &\quad r(Right) + V_2([Listen, HL, Right, T]), \\ &\quad r(Listen) + V_2([Listen, HL, Listen, T])\} \end{aligned}$$

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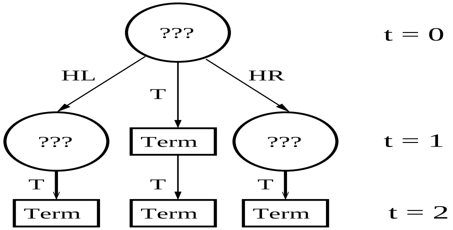

$$
\begin{aligned}
V_2(I) &= \sum_{s \in S} p(s \mid I) r(s, \cdot) \\
&= 0
\end{aligned}
$$

Open Problems

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   $\Rightarrow$ Formulas needed for $p(o_t \mid a_t, I_t)$ and $p(s_t \mid I_t)$
2. Length of information states grows linearly with horizon T_F
3. Number of information states grows exponentially with horizon T_F

Belief States

How can we represent information states by a (data)-structure of constant size?

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Definition (Belief State)

The belief state b_t is an $|S|$-dimensional vector such that $b_t(s) := p(s_t = s \mid I_t)$. Belief states therefore form probability distributions over states.

Value Functions on Belief States

Theorem (Equivalence of Information States and Belief States)

Given a POMDP $(T, S, A, O, P_S, P_O, r)$ *and a finite horizon* T_F*, it is possible to rewrite the sequence of (optimal) value functions* $(V_n^*)_{(0 \leq n \leq T_F)}$ *in terms of belief states*

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$$V_n^*(b) := \max_{a \in A} [\sum_{s \in S} b(s) r(s, a) + \beta \sum_{o \in O} \sum_{s' \in s, s'' \in S} P_O(o \mid s'', a) \cdot P_S(s'' \mid s', a) b(s') V_{n+1}^*(b_o^a)]$$

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The belief state b_o^a denotes the successor belief of b after executing action $a \in A$ and making observation $o \in O$

$$b_o^a(s) := \frac{P_O(o \mid s, a) \sum_{s' \in S} P_S(s \mid s', a) b(s')}{\sum_{s' \in S, s'' \in S} P_S(s'' \mid s', a) P_O(o \mid s'', a) b(s')}$$

Successor Beliefs

Lemma

Given that $b := b_t$, $a := a_t$, $o := o_{t+1}$, it holds that

$$b_o^a(s) = b_{t+1} = \frac{P_O(o \mid s, a) \sum_{s' \in S} P_S(s \mid s', a) b_t(s')}{\sum_{s' \in S, s'' \in S} P_S(s'' \mid s', a) P_O(o \mid s'', a) b_t(s')}$$

Successor Beliefs (2)

Proof.

$$b_{t+1}(s) = p(s_{t+1} = s \mid I_{t+1})$$

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Successor Beliefs (3)

Proof.

Numerator

$$\sum_{s' \in S} b_t(s')$$

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Value Functions on Belief States (2)

Theorem

The sequence of (optimal) value functions $(V_n^*)_{(0 \leq n \leq T_F)}$ can be rewritten in terms of belief states such that
$\forall\, 0 \leq n \leq T_F,\ \forall t \in T : V_n^*(I_t) = V_n^*(b_t)$

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&= V_n^*(b_t)
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Value Iteration on Belief States

Value Functions on Belief States

Compute sequence of value functions $(V_n)_{0 \leq n \leq T_F}$ defined on belief states, $V_n : B \longrightarrow \mathbb{R}$

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$$V_{T_F}(b) = \sum_{s \in S} b(s) r(s, \cdot)$$

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Algorithm

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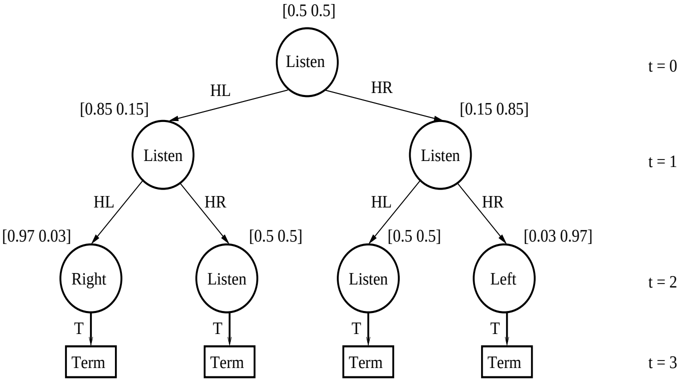
$$V_{T_F}(b) = \sum_{s \in S} b(s) r(s, \cdot)$$

2. Bellman Equation

$$V_n^*(b) := \max_{a \in A} [\sum_{s \in S} b(s) r(s, a) + \beta \sum_{o \in O} \sum_{s' \in s, s'' \in S} P_O(o \mid s'', a) \cdot P_S(s'' \mid s', a) b(s') V_{n+1}^*(b_o^a)]$$

Optimal Policy for $T_F = 3$

Optimal Policy Tree



General Value Functions for POMDPs

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General Value Functions

Let V_r be the optimal value function for policy tree P_r

$$V^*(b) = \max_{B_r} V_r(b)$$

Representing Value Functions by Vector Sets

Lemma

Let b_s be the belief state assigning probability $p = 1$ to state $s \in S$. Thus, it holds that

$$V(b) = \sum_{s \in S} b(s) V(b_s)$$

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General Value Iteration for POMDPs

Theorem

Given a POMDP $(T, S, A, O, P_S, P_O, r)$ *and a finite horizon* T_F*, each value function from the sequence of optimal value functions* $(V_n^*)_{(0 \leq n < T_F)}$ *can be represented by a finite set of vectors* Γ_n*.*

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$$\Gamma_n := \{\sum_{o \in O} \alpha_{f(o)}^{o,a} \mid f \in f(O, \Gamma_{n+1}),\ a \in A\}$$

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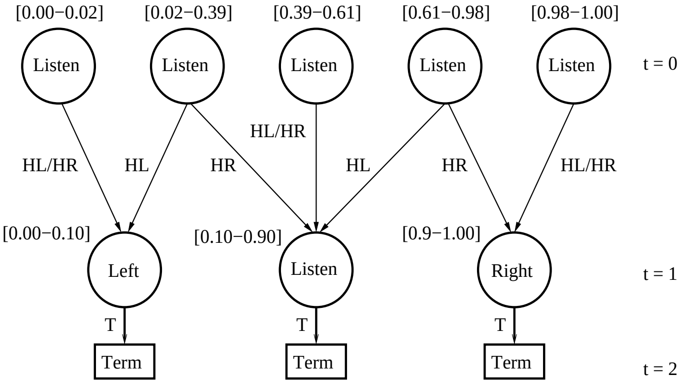
$$\forall \gamma \in \Gamma_{n+1} : \alpha_\gamma^{o,a}(s') := \frac{r(s', a)}{|O|} + \beta \sum_{s \in S} P_O(o \mid s, a) P_S(s \mid s', a) \gamma(s)$$

General Optimal Policy for $T_F = 2$



Computational Complexity

Theorem

There exists a family of POMDPs such that, for every m, $|S| = 2m$, $|A| = 1$, $|O| = m$, *it exists* $|\Gamma_n| = 2$ *and* $|\Gamma_{n-1}| = 2^m$

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$\Rightarrow$ Solving POMDPs exactly is fundamentally inefficient!