Module-1: Calculus
Partial differentiation, total derivative, differentiation of composite functions, Jacobian, Statement of Taylor's and Maclaurin's series expansion for two variables. Maxima and minima for the function of two variables. **(RBT Levels: L1, L2 and L3)**

LECTURE 1:
Partial differentiation

Recall:

1. Definition of a derivative.
2. What is Algebra of derivative of functions?
3. What is Leibnitz rule?
4. Which are the standard derivatives?

Partial derivatives :

Let $z = f(x,y)$ be a function of two variables in x and y.

The first order partial derivative of z w.r.t. x, denoted by $\frac{\partial z}{\partial x}$ or z_x (i.e. Derivative of z w.r.to x keeping 'y' fixed).

Similarly $\frac{\partial z}{\partial y}$ or z_y is the derivative of $\boldsymbol{z}$ w.r.to y keeping 'x' fixed.

$$\frac{\partial z}{\partial x} = \underset{h\to 0}{Lt} \frac{f(x+h,\ y)-f(x,y)}{h} \quad . \quad \text{And } \frac{\partial z}{\partial y} = \underset{h\to 0}{Lt} \frac{f(x,\ y+h)-f(x,y)}{h}$$

Higher order partial derivatives also obtained in the same way.

In all ordinary cases, it can be verified that $z_{xy} = z_{yx}$.

Examples:

1. Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.

Solution: Differentiating z partially w.r.t. x we get,

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay$$

Again differentiating z partially w.r.t. x we get,

$$\frac{\partial^2 z}{\partial^2 x} = 6x.$$

2. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$.

Solution: Given $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Differentiating u partially w.r.t. y we get,

$$\frac{\partial u}{\partial y} = x^2.\frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{1}{x} - \left[y^2.\frac{1}{1+\left(\frac{x}{y}\right)^2}.\left(\frac{-x}{y^2}\right) + 2y\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

Again differentiating w.r.t x we get,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 1 - 2y.\frac{1}{1+\left(\frac{x}{y}\right)^2}.\frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

3. If $z = \sin^{-1}(\frac{y}{x})$. Verify that $z_{xy} = z_{yx}$.

Solution: Given $z = \sin^{-1}(\frac{y}{x})$

Differentiating z partially w.r.t. x we get,

$$z_x = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating z partially w.r.t. y we get,

$$z_y = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Differentiating (2) partially w.r.t. x we get,

$$z_{xy} = \frac{-1}{2}(x^2-y^2)^{\frac{-3}{2}} . 2x = \frac{-x}{(x^2-y^2)^{\frac{3}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Differentiating (1) partially w.r.t. y we get,

$$z_{yx} = \frac{-1}{x}\left(\frac{\sqrt{x^2-y^2}.1 - y.\frac{-2y}{2\sqrt{x^2-y^2}}}{x^2-y^2}\right) = \frac{-x}{(x^2-y^2)^{\frac{3}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From (3) and (4) we have $z_{xy} = z_{yx}$.

4. If $u = \log(x^3+y^3+z^3-3xyz)$, show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -9(x+y+z)^{-2}$.

Solution: Given $u = \log(x^3+y^3+z^3-3xyz)$.

Consider $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$ …………(1)

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} . 3x^2-3yz \qquad \ldots\ldots\ldots..(2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} . 3y^2-3xz \qquad \ldots\ldots\ldots(3)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} . 3z^2-3xy \qquad \ldots\ldots\ldots(4)$$

Adding (2), (3) and (4) we get,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}$$

……………(5)

(5) in (1) , $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right).\frac{3}{x+y+z} = \frac{-9}{(x+y+z)^2}$.

5. If $u = f(r)$, where $r^2 = x^2+y^2+z^2$. Then show that $u_{xx}+u_{yy}+u_{zz} = f''(r)+\frac{2}{r}f'(r)$.

Solution: $u = f(r)$ where r is a function of x, y, z.

$\therefore u_x = f^1(r)\frac{\partial r}{\partial x} = f^1(r)\frac{x}{r}$ $\qquad \left(\because r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}\right)$

$$u_{xx} = f^1(r)\left(\frac{r.1 - x\frac{\partial r}{\partial x}}{r^2}\right) + f^{11}(r)\frac{\partial r}{\partial x}\frac{x}{r} \qquad \text{(Using Chain rule)}$$

$$= \frac{f^1(r)}{r^3}(r^2-x^2) + f^{11}(r)\frac{x^2}{r^2}$$

Similarly $\quad u_{yy} = \frac{f^1(r)}{r^3}(r^2-y^2) + f^{11}(r)\frac{y^2}{r^2}$

$$u_{zz} = \frac{f^1(r)}{r^3}(r^2-z^2) + f^{11}(r)\frac{z^2}{r^2}$$

Then $u_{xx}+u_{yy}+u_{zz} = \frac{f^1(r)}{r^3}[(r^2-x^2)+(r^2-y^2)+(r^2-z^2)] + \frac{f^{11}(r)}{r^2}(x^2+y^2+z^2)$

$$= \frac{f^1(r)}{r^3}(3r^2-r^2) + f^{11}(r)$$

$$= \frac{2}{r} f'(r) + f''(r)$$

6. If $u = e^{ax+by} f(ax - by)$, show that $bu_x + au_y = 2abu$.

Solution: $u = e^{ax+by} f(ax - by)$

$\Longrightarrow u_x = ae^{ax+by} f'(ax - by) + ae^{ax+by} f(ax - by)$

$= ae^{ax+by} f'(ax - by) + au$

$\Longrightarrow bu_x = abe^{ax+by} f'(ax - by) + abu$

And $u_y = -be^{ax+by} f'(ax - by) + be^{ax+by} f(ax - by)$

$= -be^{ax+by} f'(ax - by) + bu$

$\Longrightarrow au_y = -abe^{ax+by} f'(ax - by) + abu$

$\therefore bu_x + au_y = 2abu$

Review:

1. For $f(x, y) = e^{xy}$ compute the partial derivative of f with respect to y.
2. Find u_x,u_y when $u = x^y$.
3. Given $f(x, y) = x^2 + y^2$,evaluate f_{xx},f_{yy} and f_{xy} .
4. Let $z = r\cos\theta$ verify $z_{r\theta} = z_{\theta r}$.
5. If $u = e^{x+y+z}$ then find $u_x + u_y + u_z$

LECTURE 2:
Total derivative - differentiation of composite functions

Recall:

1. What is a composite function.
2. How you describe partial derivative?
3. The temperature T(x,y) in degrees Celsius at a point (x,y) on a metal plate is given by $\mathrm{T(x, y) = 3x^2 - 2xy + y^2}$. Find the rate of change of temperature with respect to y at the point $(2,1)$.
4. A fluid flows through a pipe with velocity function $\mathrm{v(x,y) = x^2 - xy + y^2}$, where x and y are spatial coordinates in the pipe. Calculate the rate of change of the velocity with respect to x at the point $(1,1)$.

Total derivatives:

1. If $u = f(x, y)$ $and\ x = g(t),\ \ y = h(t)\ \ then$ $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.
2. If $f(x,y) = constant$, then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.
3. If $u = f(x, y)$ subject to $\varphi(x, y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.

Problems:

1. If $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$ and $\mathrm{y = t^2}$ find $\frac{du}{dt}$ $as\,a function\ of\ t.$

Solution: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t$

$$= \frac{1}{\mathrm{t}^2}\cos\left(\frac{e^t}{\mathrm{t}^2}\right)e^t - \frac{e^t}{t^4}\cos\left(\frac{e^t}{\mathrm{t}^2}\right)2t$$

$$= e^t\cos\left(\frac{e^t}{\mathrm{t}^2}\right)\left[\frac{1}{\mathrm{t}^2} - \frac{2}{\mathrm{t}^3}\right]$$

2. If x increases at the rate of 2 cm/sec at the instant when $x = 3\ cm.$ and $y = 1\ cm.$, at what rate must y changing in order that the function $2xy - 3x^2y$ shell be neither increasing nor decreasing?

Solution: Let $u = 2xy - 3x^2y$, given that $\frac{dx}{dt} = 2,\ \ \frac{du}{dt} = 0,\ \ x = 3$ and $y = 1$.

So that $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt}$

$\Longrightarrow\quad 0 = 2(2 - 18) + (6 - 27)\ \frac{dy}{dt}\quad \Longrightarrow \frac{dy}{dt} = -\frac{32}{21}$ cm/sec.

Thus y is decreasing at the rate of $\frac{32}{21}$ cm/sec.

3. If $u = x\log xy$ where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$.

Solution: If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.

Given that $u = x\log xy$, $\varphi(x,y) = x^3 + y^3 + 3xy$

Clearly $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = -\frac{3x^2+3y}{3y^2+3x} = -\frac{x^2+y}{y^2+x}$, $\frac{\partial u}{\partial x} = \log xy + 1$, $\frac{\partial u}{\partial y} = \frac{x}{y}$.

Hence $\frac{du}{dx} = \log xy + 1 - \frac{x(x^2+y)}{y(y^2+x)}$

4. If $u = \sqrt{x^2+y^2}\ and\ x^3 + y^3 + 3axy = 5a^2, find\ \frac{du}{dx}$ when $x = y = a$.

Solution: If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.

Hear $u = \sqrt{x^2+y^2}\ and\ \varphi = x^3 + y^3 + 3axy = 5a^2$

$\Longrightarrow \frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = -\frac{3x^2+3ay}{3y^2+3ax} = -\frac{x^2+ay}{y^2+ax} = -1$ at $x = y = a$

Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\left(-\frac{x^2+ay}{y^2+ax}\right)$

$= \frac{a}{\sqrt{2a^2}} - \frac{a}{\sqrt{2a^2}} = 0$, at $x = y = a$.

Review:

1. When to use total derivative rule?
2. If $u(x,y) = c$ is an implicit function then express $\frac{dy}{dx}$ in terms of its partial derivatives.
3. The height of a particle is given by $\mathrm{z} = \mathrm{x}^2 + \mathrm{y}^2$, where $\mathrm{x} = \cos \mathrm{t}$ and $\mathrm{y} = \sin \mathrm{t}$. Find the rate of change of height with respect to time t.

LECTURE 3:
Differentiation of composite functions-Chain Rule

Recall:

1. What is Composite function?
2. How does the total derivative differ from a partial derivative?
3. Extend the total derivative to $f(x,y,z)$ where each variable depends on T.
4. Let $u(x,y) = xy$ where $= e^t$, $y = \sin t$. Compute $\frac{du}{dt}$.

If $u = f(r,s,t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

$u_x = u_r r_x + u_s s_x + u_t t_x$, $u_y = u_r r_y + u_s s_y + u_t t_y$ and $u_z = u_r r_z + u_s s_z + u_t t_z$.

1. If $u = f(y-z, z-x, x-y), then\ prove\ that\ u_x + u_y + u_z = 0$.

Solution: Let $r = y - z$, $s = z - x$, $t = x - y$

Then $r_x = 0$, $r_y = 1$, $r_z = -1$, $s_x = -1$, $s_y = 0$, $s_z = 1$, $t_x = 1$, $t_y = -1$, $t_z = 0$.

If $u = f(r,s,t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

$u_x = u_r r_x + u_s s_x + u_t t_x$, $u_y = u_r r_y + u_s s_y + u_t t_y$ and $u_z = u_r r_z + u_s s_z + u_t t_z$.

$\Longrightarrow u_x = 0 - u_s + u_t$, $u_y = u_r + 0 - u_t$ and $u_z = -u_r + u_s + 0$.

$\therefore u_x + u_y + u_z = -u_s + u_t + u_r - u_t - u_r + u_s = 0$.

2. If $u = f(2x-3y, 3y-4z, 4z-2x), then\ find\ the\ value\ of\ \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$.

Solution: Let $r = 2x - 3y$, $s = 3y - 4z$, $t = 4z - 2x$

Then $r_x = 2$, $r_y = -3$, $r_z = 0$, $s_x = 0$, $s_y = 4$, $s_z = -4$, $t_x = -2$, $t_y = 0$, $t_z = 4$.

If $u = f(r,s,t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

$u_x = u_r r_x + u_s s_x + u_t t_x$, $u_y = u_r r_y + u_s s_y + u_t t_y$ and $u_z = u_r r_z + u_s s_z + u_t t_z$.

$\Longrightarrow u_x = 2u_r + 0 - 2u_t$, $u_y = -3u_r + 3u_s + 0$ and $u_z = 0 - 4u_s + 4u_t$.

$$\therefore \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = u_r - u_t - u_r + u_s - u_s + u_t = 0.$$

3. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Solution: Let $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$, $s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$.

Then $r_x = -\frac{1}{x^2}$, $r_y = \frac{1}{y^2}$, $r_z = 0$, $s_x = -\frac{1}{x^2}$, $s_y = 0$, $s_z = \frac{1}{z^2}$,

If $u = f(r,s)$ where $r,\ and\ s\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

$u_x = u_r r_x + u_s s_x$, $u_y = u_r r_y + u_s s_y$ and $u_z = u_r r_z + u_s s_z$

$\Longrightarrow u_x = -\frac{1}{x^2}u_r - \frac{1}{x^2}u_s$, $u_y = \frac{1}{y^2}u_r + 0$ and $u_z = 0 + \frac{1}{z^2}u_s$

$\Longrightarrow x^2\frac{\partial u}{\partial x} = -u_r - u_s$, $y^2\frac{\partial u}{\partial y} = u_r$ and $z^2\frac{\partial u}{\partial z} = u_s$.

Therefore $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Review:

1. Explain how the chain rule connects Cartesian coordinates with Polar coordinate?
2. When to use chain rule and total derivative rule?
3. If $u = x^2 + y^2 + z^2$ where $x = r\cos\theta, y = r\sin\theta$ and $z = h$.Compute u_h

LECTURE 4:
Jacobian and problems

Recall:

1. If $u(x,y) = x^2y^2$ where $x = e^t$ and $y = e^{-t}$ find the total derivative $\frac{du}{dt}$.
2. If $z = x^2 + y^2$ and $= r cos\theta$, $y = r sin\theta$ find $\frac{\partial z}{\partial r}$.
3. If $u = f(r,s,t)$ where $r,s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$ then give u_x, u_y and u_z.using chain rule.
4. Find $\frac{dy}{dx}$, given $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Jacobian:

$$\boldsymbol{J} = \frac{\partial(\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w})}{\partial(\boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z})} = \begin{vmatrix} \boldsymbol{u_x} & \boldsymbol{u_y} & \boldsymbol{u_z} \\ \boldsymbol{v_x} & \boldsymbol{v_y} & \boldsymbol{v_z} \\ \boldsymbol{w_x} & \boldsymbol{w_y} & \boldsymbol{w_z} \end{vmatrix}.$$

Problems:

1. If $x = r\cos\theta$, $y = r\sin\theta$, then verify that $JJ' = 1$.

Solution: $x = r\cos\theta$, $y = r\sin\theta$

$$J = \frac{\partial(x,\ y)}{\partial(r,\ \theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r\ .$$

$x = r\cos\theta$, $y = r\sin\theta$

$$\Longrightarrow \quad r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{And} \quad J' = \frac{\partial(r,\ \theta)}{\partial(x,\ y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) & \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{r(x^2+y^2)} + \frac{y^2}{r(x^2+y^2)} = \frac{1}{r} \therefore JJ' = 1.$$

2. If $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$, then find $\boldsymbol{J} = \frac{\partial(x\ y,\ z)}{\partial(r,\ \varphi,\ z)}$

Solution: $J = \frac{\partial(x\ y,\ z)}{\partial(r,\ \varphi,\ z)} = \begin{vmatrix} x_r & x_\varphi & x_z \\ y_r & y_\varphi & y_z \\ z_r & z_\varphi & z_z \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$.

3. If $x = u(1+v)$, $y = v(1+u)$, *show that* $\frac{\partial(x,\ y)}{\partial(u,\ v)} = 1 + \mathrm{u} + \mathrm{v}$.

Solution: Given that $x = u(1+v), y = v(1+u)$

$$\frac{\partial(x,\ y)}{\partial(u,\ v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} (1+v) & u \\ v & (1+u) \end{vmatrix} = (1+v)(1+u) - uv = 1 + u + v + uv - uv = 1 + u + v.$$

4. If $u = x + y + z$, $v = x^2 + y^2 + z^2$ *and* $w = xy + yz + zx$, *then find* $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$.

Solution: Given that, $u = x + y + z$, $v = x^2 + y^2 + z^2$ *and* $w = xy + yz + zx$

$$\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

5. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $= r\cos\theta$, $y = r\sin\theta$, then show that $\frac{\partial(u,\ v)}{\partial(r,\ \theta)} = 6r^3\sin 2\theta$.

Solution: Since $\frac{\partial(u,\ v)}{\partial(r,\ \theta)} = \frac{\partial(u,\ v)}{\partial(x,\ y)} \times \frac{\partial(x,\ y)}{\partial(r,\ \theta)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \times \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$

$$= \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 12xyr = 12r^3\sin\theta\cos\theta = 6r^3\sin 2\theta.$$

6. Prove that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ for $x = r\cos\theta$, $y = r\sin\theta$.

Solution: Clearly $\frac{\partial x}{\partial r} = \cos\theta$ and

Since, $r = \sqrt{x^2 + y^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\therefore \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

7. If $z = f(x,y)$ *and* $x = r\cos\theta$, $y = r\sin\theta$ then show that, $(z_x)^2 + (z_y)^2 = (z_r)^2 + \frac{1}{r^2}(z_\theta)^2$.

Solution: $z_r = z_x x_r + z_y y_r = z_x\cos\theta + z_y\sin\theta \qquad \cdots\cdots\cdots(1)$

And $z_\theta = z_x x_\theta + z_y y_\theta = -rz_x\sin\theta + \mathrm{r}z_y\cos\theta$

$\Longrightarrow \frac{1}{r} z_\theta = -z_x\sin\theta + z_y\cos\theta \qquad \cdots\cdots\cdots(2)$

$(1)^2 + (2)^2 \qquad \Longrightarrow (z_r)^2 + \frac{1}{r^2}(z_\theta)^2 = (z_x)^2\cos^2\theta + (z_y)^2\sin^2\theta + 2z_x z_y\cos\theta\sin\theta$

$$+(z_x)^2\sin^2\theta + (z_y)^2\cos^2\theta - 2z_x z_y\cos\theta\sin\theta$$

$$= (z_x)^2 + (z_y)^2.$$

Review:

1. If $u = u(x,y)$, $v = v(x,y)$ then what is $J\left(\frac{u,v}{x,y}\right)$? ?
2. What is null Jacobian ?
3. Suppose $\frac{\partial(u,\ v)}{\partial(x,\ y)} = 0$ where u, v are functions of x and y. What inference about u and v can we get?
4. If $J\left(\frac{u,v}{x,y}\right) = \frac{1}{x+y}$ then what is $J'\left(\frac{x,y}{u,v}\right)$?
5. Given a transformation $x = r\cos\theta$, $y = r\sin\theta$. Find the Jacobian $\frac{\partial(x,\ y)}{\partial(r,\theta\ )}$.s

TUTORIAL 1:
Problems on Partial differentiation and Jacobian

1. If $u = f(x+ct) + g(x-ct)$, show that $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.
2. If $u = \tan^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = 1$.
3. At a given instant the sides of a rectangle are 4ft and 3 ft, and they are increasing at the rate of 1.5ft/sec and 0.5ft/sec respectively. Find the rate at which the area is increasing at that instant.
4. If $u = x^2 + y^2, x = e^{2t}\cos 3t$ and $y = e^{2t}\sin 3t$. find $\frac{du}{dt}$ as a function of t
5. If $u = \sin(x^2 + y^2)$ where $a^2x^2 + b^2y^2 = c^2$ find $\frac{du}{dx}$.
6. If $u = f(ax - by, by - cz, cz - ax)$, then show that $\frac{1}{a}u_x + \frac{1}{b}u_y + \frac{1}{c}u_z = 0$.
7. If $u = x + y + z,\ v = x^2 + y^2 + z^2\ and\ w = xy + yz + zx, then\ find\ \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$.
8. If $x = u(1-v)$, $y = uv$ then verify that $JJ' = 1$.
9. If $u = x + 3y^2 - z^3,\ v = 4x^2yz\ and\ w = 2z^2 - xy, then\ find\ \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}\ at\ (1,-1,0)$.
10. If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

TUTORIAL 2:
Lab Activity 1: Finding partial derivatives and Jacobian

Objectives:

Use python

- To find partial derivatives of functions of several variables.
- To find Jacobian of function of two and three variables.

1. Prove that if $u = e^x f(xcosy - ysiny)$, show that $u_{xx} + u_{yy} = 0$.

```
from sympy import *
x, y = symbols ('x y')
u = exp (x)*(x* cos (y)-y* sin (y))
display (u)
ux = diff (u, x)
uy = diff (u, y)
uxx = diff (ux , x)
# uxx = diff (u, x, x)
uyy = diff (uy , y)
# uyy = diff (u, y, y)
w = uxx +uyy
w1 = simplify (w)
print ('RHS :', w1)
```

2. If $ux = yz,\ vy = zx,\ wz = xy$, then show that $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = 4$

```python
from sympy import *
x, y, z = symbols ('x y z')
u = x*y/z
v = y*z/x
w = z*x/y
ux = diff (u, x)
uy = diff (u, y)
uz = diff (u, z)
vx = diff (v, x)
vy = diff (v, y)
vz = diff (v, z)
wx = diff (w, x)
wy = diff (w, y)
wz = diff (w, z)
J = Matrix ([[ ux , uy , uz ], [vx , vy , vz], [wx , wy , wz ]])
print (" The Jacobian matrix is")
display (J)
Jac = det (J). doit ()
# Jac = Determinant (J). doit ()
print ('J = ', Jac )
```

3. If $u = x + 3y^2 - z^3,\ v = 4x^2yz$ *and* $w = 2z^2 - xy$, *then find* $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ *at* $(1, -1, 0)$.

LECTURE 5:

Statement of Taylor's series expansion for two variables-Problems.

Recall:

1. What is a sequence.
2. What is a series.
3. What are the different types of progressions.
4. When we say a function is continuous at any given point.
5. Define Taylor's series expansion for $y = f(x)$ about the point $x = a$.

Taylor's expansion of a function of two variables:

Expansion of $f(x,\ y)$ about $(a,\ b)$ or (in the powers of $(x - a)$ and $(y - b)$) is

$$f(x,\ y) = f + (x-a)f_x + (y-b)f_y + \frac{1}{2!}\left[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right] + \cdots$$

Where $f = f(a,\ b)$, $f_x = f_x(a,\ b), f_y = f_y(a,\ b)$ and so on.

Problems:

1. Use Taylor's formula to expand the function f defined by $f(x,\ y) = x^2 + xy + y^2$ in powers of $(x - 1)$ and $(y - 2)$ up to second degree terms .

   Solution:

Function	Value at $(1,\ 2)$
$f(x,\ y) = x^2 + xy + y^2$	7
$f_x = 2x + y$ $f_y = x + 2y$	4 5
$f_{xx} = 2$ $f_{xy} = 1$ $f_{yy} = 2$	2 1 2

Substituting in Taylor's expansion

$f(x,\ y) = f + (x-a)f_x + (y-b)f_y + \frac{1}{2!}\left[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right] + \cdots$

Put $a = 1,\ \mathrm{b} = 2$ and the functional values, we get

$$x^2 + xy + y^2 = 7 + 4(x-1) + 5(y-2) + \frac{1}{2}[2(x-1)^2 + 2(x-1)(y-2) + 2(y-2)^2]$$
$$= 7 + 4(x-1) + 5(y-2) + (x-1)^2 + (x-1)(y-2) + (y-2)^2.$$

2. Expand $f(x,\ y) = e^x \cos y$ in Taylor's Series at $\left(1, \frac{\pi}{2}\right)$ up to second degree.

Solution:

Function	Value at $\left(1,\ \frac{\pi}{2}\right)$
$f(x,\ y) = e^x \cos y$	0
$f_x = e^x \cos y$ $f_y = -e^x \sin y$	0 $-$ e
$f_{xx} = e^x \cos y$ $f_{xy} = -e^x \sin y$ $f_{yy} = -e^x \cos y$	0 $-$ e 0

Substituting in Taylor's expansion

$f(x,\ y) = f + (x-a)f_x + (y-b)f_y + \frac{1}{2!}\left[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right] + \cdots$

Put $a = 1,\ \mathrm{b} = \frac{\pi}{2}$ and the functional values, we get

$$e^x \cos y = -e\left(y - \frac{\pi}{2}\right) + \frac{1}{2}\left[-2e(x-1)\left(y-\frac{\pi}{2}\right)\right] = -e\left(y-\frac{\pi}{2}\right) - e(x-1)\left(y-\frac{\pi}{2}\right).$$

3. Expand the function $\sin xy$ in powers of $x-1$ and $\mathrm{y} - \frac{\pi}{2}$ up to second degree terms.

Solution:

Function	Value at $\left(1,\ \frac{\pi}{2}\right)$
$f(x,\ y) = \sin xy$	1
$f_x = y\cos xy$ $f_y = x\cos xy$	0 0
$f_{xx} = -y^2 \sin xy$ $f_{xy} = -xy\sin xy + \cos xy$ $f_{yy} = -x^2 \sin xy$	$\frac{-\pi^2}{4}$ $\frac{-\pi}{2}$ -1

Substituting in Taylor's expansion

$f(x,\ y) = f + (x-a)f_x + (y-b)f_y + \frac{1}{2!}\left[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right] + \cdots$

Put $a = 1,\ \mathrm{b} = \frac{\pi}{2}$ and the functional values, we get

$$e^x \cos y = 1 + \frac{1}{2}\left[\frac{-\pi^2}{4}(x-1)^2 - \pi(x-1)\left(y-\frac{\pi}{2}\right) - \left(y-\frac{\pi}{2}\right)^2\right]$$
$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right) - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2.$$

Review:

1. What is the general form of Taylor's series expansion of a function $f(x,y)$ about $x = a$ and $y = b$.
2. When is Taylor's series referred to as a Maclaurin's series?
3. What is the general form of Taylor's series expansion of a function $f(x,y)$ about (1,0).
4. Give the series expansion of e^{x+y} in powers of $x-1$ and y.

LECTURE 6:
Statement of Maclaurin's series expansion for two variables-Problems.

Recall:

1. State Taylors' series expansion for function of two variables about (a,b)?
2. What is the difference between Taylors' and Maclaurin's' expansion?
3. Expand $\sin(x+y)$ in powers of $x-1$ and $y-\frac{\pi}{2}$.

Maclaurin's expansion of a function of two variables:

Expansion of $f(x,\ y)$ about $(0,\ 0)$ or (in the powers of and y) is

$$f(x,\ y) = f + xf_x + yf_y + \frac{1}{2!}\left[x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right] + \frac{1}{3!}\left[x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right]\cdots$$

Where $f = f(0,\ 0)$, $f_x = f_x(0,\ 0), f_y = f_y(0,\ 0)$ and so on.

Problems:

1. Use Maclaurin's formula to expand the function f defined by $f(x,\ \ y) = x^2y + 3y - 2$ in powers of x and y.

   Solution:

Function	Value at $(0,\ \ 0)$
$f(x,\ \ y) = x^2y + 3y - 2$	-2
$f_x = 2xy$ $f_y = x^2 + 3$	0 3
$f_{xx} = 2y$ $f_{xy} = 2x$ $f_{yy} = 0$	0 0 0
$f_{xxx} = 0$ $f_{xxy} = 2$ $f_{xyy} = 0$ $f_{yyy} = 0$	0 2 0 0

Substituting in Maclaurin's expansion

$$f(x,\ y) = f + xf_x + yf_y + \frac{1}{2!}\left[x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right] + \frac{1}{3!}\left[x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right]\cdots$$

$$x^2y + 3y - 2 = -2 + 3y + x^2y.$$

2. Expand $f(x,\ \ y) = e^x \log(1+y)$ as Maclaurin's Series up to second degree terms.

   Solution:

Function	Value at $(0,\ \ 0)$
$f(x,\ \ y) = e^x \log(1+y)$	0
$f_x = e^x \log(1+y)$ $f_y = \dfrac{e^x}{1+y}$	0 1
$f_{xx} = e^x \log(1+y)$ $f_{xy} = \dfrac{e^x}{1+y}$ $f_{yy} = \dfrac{-e^x}{(1+y)^2}$	0 1 -1

Substituting in Maclaurin's expansion

$$f(x,\ y) = f + xf_x + yf_y + \frac{1}{2!}\left[x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right] + \cdots$$

$e^x \log(1+y) = y + \frac{1}{2}(2xy - y^2) = y + xy - \frac{y^2}{2}.$

3. Expand the function $e^x \cos y$ in powers of x and y up to third degree terms.

Solution:

Function	Value at $(0,\ 0)$
$f(x,\ y) = e^x \cos y$	1
$f_x = e^x \cos y$ $f_y = -e^x \sin y$	1 0
$f_{xx} = e^x \cos y$ $f_{xy} = -e^x \sin y$ $f_{yy} = -e^x \cos y$	1 0 -1
$f_{xxx} = e^x \cos y$ $f_{xxy} = -e^x \sin y$ $f_{xyy} = -e^x \cos y$ $f_{yyy} = e^x \sin y$	1 0 -1 0

Substituting in Maclaurin's expansion

$f(x,\ y) = f + xf_x + yf_y + \frac{1}{2!}\left[x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right] + + \frac{1}{3!}\left[x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right] \cdots$

$e^x \cos y = 1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}(x^3 - 3xy^2) = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} - \frac{xy^2}{2}.$

Review:

1. What is the general form of Maclaurin's series expansion of a function $f(x, y)$ about origin.
2. Give the series expansion of e^{x+y} in powers of x and y.

LECTURE 7:
Maxima and minima for a function of two variables.

Recall:

1. If $u = x + y$ and $v = x - y$, determine the value of the Jacobian $\frac{\partial(x,\ y)}{\partial(u,v)}$.
2. Given $= e^{x+y}$, $v = e^{-x-y}$ compute Jacobian J of u, v with respect to x, y.
3. If the given transformations are associated to each other, then what will be the Jacobian determinant value?
4. What does the Jacobian matrix represent in a multivariable transformation?
5. Determine Jacobian of the transformation $x = u + v$ and $y = \frac{1}{u+v}$.

Maxima and minima of functions of two variables:

1. $f(x,\ y)$ is stationary at $(a,\ b)$ i.e. $f(a,\ b)$ is the stationary value of f if $f_x = 0 = f_y$ at $(a,\ b)$.

2. $f(x,\ y)$ is maximum at $(a,\ b)$ i.e. $f(a,\ b)$ is the maximum value of f

If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} < 0$.

3. $f(x,\ y)$ is minimum at $(a,\ b)$ i.e. $f(a,\ b)$ is the minimum value of f

If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} > 0$.

4. $(a,\ b)$ is said to be saddle point of $f(x,\ y)$ if i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$.

5. If $f_x = 0 = f_y$ and $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$, then by discussion find maxima and minima.

Examples:

1. Discuss the maxima and minima of $f(x,\ y) = x^3 + y^3 - 3x - 3y + 20$.

Solution: Given $f(x,\ y) = x^3 + y^3 - 3x - 3y + 20$

Differentiating partially we get,

$f_x = 3x^2 - 3,\ f_y = 3y^2 - 3,\ f_{xx} = 6x, f_{yy} = 6y$ and $f_{xy} = 0$.

Now for extreme values

$f_x = 0$, $f_y = 0$ $\Longrightarrow$ $x = \pm 1,\ y = \pm 1$

Point	$f_{xx} = 6x$	$f_{yy} = 6y$	$f_{xy} = 0$	$f_{xx}f_{yy} - f_{xy}^2 = 36xy$	Conclusion
$(1,\ 1)$	$6 > 0$	6	0	$36 > 0$	$f(1,\ 1) = 16$ is minimum
$(1,\ -1)$	$6 > 0$	-6	0	$-36 < 0$	$(1,\ -1)$ is saddle point
$(-1,\ 1)$	$-6 < 0$	6	0	$-36 < 0$	$(-1,\ 1)$ is saddle point
$(-1,\ -1)$	$-6 < 0$	-6	0	$36 > 0$	$f(-1,\ -1) = 24$ is maximum

2. Examine the function $f(x,\ y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.

Solution: $f(x,\ y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Differentiating partially we get, $f_x = 4x^3 - 4x + 4y$, $f_y = 4y^3 + 4x - 4y$,

$f_{xx} = 12x^2 - 4$, $f_{yy} = 12y^2 - 4$ and $f_{xy} = 4$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Longrightarrow 4x^3 - 4x + 4y = 0$ and $4y^3 + 4x - 4y = 0$.

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Put $y = -x$ in $x^3 - x + y = 0$, we get $x^3 - 2x = 0$

$\Longrightarrow$ $x = 0, \sqrt{2}, -\sqrt{2}$ and corresponding values of y are $0, -\sqrt{2}, \sqrt{2}$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$f(\sqrt{2}, -\sqrt{2}) = -8$ is minimum
$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$f(-\sqrt{2},\ \sqrt{2}) = -8$ is minimum
$(0, 0)$	$-4 < 0$	-4	4	0	Since $f_{xx}f_{yy} - f_{xy}^2 = 0$ Further investigation is needed.

Clearly $f(0,\ 0) = 0$, $f(0.1,\ 0) = -0.0199$, $f(0.1,\ 0.1) = 0.0002$.

Thus in the neighborhood of $(0,\ 0)$, $f > f(0,\ 0)$ at some points and $f < f(0,\ 0)$ at some points.

Hence $f(0,\ 0)$ is not an extreme value. The point $(0,\ 0)$ is saddle point.

3. Discuss the maxima and minima of $f(x,\ y) = x^3y^2(1 - x - y)$.

Solution: $f(x,\ y) = x^3y^2 - x^4y^2 - x^3y^3$.

Differentiating partially we get, $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$, $f_y = 2x^3y - 2x^4y - 3x^3y^2$,

$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$, $f_{yy} = 2x^3 - 2x^4 - 6x^3y$ and $f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Longrightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ and $2x^3y - 2x^4y - 3x^3y^2 = 0$.

$\Longrightarrow 3 - 4x - 3y = 0$ and $2 - 2x - 3y = 0$.

Therefore stationary points are $\left(\frac{1}{2},\ \frac{1}{3}\right)$ and $(0,\ 0)$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$\left(\frac{1}{2},\ \frac{1}{3}\right)$	$-\frac{1}{9} < 0$	$-\frac{1}{8}$	$-\frac{1}{12}$	$\frac{1}{144} > 0$	$f\left(\frac{1}{2},\ \frac{1}{3}\right) = \frac{1}{432}$ is maximum
$(0, 0)$	0	0	0	0	Since $f_{xx}f_{yy} - f_{xy}^2 = 0$ Further investigation is needed.

Clearly $f(0,\ 0) = 0$, $f(0.1,\ 0.1) > 0$, $f(-0.1, -0.1) < 0$.

Thus in the neighborhood of $(0,\ 0)$, $f > f(0,\ 0)$ at some points and $f < f(0,\ 0)$ at some points. Hence $f(0,\ 0)$ is not an extreme value. The point $(0,\ 0)$ is saddle point.

LECTURE 8:

Maxima and minima for a function of two variables-Problems.

Recall:

1. Show that $u = x^3 + y^3 - 3xy$ has a saddle point at the origin.
2. Determine whether $\cos x + \cos y$, $v = e^{-x-y}$ has a maxima or minima at (0,0).
3. Find lacal maxima, minima or saddle point for $u = x^3 - 3xy^2$.

4. Find the maximum and minimum values of the function $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Solution: Given $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Differentiating partially we get, $f_x = 3x^2 + 3y^2 - 30x + 72$, $f_y = 6xy - 30y$,

$f_{xx} = 6x - 30x$, $f_{yy} = 6x - 30$ and $f_{xy} = 6y$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Rightarrow\ 3x^2 + 3y^2 - 30x + 72 = 0 \ldots\ldots\ldots\ldots.(1)$,$6xy - 30y = 0 \ldots\ldots\ldots\ldots(2)$

Solving (2) we get $y = 0, x = 5$.

Substituting $y = 0, x = 5$ in (1) we get $(4,0), (6,0), (5,1), (5,-1)$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(4,0)$	-6 $<$ 0	-6	0	36 $>$ 0	$f(4,0)$=112 is maximum.
$(6,0)$	6 $>$ 0	6	0	36 $>$ 0	$f(6,0)$= 108 is minimum.
$(5,1)$	0	0	6	$-36 < 0$	$(5,1)$ is a saddle point.
$(5,-1)$	0	0	-6	$-36 < 0$	$(5,-1)$ is a saddle point.

Therefore, Maximum value is 112 and minimum value is 108.

5. Find the extrem values of $f(x,\ y) = x^3 + y^3 - 3axy$, $a \geq 0$.

Solution: $f(x,\ y) = x^3 + y^3 - 3axy$.

Differentiating partially we get, $f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$,

$f_{xx} = 6x$, $f_{yy} = 6y$ and $f_{xy} = -3a$.

$f_x = 0$, $f_y = 0$ $\Longrightarrow 3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0$.

$\Longrightarrow x^2 = ay$ and $y^2 = ax \Longrightarrow x^4 = a^3x$.

$x = 0,\ y = 0$ and $x = a,\ y = a$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(a,\ a)$	$6a > 0$	$6a$	$-3a$	$27a^2 > 0$	$f(a,\ a) = -a^3$ is minimum.
$(0,\ 0)$	0	0	$-3a$	$-9a^2$	Since $f_{xx}f_{yy} - f_{xy}^2 < 0$ The point $(0,\ 0)$ is saddle point

6. Examine the function $\mathrm{f(x,\ y) = xy(1 - x - y)}$ for extreme values.

Solution: Given $f(x,\ y) = xy - x^2y - xy^2$

Differentiating partially we get,

$f_x = y - 2xy - y^2$, $f_y = x - x^2 - 2xy$,$f_{xx} = -2y$, $f_{yy} = -2x$ and $f_{xy} = 1 - 2x - 2y$.

Now for extreme values $\Longrightarrow y - 2xy - y^2 = 0$ and $x - x^2 - 2xy = 0$.

$\Longrightarrow y = 0,\ x = 1\ or\ 0$,and $x = 0,\ y = 0$ or 1. $x = y = \frac{1}{3}$.

Therefore stationary points are $\left(\frac{1}{3},\ \frac{1}{3}\right)$, $(1,\ 0)$, $(0,\ 1)$and $(0,\ 0)$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy}-f_{xy}^2$	Conclusion
$\left(\frac{1}{3},\ \frac{1}{3}\right)$	$-\frac{2}{3}<0$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}>0$	$f\left(\frac{1}{3},\ \frac{1}{3}\right)=\frac{1}{27}$ is maximum
$(0,\ 0)$	0	0	1	-1	Since $f_{xx}f_{yy}-f_{xy}^2<0$ The point $(0,\ 0)$ is saddle point
$(1,\ 0)$	0	-2	-1	-1	Since $f_{xx}f_{yy}-f_{xy}^2<0$ The point $(1,\ 0)$ is saddle point
$(0,\ 1)$	-2	0	-1	-1	Since $f_{xx}f_{yy}-f_{xy}^2<0$ The point $(0,\ 1)$ is saddle point

Review:

1. When we say $(a,\ b)$ is a stationary value for the function $f(x,\ y)$?
2. What is the condition for $f(x,\ y)$ to be maximum at $(a,\ b)$?
3. When $(a,\ b)$ is said to be saddle point of $f(x,\ y)$?
4. If $f(x,\ y)=x^2+y^2-x+y$ then find the critical point
5. Given $f(x,\ y)=x^2+y^2-4x+6y$ then find the minimum value of $f(x,\ y)$.

TUTORIAL 3:

Problems on Taylor's ,Maclaurin's Series and Maxima and minima for a function of two variables.

1. Find the Taylor's series expansion of $e^x \sin y$ at the point $\left(-1, \frac{\pi}{4}\right)$ up to third degree terms.
2. Expand $f(x,\ y)= e^{xy}$ in Taylor's Series at (1,1) up to second degree.
3. Express $\cos x \cos y$ in powers of x and y up to third degree terms.
4. Expand $\log(1+x-y)$ up to second degree term about the origin.
5. Find the maximum and minimum values of

a) $x^3+y^3-3x-12y+20$ b) $x^3+3xy^2-3x^2-3y^2+4$ c) $xy(1-x-y)$

6. Show that $f(x,y)=x^3+y^3-3xy+1$ is minimum at (1, 1).

TUTORIAL 4:

Lab Activity 2: Expansion of Taylor's and Maclaurin's series

Objectives: Use python

- Use python to expand the given single variable function as Taylor's series.
- Use python to expand the given single variable function as Maclaurin's series

1. Use Taylor's formula to expand the function f defined by $f(x,\ y)=x^2+xy+y^2$ in powers of $(x-1)$ and $(y-2)$

```
from sympy import *
x, y = symbols('x y')
f = x**2 + x*y + y**2
a, b = 1, 2
fx = diff(f, x)
fy = diff(f, y)
fxx = diff(f, x, 2)
fyy = diff(f, y, 2)
fxy = diff(f, x, y)
f0 = f.subs({x: a, y: b})1
fx0 = fx.subs({x: a, y: b})
fy0 = fy.subs({x: a, y: b})
fxx0 = fxx.subs({x: a, y: b})
fyy0 = fyy.subs({x: a, y: b})
fxy0 = fxy.subs({x: a, y: b})
taylor_expansion = ( f0+ fx0*(x-a) + fy0*(y-b)
+ 1/2*( fxx0*(x-a)**2 + 2*fxy0*(x-a)*(y-b) + fyy0*(y-b)**2 ))
print("Taylor expansion of f(x,y)=x^2+xy+y^2 about (1,2): \n")
print(simplify(taylor_expansion))
```

2. Expand the function $\cos x \cos y$ in powers of x and y up to third degree terms.

```python
from sympy import *
x, y = symbols('x y')
f = cos(x) * cos(y)
fx = diff(f, x)
fy = diff(f, y)
fxx = diff(f, x, 2)
fyy = diff(f, y, 2)
fxy = diff(f, x, y)
f0 = lambdify((x, y), f)
fx0 = lambdify((x, y), fx)
fy0 = lambdify((x, y), fy)
fxx0 = lambdify((x, y), fxx)
fyy0 = lambdify((x, y), fyy)
fxy0 = lambdify((x, y), fxy)
a, b = 0, 0
maclaurin = ( f0(a, b)+ fx0(a, b)*(x-a) + fy0(a, b)*(y-b)
        + (1/2)*( fxx0(a, b)*(x-a)**2+ 2*fxy0(a, b)*(x-a)*(y-b)
              + fyy0(a, b)*(y-b)**2 ) )
print("Maclaurin series expansion of cos(x)cos(y) up to 2nd order:\n")
print(simplify(maclaurin))
```

PRACTICE QUESTION BANK

MODULE 1: CALCULUS

Partial differentiation, total derivative - differentiation of composite functions. Jacobian:

1. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$.
2. If $z = \sin^{-1}(\frac{y}{x})$. Verify that $z_{xy} = z_{yx}$.
3. If x increases at the rate of 2 cm/sec at the instant when $x = 3$ cm and $y = 1$ cm. , at what rate must y changing in order that the function $2xy - 3x^2y$ shall neither be increasing nor decreasing?
4. At a given instant the sides of a rectangle are 4ft and 3 ft, and they are increasing at the rate of 1.5ft/sec and 0.5ft/sec respectively. Find the rate at which the area is increasing at that instant.
5. If $u = e^{ax+by} f(ax - by), prove\ that\ bu_x + au_y = 2abu.$
6. If $u = \log(tanx + tany + tanz)$, show that $\sin(2x)u_x + \sin(2y)u_y + \sin(2z)u_z = 2$.
7. If $z(x + y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.
8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -9(x + y + z)^{-2}$.
9. If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$.
10. If $u = \sin(x^2 + y^2)$ where $a^2x^2 + b^2y^2 = c^2$ find $\frac{du}{dx}$.
11. If $u = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find $\frac{du}{dx}$ when $x = y = a$.
12. If $u = f(ax - by, by - cz, cz - ax)$, then show that $\frac{1}{a}u_x + \frac{1}{b}u_y + \frac{1}{c}u_z = 0$.
13. If $u = f(y - z, z - x, x - y)$, then prove that $u_x + u_y + u_z = 0$.
14. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, then find the value $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$.
15. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.
16. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

17. If $u = \tan^{-1}\left(\frac{y}{x}\right)$, $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$. find $\frac{du}{dt}$ as a function of t.
18. If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,\ v)}{\partial(r,\ \theta)}$.
19. If $ux = yz$, $vy = zx$, $wz = xy$, then show that $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = 4$.
20. If $x = r\cos\theta$, $y = r\sin\theta$, then verify that $JJ' = 1$.
21. If $u = x + y + z$, $uv = y + z$ and $uvw = z$, then find $\frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}$.
22. If $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$ and $z = r\cos\theta$, then find $\frac{\partial(x\ y,\ z)}{\partial(r,\ \theta,\ \emptyset)}$.
23. If $u = x + y + z,\ v = x^2 + y^2 + z^2\ and\ w = xy + yz + zx, then\ find$ $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$.
24. If $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$, then find $J = \frac{\partial(x\ y,\ z)}{\partial(r,\ \varphi,\ z)}$.
25. If $x = u(1 + v),\ y = v(1 + u), show\ that$ $\frac{\partial(x,\ y)}{\partial(u,\ v)} = 1 + u + v$.
26. If $u = \frac{2yz}{x},\ v = \frac{3xz}{y}\ and\ w = \frac{4xy}{z}, then\ find$ $J\left(\frac{u,v,w}{x,y,z}\right)$.
27. If $u = x + 3y^2 - z^3,\ v = 4x^2yz\ and\ w = 2z^2 - xy, then\ find$ $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ at $(1,-1,0)$.

<u>Taylor's and Maclaurin's series expansion for two variables:</u>

28. Find the Taylor's series expansion of $e^x \sin y$ at the point $\left(-1, \frac{\pi}{4}\right)$ up to third degree terms.
29. Use Taylor's formula to expand the function f defined by $f(x,\ y) = x^2 + xy + y^2$ in powers of $(x - 1)$ and $(y - 2)$.
30. Expand $f(x,\ y) = xy$ in Taylor's Series at $(1,1))$ up to first degree.
31. Expand the function $\sin xy$ in powers of $x - 1$ and $y - \frac{\pi}{2}$ up to second degree terms.
32. Use Maclaurin's formula to expand the function f defined by $f(x,\ y) = x^2y + 3y - 2$ in powers of x and y.
33. Expand $f(x,\ y) = e^x\log(1 + y)$ as Maclaurin's Series up to second degree terms.
34. Expand the function $e^x\cos y$ in powers of x and y up to third degree terms.
35. Expand $\log(1 + x - y)$ up to third degree term about the origin.

<u>Maxima and minima for a function of two variables:</u>

36. Find the maximum and minimum values of
    - a. $x^3 + y^3 - 2x^2 - 3axy$
    - b. $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
    - c. $\sin x \sin y \sin(x + y)$
    - d. $x^3 + y^3 - 3axy,\ a \geq 0$
    - e. $xy(1 - x - y)$
    - f. $x^3 + y^3 - 3x - 12y + 20$
    - g. $x^2 + y^2 + 6x - 12$
    - h. $x^3 + 3x^2 + 4xy + y^2$
37. Examine the function $f(x,\ y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.
38. Discuss the maxima and minima of $f(x,\ y) = x^3 + y^3 - 3x - 3y + 20$.
39. Find the points on which the function $f(x,\ y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ is extreme.
40. Show that the function of $f(x,\ y) = x^3 + y^3 - 3xy + 1$ is minimum at the point $(1,\ 1)$.
41. Discuss the maxima and minima of $f(x,\ y) = x^3y^2(1 - x - y)$.