We know that the calculation of a third-order determinant is quite long. The number of operations is high and it will be even higher when fourth-order determinants or larger must be calculated.

We also know that Sarrus' rule is valid only for the calculation of third-order determinants. Let’s now see a more general method.

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 4 & 5 \\ 1 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} =$$

$$= 1 \cdot (4 \cdot 2 - 5 \cdot 0) - 0 + 2 \cdot (2 \cdot 0 - 4 \cdot 1) = 8 - 0 - 8 = 0$$

Let's start with some examples of 3×3 determinants to see if your intuition helps you to find the general rule for calculating determinants.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} =$$

$$= 1 \cdot (5 \cdot 9 - 6 \cdot 8) - 2 \cdot (4 \cdot 9 - 6 \cdot 7) + 3 \cdot (4 \cdot 8 - 5 \cdot 7) = -3 + 12 - 9 = 0$$

As you can see, the procedure is simple. The determinant of a matrix is the sum of the products of the elements of one row by their cofactors.

In these two solved cases, we have chosen the first row and each of its elements has been multiplied by its corresponding cofactor.

And so on with the following elements.

4. The second term is chosen, that is 2.
5. Its cofactor is calculated.

$$\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{vmatrix}$$

Do not forget the minus sign!!! ($1 + 2 = 3$, which is odd).

$$-2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -2 \cdot (4 \cdot 9 - 6 \cdot 7) = 12$$

6. We choose the last term and repeat the process, now with the positive sign.

$$\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & 5 & \cancel{6} \\ 7 & 8 & \cancel{9} \end{vmatrix} = 3 \cdot \begin{vmatrix} 4 & \\ 7 & 8 \end{vmatrix} = 3 \cdot (4 \cdot 8 - 5 \cdot 7) = 3 \cdot (4 \cdot 8 - 5 \cdot 7) = 3 \cdot (-3) = -9$$

7. Finally we add up all three cases. In this example the result is zero.

To summarize:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{vmatrix} - \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{vmatrix} + \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & 5 & \cancel{6} \\ 7 & 8 & \cancel{9} \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & \\ 7 & 8 \end{vmatrix}$$

and using what we have learned about the second-order determinants, we solve this case.

Let's take a look at the same example and proceed step by step:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

1. We choose one row, for instance, the first one (123), and we begin with the first element: 1.
2. Its cofactor is calculated.

$$\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{vmatrix} = 5 \cdot 9 - 8 \cdot 6 = -3$$

(we must not change the sign, since $1 + 1 = 2$, a pair.)

3. We multiply the selected term, that is 1, by its corresponding cofactor.

$$1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 1 \cdot (5 \cdot 9 - 6 \cdot 8) = -3$$

Rules for the calculation of higher order determinants

We will use the same method. The elements of a row are multiplied by their corresponding cofactors. That is:

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 3 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{2} & \cancel{1} \\ \cancel{0} & 3 & 2 & 1 \\ \cancel{2} & 4 & 1 & 2 \\ \cancel{3} & 1 & 0 & 2 \end{vmatrix} - \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{2} & \cancel{1} \\ 0 & \cancel{3} & 2 & 1 \\ 2 & \cancel{4} & 1 & 2 \\ 3 & \cancel{1} & 0 & 2 \end{vmatrix} + \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{2} & \cancel{1} \\ 0 & 3 & \cancel{2} & 1 \\ 2 & 4 & \cancel{1} & 2 \\ 3 & 1 & \cancel{0} & 2 \end{vmatrix} - \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{2} & \cancel{1} \\ 0 & 3 & 2 & \cancel{1} \\ 2 & 4 & 1 & \cancel{2} \\ 3 & 1 & 0 & \cancel{2} \end{vmatrix}$$

As in the case of third-order determinants, this boils down to multiplying each element of the first row by the determinants of "what is not crossed out", remembering always the corresponding sign. That is,

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 3 & 1 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 1 & 0 \end{vmatrix}$$

So, solving four third-order determinants and using the previous expression, we can compute the determinant of the 4×4 matrix.

The same method can be used to calculate higher determinants. Thus, to calculate a 5th-order determinant, you should calculate 5 fourth-order determinants, which in turn require the calculation of 4 third-order determinants, and so on. We can see, then, that the method is sufficiently slow and tiresome to ensure that the use of powerful calculators is completely indispensable.