Notation

It is known that a matrix 3×3 is written as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where the subscripts indicate the row and the column respectively.

If we write:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

it means that the we want to calculate the determinant of this matrix.

Obviously this has been written for a 3×3 matrix, –because this one is the most common–, but the determinants of matrices 2×2, 4×4 or N×N can also be computed. It only makes sense to speak of determinants of square matrices.

Complementary minors

Let's consider the 3×3 matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

The complementary minor of the element a11 is the determinant of order 2 that survives when row 1 and column 1 are eliminated.

In other words, the complementary minor we are looking for is:

$$M_{11} = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{vmatrix} = 5\cdot 9 - 8\cdot 6 = -3$$

Now let's calculate the complementary minor of the element a23, in other words, the determinant of order 2 that survives when we eliminate the second row and the third column.

$$M_{23} = \begin{vmatrix} 1 & 2 & \cancel{3} \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & \cancel{9} \end{vmatrix} = 1\cdot 8 - 7\cdot 2 = -6$$

Generally, the complementary minor of an element aij is written as Mij and it is the determinant of lower order that survives when row i and column j are eliminated.

Cofactors

We call the cofactor of an element of a matrix, its complementary minor but placing before it:

1. The sign + when i+j is even
2. The sign − when i+j is odd

Following the previous examples, the cofactor of the element a11 is written as C11 and must have the sign + (1+1=2, which is even), while the cofactor of the element a23 is written as C23 and must have the sign − (2+3=5, which is odd).

Using the precise notation, we conclude C11=+M11=−3 and C23=−M23=6.

Let's see another example:

Consider the matrix:

$$\begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & -3 \\ 0 & 2 & 4 \end{pmatrix}$$

We want to find the cofactor of the element a_{11}.

First we calculate the complementary minor:

$$M_{11} = \begin{vmatrix} \cancel{-1} & \cancel{0} & \cancel{2} \\ \cancel{1} & 1 & -3 \\ \cancel{0} & 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot (-3) = 10$$

We check which sign corresponds: $1+1=2$, pair, and therefore the sign is positive. Then the cofactor of a_{11} is $C_{11} = 10$.

Now let's find the cofactor of the element a_{22}:

$$M_{22} = \begin{vmatrix} -1 & \cancel{0} & 2 \\ \cancel{-1} & \cancel{1} & \cancel{-3} \\ 0 & \cancel{2} & 4 \end{vmatrix} \rightarrow \begin{vmatrix} -1 & 2 \\ 0 & 4 \end{vmatrix} = (-1) \cdot 4 - 2 \cdot 0 = -4$$

We verify the sign: $2+2=4$, pair, and therefore the sign does not change, so $C_{22} = M_{22}$.

And this way, we can successively find the cofactors of all the elements a_{ij} of the matrix.

Adjoint matrix

If we replace every element of the matrix A by its cofactor we obtain the adjoint matrix, which is written as Adj(A).

Let's calculate it using the previous example, starting with the complementary minors:

$$M_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} = 10 \quad M_{12} = \begin{vmatrix} 1 & -3 \\ 0 & 4 \end{vmatrix} = 4 \quad M_{13} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 3$$

$$M_{21} = \begin{vmatrix} 0 & 2 \\ 2 & 4 \end{vmatrix} = -4 \quad M_{22} = \begin{vmatrix} -1 & 2 \\ 0 & 4 \end{vmatrix} = -4 \quad M_{23} = \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2$$

$$M_{31} = \begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} = -2 \quad M_{32} = \begin{vmatrix} -1 & 2 \\ 0 & 4 \end{vmatrix} = -4 \quad M_{33} = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

Nine complementary minors have been found, but the signs of each one must be added depending on the sum i+j being even or odd. Adding up, the signs will be as follows:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

and therefore the adjoint matrix will be:

$$Adj \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & -3 \\ 0 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 10 & -4 & 3 \\ 4 & -4 & 2 \\ -2 & 4 & -1 \end{pmatrix}$$