The necessary and sufficient condition for a system of m equations and n unknowns to have a solution is that the rank of its coefficient matrix and that of its augmented matrix are equal.

1. r=r′ Compatible System.
   - r=r′=n Determinate Compatible System.
   - r=r′≠n Indeterminate Compatible System.
2. r≠r′ Incompatible System.

where, as we said, r is the Rank of the matrix of the system and r′ is the Rank of the augmented matrix of the system.

Obviously, for the correct use of the above mentioned theorem one must have assimilated what is the rank of a matrix and how it is calculated.

When the technical part is not a problem, this theorem allows us to discuss the systems of equations.

See the following example:

Let the system of equations be:

$$\begin{cases} 2x - y - 2z = -2 \\ -x + y + z = 0 \\ x - 2y + z = 8 \\ 2x - 2y = 6 \end{cases}$$

1) The coefficient matrix and the rank are

$$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -2 & 0 \end{pmatrix}$$

(and the rank is calculated)

$$|2| = 2 \neq 0; \quad \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1 \neq 0; \quad \begin{vmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 2 \neq 0$$

i.e., $r(A) = 3$.

2) The rank of the augmented matrix is computed.

We look at the matrix of order 4 (it has already been proved that up to order 3 we can find some matrix with a nonzero determinant)

$$|A'| = \begin{vmatrix} 2 & -1 & -2 & -2 \\ -1 & 1 & 1 & 0 \\ 1 & -2 & 1 & 8 \\ 2 & -2 & 0 & 6 \end{vmatrix} = 0$$

i.e., $r(A') = 3 = r(A)$.

3) The Rouché theorem is applied: $r(A) = r(A') = n$, so it is a Determinate compatible system.

4) The system is solved, if it is not incompatible, by Cramer's rule or by the Gaussian elimination method.

We take the system that corresponds to the submatrix of order 3, which has rank 3, and solve it.

$$\begin{cases} 2x - y - 2z = -2 \\ -x + y + z = 0 \\ x - 2y + z = 8 \end{cases}$$

We solve it using Cramer's rule:

$$x = \frac{\begin{vmatrix} -2 & -1 & -2 \\ 0 & 1 & 1 \\ 8 & -2 & 1 \end{vmatrix}}{2} = \frac{2}{2} = 1; \quad y = \frac{\begin{vmatrix} 2 & -2 & -2 \\ -1 & 0 & 1 \\ 1 & 8 & 1 \end{vmatrix}}{2} = \frac{-4}{2} = -2$$

$$z = \frac{\begin{vmatrix} 2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & -2 & 8 \end{vmatrix}}{2} = \frac{6}{2} = 3$$