System Optimization Method Project
Title: Optimization of PID controller for Interval Plant

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Introduction:

- This Project involves in Optimization of PID controller for interval plants. The optimization based design of robust PID controller guarantees both the stability and the performance of an interval plant. The objective is to find the optimum Kp and Ki value for stable First Order Plus Time Delay (FOPTD) process. Optimization is done by using Non-Linear Programming (NLP). Two methods are used for solving the optimization problem in this project. One is done by Lagrange Multipliers method and other is done by converting the nonlinear equations to linear equation and solving it by simplex method, along with their MATLAB simulation.

PID controller design:

- Let us consider a control system as shown
- *The transfer function is given as*
- $G(s) = \frac{\mathrm{N(s)}}{\mathrm{D(s)}}$
- $\mathrm{C(s)} = Kp + \frac{Ki}{s} + Kds$
- Let us consider the system parameters are bound to vary with a closed limits ie parametric uncertainties. For these system the stability can be analysed by Kharitonov theorem.

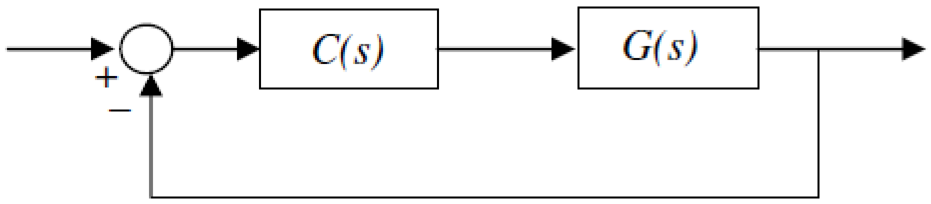


Fig. 1. A SISO control system.

PID controller design(Cont'd)

- The Kharitonov polynomials are given by

$$P(s) = P_0 + P_1 s + P_2 s^2 + \ldots\ldots\ldots + P_n s^n$$

Where the coefficients are given by

$$P_0 \in [a_0,b_0],\ P_1 \in [a_1,b_1], \ldots\ldots P_n \in [a_n,b_n]$$

$$P_1(s) = a_0 + a_1 s + b_2 s^2 + \ldots\ldots + a_n s^n$$

$$P_2(s) = a_0 + b_1 s + b_2 s^2 + \ldots\ldots + a_n s^n$$

$$P_3(s) = b_0 + a_1 s + a_2 s^2 + \ldots\ldots + b_n s^n$$

$$P_4(s) = b_0 + b_1 s + a_2 s^2 + \ldots\ldots + b_n s^n$$

Stable FOPTD Process

- Let us consider a stable FOTPD process with given uncertainties,

$$G(s) = \frac{Ke^{-Ls}}{(Ts+1)}$$

- Where the nominal value be $K_p^o = 1.4375$, $K_i^o = 1.25$,
- T = [0.9, 1.1], K = [0.8, 1.2], L = [0.29, 0.31]

- The time delay is given as
- $e^{-Ls} = \frac{1}{1+Ls}$

Objective Function:

- To obtain the objective function for an interval block the controller parameters (K_p^0, K_i^0, K_d^0) are obtained by pole placement method.
- $J = \min ||K-K^0||^2$
- $J = (K_p - K_p^o)^2 + (K_i - K_i^o)^2 + (K_d - K_d^o)^2$
- Since K_d is fixed we can neglect it

$$\mathbf{J = (K_p - K_p^o)^2 + (K_i - K_i^o)^2}$$

Constraints:

- Hurwitz Criteria are used to frame the Constraints. These are designed to **minimize** the objective function. Therefore the constraints can be formulated as

$$-0.952\,K_p + 0.4092\,K_i - 1.19 + \in \leq 0$$

$$-0.952\,K_p + 0.8790\,K_i - 1.19 + \in \leq 0$$

$$-0.8\,K_p - 1 + \in \leq 0$$

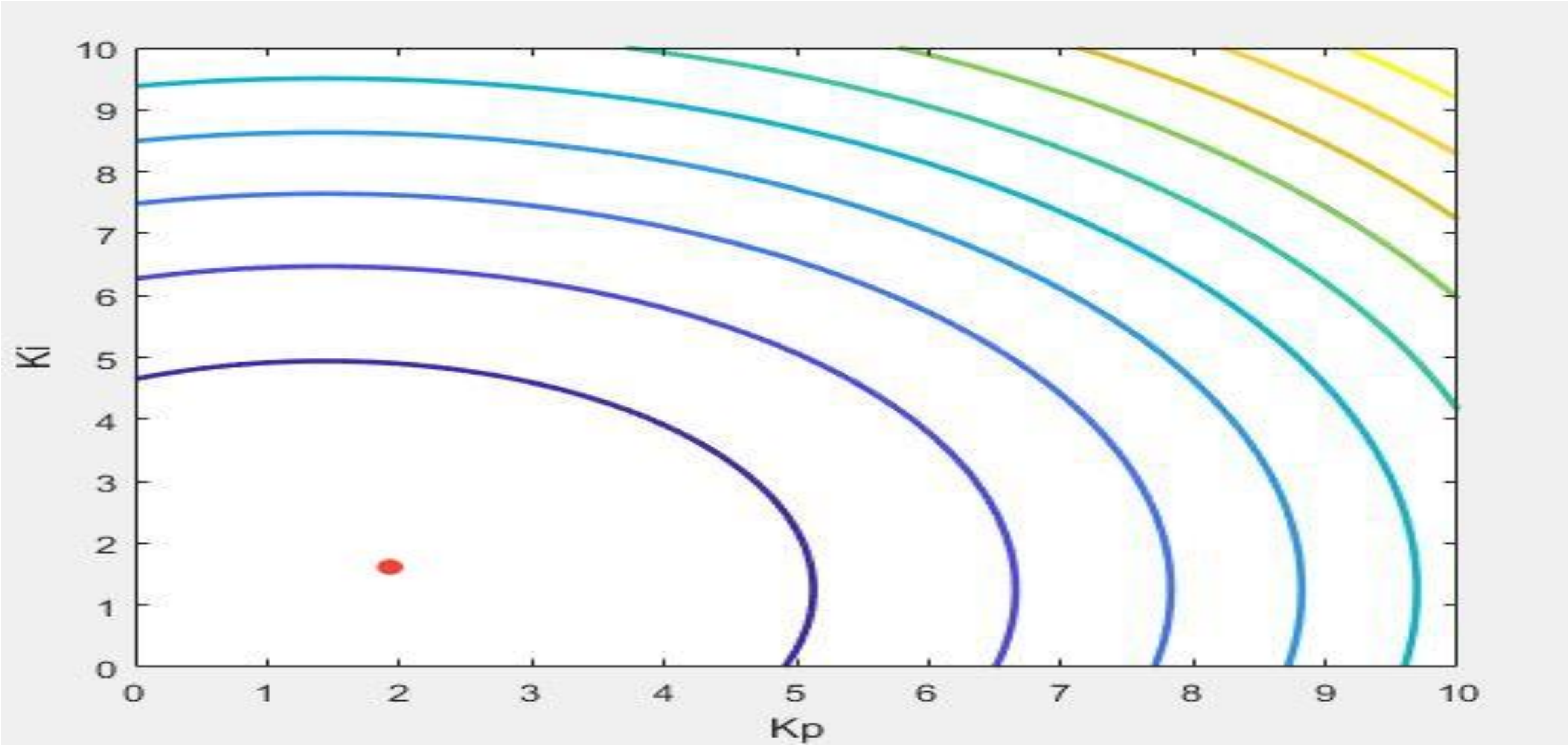
$$-1.2\,K_p - 1 + \in \leq 0$$

$$-0.8\,K_i + \in \leq 0$$

$$-1.2\,K_i + \in \leq 0$$

- Where $\in$ is the small variation in the uncertainties. Let $\in = 1.5$.

Contour of the Objective Function:



Method Of Lagrange Multipliers:

- The method of Lagrange multipliers is used for finding the local minima and maxima of a objective function given the equality or inequality, linear or non-linear constraints.
- Let let us consider the objective function and constraints,

$$\text{maximize } f(x)$$

$$\text{subject to } Ax \geq 0$$

- where i = 1….m, m = number of constraints

(Cont'd)

- It can be changed to

$$\min f(x)$$

subject to $Ax = b$, all the active constraints

- For the inactive constraints, we associate with zero Lagrange multipliers:
- $\lambda_i (a^T x_* - b_i) = 0$, I = 0,1, 2….,

Necessary Conditions

- For the points to be optimum it should satisfy the necessary and sufficient conditions (KKT)
- Necessary conditions:
- $\nabla f(x) = A^T \lambda_*$
- $\lambda_* \geq 0$
- $\lambda_*^T (Ax_* - b) = 0$
- $Z^T \nabla^2 f(x_*) Z$ is a positive semi-definite
- Where Z is the null space matrix of active constraints

Sufficient Conditions:

- It should satisfy necessary conditions.
- $A_i^T x_* - b_i = 0$ or $\lambda_* = 0$
- $Z^T \nabla^2 f(x_*) Z$ is a positive definite
- If the constraints are degenrative
- It should satisfy the necessary conditions
- $\lambda_*^T (Ax_* - b) = 0$
- $Z^T \nabla^2 f(x_*) Z$ is a positive definite, where Z is the null space with respect to non-degenerate active constraints

Calculations

- After simplification of the Objective junction and Constraints,

$$\textbf{Min } J = (K_p – 1.435)^2 + (K_i – 1.25)^2$$

- **Subject to**

$$0.952\, K_p - 0.4092\, K_i \geq 0.31$$

$$0.952\, K_p - 0.8790\, K_i \geq 0.31$$

$$0.8\, K_p \geq 0.5$$

$$1.2\, K_p \geq 0.5$$

$$0.8\, K_i \geq 1.5$$

$$1.2\, K_i \geq 1.5$$

Since there are 6 lagrangain variable, $2^n = 2^6 = 64$ combinations are possible

$\boxed{\text{Method ①}}$

SYSTEM OPTIMIZATION METHOD

method of lagrange multipliers

Objective function

$$\min \quad J = (K_p - K_p^o)^2 + (K_i - K_i^o)^2$$

Let $K_p^o = 1.4375$, $K_i^o = 1.25$

$$\min \quad J = (K_p - 01.4375)^2 + (K_i - 1.25)^2$$

$$\boxed{\min \quad J = K_p^2 + K_i^2 - 2.8750\, K_p - 2.5 K_i + 3.6289}$$

Let $\xi = 1.5$

Subject to

$$-0.952 K_p + 0.4092 K_i \leq -0.31$$
$$-0.952 K_p + 0.8790 K_i \leq -0.31$$
$$-0.8 K_p \leq -0.5$$
$$-1.2 K_p \leq -0.5$$
$$-0.8 K_i \leq -1.5$$
$$-1.2 K_i \leq -1.5$$

$$\Rightarrow \boxed{\begin{array}{l} 0.952 K_p - 0.4092 K_i \geq 0.31 \\ 0.952 K_p - 0.8790 K_i \geq 0.31 \\ 0.8 K_p \geq 0.5 \\ 1.2 K_p \geq 0.5 \\ 0.8 K_i \geq 1.5 \\ 1.2 K_i \geq 1.5 \end{array}}$$

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Necessary conditions (KKT)

$$\nabla J(K_*) = A^T \lambda_*$$

$$\lambda_* \geq 0$$

$$\lambda_*^T (A K_* - b) = 0$$

$Z^T \nabla^2 J(K_*) Z \longrightarrow$ P.S.D   Z-null matrix of active constrai[nts]

if $\lambda = 0$ degenerative

$\therefore Z_*^T \nabla^2 f(x_*) Z_* \longrightarrow$ P.D

$$\therefore \quad A = \begin{bmatrix} 0.952 & -0.4092 \\ 0.952 & -0.8790 \\ 0.8 & 0 \\ 1.2 & 0 \\ 0 & 0.8 \\ 0 & 1.2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0.952 & 0.952 & 0.8 & 1.2 & 0 & 0 \\ -0.4092 & -0.8790 & 0 & 0 & 0.8 & 1.2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.31 \\ 0.31 \\ 0.5 \\ 0.5 \\ 1.5 \\ 1.5 \end{bmatrix}$$

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Necessary conditions:

1)
$$0.952k_p - 0.4092\,k_i \geq 0.31$$
$$0.952k_p - 0.8790\,k_i \geq 0.31$$
$$0.8\,k_p \geq 0.5$$
$$1.2\,k_p \geq 0.5$$
$$0.8\,k_i \geq 1.5$$
$$1.2\,k_i \geq 1.5$$

2) $\nabla f(x) = A^T \lambda$

$$\begin{bmatrix} 2k_p - 2.8750 \\ 2k_i - 2.5 \end{bmatrix} = \begin{bmatrix} 0.952 \\ -0.4092 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0.952 \\ -0.8790 \end{bmatrix} \lambda_2 + \begin{bmatrix} 0.8 \\ 0 \end{bmatrix} \lambda_3$$
$$+ \begin{bmatrix} 1.2 \\ 0 \end{bmatrix} \lambda_4 + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} \lambda_5 + \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \lambda_6$$

3)
$$\lambda_1 (0.952\,k_p - 0.4092k_i - 0.31) = 0$$
$$\lambda_2 (0.952k_p - 0.8790\,k_i - 0.31) = 0$$
$$\lambda_3 (0.8\,k_p - 0.5) = 0$$
$$\lambda_4 (1.2k_p - 0.5) = 0$$
$$\lambda_5 (0.8\,k_i - 1.5) = 0$$
$$\lambda_6 (1.2k_i - 1.5) = 0$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0$$

$2^n = 64$ combinations possible

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Case ①:

Let λ_2 and λ_5 are active $\lambda_1, \lambda_3, \lambda_4, \lambda_6 = 0$

$$\begin{bmatrix} 2K_p - 2.8750 \\ 2k_i - 2.5 \end{bmatrix} = \begin{bmatrix} 0.952 \\ -0.8790 \end{bmatrix} \lambda_2 + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} \lambda_5$$

$\lambda_2 (0.952\, k_p - 0.8790 k_i - 0.31) = 0$

$\lambda_5 (0.8 k_i - 1.5) = 0$

$\lambda_5 (0.8 K_i - 1.5) = 0$

$0.8 K_i - 1.5 = 0$

$0.8 K_i - 1.5 = 0 \Rightarrow 0.8 K_i = 1.5 = K_i = 1.5 / 0.8$

$\boxed{K_i = 1.875}$

$\lambda_2 (0.952 k_p - 0.8790 k_i - 0.31) = 0$

$0.952 k_p - 0.8790 k_i - 0.31 = 0$

$0.952 k_p - 0.8790(1.875) - 0.31 = 0$

$0.952 k_p = 1.958125$

$\boxed{K_p = 2.056853}$

$K_i = 1.875$

$K_p = 2.056853$

Sub in necessary condition 2

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$K_p = 2.0569 \qquad K_i = 1.875$

$$\begin{bmatrix} 2K_p - 2.8750 \\ 2K_i - 2.5 \end{bmatrix} = \begin{bmatrix} 0.952 \\ -0.8790 \end{bmatrix} \lambda_2 + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} \lambda_5$$

$$\begin{bmatrix} 1.2388 \\ 1.25 \end{bmatrix} = \begin{bmatrix} 0.952 \\ -0.8790 \end{bmatrix} \lambda_2 + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} \lambda_5$$

$$1.2388 = 0.952\, \lambda_2 + (0)\, \lambda_5$$

$$1.25 = -0.8790\, \lambda_2 + 0.8\, \lambda_5$$

$$\frac{1.2388}{0.952} = \lambda_2$$

$$\therefore \boxed{\lambda_2 = 1.301}$$

$$1.25 = -0.8790\, \lambda_2 + 0.8\, \lambda_5$$

$$1.25 = -0.8790(1.301) + 0.8\, \lambda_5$$

$$\boxed{\lambda_5 = 2.991}$$

$\lambda_2 \geq 0$

$\lambda_2 \geq 0$

feasible.

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$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Z is a null space matrix for active constraint

then

$$Z = \begin{bmatrix} 0.952 & -0.8790 \\ 0 & 0.8 \end{bmatrix}$$

$$Z^T \nabla^2 f(x) Z = ?$$

$$Z^T = \begin{bmatrix} 0.952 & 0 \\ -0.8790 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.952 & 0 \\ -0.8790 & 0.8 \end{bmatrix} \begin{bmatrix} 0.952 & -0.8790 \\ 0 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8126 & -1.6736 \\ 16736 & -1.545 \end{bmatrix}$$

$P_{11} = 1.8126 > 0$

$P_{22} = -2.8009 + 2.8009 = 0$

$\therefore Z^T \nabla^2 f(x) Z$ is a P.S.D Positive Semi definite

$\therefore$ The point $(2.056853, 1.875)$ is the optimum point

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- Since there are 6 lagrangain variable, $2^n = 2^6 = 64$ combinations are possible
- By using the method of lagrange the optimum value can be found at point Kp = 2.0569, KI = 1.8750

MATLAB

- MATLAB code

- fun = @(k)100*(k(1)-1.4375)^2 + (k(2)-1.25)^2;
- k0 = [0,0];
- A = [-0.952, 0.4092; -0.952, 0.8790; -0.8, 0; -1.2, 0; 0 -0.8; 0, -1.2];
- b = [-0.31; -0.31; -0.5; -0.5; -1.5; -1.5];
- k = fmincon(fun,k0,A,b)

MATLAB:

```
>> Untitled

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the optimality tolerance,
and constraints are satisfied to within the default value of the constraint tolerance.

<stopping criteria details>


k =

    2.0569    1.8750

>>
```

Quadratic Programming:

- In this method the nonlinear objective method is converted to linear equations using KKT (Karush-Kuhn-Tucker) formulas. Then the linear problem is solved using Simplex method (Two-Phase Simplex method).
- Objective Function,

$$\text{Max } Z = CX + X^T DX$$

$$AX \geq B$$

$$X \geq 0$$

$$\lambda \geq 0$$

- Conversion to linear programming can be done using the below formulas

$$\begin{bmatrix} -2D & A' & -I & 0 \\ A & 0 & 0 & I \end{bmatrix} \begin{bmatrix} X \\ \lambda \\ \mu \\ s \end{bmatrix} = \begin{bmatrix} C' \\ B \end{bmatrix}$$

λ_{i}, i = 1,2,….m, m = number of constraints (6 in this problem)
μ_j, j = 1,2,……n, n = number of variables (2 in this problem)
S = Slack variables

- 2 phase method:
- Min P = $C^T x$
  subject to Ax=b,
  x≥0
- Phase 1
- Min P = $\sum_i ai$
  Subject to Ax + a = b,
  x, a ≥ 0

Method 2: Solving by converting to linear problem

$$\text{Max} \quad Z = Cx + x^T D x \quad — (1)$$

$Ax \le b$ $\qquad \lambda \to m$ constraints

$x \ge 0$ $\qquad \mu \to n$ constraints

$\lambda \ge 0$

$$\text{Max } J = -(K_p - 1.4375)^2 - (K_i - 1.25)^2$$

$$\boxed{\text{max } J = -K_p^2 - K_i^2 + 2.8750 K_p + 2.5 K_i - 3.6289} \quad — (2)$$

Comparing (1) & (2)

$$\boxed{\begin{pmatrix} -2D & A^T & -I & 0 \\ A & 0 & 0 & I \end{pmatrix} \begin{pmatrix} x \\ \lambda \\ \mu \\ S \end{pmatrix} = \begin{pmatrix} C^T \\ b \end{pmatrix}}$$

$$x = \begin{pmatrix} K_p \\ K_i \end{pmatrix}$$

$$C = \begin{pmatrix} 2.8750 & 2.5 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

6 constraints $\quad \lambda_1 \; \lambda_2 \; \lambda_3 \; \lambda_6 \; \lambda_5 \; \lambda_6$

2 variables $\quad \mu_1 \; \mu_2$

From constraint

$$A = \begin{bmatrix} -0.952 & 0.4092 \\ -0.952 & 0.8790 \\ -0.8 & 0 \\ -1.2 & 0 \\ 0 & 0.8 \\ 0 & -1.2 \end{bmatrix} \qquad B = \begin{bmatrix} -0.31 \\ -0.31 \\ -0.5 \\ +0.5 \\ -1.5 \\ -1.5 \end{bmatrix}$$

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$$
\begin{bmatrix}
+2 & 0 & -0.952 & -0.952 & -0.8 & -1.2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0.4092 & 0.8790 & 0 & 0 & -0.8 & -1.2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.952 & 0.4092 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-0.952 & 0.8790 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\times
\begin{bmatrix}
K_p \\ K_i \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \mu_1 \\ \mu_2 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6
\end{bmatrix}
=
\begin{bmatrix}
2.8750 \\ 2.5 \\ -0.31 \\ -0.31 \\ -0.5 \\ -0.5 \\ -1.5 \\ -1.5
\end{bmatrix}
$$

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$\therefore$ $2K_p - 0.952\lambda_1 - 0.952\lambda_2 - 0.8\lambda_3 - 1.2\lambda_4 - \mu_1 = 2.8750$

$2K_i + 0.4092\lambda_1 + 0.8790\lambda_2 - 0.8\lambda_5 - 1.2\lambda_6 - \mu_2 = 2.5$

$-0.952K_p + 0.4092K_i + S_1 = -0.31$

$-0.952K_p + 0.8790K_i + S_2 = -0.31$

$-0.8K_p + S_3 = -0.5$

$-1.2K_p + S_4 = -0.5$

$-0.8K_i + S_5 = -1.5$

$-1.2K_i + S_6 = -1.5$

By 2 phase method we define the objective function as Min $P = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8$

after solving we get the optimum value

for $K_p = 2.0568$

$K_i = 1.875$

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Find solution using Simplex(BigM) method
MIN Z = 0x1 + 0x2 + 0x3 + 0x4 + 0x5 + 0x6 + 0x7 + 0x8 + 0x9 + 0x10 + 0x11 + 0x12 + 0x13 + 0x14 + 0x15 + 0x16
subject to
2x1 - 0.952x3 - 0.952x4 - 0.8x5 - 1.2x6 - x9 = 2.8750
2x2 + 0.4092x3 + 0.8790x4 - 0.8x7 - 1.2x8 - x10 = 2.5
-0.952x1 + 0.8790x2 + x11 = -0.31
-0.952x1 + 0.8790x2 + x12 = -0.31
-0.8x1 + x13 = -0.5
-1.2x1 + x14 = -0.5
-0.8x2 + x15 = -1.5
-1.2x2 + x16 = -1.5
and x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,x14,x15,x16 >= 0

Solution:
Problem is

Min Z =

subject to

$2x_1 - 0.952x_3 - 0.952x_4 - 0.8x_5 - 1.2x_6 - x_9 = 2.875$

$2x_2 + 0.4092x_3 + 0.879x_4 - 0.8x_7 - 1.2x_8 - x_{10} = 2.5$

$-0.952x_1 + 0.879x_2 + x_{11} = -0.31$

Here $b_3 = -0.31 < 0$,

so multiply this constraint by -1 to make $b_3 > 0$.

$0.952x_1 - 0.879x_2 - x_{11} = 0.31$

$-0.952x_1 + 0.879x_2 + x_{12} = -0.31$

Here $b_4 = -0.31 < 0$,

so multiply this constraint by -1 to make $b_4 > 0$.

$0.952x_1 - 0.879x_2 - x_{12} = 0.31$

$-0.8x_1 + x_{13} = -0.5$

Here $b_5 = -0.5 < 0$,

so multiply this constraint by -1 to make $b_5 > 0$.

$0.8x_1 - x_{13} = 0.5$

$-1.2x_1 + x_{14} = -0.5$

Here $b_6 = -0.5 < 0$,

so multiply this constraint by -1 to make $b_6 > 0$.

$1.2x_1 - x_{14} = 0.5$

$-0.8x_2 + x_{15} = -1.5$

Here $b_7 = -1.5 < 0$,

so multiply this constraint by -1 to make $b_7 > 0$.

$0.8x_2 - x_{15} = 1.5$

$-1.2x_2 + x_{16} = -1.5$

Here $b_8 = -1.5 < 0$,

so multiply this constraint by -1 to make $b_8 > 0$.

$1.2x_2 - x_{16} = 1.5$

and $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \geq 0$;

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropiate

1. As the constraint 1 is of type ' = ' we should add artificial variable A_1

2. As the constraint 2 is of type ' = ' we should add artificial variable A_2

3. As the constraint 3 is of type ' = ' we should add artificial variable A_3

4. As the constraint 4 is of type ' = ' we should add artificial variable A_4

5. As the constraint 5 is of type ' = ' we should add artificial variable A_5

6. As the constraint 6 is of type ' = ' we should add artificial variable A_6

7. As the constraint 7 is of type ' = ' we should add artificial variable A_7

8. As the constraint 8 is of type ' = ' we should add artificial variable A_8

After introducing artificial variables

Min $Z = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8 + 0x_9 + 0x_{10} + 0x_{11} + 0x_{12} + 0x_{13} + 0x_{14} + 0x_{15} + 0x_{16} + MA_1 + MA_2 + MA_3 +$

subject to

$2x_1 - 0.952x_3 - 0.952x_4 - 0.8x_5 - 1.2x_6 - x_9 + A_1$

$2x_2 + 0.4092x_3 + 0.879x_4 - 0.8x_7 - 1.2x_8 - x_{10} + A_2$

$0.952x_1 - 0.879x_2 - x_{11} + A_3$

$0.952\,x_1\; -\; 0.879\,x_2 \qquad - x_{12} \qquad + A_4$

$0.8 \quad x_1 \qquad - x_{13} \qquad +$

$1.2 \quad x_1 \qquad - x_{14}$

$0.8 \quad x_2 \qquad - x_{15}$

$1.2 \quad x_2 \qquad - x_{16}$

and $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8 \geq 0$

Iteration-1		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	M	M
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**	**x_{13}**	**x_{14}**	**x_{15}**	**x_{16}**	**A_1**	**A_2**	**A_3**
A_1	M	2.875	2	0	-0.952	-0.952	-0.8	-1.2	0	0	-1	0	0	0	0	0	0	0	1	0	0
A_2	M	2.5	0	2	0.4092	0.879	0	0	-0.8	-1.2	0	-1	0	0	0	0	0	0	0	1	0
A_3	M	0.31	0.952	-0.879	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
A_4	M	0.31	**(0.952)**	-0.879	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0
A_5	M	0.5	0.8	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
A_6	M	0.5	1.2	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0
A_7	M	1.5	0	0.8	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0
A_8	M	1.5	0	1.2	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0
$Z = 0$		**Z_j**	**$5.904M$**	**$2.242M$**	**$-0.5428M$**	**$-0.073M$**	**$-0.8M$**	**$-1.2M$**	**$-0.8M$**	**$-1.2M$**	**$-M$**	**$-M$**	**$-M$**	**$-M$**	**$-M$**	**$-M$**	**$-M$**	**$-M$**	**M**	**M**	**M**
		$C_j - Z_j$	$-5.904M \uparrow$	$-2.242M$	$0.5428M$	$0.073M$	$0.8M$	$1.2M$	$0.8M$	$1.2M$	M	M	M	M	M	M	M	M	0	0	0

Negative minimum $C_j - Z_j$ is $-5.904M$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 0.3256 and its row index is 4. So, the leaving basis variable is A_4.

$\therefore$ The pivot element is 0.952.

Entering $= x_1$, Departing $= A_4$, Key Element $= 0.952$

$R_4(\text{new}) = R_4(\text{old}) \times 1.0504$

$R_1(\text{new}) = R_1(\text{old}) - 2R_4(\text{new})$

$R_2(\text{new}) = R_2(\text{old})$

$R_3(\text{new}) = R_3(\text{old}) - 0.952R_4(\text{new})$

$R_5(\text{new}) = R_5(\text{old}) - 0.8R_4(\text{new})$

$R_6(\text{new}) = R_6(\text{old}) - 1.2R_4(\text{new})$

$R_7(\text{new}) = R_7(\text{old})$

$R_8(\text{new}) = R_8(\text{old})$

Iteration-2		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	M	M
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**	**x_{13}**	**x_{14}**	**x_{15}**	**x_{16}**	**A_1**	**A_2**	**A_3**
A_1	M	2.2237	0	1.8466	-0.952	-0.952	-0.8	-1.2	0	0	-1	0	0	2.1008	0	0	0	0	1	0	0
A_2	M	2.5	0	2	0.4092	0.879	0	0	-0.8	-1.2	0	-1	0	0	0	0	0	0	0	1	0
A_3	M	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	1
x_1	0	0.3256	1	-0.9233	0	0	0	0	0	0	0	0	0	-1.0504	0	0	0	0	0	0	0

A_5	M	0.2395	0	0.7387	0	0	0	0	0	0	0	0	0	0.8403	-1	0	0	0	0	0	0
A_6	M	0.1092	0	**(1.108)**	0	0	0	0	0	0	0	0	0	1.2605	0	-1	0	0	0	0	0
A_7	M	1.5	0	0.8	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0
A_8	M	1.5	0	1.2	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0
$Z = 0$		Z_j	**0**	**$7.6933M$**	**$-0.5428M$**	**$-0.073M$**	**$-0.8M$**	**$-1.2M$**	**$-0.8M$**	**$-1.2M$**	**$-M$**	**$-M$**	**$-M$**	**$5.2017M$**	**$-M$**	**$-M$**	**$-M$**	**$-M$**	**M**	**M**	**M**
		$C_j - Z_j$	0	$-7.6933M$ ↑	$0.5428M$	$0.073M$	$0.8M$	$1.2M$	$0.8M$	$1.2M$	M	M	M	$-5.2017M$	M	M	M	M	0	0	0

Negative minimum $C_j - Z_j$ is $-7.6933M$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 0.0986 and its row index is 6. So, the leaving basis variable is A_6.

∴ The pivot element is 1.108.

Entering $= x_2$, Departing $= A_6$, Key Element $= 1.108$

$R_6(\text{new}) = R_6(\text{old}) \times 0.9025$

$R_1(\text{new}) = R_1(\text{old}) - 1.8466 R_6(\text{new})$

$R_2(\text{new}) = R_2(\text{old}) - 2 R_6(\text{new})$

$R_3(\text{new}) = R_3(\text{old})$

$R_4(\text{new}) = R_4(\text{old}) + 0.9233 R_6(\text{new})$

$R_5(\text{new}) = R_5(\text{old}) - 0.7387 R_6(\text{new})$

$R_7(\text{new}) = R_7(\text{old}) - 0.8 R_6(\text{new})$

$R_8(\text{new}) = R_8(\text{old}) - 1.2 R_6(\text{new})$

Iteration-3		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	M	M
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**	**x_{13}**	**x_{14}**	**x_{15}**	**x_{16}**	**A_1**	**A_2**	**A_3**
A_1	M	2.0417	0	0	-0.952	-0.952	-0.8	-1.2	0	0	-1	0	0	0	0	1.6667	0	0	1	0	0
A_2	M	2.3028	0	0	0.4092	0.879	0	0	-0.8	-1.2	0	-1	0	-2.2753	0	1.8051	0	0	0	1	0
A_3	M	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	1
x_1	0	0.4167	1	0	0	0	0	0	0	0	0	0	0	0	0	-0.8333	0	0	0	0	0
A_5	M	0.1667	0	0	0	0	0	0	0	0	0	0	0	0	-1	**(0.6667)**	0	0	0	0	0
x_2	0	0.0986	0	1	0	0	0	0	0	0	0	0	0	1.1377	0	-0.9025	0	0	0	0	0
A_7	M	1.4211	0	0	0	0	0	0	0	0	0	0	0	-0.9101	0	0.722	-1	0	0	0	0
A_8	M	1.3817	0	0	0	0	0	0	0	0	0	0	0	-1.3652	0	1.083	0	-1	0	0	0
$Z = 0$		Z_j	**0**	**0**	**$-0.5428M$**	**$-0.073M$**	**$-0.8M$**	**$-1.2M$**	**$-0.8M$**	**$-1.2M$**	**$-M$**	**$-M$**	**$-M$**	**$-3.5506M$**	**$-M$**	**$5.9435M$**	**$-M$**	**$-M$**	**M**	**M**	**M**
		$C_j - Z_j$	0	0	$0.5428M$	$0.073M$	$0.8M$	$1.2M$	$0.8M$	$1.2M$	M	M	M	$3.5506M$	M	$-5.9435M$ ↑	M	M	0	0	0

Negative minimum $C_j - Z_j$ is $-5.9435M$ and its column index is 14. So, the entering variable is x_{14}.

Minimum ratio is 0.25 and its row index is 5. So, the leaving basis variable is A_5.

∴ The pivot element is 0.6667.

Entering $= x_{14}$, Departing $= A_5$, Key Element $= 0.6667$

$R_5(\text{new}) = R_5(\text{old}) \times 1.5$

R_1(new) = R_1(old)- 1.6667R_5(new)

R_2(new) = R_2(old)- 1.8051R_5(new)

R_3(new) = R_3(old)

R_4(new) = R_4(old)+ 0.8333R_5(new)

R_6(new) = R_6(old)+ 0.9025R_5(new)

R_7(new) = R_7(old)- 0.722R_5(new)

R_8(new) = R_8(old)- 1.083R_5(new)

Iteration-4		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	M	M
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**	**x_{13}**	**x_{14}**	**x_{15}**	**x_{16}**	**A_1**	**A_2**	**A_3**
A_1	M	1.625	0	0	-0.952	-0.952	-0.8	-1.2	0	0	-1	0	0	-0	**(2.5)**	0	0	0	1	0	0
A_2	M	1.8515	0	0	0.4092	0.879	0	0	-0.8	-1.2	0	-1	0	-2.2753	2.7076	0	0	0	0	1	0
A_3	M	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	1
x_1	0	0.625	1	0	0	0	0	0	0	0	0	0	0	0	-1.25	0	0	0	0	0	0
x_{14}	0	0.25	0	0	0	0	0	0	0	0	0	0	0	0	-1.5	1	0	0	0	0	0
x_2	0	0.3242	0	1	0	0	0	0	0	0	0	0	0	1.1377	-1.3538	0	0	0	0	0	0
A_7	M	1.2406	0	0	0	0	0	0	0	0	0	0	0	-0.9101	1.083	0	-1	0	0	0	0
A_8	M	1.1109	0	0	0	0	0	0	0	0	0	0	0	-1.3652	1.6246	0	0	-1	0	0	0
$Z = 0$		**Z_j**	**0**	**0**	**$-0.5428M$**	**$-0.073M$**	**$-0.8M$**	**$-1.2M$**	**$-0.8M$**	**$-1.2M$**	**$-M$**	**$-M$**	**$-M$**	**$-3.5506M$**	**$7.9152M$**	**0**	**$-M$**	**$-M$**	**M**	**M**	**M**
		$C_j - Z_j$	0	0	$0.5428M$	$0.073M$	$0.8M$	$1.2M$	$0.8M$	$1.2M$	M	M	M	$3.5506M$	$-7.9152M$ ↑	0	M	M	0	0	0

Negative minimum $C_j - Z_j$ is $-7.9152M$ and its column index is 13. So, the entering variable is x_{13}.

Minimum ratio is 0.65 and its row index is 1. So, the leaving basis variable is A_1.

∴ The pivot element is 2.5.

Entering $= x_{13}$, Departing $= A_1$, Key Element $= 2.5$

R_1(new) = R_1(old) × 0.4

R_2(new) = R_2(old)- 2.7076R_1(new)

R_3(new) = R_3(old)

R_4(new) = R_4(old)+ 1.25R_1(new)

R_5(new) = R_5(old)+ 1.5R_1(new)

R_6(new) = R_6(old)+ 1.3538R_1(new)

R_7(new) = R_7(old)- 1.083R_1(new)

R_8(new) = R_8(old)- 1.6246R_1(new)

Iteration-5		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**	**x_{13}**	**x_{14}**	**x_{15}**	**x_{16}**	**A_2**
x_{13}	0	0.65	0	0	-0.3808	-0.3808	-0.32	-0.48	0	0	-0.4	0	0	-0	1	0	0	0	0
A_2	M	0.0916	0	0	1.4403	**(1.9101)**	0.8664	1.2997	-0.8	-1.2	1.083	-1	0	-2.2753	0	0	0	0	1
A_3	M	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0
x_1	0	1.4375	1	0	-0.476	-0.476	-0.4	-0.6	0	0	-0.5	0	0	0	0	0	0	0	0

x_{14}	0	1.225	0	0	-0.5712	-0.5712	-0.48	-0.72	0	0	-0.6	0	0	0	0	1	0	0	0
x_2	0	1.2042	0	1	-0.5155	-0.5155	-0.4332	-0.6498	0	0	-0.5415	0	0	1.1377	0	0	0	0	0
A_7	M	0.5366	0	0	0.4124	0.4124	0.3466	0.5199	0	0	0.4332	0	0	-0.9101	0	0	-1	0	0
A_8	M	0.0549	0	0	0.6186	0.6186	0.5199	0.7798	0	0	0.6498	0	0	-1.3652	0	0	0	-1	0
Z = 0		Z_j	**0**	**0**	**2.4713*M***	**2.9411*M***	**1.7329*M***	**2.5993*M***	**-0.8*M***	**-1.2*M***	**2.1661*M***	**-*M***	**-*M***	**-3.5506*M***	**0**	**0**	**-*M***	**-*M***	***M***
		$C_j - Z_j$	0	0	-2.4713M	-2.9411M ↑	-1.7329M	-2.5993M	0.8M	1.2M	-2.1661M	M	M	3.5506M	0	0	M	M	0

Negative minimum $C_j - Z_j$ is -2.9411M and its column index is 4. So, the entering variable is x_4.

Minimum ratio is 0.0479 and its row index is 2. So, the leaving basis variable is A_2.

∴ The pivot element is 1.9101.

Entering $= x_4$, Departing $= A_2$, Key Element $= 1.9101$

R_2(new) = R_2(old) × 0.5235

R_1(new) = R_1(old)+0.3808R_2(new)

R_3(new) = R_3(old)

R_4(new) = R_4(old)+0.476R_2(new)

R_5(new) = R_5(old)+0.5712R_2(new)

R_6(new) = R_6(old)+0.5155R_2(new)

R_7(new) = R_7(old)-0.4124R_2(new)

R_8(new) = R_8(old)-0.6186R_2(new)

Iteration-6		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
x_{13}	0	0.6683	0	0	-0.0937	0	-0.1473	-0.2209	-0.1595	-0.2392	-0.1841	-0.1994	0	-0.4536	1	0	0	0
x_4	0	0.0479	0	0	0.754	1	0.4536	0.6804	-0.4188	-0.6283	0.567	-0.5235	0	-1.1912	0	0	0	0
A_3	M	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
x_1	0	1.4603	1	0	-0.1171	0	-0.1841	-0.2761	-0.1994	-0.299	-0.2301	-0.2492	0	-0.567	0	0	0	0
x_{14}	0	1.2524	0	0	-0.1405	0	-0.2209	-0.3313	-0.2392	-0.3589	-0.2761	-0.299	0	-0.6804	0	1	0	0
x_2	0	1.2289	0	1	-0.1268	0	-0.1994	-0.299	-0.2159	-0.3239	-0.2492	-0.2699	0	0.5235	0	0	0	0
A_7	M	0.5169	0	0	0.1014	0	0.1595	0.2392	0.1727	0.2591	0.1994	0.2159	0	-0.4188	0	0	-1	0
A_8	M	0.0253	0	0	0.1522	0	0.2392	0.3589	0.2591	**(0.3887)**	0.299	0.3239	0	-0.6283	0	0	0	-1
Z = 0		Z_j	**0**	**0**	**0.2536*M***	**0**	**0.3987*M***	**0.5981*M***	**0.4318*M***	**0.6478*M***	**0.4984*M***	**0.5398*M***	**-*M***	**-0.0471*M***	**0**	**0**	**-*M***	**-*M***
		$C_j - Z_j$	0	0	-0.2536M	0	-0.3987M	-0.5981M	-0.4318M	-0.6478M ↑	-0.4984M	-0.5398M	M	0.0471M	0	0	M	M

Negative minimum $C_j - Z_j$ is -0.6478M and its column index is 8. So, the entering variable is x_8.

Minimum ratio is 0.0651 and its row index is 8. So, the leaving basis variable is A_8.

∴ The pivot element is 0.3887.

Entering $= x_8$, Departing $= A_8$, Key Element $= 0.3887$

R_8(new) = R_8(old) × 2.5729

R_1(new) = R_1(old)+0.2392R_8(new)

R_2(new) = R_2(old)+0.6283R_8(new)

R_3(new) = R_3(old)

R_4(new) = R_4(old)+0.299R_8(new)

R_5(new) = R_5(old)+0.3589R_8(new)

R_6(new) = R_6(old)+0.3239R_8(new)

R_7(new) = R_7(old)-0.2591R_8(new)

Iteration-7		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	M	
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**	**x_{13}**	**x_{14}**	**x_{15}**	**x_{16}**	**A_3**	**A_7**	**MinR** $\frac{X_B}{x_{12}}$
x_{13}	0	0.6838	0	0	0	0	0	0	0	0	-0	0	0	-0.8403	1	0	0	-0.6155	0	0	---
x_4	0	0.0888	0	0	1	1	0.8403	1.2605	0	0	1.0504	0	0	-2.2068	0	0	0	-1.6165	0	0	---
A_3	M	0	0	0	0	0	0	0	0	0	0	0	-1	(1)	0	0	0	0	1	0	$\frac{0}{1}=0$
x_1	0	1.4798	1	0	0	0	0	0	0	0	0	0	0	-1.0504	0	0	0	-0.7694	0	0	---
x_{14}	0	1.2757	0	0	0	0	0	0	0	0	-0	0	0	-1.2605	0	1	0	-0.9233	0	0	---
x_2	0	1.25	0	1	0	0	-0	-0	0	0	-0	0	0	-0	0	0	0	-0.8333	0	0	---
A_7	M	0.5	0	0	-0	0	0	0	-0	0	0	-0	0	0	0	0	-1	0.6667	0	1	$\frac{0.5}{0}=900719$
x_8	0	0.0651	0	0	0.3915	0	0.6155	0.9233	0.6667	1	0.7694	0.8333	0	-1.6165	0	0	0	-2.5729	0	0	---
$Z=0$		**Z_j**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**$-M$**	**M**	**0**	**0**	**$-M$**	**$0.6667M$**	**M**	**M**	
		C_j-Z_j	0	0	0	0	0	0	0	0	0	0	M	$-M\uparrow$	0	0	M	$-0.6667M$	0	0	

Negative minimum $C_j - Z_j$ is $-M$ and its column index is 12. So, the entering variable is x_{12}.

Minimum ratio is 0 and its row index is 3. So, the leaving basis variable is A_3.

∴ The pivot element is 1.

Entering $= x_{12}$, Departing $= A_3$, Key Element $= 1$

R_3(new) = R_3(old)

R_1(new) = R_1(old)+0.8403R_3(new)

R_2(new) = R_2(old)+2.2068R_3(new)

R_4(new) = R_4(old)+1.0504R_3(new)

R_5(new) = R_5(old)+1.2605R_3(new)

R_6(new) = R_6(old)

R_7(new) = R_7(old)

R_8(new) = R_8(old)+1.6165R_3(new)

Iteration-8		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	**x_8**	**x_9**	**x_{10}**	**x_{11}**	**x_{12}**	**x_{13}**	**x_{14}**	**x_{15}**	**x_{16}**	**A_7**	**MinRatio** $\frac{X_B}{x_{16}}$
x_{13}	0	0.6838	0	0	0	0	0	0	0	0	-0	0	-0.8403	0	1	0	0	-0.6155	0	---
x_4	0	0.0888	0	0	1	1	0.8403	1.2605	0	0	1.0504	0	-2.2068	0	0	0	0	-1.6165	0	---
x_{12}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	---
x_1	0	1.4798	1	0	0	0	0	0	0	0	0	0	-1.0504	0	0	0	0	-0.7694	0	---
x_{14}	0	1.2757	0	0	0	0	0	0	0	0	-0	0	-1.2605	0	0	1	0	-0.9233	0	---
x_2	0	1.25	0	1	0	0	-0	-0	0	0	-0	0	-0	0	0	0	0	-0.8333	0	---
A_7	M	0.5	0	0	-0	0	0	0	-0	0	0	-0	0	0	0	0	-1	(0.6667)	1	$\frac{0.5}{0.6667}=0.75$ -
x_8	0	0.0651	0	0	0.3915	0	0.6155	0.9233	0.6667	1	0.7694	0.8333	-1.6165	0	0	0	0	-2.5729	0	---

$Z = 0$		Z_j	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**$-M$**	**$0.6667M$**	**M**	
		$C_j - Z_j$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	$-0.6667M$ ↑	0	

Negative minimum $C_j - Z_j$ is $-0.6667M$ and its column index is 16. So, the entering variable is x_{16}.

Minimum ratio is 0.75 and its row index is 7. So, the leaving basis variable is A_7.

∴ The pivot element is 0.6667.

Entering $= x_{16}$, Departing $= A_7$, Key Element $= 0.6667$

$R_7(\text{new}) = R_7(\text{old}) \times 1.5$

$R_1(\text{new}) = R_1(\text{old}) + 0.6155 R_7(\text{new})$

$R_2(\text{new}) = R_2(\text{old}) + 1.6165 R_7(\text{new})$

$R_3(\text{new}) = R_3(\text{old})$

$R_4(\text{new}) = R_4(\text{old}) + 0.7694 R_7(\text{new})$

$R_5(\text{new}) = R_5(\text{old}) + 0.9233 R_7(\text{new})$

$R_6(\text{new}) = R_6(\text{old}) + 0.8333 R_7(\text{new})$

$R_8(\text{new}) = R_8(\text{old}) + 2.5729 R_7(\text{new})$

Iteration-9		C_j	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
B	***C_B***	***X_B***	***x_1***	***x_2***	***x_3***	***x_4***	***x_5***	***x_6***	***x_7***	***x_8***	***x_9***	***x_{10}***	***x_{11}***	***x_{12}***	***x_{13}***	***x_{14}***	***x_{15}***	***x_{16}***	**MinRatio**
x_{13}	0	1.1455	0	0	-0	0	0	0	-0	0	-0	0	-0.8403	0	1	0	-0.9233	0	
x_4	0	1.3012	0	0	1	1	0.8403	1.2605	-0	0	1.0504	0	-2.2068	0	0	0	-2.4247	0	
x_{12}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	
x_1	0	2.0569	1	0	-0	0	0	0	-0	0	0	0	-1.0504	0	0	0	-1.1541	0	
x_{14}	0	1.9682	0	0	-0	0	0	0	-0	0	-0	0	-1.2605	0	0	1	-1.385	0	
x_2	0	1.875	0	1	-0	0	-0	-0	-0	0	-0	-0	-0	0	0	0	-1.25	0	
x_{16}	0	0.75	0	0	-0	0	0	0	-0	0	0	-0	0	0	0	0	-1.5	1	
x_8	0	1.9948	0	0	0.3915	0	0.6155	0.9233	0.6667	1	0.7694	0.8333	-1.6165	0	0	0	-3.8594	0	
$Z = 0$		Z_j	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	
		$C_j - Z_j$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

Since all $C_j - Z_j \geq 0$

Hence, optimal solution is arrived with value of variables as :
$x_1 = 2.0569, x_2 = 1.875, x_3 = 0, x_4 = 1.3012, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 1.9948, x_9 = 0, x_{10} = 0, x_{11} = 0, x_{12} = 0, x_{13} = 1.1455, x_{14} = 1.9682, x_{15} = 0, x_{16} = 0.75$

Min $Z = 0$

Solution is provided by AtoZmath.com

MATLAB

```
[Kp,Ki] = meshgrid(0:.1:10, 0:.1:10);

J=(Kp-1.4375).^2+(Ki-1.25).^2;
%J=(Kp)^2+(Ki)^2-2.8750*Kp-2.5*Ki;
figure(1)
contour(Kp,Ki,J,10,'LineWidth', 2);
xlabel('Kp')
ylabel('Ki')
zlabel('J')
hold on

%plot([3.25;0],[0;-7.576],'k', 'LineWidth', 2)
%plot([0;-3.5267],[3.25;0],'k', 'LineWidth', 2)

H = [2 0; 0 2];
f = [-2.875; -2.5];
A = [-0.952 0.4092; -0.952 0.8790; -0.8 0; -1.2 0; 0 -0.8; 0 -1.2];
b = [-0.31; -0.31; -0.5; -0.5; -1.5; -1.5];
LB =[0,0];
UB =[10,10];

[K,P] = quadprog(H,f, A, b,[],[],LB,UB)
```

Output:

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the optimality tolerance,
and constraints are satisfied to within the default value of the constraint tolerance.

<stopping criteria details>


K =

    2.0569
    1.8750


P =

   -2.8547

>>
```

Conclusion:

- Thus by using Nonlinear Programming on the objective function for a control system such as in FOPTD process we can improve the stability and also the performance of an interval block. Thus Optimization techniques play an important role in control systems. Also it can be inferred that the nonlinear programming methods are much simpler in computation than the linear programming methods for a larger problems.

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