Probabilistic Robotics

Planning and Control:

Partially Observable Markov Decision Processes

POMDPs

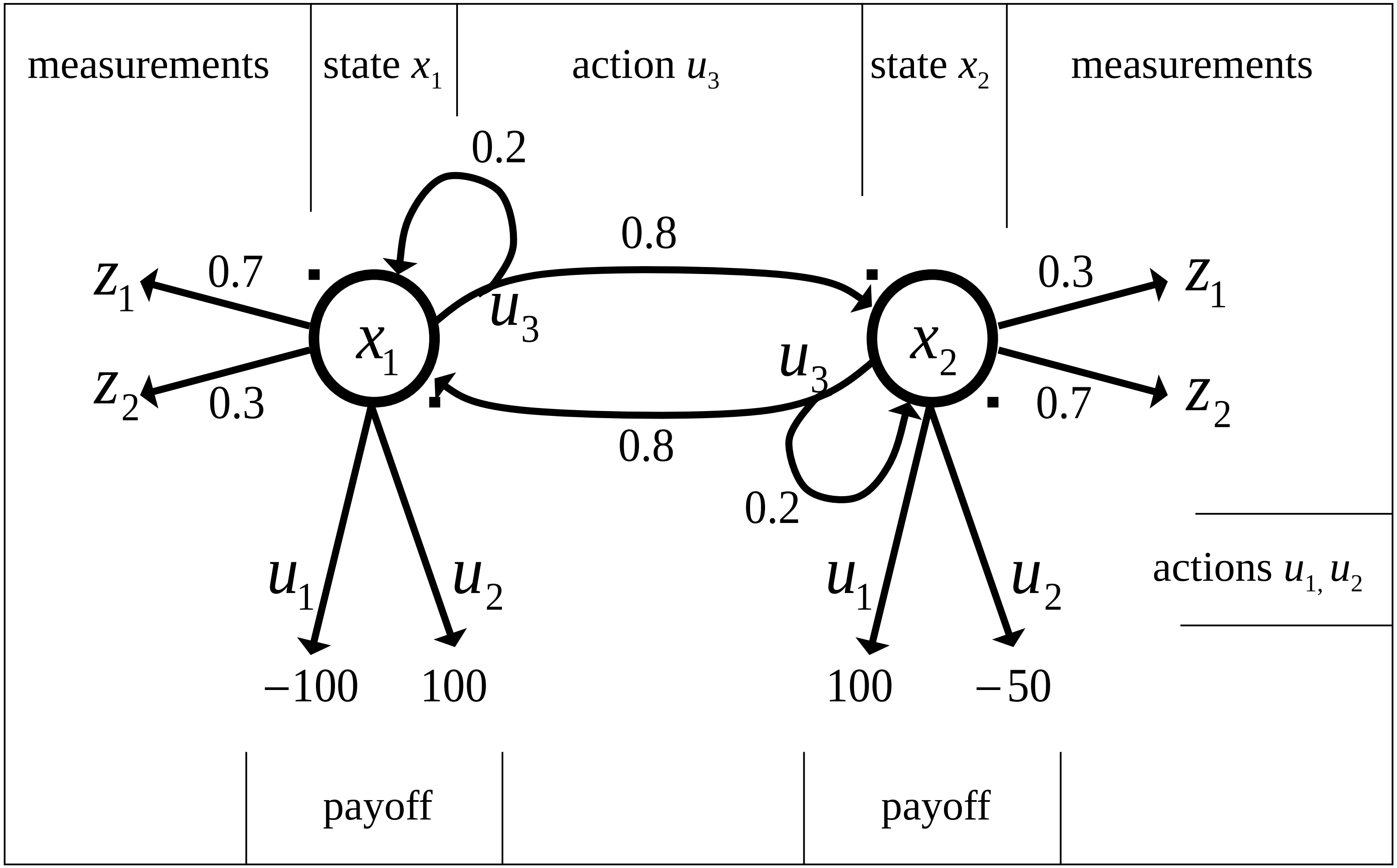
- In POMDPs we apply the very same idea as in MDPs.
- **Since the state is not observable**, the agent has to **make its decisions based on** the belief state which is a **posterior distribution over states**.
- Let b be the belief of the agent about the state under consideration.
- POMDPs compute a **value function over belief space**:

$$V_T(b) \;=\; \gamma \max_u \left[r(b,u) + \int V_{T-1}(b') p(b' \mid u, b) \; db' \right]$$

Problems

- Each belief is a probability distribution, thus, each value in a **POMDP is a function of an entire probability distribution**.
- **This is problematic, since probability distributions are continuous**.
- Additionally, we have to deal with the **huge complexity of belief spaces**.
- For **finite worlds** with finite state, action, and measurement spaces and finite horizons, however, we can **effectively represent the value functions by piecewise linear functions**.

An Illustrative Example



The Parameters of the Example

- The actions u_1 and u_2 are terminal actions.
- The action u_3 is a sensing action that potentially leads to a state transition.
- The horizon is finite and γ=1.

$$
\begin{aligned}
r(x_1, u_1) &= -100 & \qquad r(x_2, u_1) &= +100 \\
r(x_1, u_2) &= +100 & \qquad r(x_2, u_2) &= -50 \\
r(x_1, u_3) &= -1 & \qquad r(x_2, u_3) &= -1
\end{aligned}
$$

$$
\begin{aligned}
p(x'_1 | x_1, u_3) &= 0.2 & \qquad p(x'_2 | x_1, u_3) &= 0.8 \\
p(x'_1 | x_2, u_3) &= 0.8 & \qquad p(z'_2 | x_2, u_3) &= 0.2
\end{aligned}
$$

$$
\begin{aligned}
p(z_1 | x_1) &= 0.7 & \qquad p(z_2 | x_1) &= 0.3 \\
p(z_1 | x_2) &= 0.3 & \qquad p(z_2 | x_2) &= 0.7
\end{aligned}
$$

Payoff in POMDPs

- In MDPs, the payoff (or return) depended on the state of the system.
- In POMDPs, however, the true state is not exactly known.
- Therefore, we compute the **expected payoff** by **integrating over all states**:

$$
\begin{aligned}
r(b,u) &= E_x[r(x,u)] \\
&= \int r(x,u)p(x)\;dx \\
&= p_1\,r(x_1,u) + p_2\,r(x_2,u)
\end{aligned}
$$

Payoffs in Our Example (1)

- If we are totally certain that we are in state x_1 and execute action u_1, we receive a reward of -100
- If, on the other hand, we definitely know that we are in x_2 and execute u_1, the reward is +100.
- In between it is the linear combination of the extreme values weighted by the probabilities

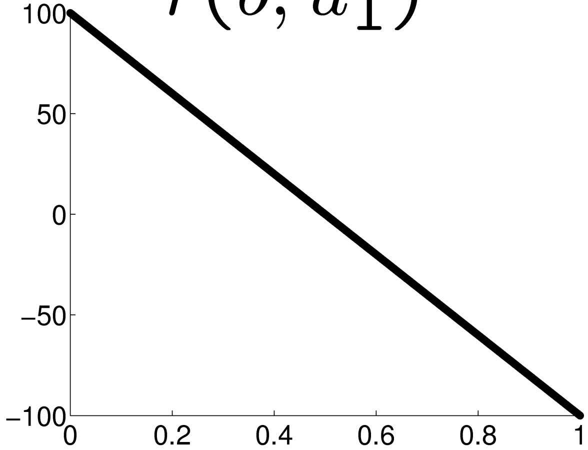
$$\begin{aligned} r(b, u_1) &= -100\, p_1 + 100\, p_2 \\ &= -100\, p_1 + 100\, (1 - p_1) \end{aligned}$$

$$r(b, u_2) = 100\, p_1 - 50\, (1 - p_1)$$

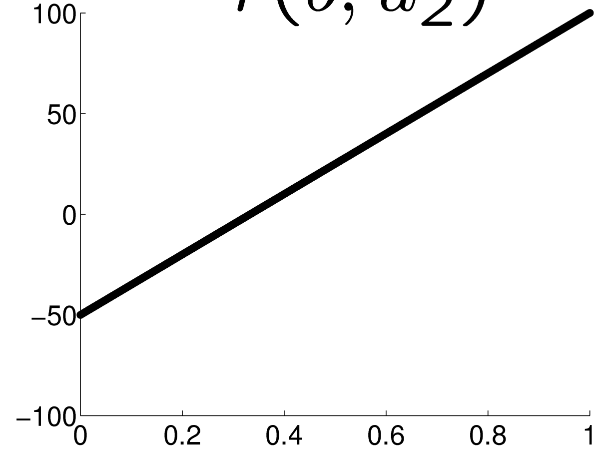
$$r(b, u_3) = -1$$

Payoffs in Our Example (2)

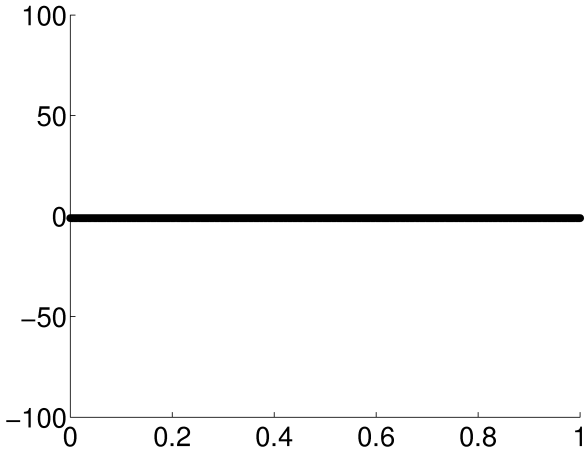
$$r(b, u_1)$$



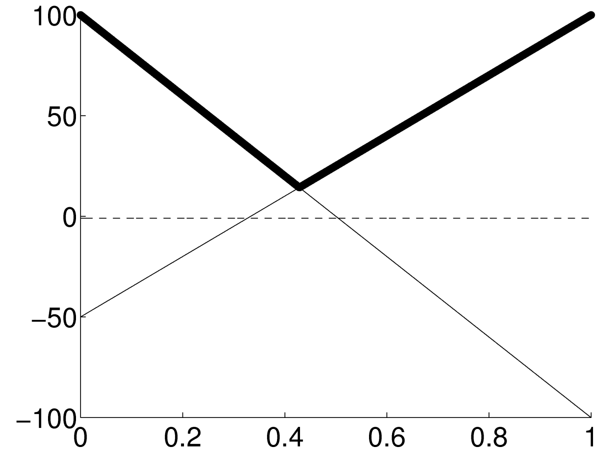
$$r(b, u_2)$$



$$r(b, u_3)$$



$$V_1(b) = max_u r(b, u)$$



The Resulting Policy for T=1

- Given we have a finite POMDP with T=1, we would use $V_1(b)$ to determine the optimal policy.
- In our example, the optimal policy for T=1 is

$$\pi_1(b) = \begin{cases} u_1 & \text{if } p_1 \leq \frac{3}{7} \\ u_2 & \text{if } p_1 > \frac{3}{7} \end{cases}$$

- This is the upper thick graph in the diagram.

Piecewise Linearity, Convexity

- The resulting value function $V_1(b)$ is the maximum of the three functions at each point

$$
\begin{aligned}
V_1(b) &= \max_u r(b,u) \\
&= \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ -1 & \end{array} \right\}
\end{aligned}
$$

- It is piecewise linear and convex.

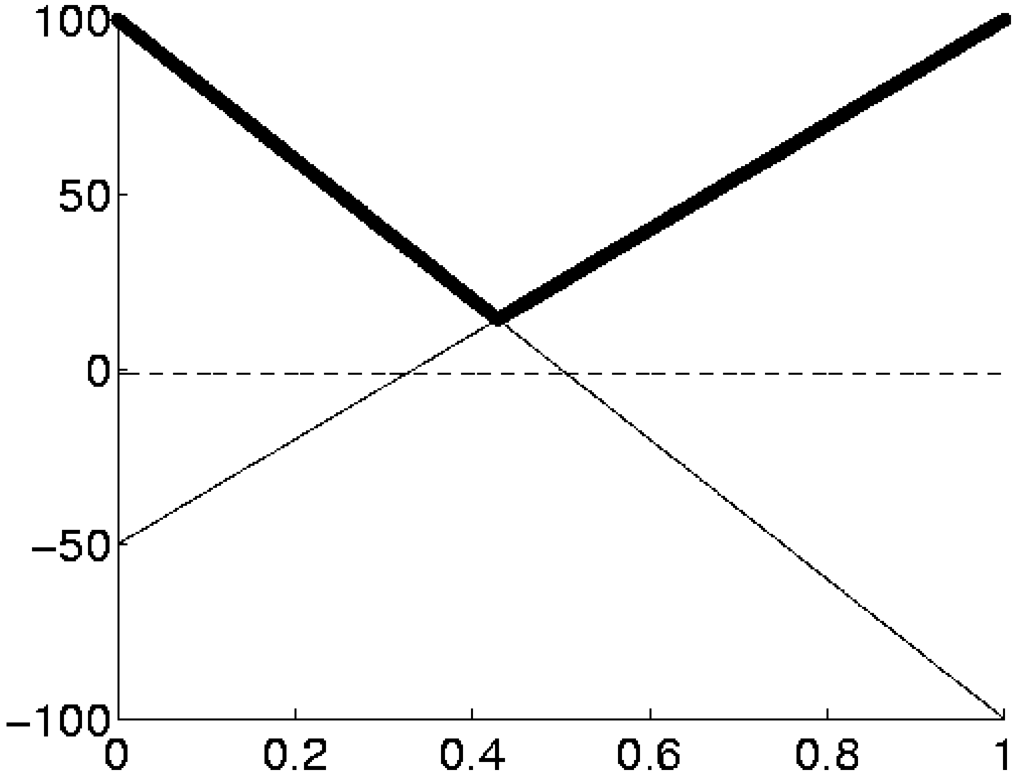
Pruning

- If we carefully consider $V_1(b)$, we see that only the first two components contribute.
- The third component can therefore safely be pruned away from $V_1(b)$.

$$V_1(b) \;=\; \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \end{array} \right\}$$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.



$V_1(b)$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.

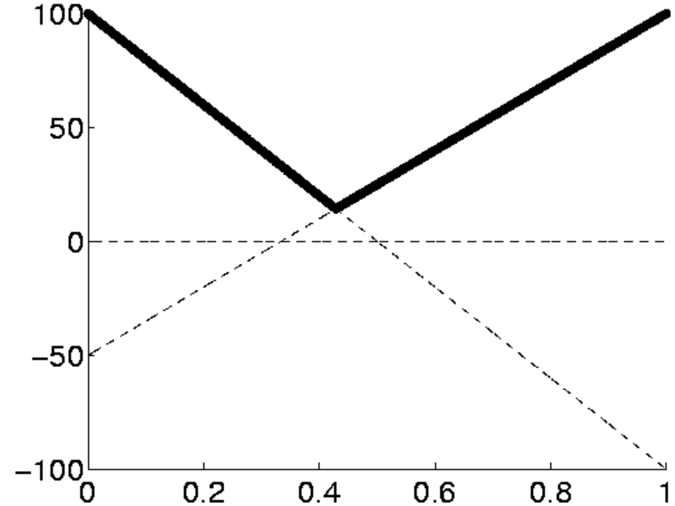
$$p'_1 = \frac{0.7\, p_1}{p(z_1)}$$

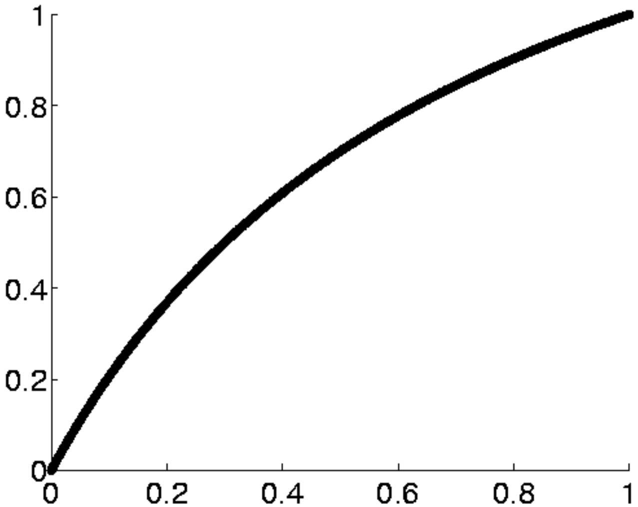
$$p'_2 = \frac{0.3(1 - p_1)}{p(z_1)}$$

$$p(z_1) = 0.7\, p_1 + 0.3(1 - p_1) = 0.4\, p_1 + 0.3$$

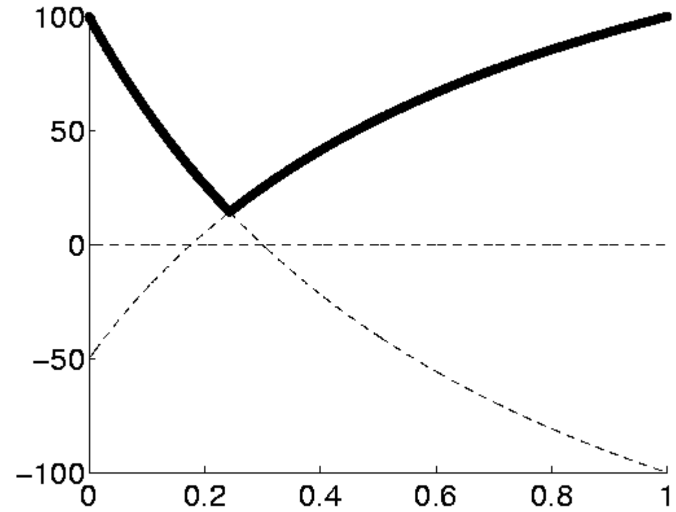
Value Function

$V_1(b)$





$b'(b|z_1)$



$V_1(b|z_1)$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.
- Thus $V_1(b \mid z_1)$ is given by

$$
\begin{aligned}
V_1(b \mid z_1) &= \max \left\{ \begin{array}{llll} -100 & \cdot \frac{0.7\, p_1}{p(z_1)} & +100 & \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \\ \\ 100 & \cdot \frac{0.7\, p_1}{p(z_1)} & -50 & \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \end{array} \right\} \\
&= \frac{1}{p(z_1)} \max \left\{ \begin{array}{ll} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$\bar{V}_1(b) = E_z[V_1(b \mid z)] = \sum_{i=1}^{2} p(z_i) V_1(b \mid z_i)$$

$$= \sum_{i=1}^{2} p(z_i) V_1\left(\frac{p(z_i \mid x_1)\, p_1}{p(z_i)} \right)$$

$$= \sum_{i=1}^{2} V_1\left(p(z_i \mid x_1)\, p_1 \right)$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

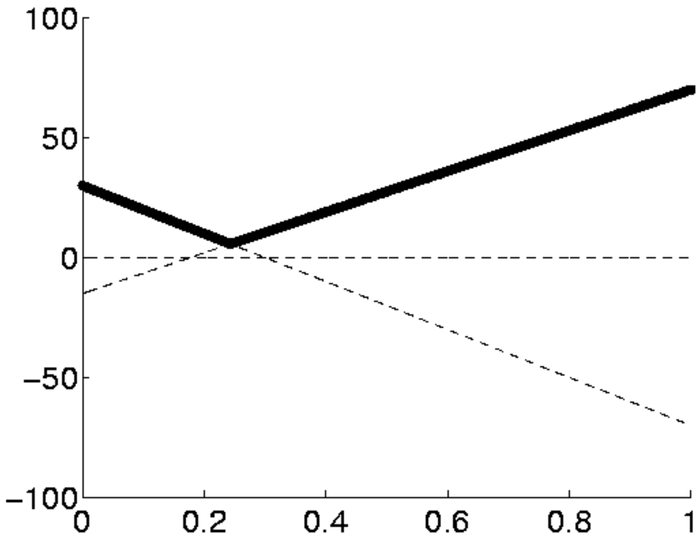
$$
\begin{aligned}
\bar{V}_1(b) &= E_z[V_1(b \mid z)] \\
&= \sum_{i=1}^{2} p(z_i)\, V_1(b \mid z_i) \\
&= \max \left\{ \begin{array}{rr} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\} \\
&\quad + \max \left\{ \begin{array}{rr} -30\, p_1 & +70\,(1-p_1) \\ 30\, p_1 & -35\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

Resulting Value Function

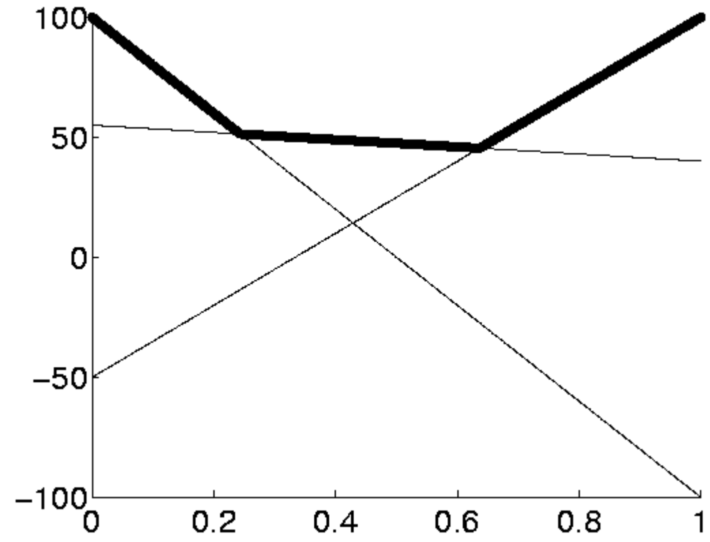
- The four possible combinations yield the following function which then can be simplified and pruned.

$$
\begin{aligned}
\bar{V}_1(b) &= \max \left\{ \begin{array}{llll}
-70\, p_1 & +30\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\
-70\, p_1 & +30\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \\
+70\, p_1 & -15\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\
+70\, p_1 & -15\,(1-p_1) & +30\, p_1 & -35\,(1-p_1)
\end{array} \right\} \\
&= \max \left\{ \begin{array}{ll}
-100\, p_1 & +100\,(1-p_1) \\
+40\, p_1 & +55\,(1-p_1) \\
+100\, p_1 & -50\,(1-p_1)
\end{array} \right\}
\end{aligned}
$$

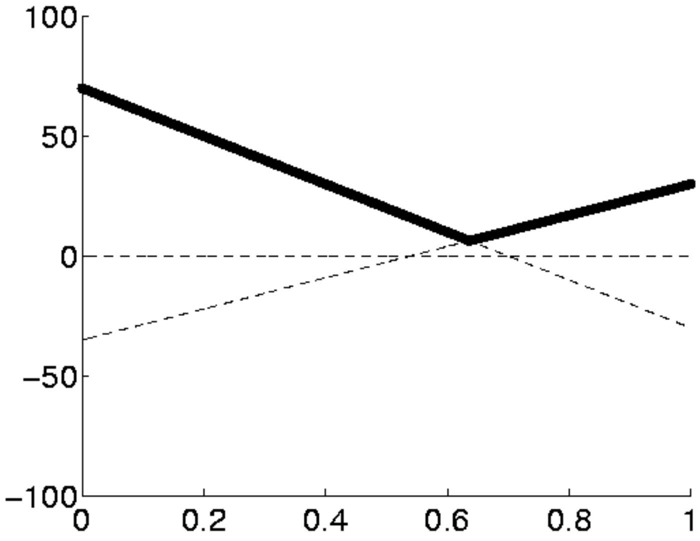
Value Function



$p(z_1)\ V_1(b|z_1)$



$\bar{V}_1(b)$



$p(z_2)\ V_2(b|z_2)$

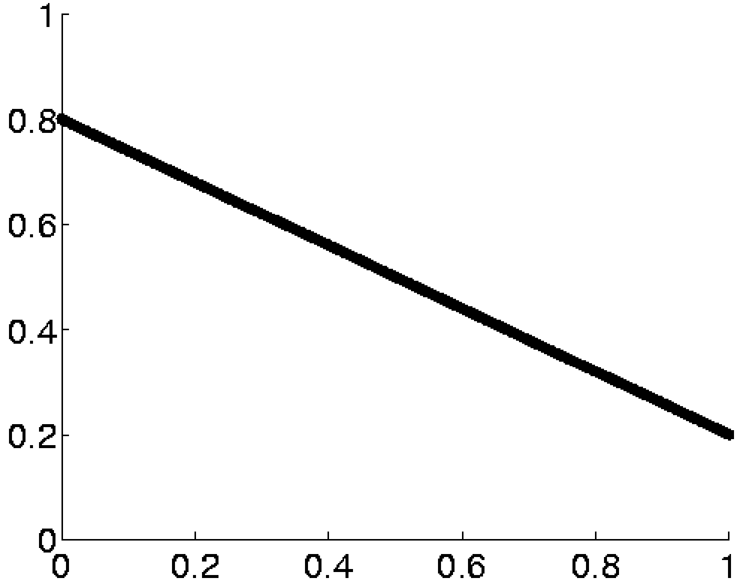
State Transitions (Prediction)

- When the agent selects u_3 its state potentially changes.
- When computing the value function, we have to take these potential state changes into account.

$$
\begin{aligned}
p_1' &= E_x[p(x_1 \mid x, u_3)] \\
&= \sum_{i=1}^{2} p(x_1 \mid x_i, u_3) p_i \\
&= 0.2 p_1 + 0.8(1 - p_1) \\
&= 0.8 - 0.6 p_1
\end{aligned}
$$

State Transitions (Prediction)

$$
\begin{aligned}
p_1' &= E_x[p(x_1 \mid x, u_3)] \\
&= \sum_{i=1}^{2} p(x_1 \mid x_i, u_3) p_i \\
&= 0.2 p_1 + 0.8(1 - p_1) \\
&= 0.8 - 0.6 p_1
\end{aligned}
$$



Resulting Value Function after executing u_3

- Taking the state transitions into account, we finally obtain.

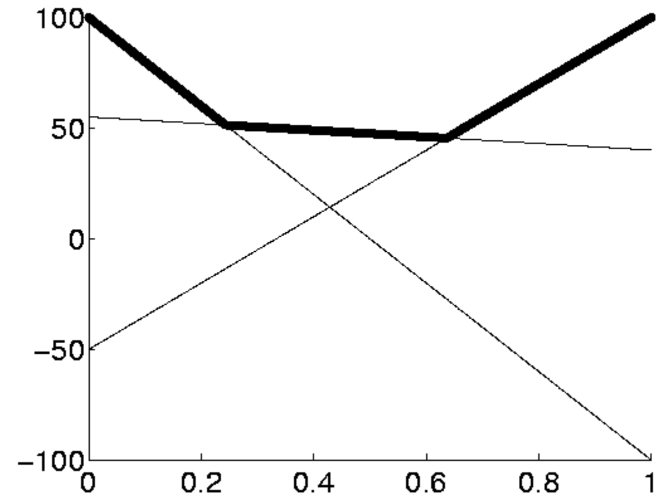
$$\bar{V}_1(b) = \max \left\{ \begin{array}{llll} -70\,p_1 & +30\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ -70\,p_1 & +30\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \end{array} \right\}$$

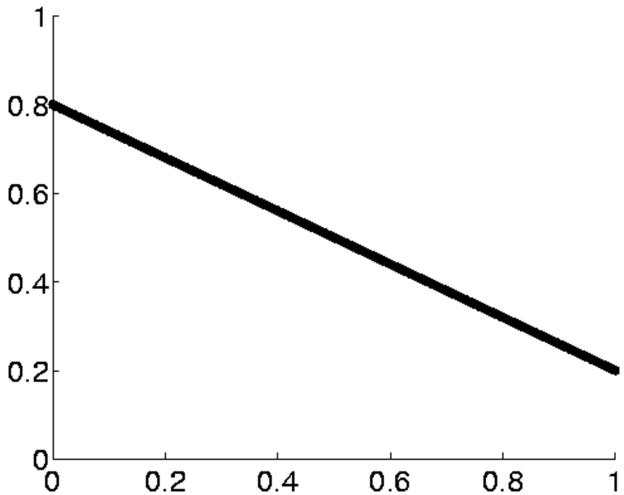
$$= \max \left\{ \begin{array}{ll} -100\,p_1 & +100\,(1-p_1) \\ +40\,p_1 & +55\,(1-p_1) \\ +100\,p_1 & -50\,(1-p_1) \end{array} \right\}$$

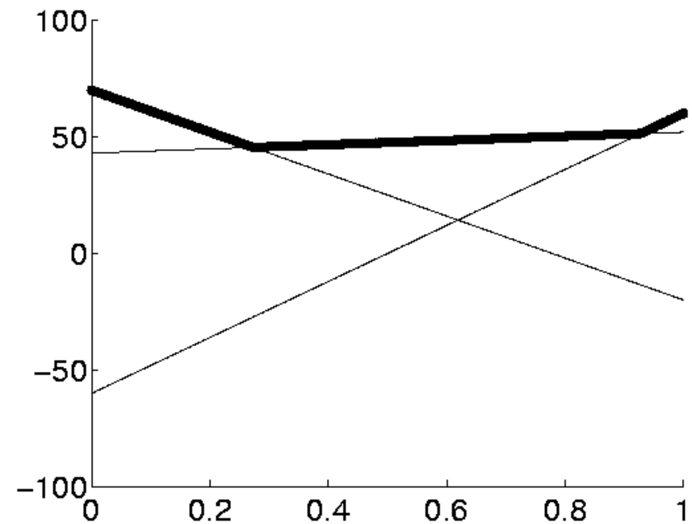
$$\bar{V}_1(b \mid u_3) = \max \left\{ \begin{array}{ll} 60\,p_1 & -60\,(1-p_1) \\ 52\,p_1 & +43\,(1-p_1) \\ -20\,p_1 & +70\,(1-p_1) \end{array} \right\}$$

Value Function after executing u_3

$$\bar{V}_1(b)$$







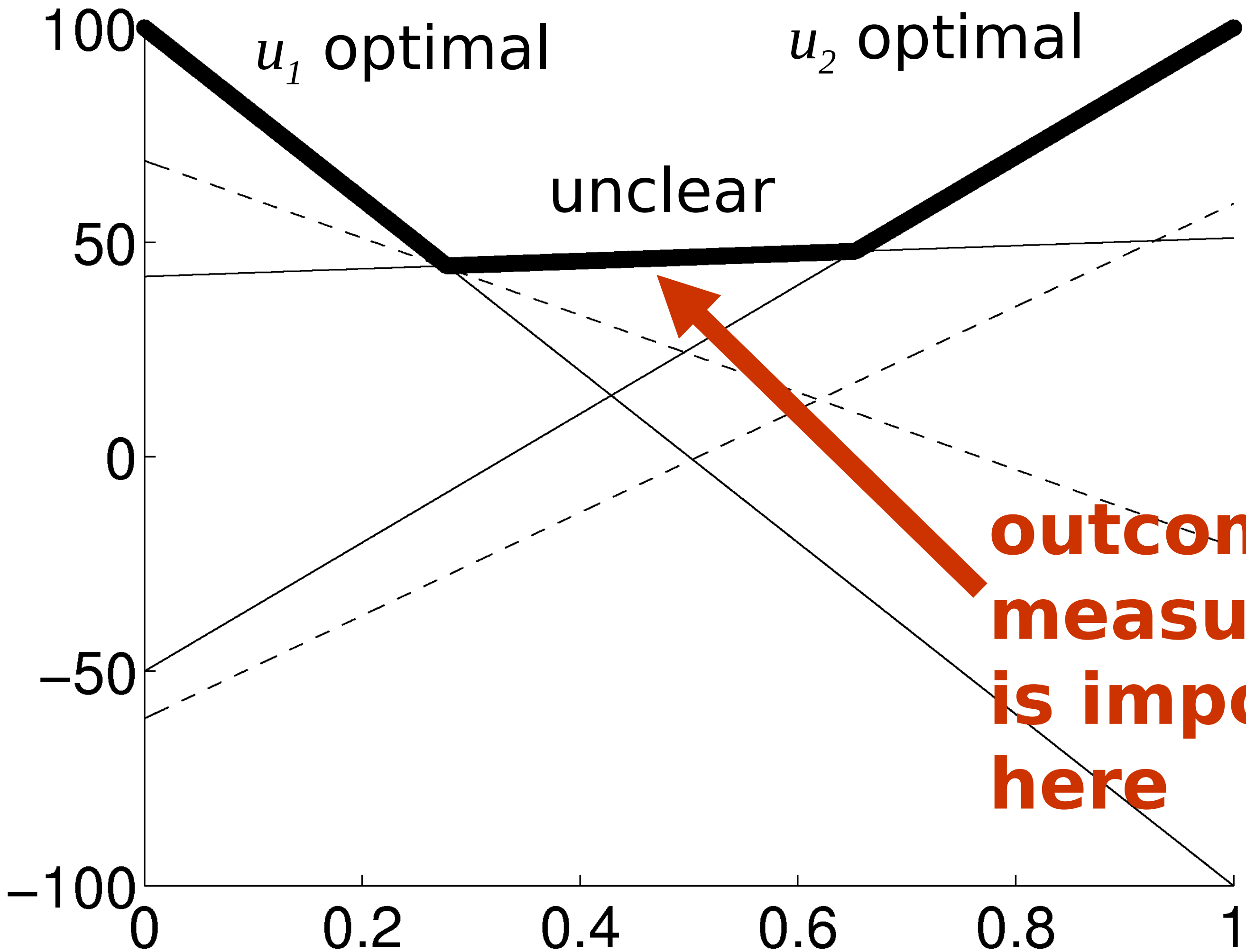
$$\bar{V}_1(b \mid u_3)$$

Value Function for T=2

- Taking into account that the agent can either directly perform u_1 or u_2 or first u_3 and then u_1 or u_2, we obtain (after pruning)

$$\bar{V}_2(b) \;=\; \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ 51\, p_1 & +42\,(1-p_1) \end{array} \right\}$$

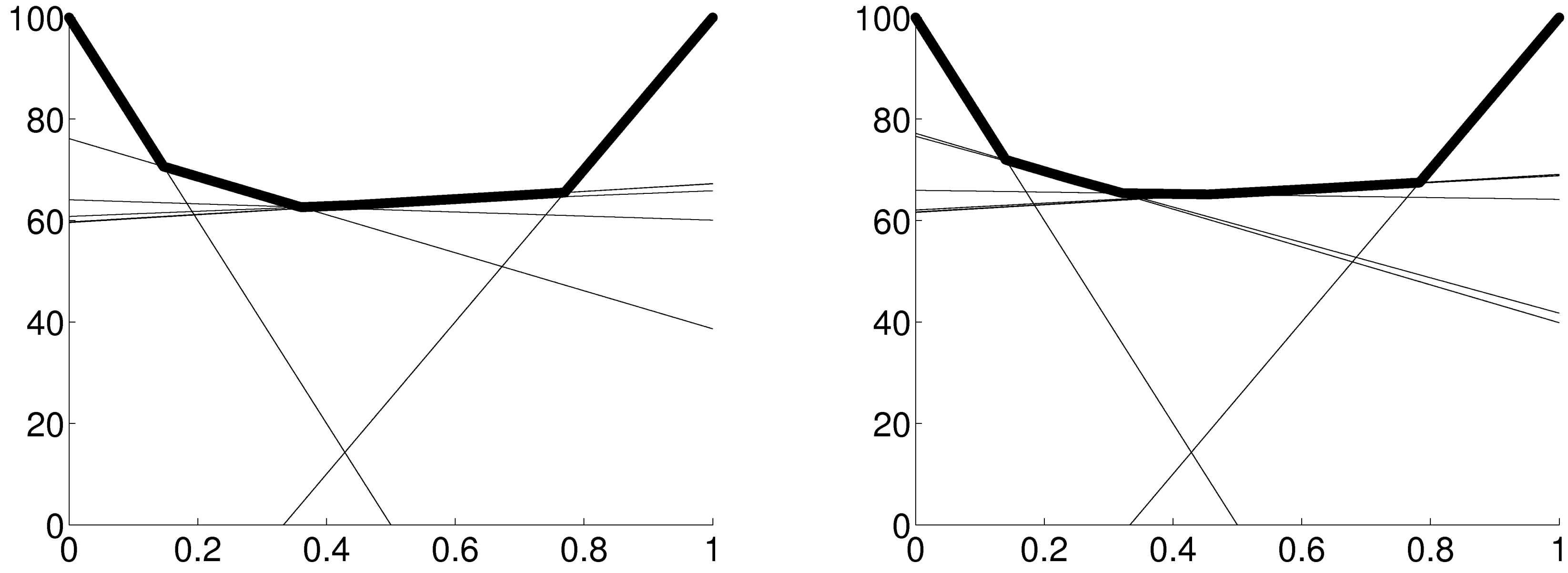
Graphical Representation of $V_2(b)$



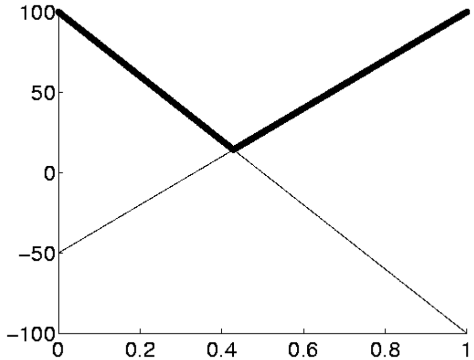
outcome of measurement is important here

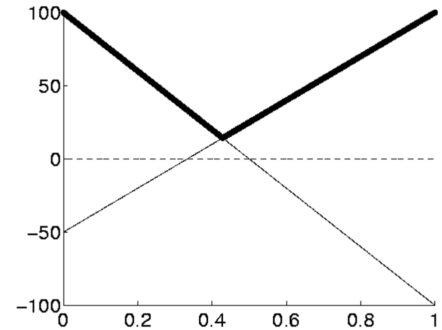
Deep Horizons and Pruning

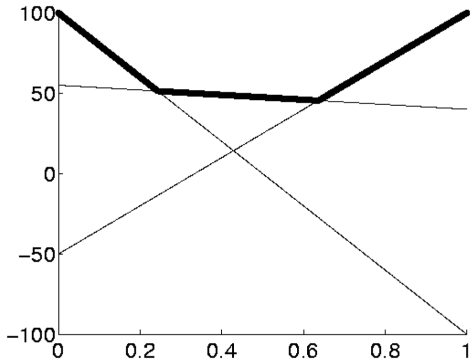
- We have now completed a full backup in belief space.
- This process can be applied recursively.
- The value functions for T=10 and T=20 are

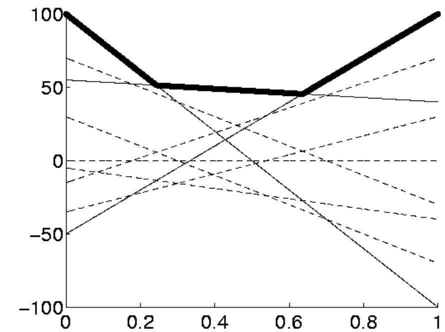


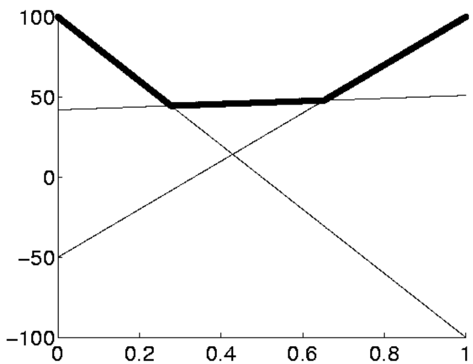
Deep Horizons and Pruning

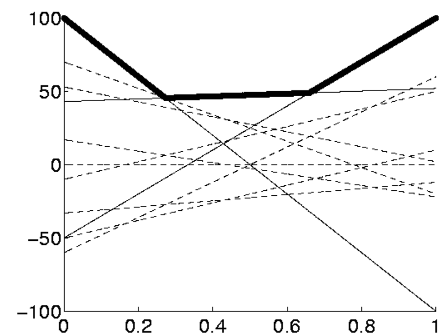












Algorithm POMDP(T):

$\Upsilon = (0, \ldots, 0)$
for $\tau = 1$ *to* T *do*
$\Upsilon' = \emptyset$
for all $(u'; v_1^k, \ldots, v_N^k)$ *in* Υ *do*
for all control actions u *do*
for all measurements z *do*
for $j = 1$ *to* N *do*
$$v_{j,u,z}^k = \sum_{i=1}^{N} v_i^k \; p(z \mid x_i) \; p(x_i \mid u, x_j)$$
endfor
endfor
endfor
endfor
for all control actions u *do*
for all $k(1), \ldots, k(M) = (1, \ldots, 1)$ *to* $(|\Upsilon|, \ldots, |\Upsilon|)$ *do*
for $i = 1$ *to* N *do*
$$v_i' = \gamma \left[r(x_i, u) + \sum_z v_{u,z,i}^{k(z)} \right]$$
endfor
add $(u; v_1', \ldots, v_N')$ *to* Υ'
endfor
endfor
optional: prune Υ'
$\Upsilon = \Upsilon'$
endfor
return Υ

Why Pruning is Essential

- Each **update introduces additional linear components** to V.
- Each **measurement squares the number of linear components**.
- Thus, an un-pruned value function for T=20 includes more than $10^{547,864}$ linear functions.
- At T=30 we have $10^{561,012,337}$ linear functions.
- The pruned value functions at T=20, in comparison, contains only 12 linear components.
- The combinatorial explosion of linear components in the value function are the major reason why **POMDPs are impractical for most applications**.

POMDP Summary

- POMDPs compute the optimal action in partially observable, stochastic domains.
- For finite horizon problems, the resulting value functions are piecewise linear and convex.
- In each iteration the number of linear constraints grows exponentially.
- POMDPs so far have only been applied successfully to very small state spaces with small numbers of possible observations and actions.

POMDP Approximations

- Point-based value iteration
- QMDPs
- AMDPs

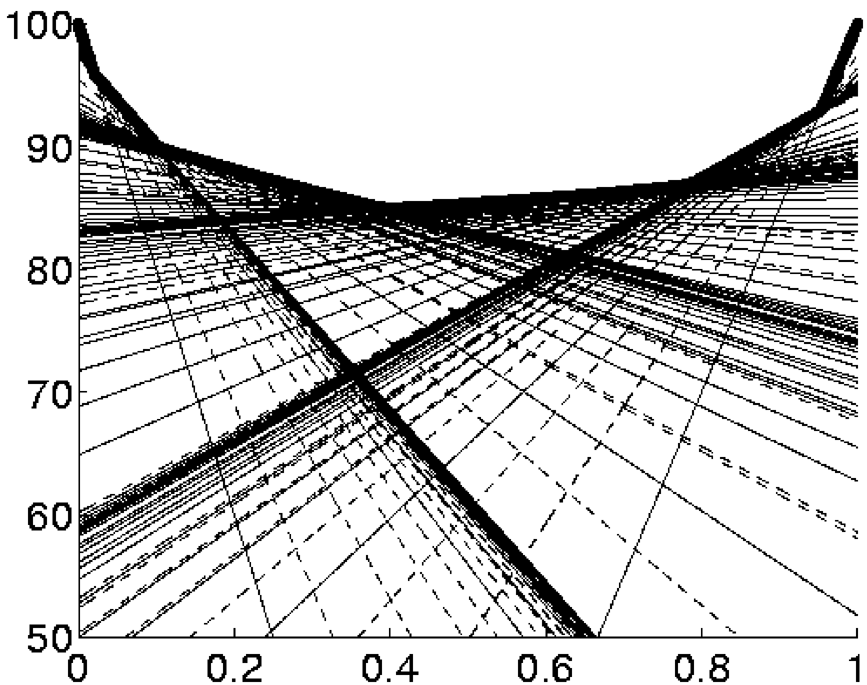
Point-based Value Iteration

- Maintains a set of example beliefs

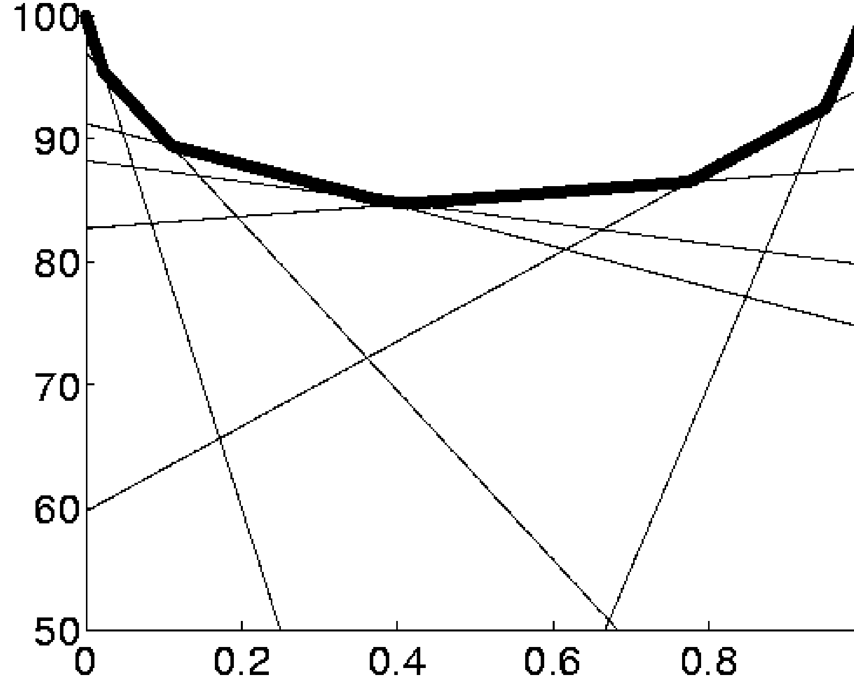
- Only considers constraints that maximize value function for at least one of the examples

Point-based Value Iteration

Value functions for T=30

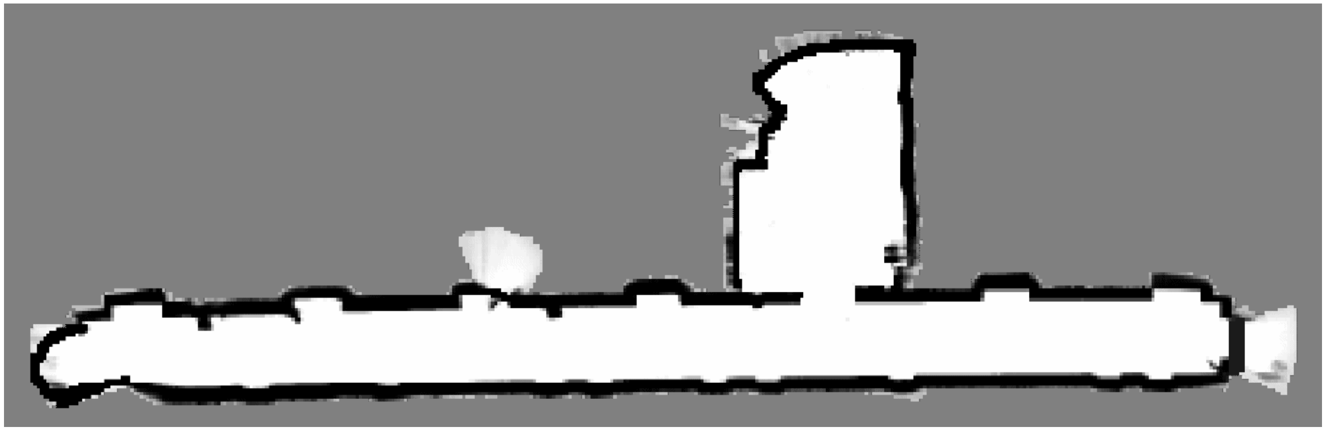


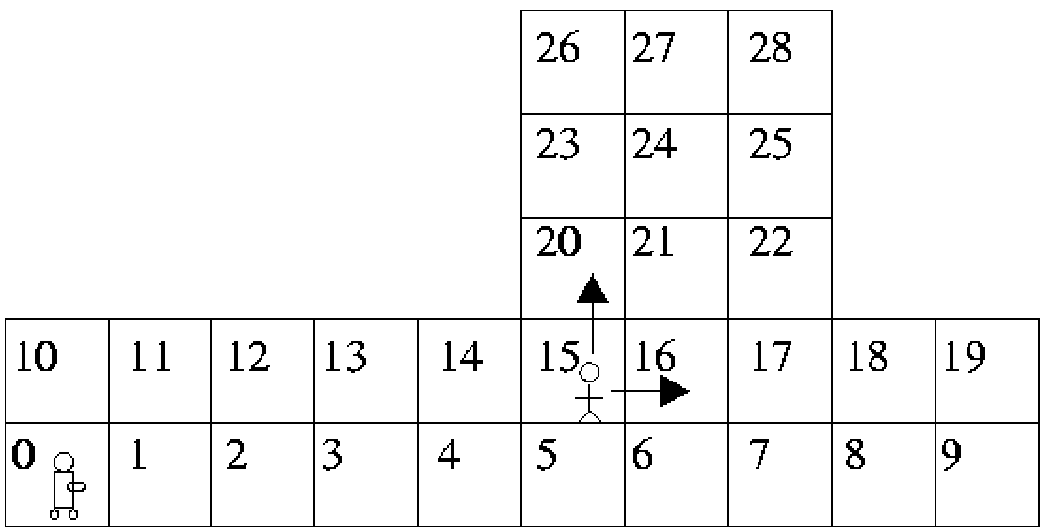
Exact value function



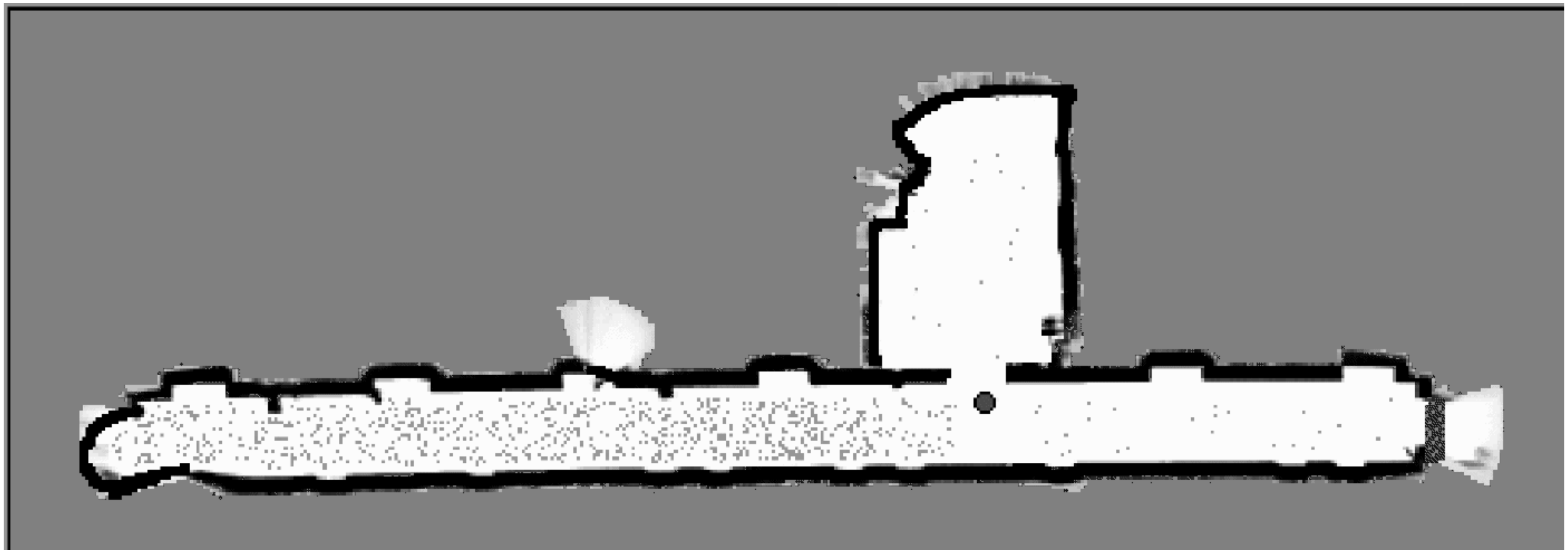
PBVI

Example Application





Example Application



QMDPs

- QMDPs only consider state uncertainty in the first step
- After that, the world becomes fully observable.

Algorithm QMDP($b = (p_1, \ldots, p_N)$)**:**

$\hat{V}$ = MDP_discrete_value_iteration() // *see page 504*

for all control actions u *do*

$$Q(x_i, u) = r(x_i, u) + \sum_{j=1}^{N} \hat{V}(x_j)\, p(x_j \mid u, x_i)$$

endfor

return $\displaystyle \operatorname*{arg\,max}_u \sum_{i=1}^{N} p_i\, Q(x_i, u)$

Augmented MDPs

- Augmentation adds uncertainty component to state space, e.g.,

$$\bar{b} = \begin{pmatrix} \arg\max_x b(x) \\ H_b(x) \end{pmatrix}, \qquad H_b(x) = -\int b(x) \log b(x)\, dx$$

- Planning is performed by MDP in augmented state space
- Transition, observation and payoff models have to be learned

Algorithm AMDP_value_iteration( ):

for all $\bar{b}$ *do* // *learn model*
    for all u *do*
        for all $\bar{b}$ *do* // *initialize model*
            $\hat{\mathcal{P}}(\bar{b}, u, \bar{b}') = 0$
        endfor
            $\hat{\mathcal{R}}(\bar{b}, u) = 0$
        repeat n *times* // *learn model*
            generate b *with* $f(b) = \bar{b}$
            sample $x \sim b(x)$ // *belief sampling*
            sample $x' \sim p(x' \mid u, x)$ // *motion model*
            sample $z \sim p(z \mid x')$ // *measurement model*
            calculate $b' = B(b, u, z)$ // *Bayes filter*
            calculate $\bar{b}' = f(b')$ // *belief state statistic*
            $\hat{\mathcal{P}}(\bar{b}, u, \bar{b}') = \hat{\mathcal{P}}(\bar{b}, u, \bar{b}') + \frac{1}{n}$ // *learn transitions prob's*
            $\hat{\mathcal{R}}(\bar{b}, u) = \hat{\mathcal{R}}(\bar{b}, u) + \frac{r(u,s)}{n}$ // *learn payoff model*
        endrepeat
    endfor
endfor
for all $\bar{b}$ // *initialize value function*
    $\hat{V}(\bar{b}) = r_{\min}$
endfor
repeat until convergence // *value iteration*
    for all $\bar{b}$ *do*

$$\hat{V}(\bar{b}) = \gamma \max_u \left[\hat{\mathcal{R}}(u, \bar{b}) + \sum_{\bar{b}'} \hat{V}(\bar{b}') \; \hat{\mathcal{P}}(\bar{b}, u, \bar{b}') \right]$$

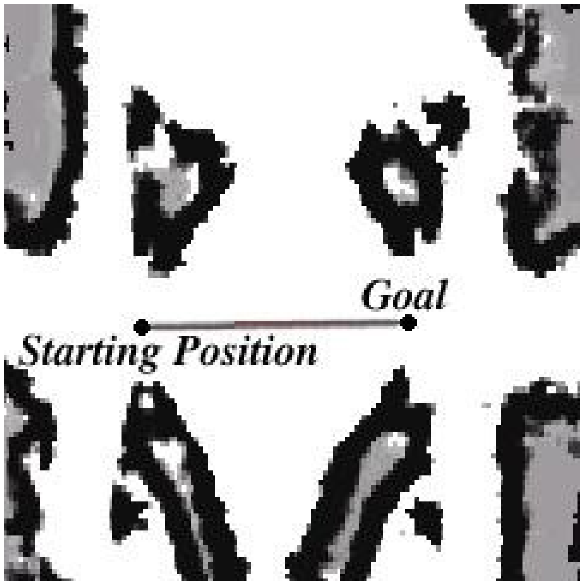
endfor
return $\hat{V}, \hat{\mathcal{P}}, \hat{\mathcal{R}}$ // *return value fct & model*

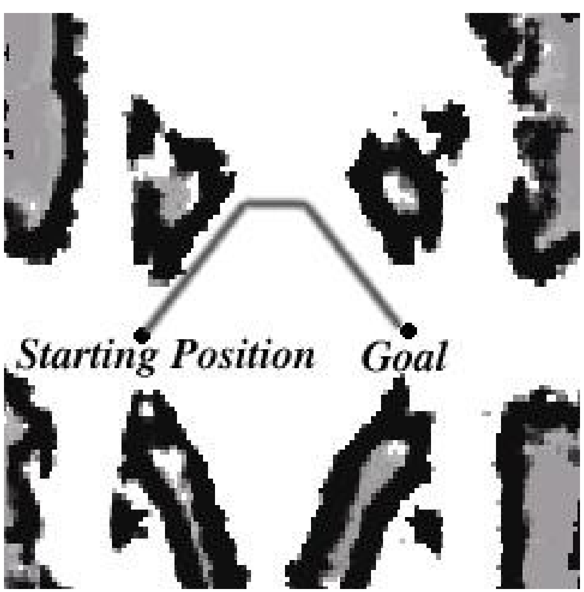
Algorithm policy_AMDP($\hat{V}$, $\hat{\mathcal{P}}$, $\hat{\mathcal{R}}$, b):

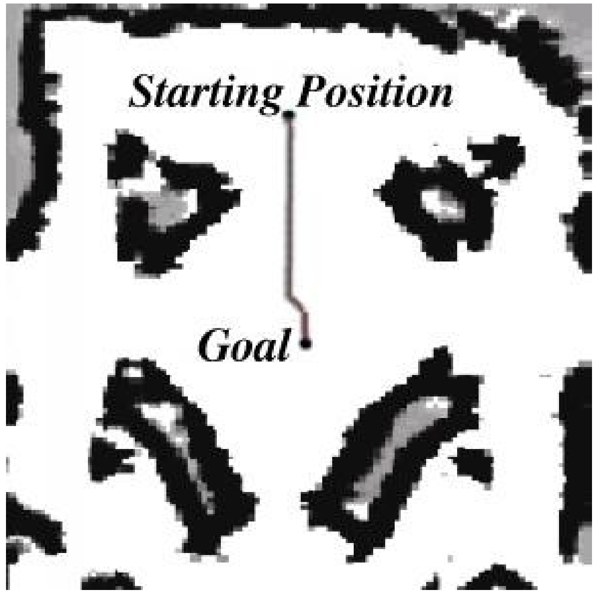
$\bar{b} = f(b)$

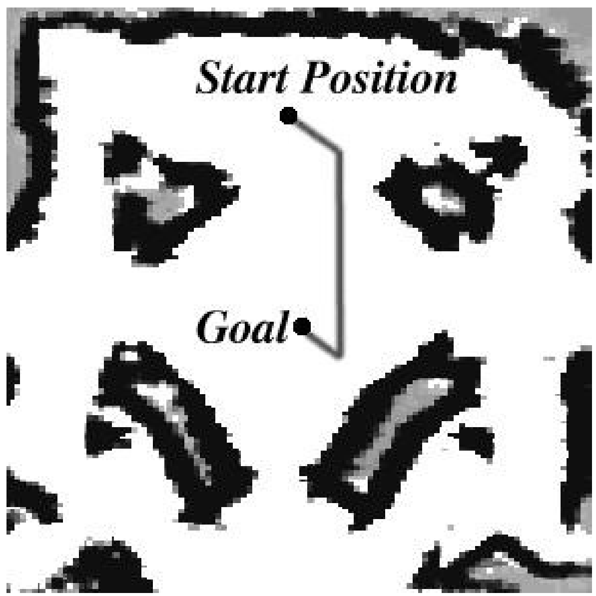
return $\displaystyle \arg\max_{u} \left[\hat{\mathcal{R}}(u, \bar{b}) + \sum_{\bar{b}'} \hat{V}(\bar{b}')\ \hat{\mathcal{P}}(\bar{b}, u, \bar{b}') \right]$

Coastal Navigation

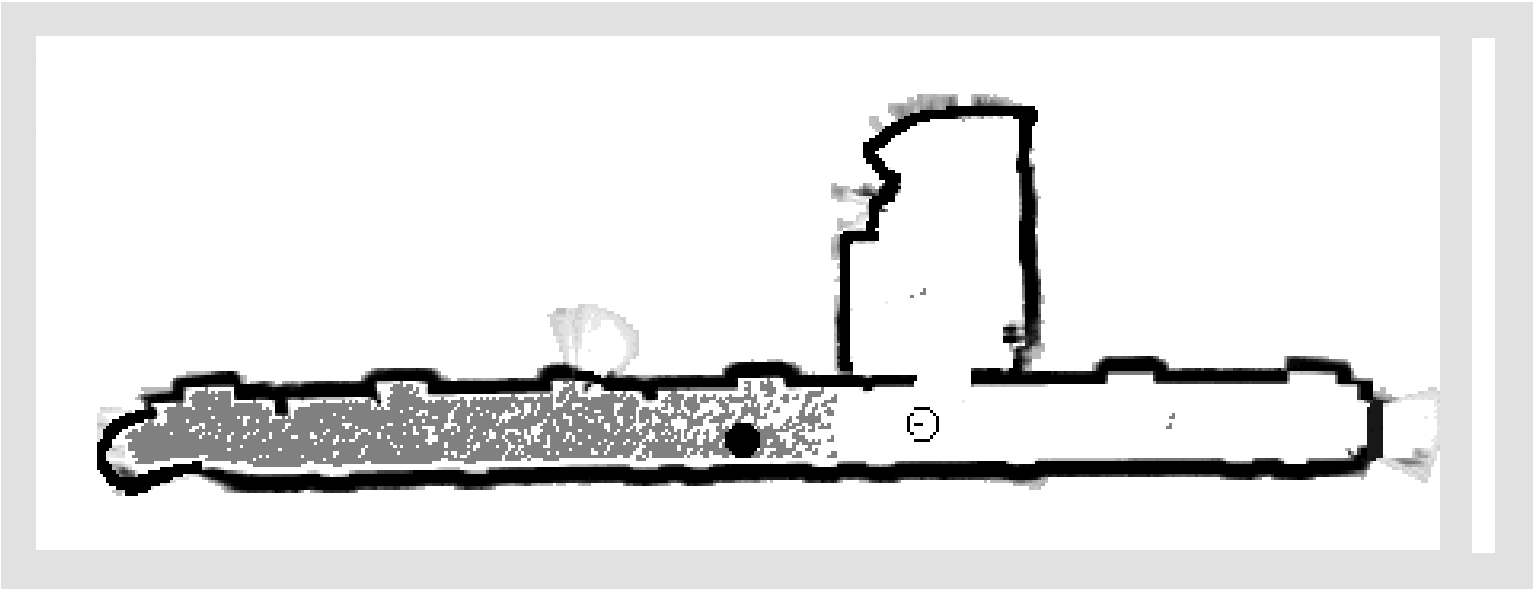








Dimensionality Reduction on Beliefs



Monte Carlo POMDPs

- Represent beliefs by samples
- Estimate value function on sample sets
- Simulate control and observation transitions between beliefs

Derivation of POMDPs Value Function Representation

$$V(b) = \sum_{i=1}^{N} v_i \, p_i$$

Piecewise linear and convex:

$$V(b) = \max_k \sum_{i=1}^{N} v_i^k \, p_i$$

Value Iteration Backup

Backup in belief space:

$$V_T(b) = \gamma \max_u r(b,u) + \int V_{T-1}(b')\, p(b' \mid u, b)\, db'$$

$$p(b' \mid u, b) = \int p(b' \mid u, b, z)\, p(z \mid u, b)\, dz$$

$$V_T(b) = \gamma \max_u r(b,u) + \int \left[\int V_{T-1}(b')\, p(b' \mid u, b, z)\, db' \right] p(z \mid u, b)\, dz$$

Belief update is a function:

$$\begin{aligned} B(b,u,z)(x') &= p(x' \mid z, u, b) \\ &= \frac{p(z \mid x', u, b)\, p(x' \mid u, b)}{p(z \mid u, b)} \\ &= \frac{1}{p(z \mid u, b)}\, p(z \mid x') \int p(x' \mid u, b, s)\, p(x \mid u, b)\, dx \\ &= \frac{1}{p(z \mid u, b)}\, p(z \mid x') \int p(x' \mid u, x)\, b(x)\, dx \end{aligned}$$

Derivation of POMDPs

$$V_T(b) \quad = \quad \gamma \, \max_u \, r(b,u) + \int V_{T-1}(B(b,u,z)) \, p(z \mid u,b) \, dz$$

Break into two components

$$\begin{aligned} V_T(b,u) &= \gamma \left[r(b,u) + \int V_{T-1}(B(b,u,z)) \, p(z \mid u,b) \, dz \right] \\ V_T(b) &= \max_u V_T(b,u) \end{aligned}$$

Finite Measurement Space

$$\begin{aligned} V_T(b,u) &= \gamma \left[r(b,u) + \sum_z V_{T-1}(B(b,u,z))\, p(z \mid u,b) \right] \\ V_T(b) &= \max_u V_T(b,u) \end{aligned}$$

$$B(b,u,z)(x') = \frac{1}{p(z \mid u,b)}\, p(z \mid x') \sum_x p(x' \mid u,x)\, b(x)$$

$$p'_j = \frac{1}{p(z \mid u,b)}\, p(z \mid x_j) \sum_{i=1}^{N} p(x_j \mid u,x_i)\, p_i$$

Starting at Previous Belief

$$
\begin{aligned}
V_{T-1}(B(b,u,z)) &= \max_k \sum_{j=1}^{N} v_j^k \, p'_j \\
&= \max_k \sum_{j=1}^{N} v_j^k \, \frac{1}{p(z \mid u,b)} \, p(z \mid x_j) \sum_{i=1}^{N} p(x_j \mid u, x_i) \, p_i \\
&= \frac{1}{p(z \mid u,b)} \max_k \sum_{j=1}^{N} v_j^k \, p(z \mid x_j) \sum_{i=1}^{N} p(x_j \mid u, x_i) \, p_i \\
&= \frac{1}{p(z \mid u,b)} \max_k \overbrace{\sum_{i=1}^{N} p_i \underbrace{\sum_{j=1}^{N} v_j^k \, p(z \mid x_j) \, p(x_j \mid u, x_i)}_{(*)}}^{(**)}
\end{aligned}
$$

*: constant

**: linear function in params of belief space

Putting it Back in

$$V_T(b,u) = \gamma \left[r(b,u) + \sum_z \max_k \sum_{i=1}^N p_i \sum_{j=1}^N v_j^k \, p(z \mid x_j) \, p(x_j \mid u, x_i) \right]$$

$$r(b,u) = E_x[r(x,u)] = \sum_{i=1}^N p_i \, r(x_i, u)$$

Maximization over Actions

$$\begin{aligned}
V_T(b) &= \max_u V_T(b,u) \\
&= \gamma \max_u \left[\sum_{i=1}^N p_i \, r(x_i,u) \right] + \sum_z \max_k \\
&\qquad \sum_{i=1}^N p_i \underbrace{\sum_{j=1}^N v_j^k \, p(z \mid x_j) \, p(x_j \mid u, x_i)}_{=: \; v_{u,z,i}^k} \\
&= \gamma \max_u \left[\sum_{i=1}^N p_i \, r(x_i,u) \right] + \underbrace{\sum_z \max_k \sum_{i=1}^N p_i \, v_{u,z,i}^k}_{(*)}
\end{aligned}$$

$$v_{u,z,i}^k = \sum_{j=1}^N v_j^k \, p(z \mid x_j) \, p(x_j \mid u, x_i)$$

Getting max in Front of Sum

$$\max\{a_1(x), \ldots, a_n(x)\} + \max\{b_1(x), \ldots, b_n(x)\}$$

$$\max_i \; \max_j \; a_i(x) + b_j(x)$$

$$\begin{aligned}
\sum_z \; \max_k \sum_{i=1}^{N} p_i \; v_{u,z,i}^{k} \quad &= \quad \max_{k(1)} \; \max_{k(2)} \cdots \max_{k(M)} \sum_z \; \sum_{i=1}^{N} p_i \; v_{u,z,i}^{k(z)} \\
&= \quad \max_{k(1)} \; \max_{k(2)} \cdots \max_{k(M)} \sum_{i=1}^{N} p_i \sum_z \; v_{u,z,i}^{k(z)}
\end{aligned}$$

Final Result

$$
\begin{aligned}
V_T(b) &= \gamma \max_u \left[\sum_{i=1}^N p_i \; r(x_i, u) \right] + \max_{k(1)} \max_{k(2)} \cdots \max_{k(M)} \sum_{i=1}^N p_i \sum_z v_{u,z,i}^{k(z)} \\
&= \gamma \max_u \max_{k(1)} \max_{k(2)} \cdots \max_{k(M)} \sum_{i=1}^N p_i \left[r(x_i, u) + \sum_z v_{u,z,i}^{k(z)} \right]
\end{aligned}
$$

Individual constraints:

$$
\left(\left[r(x_1, u) + \sum_z v_{u,z,1}^{k(z)} \right] \left[r(x_2, u) + \sum_z v_{u,z,2}^{k(z)} \right] \cdots \left[r(x_N, u) + \sum_z v_{u,z,N}^{k(z)} \right] \right)
$$

Algorithm POMDP(T):

$\Upsilon = (0, \ldots, 0)$
for $\tau = 1$ *to* T *do*
$\Upsilon' = \emptyset$
for all $(u'; v_1^k, \ldots, v_N^k)$ *in* Υ *do*
for all control actions u *do*
for all measurements z *do*
for $j = 1$ *to* N *do*

$$v_{j,u,z}^k = \sum_{i=1}^{N} v_i^k \; p(z \mid x_i) \; p(x_i \mid u, x_j)$$

endfor
endfor
endfor
endfor
for all control actions u *do*
for all $k(1), \ldots, k(M) = (1, \ldots, 1)$ *to* $(|\Upsilon|, \ldots, |\Upsilon|)$ *do*
for $i = 1$ *to* N *do*

$$v_i' = \gamma \left[r(x_i, u) + \sum_{z} v_{u,z,i}^{k(z)} \right]$$

endfor
add $(u; v_1', \ldots, v_N')$ *to* Υ'
endfor
endfor
optional: prune Υ'
$\Upsilon = \Upsilon'$
endfor
return Υ