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A Note on Fuzzy Sets

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Fuzzy sets are defined as mappings from sets into Boolean lattices. The basic set theory type results for Zadeh's fuzzy sets are shown to carry over. Some results on convex fuzzy sets, star-shaped fuzzy sets, and arcwise connected fuzzy sets are given. Fuzzy sets with "holes," bounded fuzzy sets, and connected fuzzy sets are also discussed.

1. INTRODUCTION

Zadeh, 1965, introduced the concept of *fuzzy sets* by defining them in terms of mappings from a set into the unit interval on the real line. Goguen, 1966, extended the concept by defining fuzzy sets to be functions from a set into a lattice. Fuzzy sets were introduced to provide a means of mathematically describing situations which give rise to ill-defined classes, i.e., "collections" of objects for which there is no precise criteria for membership. Collections of this type have vague or "fuzzy" boundaries; there are objects for which it is impossible to determine whether or not they belong to the collection. In real situations, especially in problems of pattern classification, fuzziness is the rule rather than the exception. It is believed that fuzzy sets can be applied at least as well and probably better to these problems than the methods now being used.

In this note fuzzy sets will be defined in terms of mappings into Boolean lattices. The properties of Boolean lattices which are assumed may be found in MacLane and Birkhoff, 1967. If B is a Boolean lattice and x and y are elements of B, the meet (greatest lower bound) of x and y is denoted by $x \wedge y$. The join (least upper bound) of x and y is denoted by $x \vee y$. For the partial order relation on B we use the ordinary symbol $\leqslant$. The zero of B is denoted by 0, and the infinity for B is denoted by I. The complement of an element x of B is denoted by x'.

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In Section 2 we show that the immediate set-theory type results satisfied by Zadeh's fuzzy sets carry over. In addition we are able to define the complement of a fuzzy set in a straightforward manner. Section 3 will deal with convex fuzzy sets. Zadeh's important separation theorem does not carry over. In Section 4 we consider α cuts, introduced in Section 3, as they apply to the properties of being (arcwise) connected, bounded, star-shaped, and "having holes."

2. Fuzzy Sets

In this section we define "fuzzy set" and derive some of the immediate properties.

Definition 1. Let X be a nonnull set and let B be a Boolean lattice. A *fuzzy set* A is a function $A : X \to B$.

Note that if B is the Boolean lattice consisting of only the points 0 and I then A is just the characteristic function of a subset of X. The same holds true if A only takes on the values 0 and I in some nontrivial Boolean lattice B.

For the following definitions assume that the set X and the Boolean lattice B are fixed.

Definition 2. A fuzzy set A is *empty* if and only if it is identically zero on X. The empty fuzzy set will be denoted by φ.

Definition 3. Two fuzzy sets C and D are *equal*, written $C = D$, if and only if $C(x) = D(x)$ for all $x \in X$.

Definition 4. The complement of a fuzzy set A is denoted by A' and is defined by $A'(x) = (A(x))'$ for all $x \in X$.

Definition 5. The fuzzy set A is contained in the fuzzy set B, written $A \subseteq B$, if and only if $A(x) \leqslant B(x)$ for all $x \in X$.

Definition 6. The *union* of two fuzzy sets A and B, written $A \cup B$, is the fuzzy set $C = A \cup B$ characterized by $C(x) = A(x) \vee B(x)$ for all $x \in X$.

Note that $A \cup B$ is the smallest fuzzy set containing both A and B. Indeed, if D is any fuzzy set containing both A and B, then for each $x \in X$, $A(x) \leqslant D(x)$ and $B(x) \leqslant D(x)$ so that $A(x) \vee B(x) \leqslant D(x)$. Hence $A \cup B \subseteq D$.

Definition 7. The *intersection* of two fuzzy sets A and B is the fuzzy set C, written $C = A \cap B$, defined by $C(x) = A(x) \wedge B(x)$ for all $x \in X$.

By a dual argument to that above we see that $A \cap B$ is the largest fuzzy set which is contained in both A and B. The next three theorems list the immediate properties of $\cup$, $\cap$, and $'$.

Theorem 1. *The operations of union and intersection are commutative, associative, and each distributes over the other. Symbolically, if A, B, C are fuzzy sets,*

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A,$$
$$A \cup (B \cup C) = (A \cup B) \cup C \text{ and } A \cap (B \cap C) = (A \cap B) \cap C,$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof. These properties follow immediately from the corresponding properties of $\wedge$ and $\vee$ in B. For example, let $D = A \cap (B \cup C)$. Then for each $x \in X$,

$$\begin{aligned} D(x) &= A(x) \wedge (B(x) \vee C(x)) \\ &= (A(x) \wedge B(x)) \vee (A(x) \wedge C(x)). \end{aligned}$$

Hence,

$$D = (A \cap B) \cup (A \cap C),$$

i.e.,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Theorem 2. *Denote the fuzzy set which is identically equal to I on X by U. If A is any fuzzy set, then*

1. $A \cup \varphi = A$;
2. $A \cap \varphi = \varphi$;
3. $A \cup U = U$;
4. $A \cap U = A$;
5. $A \cup A' = U$;
6. $A \cap A' = \varphi$;
7. $\varphi' = U$;
8. $U' = \varphi$.

Proof. These eight statements follow, respectively, from the following eight equalities. For each $x \in X$,

1. $A(x) \vee \varphi(x) = A(x) \vee 0 = A(x);$
2. $A(x) \wedge \varphi(x) = A(x) \wedge 0 = 0;$
3. $A(x) \vee U(x) = A(x) \vee I = I;$
4. $A(x) \wedge U(x) = A(x) \wedge I = A(x);$
5. $A(x) \vee A'(x) = A(x) \vee (A(x))' = I;$
6. $A(x) \wedge A'(x) = A(x) \wedge (A(x))' = 0;$
7. $\varphi'(x) = (\varphi(x))' = 0' = I;$
8. $U'(x) = (U(x))' = I' = 0.$

THEOREM 3. *The statement of DeMorgan's Laws also holds for fuzzy sets, i.e.,*

$$(A \cap B)' = A' \cup B'$$

and

$$(A \cup B)' = A' \cap B'.$$

Proof. Let A and B be fuzzy sets and let $x \in X$. If $C = (A \cap B)$ then

$$\begin{aligned} C'(x) &= (A(x) \wedge B(x))' \\ &= A'(x) \vee B'(x), \end{aligned}$$

so that

$$C' = A' \cup B',$$

i.e.,

$$(A \cap B)' = A' \cup B'.$$

The proof of the other law is dual.

We conclude this section by noting that since the set of functions from a set into a Boolean lattice is a Boolean lattice, the collection of all fuzzy sets on X into B is a Boolean lattice.

3. CONVEX FUZZY SETS

In this section we will define the concept of a convex fuzzy set and derive the immediate properties of this concept. First, we discuss the convex combination of fuzzy sets. Recalling that a convex combination of two vectors x and y usually means a linear combination of x and y of the form

$$\lambda x + (1 - \lambda) y,$$

we state the following definition. Again the set X and Boolean lattice B are assumed to be fixed.

DEFINITION 8. Let A, C, and Λ be fuzzy sets. The convex combination of A, C, and Λ is denoted by $(A, C; \Lambda)$ and is defined by the relation

$$(A, C; \Lambda) = (\Lambda \cap A) \cup (\Lambda' \cap C).$$

THEOREM 4. *Let A and C be fuzzy sets. Then for all fuzzy sets Λ,*

$$A \cap C \subseteq (A, C; \Lambda) \subseteq A \cup C.$$

Proof. By definition,

$$(A, C; \Lambda) = (\Lambda \cap A) \cup (\Lambda' \cap C).$$

But

$$\begin{aligned}(\Lambda \cap A) \cup (\Lambda' \cap C) &= ((\Lambda \cap A) \cup \Lambda') \cap ((\Lambda \cap A) \cup C)\\ &= (\Lambda \cup \Lambda') \cap (A \cup \Lambda') \cap (\Lambda \cup C) \cap (A \cup C).\end{aligned}$$

Since the right side is contained in $A \cup C$ we have,

$$(A, C; \Lambda) \subseteq A \cup C.$$

Moreover,

$$A \cap C \cap \Lambda \subseteq A \cap \Lambda \subseteq (A, C; \Lambda)$$

and

$$A \cap C \cap \Lambda' \subseteq C \cap \Lambda' \subseteq (A, C; \Lambda).$$

Therefore,

$$A \cap C \subseteq (A, C; \Lambda).$$

In order to have a meaningful definition of convexity in fuzzy sets we assume that X is a real Euclidean space E^n. Of course, any nontrivial convex subset of E^n would do as well. Henceforth we assume that the Boolean Lattice, B, is fixed.

DEFINITION 9. A fuzzy set A is convex if and only if

$$A[\gamma x_1 + (1 - \gamma) x_2] \geqslant A(x_1) \wedge A(x_2)$$

for all x_1, $x_2 \in X$ and all $\gamma \in [0, 1]$.

THEOREM 5. *A fuzzy set A is convex if and only if the sets*

$$\Gamma_\alpha = \{x \mid A(x) \geqslant \alpha\}$$

are convex in X for all $\alpha \in B$, $\alpha \neq 0$.

Proof. Assume A is a convex fuzzy set, i.e.,

$$A[\gamma x_1 + (1 - \gamma)x_2] \geqslant A(x_1) \wedge A(x_2) \tag{1}$$

for all $x_1, x_2 \in X$ and all $\gamma \in [0, 1]$. Let $\alpha \in B$, $\alpha \neq 0$ be given and consider

$$\Gamma_\alpha = \{x \mid A(x) \geqslant \alpha\}. \tag{2}$$

If Γ_α is empty or contains only one point it is obviously convex. Assume $x_1, x_2 \in \Gamma_\alpha$, i.e., $A(x_1) \geqslant \alpha$ and $A(x_2) \geqslant \alpha$. Then $A(x_1) \wedge A(x_2) \geqslant \alpha$. Hence by (1)

$$A[\gamma x_1 + (1 - \gamma)x_2] \geqslant \alpha$$

for all $\gamma \in [0, 1]$. Thus all points of the form $\gamma x_1 + (1 - \gamma)x_2$, $\gamma \in [0, 1]$ belong to Γ_α and Γ_α is convex in X.

Conversely, assume the sets Γ_α, $\alpha \in B$, $\alpha \neq 0$ are convex in X. Let $x_1, x_2 \in X$ and let $\alpha = A(x_1) \wedge A(x_2)$. If $\alpha = 0$, condition (1) is obviously satisfied. If $\alpha \neq 0$, then $A(x_1) \geqslant \alpha$ and $A(x_2) \geqslant \alpha$ so that $x_1, x_2 \in \Gamma_\alpha$. By convexity of Γ_α, $\gamma x_1 + (1 - \gamma)x_2 \in \Gamma_\alpha$ for all $\gamma \in [0, 1]$. Hence $A[\gamma x_1 + (1 - \gamma)x_2] \geqslant \alpha = A(x_1) \wedge A(x_2)$, i.e., A is a convex fuzzy set.

A simple direct argument will establish the following theorem.

THEOREM 6. *If A and C are convex fuzzy sets, so is their intersection.*

4. PROPERTIES DETERMINED BY α CUTS

In the previous section we saw that a fuzzy set A is convex if and only if its α cuts, the sets $\Gamma_\alpha = \{x \in X \mid A(x) \geqslant \alpha\}$, are convex. A natural question is "Which properties of a fuzzy set are determined by its α cuts?" We show that arcwise connectedness and the property of being star-shaped are two such properties. We also discuss the relationship between α cuts and boundedness, connectedness, and "having a hole." Throughout this section B is a fixed Boolean lattice and X is a real Euclidean space E^n.

DEFINITION 10. A fuzzy set A is arcwise connected if and only if for every pair of elements $x_1, x_2 \in X$ there is a Jordan arc $\pi : [0, 1] \to X$ such that $\pi(0) = x_1$, $\pi(1) = x_2$, and $A(\pi(\gamma)) \geqslant A(x_1) \wedge A(x_2)$ for all $\gamma \in [0, 1]$.

THEOREM 7. *A is arcwise connected if and only if the sets*

$$\Gamma_\alpha = \{x \mid A(x) \geqslant \alpha\}$$

are arcwise connected for all $\alpha \in B$, $\alpha \neq 0$.

The proof of Theorem 7 is a direct analog of the proof of Theorem 5.

DEFINITION 11. A fuzzy set A is star-shaped with respect to a point $x_0 \in X$ if and only if

$$A(\gamma x + (1 - \gamma)x_0) \geqslant A(x) \wedge A(x_0)$$

for all $x \in X$ and all $\gamma \in [0, 1]$.

THEOREM 8. *A fuzzy set A is star-shaped with respect to a point $x_0 \in X$ if and only if the sets*

$$\Gamma_\alpha = \{x \mid A(x) \geqslant \alpha\}$$

are star-shaped with respect to x_0 for all $\alpha \in B$ such that $0 < \alpha \leqslant A(x_0)$.

The proof of Theorem 8 is also a direct analog of the proof of Theorem 5. To define boundedness for fuzzy sets we use $|x|$ to mean the ordinary Euclidean norm of x in X. We give two separate definitions to include the case where A has value 0 except on a bounded subset of X.

DEFINITION 12. A fuzzy set A is strictly bounded if and only if there is a real number r such that $A(x) = 0$ for all $x \in X$ with $|x| \geqslant r$.

DEFINITION 13. A fuzzy set A is bounded with degree α_0 if and only if there is a real number r such that $A(x) < \alpha_0$ for all $x \in X$ with $|x| \geqslant r$.

Definition 13 is obviously equivalent to

THEOREM 9. *A fuzzy set A is bounded with degree α_0 if and only if the sets*

$$\Gamma_\alpha = \{x \mid A(x) \geqslant \alpha \geqslant \alpha_0\}$$

are bounded (indeed, if and only if the set $\Gamma_{\alpha_0} = \{x \mid A(x) \geqslant \alpha_0\}$ is bounded).

Connectedness and "having a hole" are not easily characterized by considering the image space of a fuzzy set. However, reasonable definitions can be obtained by considering α cuts. It is assumed that a "hole" in E^n is a Jordan region.

DEFINITION 14. A fuzzy set A has a hole if and only if the set

$$\Gamma = \{x \mid A(x) = 0\}$$

is a Jordan region.

DEFINITION 15. A fuzzy set A has a hole of degree α_0 if and only if the set

$$\bar{\Gamma}_{\alpha_0} = \{x \mid A(x) \leqslant \alpha_0\}$$

is a Jordan region.

It is clear that the fuzzy set A has a hole of degree α_0 if and only if the set $\Gamma_{\alpha_0} = \{x \mid A(x) > \alpha_0\}$ has a hole in E^n.

Connectedness could be defined by stating that a fuzzy set is connected if and only if all its α cuts are connected. Also, connectedness with degree α_0 could be defined in terms of α cuts for $\alpha \geqslant \alpha_0$.

Further interesting results on fuzzy sets should be obtainable by topologizing the Boolean lattice as has been done by Stone, 1937. We plan to look into this at some future time.

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