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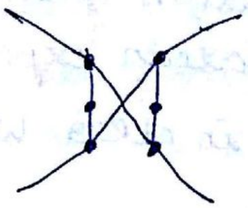
Homework Assignment-12

1. Explain why the widget in Fig 34.16 (page 1092 of CLRS3) cannot be replaced with the simpler graph.



The gadget in the book has a property and that is if the cycle enters through vertex [u,v,1], it must exit through vertex [u,v,6].

The simpler graph does not have this property. So in this graph if the cycle enters through vertex [u,v,1] then it may exit through vertex [u,v,3] or [v,u,3] if we name the vertices as in the textbook. Without this property and by the reduction method, we cannot guarantee that the hamiltonian cycle in ~~G'~~ ~~exits~~ G' exists implies the vertex cover in G exists. For example:

~~Eg~~ The graph in CLRS3 page ~~1092~~ 1093 would have a hamiltonian cycle when there is only one selector vertex, but in the corresponding graph G, there is no vertex cover with size 1.

2) On the course website you will find a paper entitled "On the NP-Completeness of Cryptarithms" Read the paper and give the cryptharithm puzzle that the reduction from 3-CNFSAT gives for the 3-CNF boolean expression on line 15 of page 1082 in CLRS3. What is the base of the puzzle?

Given a cryptarithm puzzle, it is not too hard to see that a solution to the puzzle need only be as long as the length of the base multiplied by the number of letters in the puzzle and that such a solution can be verified quickly. Thus cryptarithms are in NP and it remains to show that they are complete for NP. Reduction from 3SAT given a 3-CNF Boolean formula we will construct a puzzle which is solvable if and only if the formula is satisfiable.

Each variable and term of the Boolean formula will correspond to some contiguous set of columns of the puzzle.

Reserve the rightmost three columns for the following letters:

$$\begin{array}{ccc} o & p & o \\ o & p & o \\ \hline 1 & q & o \end{array}$$

0 and 1 is read as letters in the puzzle

For each variable v_i of the formula we set aside the following columns in which the letters v_i and $\bar{v}_i$ represent the variables and its complement.

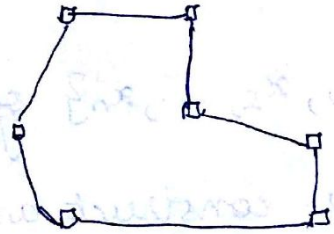
$$\begin{array}{cccccccccccccc} d_i & 0 & 1 & y_i & 0 & c_i & y_i & 0 & b_i & y_i & 0 & a_i & 0 \\ c_i & 0 & d_i & y_i & 0 & c_i & y_i & 0 & b_i & y_i & 0 & a_i & 0 \\ \hline \bar{v}_i & 0 & e_i & z_i & 0 & d_i & z_i & 0 & v_i & z_i & 0 & b_i & 0 \end{array}$$

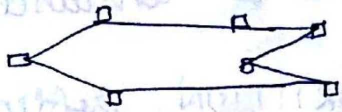
Else for each term $(v_a \vee v_b \vee v_c)$ in our 3 CNF formula.

$$
\begin{array}{cccccccccccc}
v_{ab} & 0 & v_a & 0 & 1 & r_i & 0 & g_i & w_i & 0 & f_i & 0 \\
v_c & 0 & v_b & 0 & h_i & r_i & 0 & g_i & w_i & 0 & f_i & 0 \\
\hline
t_i & 0 & u_{ab} & 0 & 4 & s_i & 0 & h_i & x_i & 0 & g_i & 0
\end{array}
$$

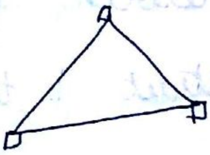
Thus the solution to the puzzle can be turned into satisfying assignment to the formula. To complete the NP completeness proof we must find a base that will give us the converse so that from a satisfying assignment to the formula we can find a solution to the puzzle. It will turn out that with the base equal to $3072n^3$, where n is the number of variable in the formula.

3) You are given a chain of n rigid struts of lengths $l_1, l_2, l_3, \ldots l_n$ linked together into a cycle by n hinges. The ~~chain~~ CHAIN FOLDING problem is to determine if a given chain can be laid out flat on a single line. We assume that hinges and struts are infinitesimally thin. For example, the following chain can be so positioned:





whereas the following chain cannot



a) Prove CHAIN FOLDING is in NP.

Given a folding method, we can check whether it is a valid folding (It can be laid out flat on a single line). The valid folding can be found in polynomial time $O(n)$. So CHAIN FOLDING is in NP.

b) Prove CHAIN FOLDING is NP-Hard by reduction from Partition which you may assume to be NP-complete.

PARTITION ($\{x_1, x_2 \dots x_n\}$) is true if and only if there exists $a_i \in \{-1, +1\}$ such that

$$\sum_{i=0}^{n} a_i x_i = 0$$

Given an instance $X = \{x_1, x_2, \dots, x_n\}$ of the PARTITION Problem, let us construct an instances, a chain with n struts, of the CHAIN-FOLDING problem by setting the length of the strut i to x_i.

If we can find a valid folding, then we set the a_i-s for each x_i based on the angle of two adjacent struts.

If the angle between x_i, $x_{(i+1) \bmod n}$ is then $a_i = 1$.

If the angle is 2π then $a_i = -1$.

So CHAIN FOLDING is ~~NP-HARD~~ NP-Hard.

c) Prove PARTITION is NP-Hard using the following reduction to SUBSET-SUM (CLRS3 Page 1097)

$$f(\langle S, t\rangle) = \begin{cases} \{1,2\} & \text{if } t > \alpha \\ \{1,2,3\} & \text{if } t = 0, \alpha \\ S & \text{if } t = \alpha/2 \\ S \cup \{\alpha + t, 2\alpha - t\} & \text{otherwise.} \end{cases}$$

where

$$\alpha = \sum_{x \in S} x$$

The reduction is well defined as the given equations.

If $t > \alpha$, it is trivial that there is no subset such that the sum of the element is t. So we can construct an impossible instance of PARTITION as above.

If $t=0$, trivial subset, empty set $\emptyset$ or S itself is the answer. So we can construct an always valid instance of PARTITION as shown.

If $t = \alpha/2$, then whether SUBSET-SUM is true is equivalent to whether there is a valid folding for the PARTITION problem constructed above.

Else we can just add two elements $\alpha+t$, $2\alpha t$ to the instances of PARTITION X to make it as the third case.

So PARTITION is NP-hard.

References:

citeseerx.ist.psu

www.ics.uci.edu