These systems can be understood as a set of three planes in the three-dimensional real space R3. Sometimes a solution will fail to be found, sometimes there will be infinite solutions (a line of points) and sometimes there will be only one solution.

To solve this type of system we will use the reduction method, so that every equation has one unknown less than the previous one. We will use the Gaussian method.

Solve:

$$\begin{cases} 3x+2y+z=1 \\ 5x+3y+4z=2 \\ x+y-z=1 \end{cases}$$

1) We place the equation that has 1 or -1 as a coefficient of x on top.

If there is none, we can look for another variable with coefficient 1 or -1, and change the order of the variables (or we can divide the first equation by the coefficent of x).

$$\begin{cases} x+y-z &= 1 \\ 3x+2y+z &= 1 \\ 5x+3y+4z &= 2 \end{cases}$$

2) We then use the reduction method for equations 1 and 2 (E_1 and E_2) in order to eliminate the variable x from the second equation:

$$E_2' = E_2 - 3 \cdot E_1$$

$$\begin{array}{r} 3x+2y+z=1 \\ -3x-3y+3z=-3 \\ \hline -y+4z=-2 \end{array}$$

3) We apply the same procedure with E_1 and E_3 to eliminate the variable x from the third equation:

$$E_3' = E_3 - 5 \cdot E_1$$

$$\begin{array}{r} 5x+3y+4z=2 \\ +\ -5x-5y+5z=-5 \\ \hline -2y+9z=-3 \end{array}$$

4) With the new equations 2 and 3 (E_2' and E_3') we use the same procedure again to eliminate the variable y of E_3':

$$E3'' = E3' - 2 \cdot E2'$$

$$\begin{array}{r} -2y+9z=-3 \\ +\ 2y-8z=4 \\ \hline z=1 \end{array}$$

5) And so, this system will be equivalent to the original one:

$$\begin{cases} x+y-z=1 \\ -y+4z=-2 \\ z=1 \end{cases}$$

6) It can be solved from the third equation up to the first one:

$$E3: z=1$$

$$E2: -y+4=-2 \Rightarrow y=6$$

$$E1: x+6-1=1 \Rightarrow x=-4$$

Namely, these three planes have only one intersection point $(-4,6,1)$.

Note: The use of matrixes is advisable for the resolution of this type of problems. The previous example would be written as:

$$\begin{cases} 3x+2y+z=1 \\ 5x+3y+4z=2 \\ x+y-z=1 \end{cases} \Rightarrow \begin{pmatrix} 3 & 2 & 1 \\ 5 & 3 & 4 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Also, the above mentioned notation offers certain advantages for the analysis of the system, since the calculation of the determinant can be useful to have an idea of the solutions that will be obtained.

- If the determinant is not zero, the system is consistent determinate, that is, it has a unique solution.
- If the determinant is zero, the system can be:
  - Consistent indeterminate: it has proportional equations and, therefore, infinite solutions.
  - Inconsistent: It has no solutions.