CS571: Artificial Intelligence

Formal Systems

Pushpak Bhattacharyya
CSE Dept.,
IIT Patna

23 August 2018

Theory of CS

- Theory A
  - Logic
- Theory B
  - Algorithm an Complexity

Concepts, Axioms, Rule

- Some foundational questions for Mechanization or Automation of Knowledge Representation and Reasoning:
  - What are symbols and concepts (well formed formulae)
  - What are the self evident and ground truths in the system (axiomatization)
  - What is the validity of the inference (soundness and consistency)
  - Is the inference system powerful enough to capture reality (completeness)
  - Can it be implemented in Turing machine (derivability and complexity)

Case study: Propositional calculus

Propositions

- Stand for facts/assertions
- Declarative statements
  - As opposed to interrogative statements (questions) or imperative statements (request, order)

Operators

AND ($\wedge$), OR ($\vee$), NOT ($\neg$), IMPLICATION (=>)

=> and $\neg$ form a minimal set (can express other operations)

- Prove it.

Tautologies are formulae whose truth value is always T, whatever the assignment is

Model

In propositional calculus any formula with n propositions has 2^n models (assignments)
- Tautologies evaluate to T in all models.

Examples:

1) $P \vee \neg P$

2) $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

-e Morgan with AND

Example

- Prove $\sim(P \wedge Q) \rightarrow (\sim P \vee \sim Q)$ is a Tautology.

Q	P	$L = \sim(P \wedge Q)$	$R = \sim P \vee \sim Q$	$L \rightarrow R$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

Formal Systems

- Rule governed
- Strict description of structure and rule application
- Constituents
  - Symbols
  - Well formed formulae
  - Inference rules
  - Assignment of semantics
  - Notion of proof
  - Notion of soundness, completeness, consistency, decidability etc.

Hilbert's formalization of propositional calculus

1. Elements are *propositions* : Capital letters
2. Operator is only one : $\rightarrow$ (called implies)
3. Special symbol F (called 'false')
4. Two other symbols : *'('* and *')'*
5. Well formed formula is constructed according to the grammar

$$WFF \rightarrow P | F | WFF \rightarrow WFF$$

6. Inference rule : only one

Given $A \rightarrow B$ and

A

write B

known as MODUS PONENS

7. Axioms : Starting structures

A1: $(A \rightarrow (B \rightarrow A))$

A2: $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$

A3 $(((A \rightarrow F) \rightarrow F) \rightarrow A)$

This formal system defines the propositional calculus

Notion of proof

1. Sequence of well formed formulae
2. Start with a set of hypotheses
3. The expression to be proved should be the last line in the sequence
4. Each intermediate expression is either one of the hypotheses or one of the axioms or the result of modus ponens
5. An expression which is proved only from the axioms and inference rules is called a THEOREM within the system

Example of proof

From P and $P \to Q$ and $Q \to R$ prove R

H1: P

H2: $P \to Q$

H3: $Q \to R$

i) P	H1
ii) $P \to Q$	H2
iii) Q	MP, (i), (ii)
iv) $Q \to R$	H3
v) R	MP, (iii), (iv)

Prove that $(P \rightarrow P)$ is a THEOREM

i) $P \rightarrow (P \rightarrow P)$ A1 : P for A and B

ii) $P \rightarrow ((P \rightarrow P) \rightarrow P)$ A1: P for A and $(P \rightarrow P)$ for B

iii) $[(P \rightarrow ((P \rightarrow P) \rightarrow P)) \rightarrow ((P \rightarrow (P \rightarrow P)) \rightarrow (P \rightarrow P))]$

A2: with P for A, $(P \rightarrow P)$ for B and P for C

iv) $(P \rightarrow (P \rightarrow P) \rightarrow (P \rightarrow P))$ MP, (ii), (iii)

v) $(P \rightarrow P)$ MP, (i), (iv)

Shorthand

1. $\neg P$ is written as $P \rightarrow F$ and called '*NOT P*'
2. $((P \rightarrow F) \rightarrow Q)$ is written as $(P \vee Q)$ and called

   '*P OR Q*'
3. $((P \rightarrow (Q \rightarrow F)) \rightarrow F)$ is written as $(P \wedge Q)$ and called

   '*P AND Q*'

Exercise: (Challenge)

- Prove that $A \rightarrow \neg(\neg(A))$

A very useful theorem (Actually a meta theorem, called deduction theorem)

Statement

If

$A_1, A_2, A_3 \ldots\ldots\ldots\ldots A_n \vdash B$

then

$A_1, A_2, A_3, \ldots\ldots\ldots\ldots\ldots A_{n-1} \vdash A_n \rightarrow B$

$\vdash$ is read as 'derives'

Given

A_1
A_2
A_3
.
.
.
.
A_n
B

Picture 1

A_1
A_2
A_3
.
.
.
.
A_{n-1}
$A_n \rightarrow B$

Picture 2

Use of Deduction Theorem

Prove

$$A \rightarrow \neg(\neg(A))$$

i.e., $$A \rightarrow ((A \rightarrow F) \rightarrow F)$$

$A, A \rightarrow F \vdash \mathrm{F}$ (M.P)

$\mathrm{A} \vdash (A \rightarrow F) \rightarrow F$ (D.T)

$\vdash A \rightarrow ((A \rightarrow F) \rightarrow F)$ (D.T)

Very difficult to prove from first principles, *i.e.*, using axioms and inference rules only

Prove $P \rightarrow (P \vee Q)$

i.e. $P \rightarrow ((P \rightarrow F) \rightarrow Q)$

$P, P \rightarrow F, Q \rightarrow F \vdash F$

$P, P \rightarrow F \vdash (Q \rightarrow F) \rightarrow F$ (D.T)

$\vdash Q$ (M.P with A3)

P $\vdash (P \rightarrow F) \rightarrow Q$

$\vdash P \rightarrow ((P \rightarrow F) \rightarrow Q)$

More proofs

$$1.\ (P \wedge Q) \rightarrow (P \vee Q)$$

$$2.\ (P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$$

$$3.\ (P \rightarrow Q) \rightarrow ((\neg Q \rightarrow P) \rightarrow Q)$$

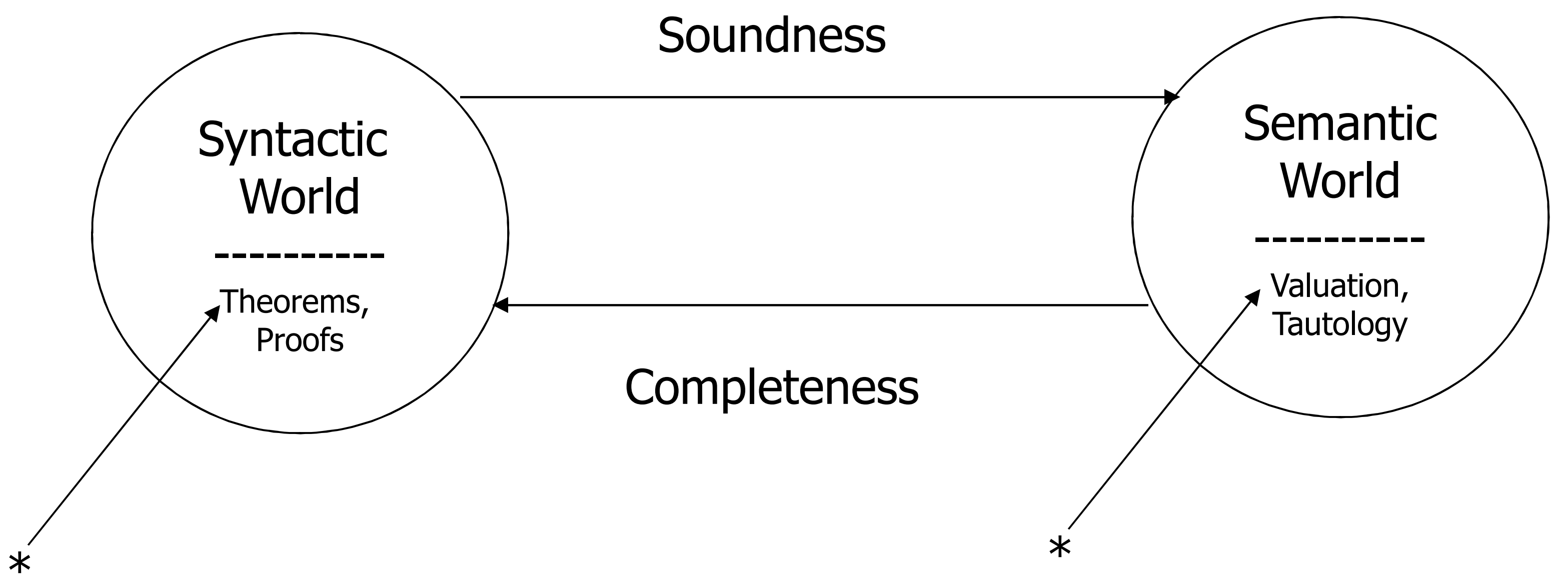
Important to note

- Deduction Theorem is a meta-theorem (statement **about** the system)
- $P \rightarrow P$ is a theorem (statement **belonging to** the system**)**
- The distinction is crucial in AI
- Self reference, diagonalization
- Foundation of Halting Theorem, Godel Theorem etc.

Example of '*of-about*' confusion

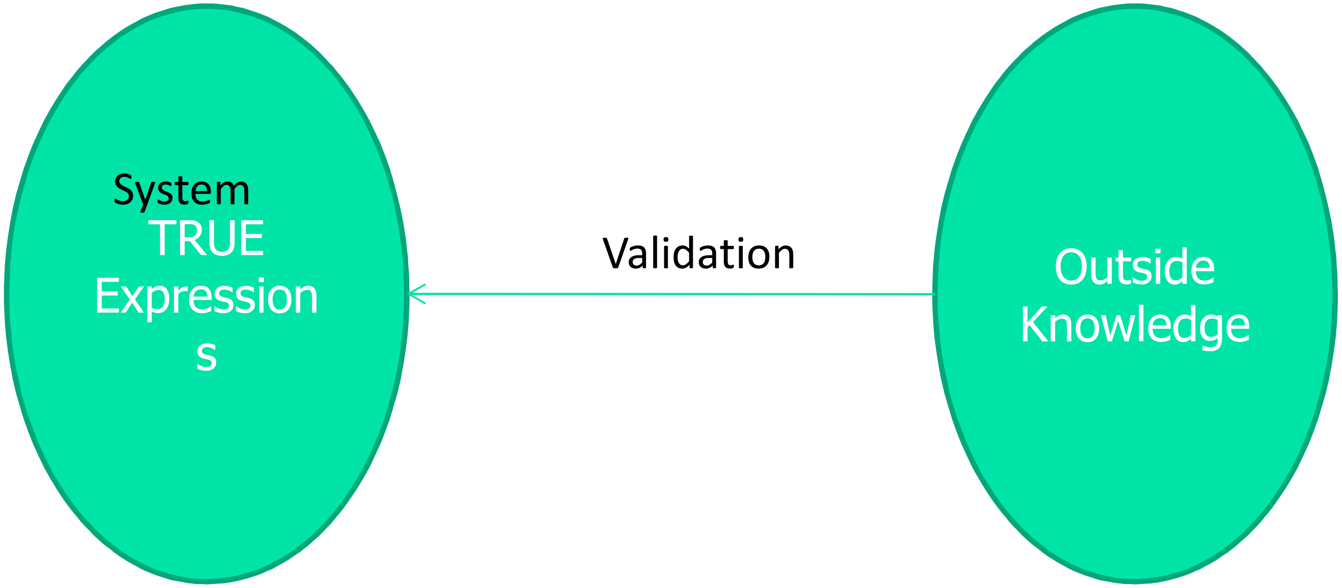
- "*This statement is false*"
- Truth of falsity cannot be decided

Soundness, Completeness & Consistency



- <u>Soundness</u> The soundness says that if a something is provable it means there has to be some meaning or truth realted to it. It should make sense. T
  - Provability $\longrightarrow$ Truth
- <u>Completeness</u> Completeness means that if there is athing which is true then it is provable in the system
  - Truth $\longrightarrow$ Provability

- <u>Soundness:</u> Correctness of the System
  - Proved entities are indeed true/valid
- <u>Completeness:</u> Power of the System
  - True things are indeed provable


System
TRUE
Expression
s
Validation
Outside
Knowledge

Consistency

The System should not be able to

prove both P and ~P, *i.e.*, should not be

able to derive

$\boxed{F}$

Examine the relation between

Soundness

&

Consistency

Soundness $\equiv$ Consistency

If a System is inconsistent, *i.e.*, can derive $\mathcal{F}$, it can prove any expression to be a theorem. Because

$\mathcal{F} \rightarrow P$ is a theorem

Inconsistency→Unsoundness

To show that

$\mathcal{F} \rightarrow P$ is a theorem

Observe that

$\mathcal{F}, \ P \rightarrow \mathcal{F} \vdash \mathcal{F}$ By D.T.

$\mathcal{F} \vdash (P \rightarrow \mathcal{F}) \rightarrow \mathcal{F}; \ \mathcal{A}3$

$\vdash P$

i.e. $\vdash \mathcal{F} \rightarrow P$

Thus, inconsistency implies unsoundness

Unsoundness→Inconsistency

- Suppose we make the Hilbert System of propositional calculus unsound by introducing *(A /| B)* as an axiom
- Now AND can be written as
  - $(A \rightarrow (B \rightarrow \mathcal{F})) \rightarrow \mathcal{F}$
- If we assign $\mathcal{F}$ to A, we have
  - $(\mathcal{F} \rightarrow (B \rightarrow \mathcal{F})) \rightarrow \mathcal{F}$
  - But $(\mathcal{F} \rightarrow (B \rightarrow \mathcal{F}))$ is an axiom (A1)
  - Hence $\mathcal{F}$ is derived

Inconsistency is a <u>Serious</u> issue.

Informal Statement of Godel Theorem:

If a sufficiently powerful system is complete it is inconsistent.

Sufficiently powerful: Can capture at least Peano Arithmetic

Introduce Semantics in Propositional logic

Valuation Function V

<u>Definition of V</u>

Things thing takes in the value of the expresison, i.e. the expression whcih is called syntax

Syntactic 'false

$$V(\boxed{\mathcal{F}}) = F$$

Semantic 'false'

Where F is called 'false' and is one of the two symbols (T, F)

$V(\mathcal{F}) = F$

$V(A \rightarrow B)$ is defined through what is called the truth table

V(A)	V(B)	V(A→B)
T	F	F
T	T	T
F	F	T
F	T	T

Tautology

An expression 'E' is a tautology if

$$V(E) = T$$

for all valuations of constituent propositions

Each 'valuation' is called a 'model'.

To see that

$(\mathscr{F} \rightarrow P)$ is a tautology

two models

$V(P) = T$
$V(P) = F$

$V(\mathscr{F} \rightarrow P) = T$ for both

$\mathcal{F} \rightarrow P$ is a theorem

Soundness $\downarrow$ $\uparrow$ Completeness

$\mathcal{F} \rightarrow P$ is a tautology

If a system is Sound & Complete, it does not

<u>matter</u> how you “Prove” or “show the validity”

Take the Syntactic Path or the Semantic Path

Syntax vs. Semantics issue

Refers to

FORM VS. CONTENT

Form

Tea

(Content)

Form & Content

painter

musician

logician

Godel, Escher, Bach

By D. Hofstadter

Problem

$$(P \wedge Q) \rightarrow (P \vee Q)$$

Semantic Proof

P	Q	A $P \wedge Q$	B $P \vee Q$	$A \rightarrow B$
T	F	F	T	T
T	T	T	T	T
F	F	F	F	T
F	T	F	T	T

To show syntactically

$$(P \wedge Q) \rightarrow (P \vee Q)$$

i.e.

$$[(P \rightarrow (Q \rightarrow \mathcal{F})) \rightarrow \mathcal{F}] \rightarrow [(P \rightarrow \mathcal{F}) \rightarrow Q]$$

If we can establish

$$(P \rightarrow (Q \rightarrow \mathcal{F})) \rightarrow \mathcal{F},$$

$$(P \rightarrow \mathcal{F}), Q \rightarrow \mathcal{F} \vdash \mathcal{F}$$

This is shown as

$$Q \rightarrow \mathcal{F} \quad \textit{hypothesis}$$

$$(Q \rightarrow \mathcal{F}) \rightarrow (P \rightarrow (Q \rightarrow \mathcal{F})) \; A1$$

$Q \rightarrow F$; hypothesis

$(Q \rightarrow F) \rightarrow (P \rightarrow (Q \rightarrow F))$; A1

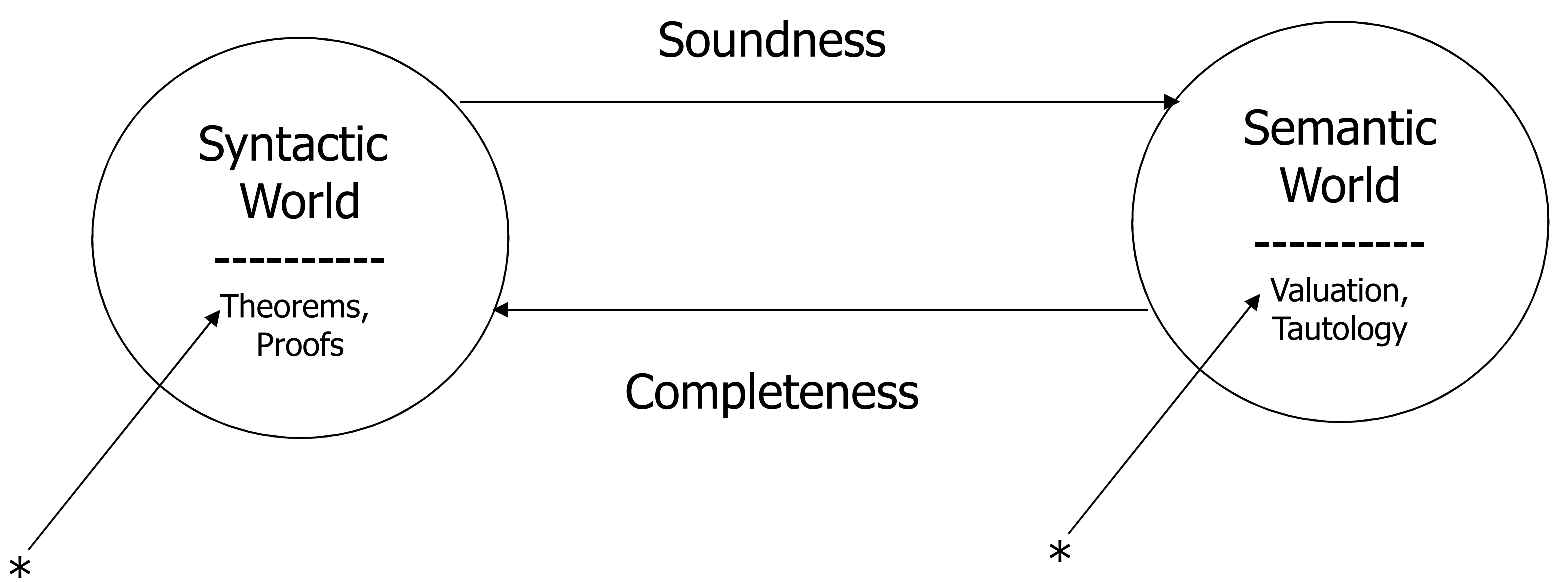
$P \rightarrow (Q \rightarrow F)$; MP

F; MP

Thus we have a proof of the line we started with

Soundness and Completeness proofs

Soundness, Completeness & Consistency



Introduce Semantics in Propositional logic

Valuation Function V

<u>Definition of V</u>

Syntactic ‘false

$$V(\mathscr{F}) = F$$

Semantic ‘false’

Where F is called ‘false’ and is one of the two symbols (T, F)

V($\mathscr{F}$) = F

V(A→B) is defined through what is called the truth table

V(A)	V(B)	V(A→B)
T	F	F
T	T	T
F	F	T
F	T	T

Tautology

An expression 'E' is a tautology if

$$V(E) = T$$

for all valuations of constituent propositions

Each 'valuation' is called a 'model'.

- <u>Soundness</u>
  - Provability $\longrightarrow$ Validity
- <u>Completeness</u>
  - Validity $\longrightarrow$ Provability

- <u>Soundness:</u> Correctness of the System
  - Proved entities are indeed valid
- <u>Completeness:</u> Power of the System
  - Valid things are indeed provable

Consistency

The System should not be able to prove both P and ~P, *i.e.*, should not be able to derive

$$\boxed{F}$$

Examine the relation between

Soundness

&

Consistency

Soundness $\equiv$ Consistency

If a System is inconsistent, *i.e.,* can derive

$\mathcal{F}$, it can prove any expression to be a

theorem. Because

$\mathcal{F} \rightarrow$ P is a theorem

If a system is Sound & Complete, it does not

<u>matter</u> how you “Prove” or “show the validity”

Take the Syntactic Path or the Semantic Path