Homework Assignment-4

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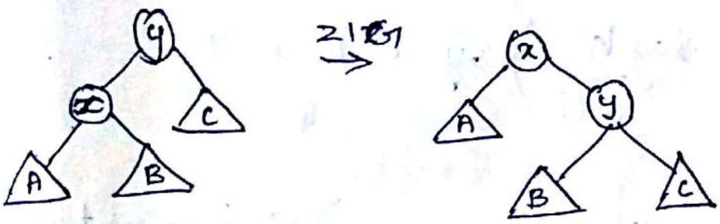
1. We could get the same effect as splaying (moving the accessed item to the root) using only zig and zag steps, not bothering with the more complicated zig-zig, zig-zag, zag-zig and zag-zag steps. If we do that, we only need case 1 on pages 4-5 in the notes. What is the amortized cost of an access with such simple splaying?

Amortizedcost = Actual cost + change in potential

$$\hat{c}_i = c_i + \Phi(T_i) - \Phi(T_i - 1)$$



The change in potential is

$$\Delta\Phi = r'(x) + r'(y) - r(x) - r(y)$$

where $r'(x)$ and $r'(y)$ denote the ranks of x and y after the ZIG operation

By case 1 of Access Lemma, the amortized cost of ZIG or ZAG at node x is at most

$$1 + r'(x) - r(x)$$

Amortized cost is $1 + r'(x) - r(x)$ where 1 is the actual cost and $r'(x) - r(x)$ is the change in potential.

Let i be a node at depth d. Then for a node at depth d, the sum of the amortized ~~amortized~~ cost is

$$d + \log \frac{W}{w(i)}$$

Where W - the total weight of the items in the tree and w(i) - weight of the node i

With such simple splaying, the amortized cost of an access is

$$O\left(d + \log \frac{W}{w(i)}\right) \text{ if } i \text{ is in } T$$

$$O\left(d + \log \frac{W}{\min\{w(i^-), w(i^+)\}}\right) \text{ if } i \text{ is not in } T.$$

The point to be noted is, the cost is proportional to the depth of the splay and it is not O(1).

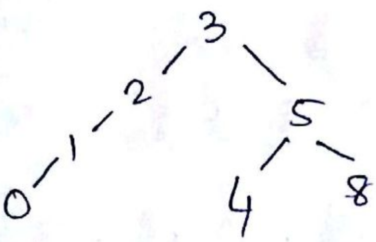
2) cut (i_1, i_2, T) that deletes all items i, $i_1 \le i \le i_2$ from the splay tree T

~~Given the tree contains many~~ many ~~nodes~~ The tree T contains many nodes.

cut (i_1, i_2, T) deletes all items i from splay tree.

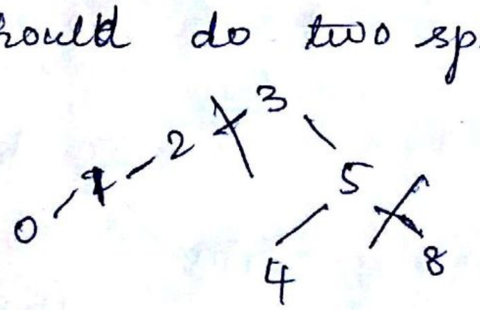
Given $i_1 \le i \le i_2$

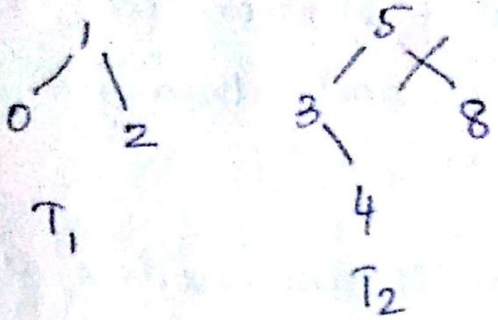
For example



cut (3, 5, T) should delete all the i that is ~~equ~~ greater than or equal to 3 and less than or equal 5 and the nodes inbetween

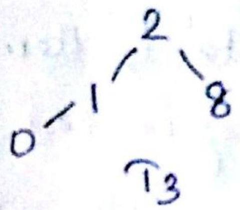
So inorder to perform cut (i_1, i_2, T) we should do two splits





after split only 8 will remain.

So 0, 1, 2 in tree T_1 and 8 in Tree T_2 has to be joined to form :

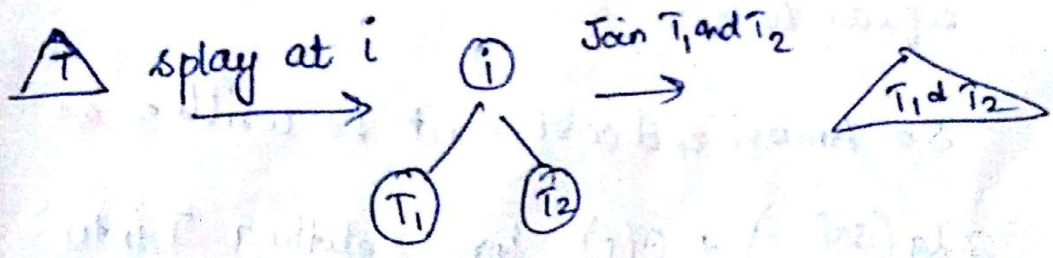


T_3 is the final tree after the cut $(3, 5, T)$.

cut (i_1, i_2, T) deletes all items i, $i_1 \le i \le i_2$

Delete : Takes a tree T and ~~an~~ deletes i from T. To perform delete we again do a split(T, i), then remove i and join the resulting subtrees.

(ie)



Inorder to perform cut operation we need to perform two splits and then join the remaining subtrees.

So split (T, i_1) takes T and node i_1 as input and creates two trees T_1 and T_2 such that T_1 contains all items that are smaller than i_1 and T_2 contains all items greater than or equal to i_1

Now split (T_2, i_2+1) takes T and nod i_2 as input and creates two trees T_3 and T_4 such that lesser than and equal to i_2 items are in T_3 and greater elements of i_2 are in T_4

Now join T_1 and T_3. then the cut (i_1, i_2, T) operation is completed.

i_2+1 is the node greater than i_2. so the split should happen in such a way T_4 has greater element of i_2 and T_3 lesser and equal to i_2.

So Amortized cost can be written as.

$3 \lg\left(\frac{W}{w(i_1)}\right) + O(1)$ for splitting T into T_1 and T_2

$3\lg\left(\frac{W}{w(i_2+1)}\right) + O(1)$ for splitting T_2 into T_3 and T_4

and Join (T_1, T_4) amortized cost is

$3\lg\left(\frac{W}{w(i)}\right) + O(1)$ when i is the last item in T_1.

So the sum of the amortized cost of these operations and change in potential that is the T_3 which has i. items that are $i_1 \le i \le i_2$ that is removed from the tree T gives the amortized cost of cut (i_1, i_2, T).

$$= 3\lg\left(\frac{W}{w(i_1)}\right) + 3\lg\left(\frac{W}{w(i_2+1)}\right) + 3\lg\left(\frac{W}{w(i)}\right) + O(1)$$

$$- 3\lg\left(\frac{W}{s(T_3)}\right)$$

$$W = s(T_1) + s(T_4)$$

This can also ~~written~~ as done. by deleting all the elements i, $i_1 \le i \le i_2$ by repeatedly doing the delete operation.

Then the amortized cost is

$$\cancel{3\lg\left(\frac{W - \sum_{i=i_1}^{i_2} w(i)}{\sum_{i=i_1}^{i_2}}\right)}$$

Then the amortized cost is

$$3 \lg \left(\frac{W - \sum_{i=i_1}^{i_2} w(i)}{\sum_{i=i_1}^{i_2} w(i^-)} \right) + 3 \lg \left(\frac{W}{\sum_{i=i_1}^{i_2} w(i)} \right) + O(1)$$

or

$$3 \lg \left(\frac{W - \sum_{i=i_1 \text{ to } i_2} w(i)}{\sum_{i=i_1 \text{ to } i_2} w(i^-)} \right) + 3 \lg \left(\frac{W}{\sum_{i=i_1 \text{ to } i_2} w(i)} \right) + O(1)$$

3) a) Given $d(x)$ be the depth of an internal node x.

Let n be the height of the tree T.

So. $S(n) = \sum_{\text{internal nodes } x \in T} 3^{-d(x)}$ for a complete binary tree with height 'n'

$$\therefore S(n) = \sum_{i=1}^{n} 2^{(i-1)} 3^{-i} \quad — ①$$

$$= \sum_{i=1}^{n} 2^{i} \cdot 2^{-1} 3^{-i}$$

$$= \sum_{i=1}^{n} 2^{i} 3^{-i} \cdot \frac{1}{2}$$

$$2S(n) = \sum_{i=1}^{n} 2^{i} 3^{-i}$$

$$= 2\left(1 - \frac{2}{3}\right)^{n} \leq 2 \quad - ②$$

Based on equality ② we can say,

$$S(n) \leq 1 \text{ for all } n.$$

Therefore for any ~~binary~~ finite binary tree we can get

$$\sum_{\text{internal nodes } x \in T} 3^{-d(x)} \leq 1$$

Hence Pseudo-Kraft inequality is proved.

b) We know that when $\lim_{n \to \infty} S(n) = 1$

Hence for complete infinite binary tree it is exactly 1.

~~Hence for incomplete ...~~

3b) $\omega(x) = 3^{-d(x)}$. Let W be total weight of items of tree.

Let $\omega(x)$ be the weight of the node x.

The amortized cost of access (x, T) is

$3\lg\left(\frac{W}{\omega(x)}\right)+1$ if x is in T

access (x,T) is $3\lg\left(\frac{W}{\min\{\omega(x^-), \omega(x^+)\}}\right)+1$ if x is not in T.

$$\text{Amortized cost} = 3\lg\left(\frac{W}{\omega(x)}\right)+1$$

$$= 3\lg\left(\frac{W}{3^{-d(x)}}\right)+1$$

$$\leq 3\lg\left(\frac{1}{3^{-d(x)}}\right)+1$$

$$= 3\lg\left(3^{d(x)}\right)+1$$

So, Amortized cost is $O\left(\lg 3^{d(x)}+1\right)$

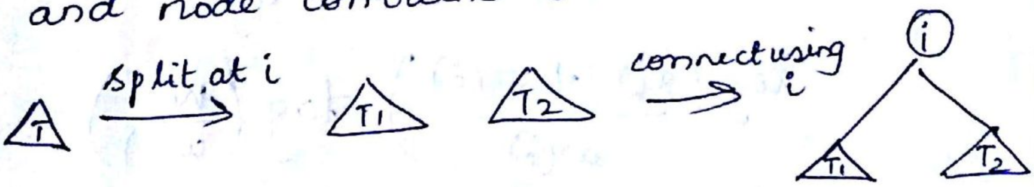
$$O\left(\lg 3^{d(x)}+1\right) = O\left(1+d(x)\right).$$

$\therefore$ Amortized cost of accessing ~~item~~ item x in a splay tree is $O(1+d(x))$.

Hence Proved.

4) Derive the bound for insert given in the table at the top of page 11 of the splay tree notes.

Insert takes a tree T and an item as an input and inserts a node containing i into T. To perform an insert, simply split(T,i) and then make a new tree whose left and right branches are the trees T_1 and T_2. This T_1 and T_2 is returned from split and node contains the item i.



Split (i, T) or Access (i, T) amortized cost is

$$3 \lg \left(\frac{W}{w(i)} \right) + O(1) \text{ if } i \text{ is in } T.$$

$$3 \log \left(\frac{W}{\min\{w(i^-), w(i^+)\}} \right) + O(1) \text{ if } i \text{ is}$$

not in T.

Then the new node weight $w(i)$ is added to the tree as root node and T_1 and T_2 are joined.

So the weight of the tree is added or increased by the weight of the node or item $w(i)$

(ie) If W is the weight of the resulting tree and $S(T)$ is the size function of the tree before insertion then W is

$$W = S(T) + w(i) \quad \text{— (1)}$$

Then the increase in potential is by adding a new root node i.

$$\therefore \log\left(\frac{S(T) + w(i)}{w(i)}\right) = \log\left(\frac{W}{w(i)}\right)$$

From (1) we can get.

$$S(T) = W - w(i) \quad \text{—— (2)}.$$

This can also be written as before insertion if the total weight is W then after insertion the total weight is $W + w(i)$.

T_1 and T_2 returned from split (i, T) should be connected using i as root node.

So the amortized cost of split and cost of inserting i should be ~~~~ added.

Amortized cost of split (i,T) can be written as

$$3\lg\left(\frac{S(T)}{w(i)}\right) + 1 \quad \text{if } i \text{ is in } T$$

(or)

$$3 \lg \left(\frac{S(T)}{\min\{w(i^-), w(i^+)\}} \right) + 1 \quad \text{when } i \text{ is not in } T$$

Substitute

$$S(T) = W - w(i).$$

then the

Amortized cost of insert

$$= 3 \lg \left(\frac{W - w(i)}{\min\{w(i^-), w(i^+)\}} \right) + O(1) + \log\left(\frac{W}{w(i)}\right)$$

= Actual cost + change in potential.