

The rank of a matrix is the number of linearly independent rows of that matrix.

A row is linearly independent from the other rows when it is not the result of a linear combination of them. So, if we can find a row that is a linear combination of other rows, we will say that this row is linearly dependent. In this case the row should be excluded from the matrix rank calculation.

Only according to the definition, then, the cases in which a row can be discarded are:

1. All the elements are empty.
2. There are two equal rows.
3. A row is proportional to another.
4. A row is a linear combination of one or more.

Let A be the matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 & -1 & 2 \\ 3 & 1 & -5 & 0 & 0 \\ 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 6 & -2 & 4 \end{pmatrix}$$

According to 1. we can discard row 4, since all its elements are null.

According to 2. we can discard row 3, since it is the same as row 1.

According to 3. we can discard row 5, since it is proportional to row 1.

Therefore, the rank of A is 2. It is written as follows: $\text{rank}(A) = 2$ or $r(A) = 2$.

The three cases presented (1, 2 and 3) are really easy. The fourth point, however, is not so easy.

Here is a small exercise to become familiar with the concept of linear combination. The goal is being able to identify or even construct linear combinations.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Is there any removable row?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 5 \end{pmatrix}$$

And now?

Indeed, in both cases the third row is a linear combination of the other two.

In the first matrix,

$$\text{row3} = \text{row1} + \text{row2},$$

while in the second one

$$\text{row3} = 2 \cdot \text{row1} + 5 \cdot \text{row2}.$$

So, in both cases, $\text{rank}(A) = 2$.

In general, the Gaussian elimination method will be used to tune a matrix as much as required, so that it is easier to apply the four norms.