Home Work Assignment - 3 Dhayalini Nagaraj

A20359686.

1. Consider an ordinary binary min heap data structure with n elements supporting the instructions INSERT and EXTRACT-MIN in $O(\lg n)$ worst case time. Give a potential function Φ such that the amortized cost of INSERT is $O(\lg n)$ and the amortized cost of EXTRACT-MIN is $O(1)$ and show that it works.

Let initial data structure be D_0. For each $i = 1, 2 \ldots, n$, c_i be the actual cost of i^{th} operation. D_i - is the resulting data structure. Amortized cost $\hat{c}_i$ with respect to potential function Φ is defined by.

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_i - 1)$$

Amortized cost = Actual cost + Change in potential due to the operation

Let k be a constant.

So each INSERT and EXTRACT-MIN ~~cost~~ operations will atmost take

<u>k lg n times.</u>

Let x be a node in D_i. Then let $d_i(x)$ denote the depth of the node x.

So,

$$\cancel{\Phi(D_i) = \Sigma}$$

$$\therefore \quad \Phi(D_i) = \sum_{x \in D_i} k(d_i(x)+1)$$

$$= k\left(n_i + \sum_{x \in D_i} d_i(x)\right)$$

$$\Phi(D_i) = k\left(n_i + \sum_{x \in D_i} d_i(x)\right)$$

D_0 is the initial data structure then $\Phi(D_0) = 0$. Initially the heap is empty so $\Phi(D_0) = 0$.

$\Phi(D_i) \geq 0$ always.

After an <u>INSERT</u> is done, the sum will change only by an amount equal to the new last node in the heap. This is equal to $[\lg n_i]$.

Thus, the change in potential due to the operation INSERT is

$$k(1 + [\lg n_i]).$$

Then the,

$$\text{Amortized cost} = O(\lg n_i) + O(\lg n_i)$$
$$= O(\lg n_i)$$
$$= O(\lg n)$$

After EXTRACT-MIN operation is done, the sum will change by the negative of the depth of old last node in the heap.

Then the potential will decrease by $k(1 + [\lg n_i])$ which is due to EXTRACT-MIN.

The Amortized cost is atmost

~~atmost is~~

$$= k \lg n_i - k(1 + [\lg n_i])$$
$$= O(1)$$

2. We want to maintain a collection of k dynamic tables in a sequential segment of memory so that the amortized cost of an insertion or deletion is $O(k)$.

Potential of T as

$$\Phi(T) = \sum_{i=1}^{k} |size(T_i) - num(T_i) - N|$$

a) Prove that the contribution of table T_i to $\Phi(T)$ is zero immediatly after it expands or contracts.

Let us assume c as the actual cost of an operation ~~and~~

α be the constant.

Given

$Size(T_i') = num(T_i') + N'$ immediately after an insertion or after the contraction.

(ie) $Size(T_i') = num(T_i') + N'$ after expansion or contraction.

Then the contribution of the table T_i is

$$|\, size(T_i') - num(T_i') - N' \,|$$

$$= |\, num(T_i') + N' - num(T_i') - N' \,|$$

$$= 0$$

Ans: 0

b) Prove that the amortized cost of an insertion that does not require an expansion is at most $k + O(1)$.

Amortized cost = Actual cost + change in potential due to the operation

$$\hat{c} = c + \Phi(T') - \Phi(T)$$

c - Actual cost.

Given: If $size(T_i) > num(T_i)$ - there is room for new element.

If $size(T_i) = num(T_i)$ - then table is full.

~~Let~~ Let us ~~co~~ consider T_j, and cost of insertion does not require ~~the~~ ~~a~~ an expansion.

~~The amortized cost~~

The actual cost here is $O(1)$

The term corresponding to the table T_j in which the insertion occurs, the change in potential is

$$|size(T_j') - num(T_j') - N'| - |size(T_j) - num(T_j) - N|$$

$$= |size(T_j) - (num(T_j)+1) - (N+1)| - |size(T_j) - num(T_j) - N|$$

$$\leq 2$$

The remaining $k-1$ tables will contribute 1 to change in potential.

(ie)

$$|size(T_i') - num(T_i') - N'| - |size(T_i) - num(T_i) - N|$$

$$= |size(T_i) - num(T_i) - (N+1)| - |size(T_i) - num(T_i) - N|$$

$$\leq 1$$

$\therefore$ Change in potential is atmost $k+1$ in this case

$$\hat{c} = c + \Phi(T') - \Phi(T)$$

T_j be the table where insertion occurs.

$$\leq O(1) + \alpha \sum_{i=1 \wedge i \neq j}^{k} \left((num(T_i') + N' - size(T_i')) - (num(T_i) + N - size(T_i)) \right) +$$

$$\alpha \left((num(T_j') + N' - size(T_j')) - (num(T_j) + N - size(T_j)) \right)$$

Scanned by CamScanner

$= O(1) + \alpha(k-1) + 2\alpha$

$= K + O(1)$.

c) Prove that the amortized cost of an insertion that requires an expansion is $O(K)$

When the insertion requires and expansion the table doubles,

(ie) The size of the table is double.

$$\hat{c} = c + \Phi(T') - \Phi(T)$$

α is a constant.

The actual cost for sequence of n insert is

$O(N)$.

Let T_j be the table where the insertion occurs then the change in potential is

$|size(T_j') - num(T_j') - N'| - |size(T_j) - num(T_j) - N|$

$= |(num(T_j) + N + 2) - (num(T_j) + 1) - (N+1)|$

$- |num(T_j) - num(T_j) - N|$

$= 0 - N = -N$

The remaining $K-1$ contribute atmost 1 to the change in potential.

Therefore change in potential is

$\Phi < K-1-N$.

Then $\hat{c} = O(N) +$ change in potential.

$\hat{c} = c + \Phi(T') - \Phi(T)$

Let T_j be the table where insertion occurs.

$\leq O(N) + \alpha \sum_{i=1 \wedge i \neq j}^{K} ((num(T_i') + N' - size(T_i')) - (num(T_i) + N - size(T_i))) + \alpha((num(T_j') + N' - size(T_j')) - (num(T_j) + N - size(T_j)))$

$= O(N) + \alpha(K-1) - \alpha \cdot N$

$= O(K)$

d.) Prove that the amortized cost of deletion is $O(K)$

(i) No contraction.

If $num(T_i) > size(T_i) - 2N + 2$

then the element is simply deleted.

Let α be a constant.

$$\hat{c} = c + \Phi(T') - \Phi(T)$$

$$\le O(1) + \alpha \sum_{i=1 \wedge i \neq j}^{k} ((size(T_i') - num(T_i') - N')$$

$$- (size(T_i) - num(T_i) - N)) +$$

$$\alpha((size(T_i') - num(T_i') - N') - (size(T_i) - num(T_i) - N))$$

$$= O(1) + \alpha(k-1) + 2\alpha$$

$$= O(k)$$

(ii) When there is no contraction where deletion occurs

The change in potential for T_i is

~~Amortized cost of the~~

$$|size(T_i') - num(T_i') - N'| - |size(T_i) - num(T_i) - N|$$

$$= |size(T_i) - (num(T_i) - 1) - (N-1)| - |size(T_i) - num(T_i) - N|$$

$$\le 2$$

For the other k-1 tables

$$|size(T_i') - num(T_i') - N'| - |size(T_i) - num(T_i) - N|$$

$$= |size(T_i) - num(T_i) - (N-1)| - |(size(T_i) - num(T_i) - N|$$

$$\le 1$$

The change in potential is at most k+1 in this case.

$\therefore$ Amortized cost is $O(K)$

ii) The table in which deletion occurs and is also contracted.

Let S be the set of indexes of tables contracted and $s = |S|$

$$\hat{c} = c + \Phi(T') - \Phi(T)$$

$$\leq O(N) + \alpha \sum_{i \notin S} ((size(T_i') - num(T_i') - N') - (size(T_i) - num(T_i) - N)) + \alpha \sum_{i \in S} ((size(T_i') - num(T_i') - N') - (size(T_i) - num(T_i) - N))$$

$$= O(N) + \alpha(K - s) - \alpha \sum_{i \in S} (size(T_i) - num(T_i) - N)$$

$\forall i \in S$, $num(T_i) \leq size(T_i) - 2N + 2$

That is

$$size(T_i) - num(T_i) - N \geq N - 2$$

$$\hat{c} \leq O(N) + \alpha(K - s) - \alpha s(N - 2)$$

$$= O(K)$$

The change in potential for T_i

$$|size(T_i') - num(T_i') - N'| - |size(T_i) - num(T_i) - N|$$

$$\leq |(num(T_i) + N - 2) - (num(T_i) - 1) - (N-1)|$$

$$- |(num(T_i) + 2N - 2) - num(T_i) - N|$$

$$= 0 - |N-2|$$

$$= 2 - N$$

Contribution of $k-n$ non shrinking tables is atmost 1.

The contribution of remaining $n-1$ tables in potential change.

$$|size(T_i') - num(T_i') - N'| - |size(T_i) - num(T_i) - N|$$

$$\leq |(num(T_i) + N - 1) - num(T_i) - (N-1)| - |(num(T_i) + 2N - 1) - num(T_i) - N|$$

$$= 0 - |N-1|$$

$$= 1 - N$$

Thus change in potential is $\leq (2-N) + (k-n) + (n-1)(1-N)$

so amortized cost is $O(N) + k + 1 - nN$.

That is $\hat{c} = O(k)$

(iii) Some tables contract but the table that is the site of the deletion is not one of them.

$$\hat{c} = c + \Phi(T') - \Phi(T)$$

$$\le O(N) + \alpha \sum_{i \notin S} ((size(T_i') - num(T_i') - N') - (size(T_i) - num(T_i) - N)) + \alpha \sum_{i \in S/\{j\}} ((size(T_i') - num(T_i') - N') - (size(T_i) - num(T_i) - N))$$

$$+ \alpha ((size(T_j') - num(T_j') - N') - (size(T_j) - num(T_j) - N))$$

$$= O(N) + \alpha(k-s) + \alpha \sum_{i \in S/\{j\}} (size(T_i) - num(T_i) - N) + 2\alpha$$

$$\le O(N) + \alpha(k-s) - \alpha(s-1)(N-2) + 2\alpha$$

$$= O(k)$$

where S be the set of indexes of tables contracted and $s = |S|$ and α be a constant.

The contribution of T_i term to the potential is 2 and there are 'n' shrinking tables and each of the table contributes at most $1-N$ to the potential. Each of the remaining $k-n-1$ non shrinking tables contribute at most 1.

Thus, change in potential is $\le 2 + n(1-N) + (k-n-1)$ $= k - nN + 1$. $\therefore$ Amortized cost is $O(k)$.

e) Give a sequence of m insertions / deletions that requires $\Omega(km)$

Given a set of empty tables then for all ~~insertion~~ T_i which belongs to T

$size(T_i) = num(T_i) = 0$.

k elements can be inserted into the the table sequentially.

So the i^{th} element is inserted into the i^{th} table and $i-1$ time to relocate tables then the sequence of insertion takes

$$\sum_{i=1}^{k} (1+i-1) + k = k(k+1)/2$$

To delete all the elements from the table such that for all T_i which belong to T..

$size(T_i) = num(T_i) = 0$ after the sequence of deletions. It takes time k.

By repeating the sequence of insertions and deletions for n times.

Let m be $2nk$

$m = 2nk$

The total time required is

$$n(k(k+1)/2 + k) = n\left(\frac{k(k+1)+2k}{2}\right)$$

$$= \cancel{n(k+3)}$$

$$= 2n\left(\frac{k^2+k+2k}{4}\right)$$

$$= 2nk\left(\frac{k+3}{4}\right)$$

$$= m\left(\frac{k+3}{4}\right)$$

$$= \Omega(km)$$

f) The load factor $\alpha(T) = N/s$ is the fraction of the allocated memory in use. Prove all tables always satisfy

$$size(T_i) \leq num(T_i) + 2N$$

and uses this to prove that $\alpha(T) \geq \frac{1}{2k+1}$

(i) When all the tables are empty, for each table T_i

$$size(T_i) = num(T_i) + 2N = 0$$

Scanned by CamScanner

ii) When a table T_i is expanded or contracted,

$$size(T_i') = num(T_i') + N' \leq num(T_i') + 2N'$$

iii) When an insertion occurs in T_i, but T_i is not expanded.

Since T_i is not expanded and

$$size(T_i) \leq num(T_i) + 2N$$

$$\therefore \quad size(T_i') = size(T_i) \leq num(T_i) + 2N$$
$$< num(T_i') + 2N'$$

iv) When a deletion occurs in T_i but T_i is not contracted.

Since T_i is not contracted,

$$num(T_i) > size(T_i) - 2N + 2.$$

Then $size(T_i') = size(T_i) < num(T_i) + 2N - 2$

$$= num(T_i') + 1 + 2(N'+1) - 2$$

$$size(T_i') < num(T_i') + 1 + 2N'$$

$$size(T_i') \leq num(T_i') + 2N'$$

Based on the above inequation

$$\sum_{i=1}^{K} size(T_i) \leq \sum_{i=1}^{K} (num(T_i) + 2N)$$

$$S \leq N + 2KN = 2(K+1)N$$

$$\frac{N}{S} \geq \frac{1}{2K+1}$$

For example.

Let's insert three elements into one of the empty table, let it be T_1

Initiall all tables are empty and size is 0.

If we insert 3 elements then the size of the table will increase.

$size(T_1) = 6$.

Delete one element in T and it does not result in contraction.

Now for the remaining K-1 ~~domeste~~ tables insert and then delete an element.

This will cause the size of each tables to become 4.

~~But~~ At the end of the sequence of operations, the load factor is $\frac{2}{(6+(k-1)4)} = \frac{1}{2k+1}$

Hence Proved.

Scanned by CamScanner

Reference:

ocw.mit.edu
www.cs.princeton.edu
www.researchgate.net

Refered these sites to study more about Efficient management of dynamic tables and Amortized Analysis