Trigonometric relationships of double-angle and half-angle

Known all the ratios of an angle, we can find all the ratios of the double of that angle and its half using the following identities:

1. $\sin(2\alpha) = 2 \cdot \sin\alpha \cdot \cos\alpha$
2. $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$
3. $\tan(2\alpha) = \dfrac{2 \cdot \tan\alpha}{1 - \tan^2\alpha}$
4. $\sin\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1 - \cos\alpha}{2}}$
5. $\cos\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1 + \cos\alpha}{2}}$
6. $\tan\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1 - \cos\alpha}{1 + \cos\alpha}}$

Given α, of which we know its trigonometric ratios, now we will be able to calculate the ratios of the double-angle and the half-angle. Bearing in mind that $\alpha = 30^\circ$, we will compute the ratios of $2\alpha = 60^\circ$ and $\frac{\alpha}{2} = 15^\circ$.

We have:

$$\sin 2\alpha = 2 \cdot \sin\alpha \cdot \cos\alpha = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$\tan(2\alpha) = \frac{2 \cdot \tan\alpha}{1 - \tan^2\alpha} = \frac{2 \cdot \frac{\sqrt{3}}{3}}{1 - \frac{1}{3}} = \sqrt{3}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1 + \cos\alpha}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}}} = \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}} =$$

$$= \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}} \cdot \sqrt{\frac{2 + \sqrt{3}}{2 + \sqrt{3}}} = \frac{\sqrt{4 - 3}}{2 + \sqrt{3}} = \frac{1}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} =$$

$$= \tfrac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}$$

Trigonometric relationships of the sum and the difference of two angles

1. $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$
2. $\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$
3. $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$
4. $\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$
5. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$
6. $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

We can calculate the trigonometrical ratios of $45^\circ = 60^\circ - 15^\circ$.

$$\sin(60 - 15) = \sin 60 \cdot \cos 15 - \cos 60 \cdot \sin 15 = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2 + \sqrt{3}}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2 - \sqrt{3}}}{2} =$$

$$= \frac{1}{4}\left(\sqrt{6 + 3\sqrt{3}} - \sqrt{2 - \sqrt{3}}\right) = \frac{\sqrt{2}}{2}$$

$$\cos(60 - 15) = \cos 60 \cdot \cos 15 + \sin 60 \cdot \sin 15 = \frac{1}{2} \cdot \frac{\sqrt{2 + \sqrt{3}}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2 - \sqrt{3}}}{2} =$$

$$= \frac{1}{4}\left(\sqrt{2 + \sqrt{3}} + \sqrt{6 - 3\sqrt{3}}\right) = \frac{\sqrt{2}}{2}$$

$$\tan(60 - 15) = \frac{\tan 60 - \tan 15}{1 + \tan 60 \cdot \tan 15} = \frac{\sqrt{3} - (2 - \sqrt{3})}{1 + \sqrt{3} \cdot (2 - \sqrt{3}} = \frac{2\sqrt{3} - 2}{1 + 2\sqrt{3} - 3} = \frac{2\sqrt{3} - 2}{2\sqrt{3} - 2} = 1$$