Module- 4 - Vector Space
Vector spaces: definition and examples subspace: definition and examples. Linear Combinations, linear span, linearly independent and dependent sets, basis and dimension, row space and column space of a matrix, Coordinates vector, inner products and Orthogonality. **(RBT Levels: L1, L2 and L3)** **Textbook-3: Chapter 4: Sections 4.1 to 4.9 and 4.11** **Chapter 7: Sections-7.1,7.2,7.5**

LECTURE 1:
Vector spaces: definition and examples

Recall:

1. What is the commutative property?
2. What is the distributive property?
3. How is scalar multiplication of a vector performed?
4. What is row reduction or Gauss elimination?
5. What does it mean for a system of equations to have one solution, infinitely many solutions, or no solution?

Vector Space:

Definition: A vector space over a field F is a nonempty set V of objects, called vectors, on which are defined two operations, called addition, and multiplication by scalars (elements of field F), subject to the ten axioms (or rules) listed below. The axioms must hold for all u, v, and $w \in V$ and for all c and $d \in F$.

1. The sum of u and v, denoted by $+ v \in V$.
2. $u + v = v + u$. (Addition is commutative)
3. $(u + v) + w = u + (v + w)$. (Associative Property)
4. There is a zero vector $0 \in V$ such that $u + 0 = u$.
5. For each $u \in V$, there is a vector $-u \in V$ such that $u + (-u) = 0$.
6. The scalar multiple of u by c, denoted by $cu \in V$.
7. $c(u + v) = cu + cv$.
8. $(c + d)u = cu + du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

(Remember 1 in axiom 10 is scalar and not the element of V)

Examples:

1. Let $= \{(x, y)|x, y \in \mathbb{R}\} = \mathbb{R}^2$, For $u = (x_1, y_1)$, $v = (x_2, y_2) \in V$, $c \in \mathbb{R}$, define

   Addition: $u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

   Scalar multiplication: $c(u) = c(x_1, y_1) = (cx_1, cy_1)$

   then V is vector space over $\mathbb{R}$ with respect to the addition and scalar multiplication defined above.

Proof: (i) $x_1, x_2, y_1, y_2 \in \mathbb{R} \Longrightarrow x_1 + x_2, y_1 + y_2 \in \mathbb{R}$.

$\therefore u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in V$, for all $u = (x_1, y_1)$, $v = (x_2, y_2) \in V$.

Hence V is closed under addition.

(ii) $u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$= (x_2 + x_1, y_2 + y_1) = (x_2, y_2) + (x_1, y_1) = v + u$, for all $u, v \in V$.

Thus, the addition is commutative in V.

(iii) $(u + v) + w = (x_1 + x_2, y_1 + y_2) + (x_3, y_3) = \big((x_1 + x_2) + x_3, (y_1 + y_2) + y_3\big)$

$= \left(x_1 + (x_2 + x_3),\ y_1 + (y_2 + y_3)\right) = (x_1,\ y_1) + (x_2 + x_3,\ y_2 +\ y_3)$

$= u + (v + w)$, for all $u = (x_1,\ y_1),\ v = (x_2,\ y_2)$ and $w = (x_3,\ y_3) \in V$.

Thus, addition is associative in V.

(iv) $u + \mathbf{0} = (x_1,\ y_1) + (0,\ 0) = (x_1 + 0,\ y_1 + 0) = (x_1,\ y_1) = u$, and $\mathbf{0} = (0,\ 0) \in V$.

Thus, the additive identity $(0,\ 0)$ exists in V.

(v) For any $u = (x_1,\ y_1) \in V,\ -u = (-x_1, -y_1) \in V$,

and $+(-u) = (x_1,\ y_1) + (-x_1, -y_1) = (x_1 + (-x_1), y_1 + (-y_1)\ ) = (0,\ 0) = \mathbf{0}$.

Therefore, every element in V has an additive inverse.

(vi) For any $u = (x_1,\ y_1) \in V$ and any scalar $c \in \mathbb{R}$ $cx_1,\ cy_1 \in \mathbb{R}$,

$c(u) = c(x_1,\ y_1) = (cx_1, cy_1) \in V$.

Thus, V is closed with respect to scalar multiplication.

(vii) $c(u + v) = c(x_1 + x_2,\ y_1 +\ y_2) = (c(x_1 + x_2),\ c(y_1 +\ y_2))$

$= (cx_1 + cx_2,\ cy_1 + cy_2) = (cx_1,\ cy_1) + (cx_2,\ cy_2)$

$= c(x_1,\ y_1) + c(x_2,\ y_2) = cu + cv.$ For all $u, v \in V$ and $c \in \mathbb{R}$.

(viii) $(c + d)u = (c + d)(x_1,\ y_1) = \left((c + d)x_1,\ (c + d)y_1\right)$

$= (cx_1 + dx_1,\ cy_1 + dy_1) = (cx_1,\ cy_1) + (dx_1,\ dy_1)$

$= cu + du$. For all $u \in V$ and $,d \in \mathbb{R}$.

(ix) $c(du) = c\left(d(x_1,\ y_1)\right) = c(dx_1,\ dy_1)$

$= (cdx_1,\ cdy_1) = cd(x_1,\ y_1) = (cd)u$. For all $u \in V$ and c $,d \in \mathbb{R}$.

(x) $1 \in \mathbb{R}$ and $1u = 1(x_1,\ y_1) = (1x_1,\ 1y_1) = (x_1,\ y_1) = u.$

Thus, V is vector space over $\mathbb{R}$.

2. In general $V = \{(x_1,\ x_2, x_3, \cdots, x_n\ ) | x_1,\ x_2, x_3, \cdots, x_n \in \mathbb{R}\} = \mathbb{R}^n$ is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

3. Set of $m \times n$ real matrices is a vector space over $\mathbb{R}$ with respect to the matrix addition and scalar multiplication.

4. $V = \{0\}$, (set containing only one element) is a vector space over $\mathbb{R}$.

5. $V = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle|\ a, b, c \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ with respect to the matrix addition and scalar multiplication.

6. Set of all polynomials with real coefficients and degree $\leq n$ is a vector space over $\mathbb{R}$.

Review:

1. What is a vector space?
2. List the axioms that a set must satisfy to be considered a vector space.
3. What is the zero vectors in a vector space? Why is it important?
4. How do we check if a given set is a vector space under a given operation?

LECTURE 2:
Subspace, definition and examples

Recall:

1. What does it mean for a vector space to be closed under addition and scalar multiplication?
2. Is the set of natural numbers under normal addition and scalar multiplication a vector space?
3. What is the dimension of a vector space? How is it determined?
4. Can a vector space have multiple zero vectors?

Subspace: A nonempty subset H of a vector space V is called subspace of V if H is also a vector space with respect to the same addition and scalar multiplication as on V.

Since $H \subseteq V$, axioms 2, 3, 7, 8, 9 and 10 of vector space follows for H from V. (Because elements of H are elements of V also). If H satisfies axioms 1, 4 and 6, then axiom 5 also follows.

Therefore, we can define subspace by only three axioms.

Definition of subspace: A nonempty subset H of a vector space V is called subspace of V if ,

i) $\mathbf{0} \in H$ (Zero vector is in H)
ii) For any $u, v \in H$, $u + v \in H$. (H is closed under addition)
iii) For any $\in H$, any scalar c, $cu \in H$. (H is closed under scalar multiplication)

Theorem1: A nonempty subset W of a vector space V over $\mathbb{R}$ is a subspace of V if and only if for any vectors $u, v \in W$, and any scalar c, $cu + v \in W$.

Proof: Suppose that W is a nonempty subset of a vector space V such that $cu + v \in W$ for all $u, v \in W$, and all scalar c.

Since W is nonempty, let $u \in W \implies u, u \in W$ and $-1 \in \mathbb{R}$, $-u + u = \mathbf{0} \in W$ (Zero vector is in W).

For any vector $u \in W$ and for any scalar c, $u, \mathbf{0} \in W \implies cu + \mathbf{0} = cu \in W$.

(W is closed under scalar multiplication)

For any vectors $u, v \in W$ and $1 \in \mathbb{R} \implies 1u + \boldsymbol{v} = u + v \in W$ (W is closed under addition)

Therefore W is a subspace of V.

Conversely,

If a nonempty subset W of a vector space V over $\mathbb{R}$ is a subspace of V, then W is vector space and hence for any vectors $u, v \in W$, and any scalar c, by the 6$^{\text{th}}$ axiom $cu \in W$, and by the first axiom $cu + v \in W$.

Examples:

1. For any vector space V, $\{\mathbf{0}\}$ and V are always subspace of V.

2. Prove that $H = \{(x, 0) |\, x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

   Proof: Let $u = (x, 0)$, $v = (y, 0) \in H$ and $\in \mathbb{R}$,
   then $cu + v = c(x, 0) + (y, 0) = (cx, 0) + (y, 0) = (cx + y, 0) \in H$.
   Hence by the theorem1, H is a subspace of $\mathbb{R}^2$.

3. Let V be the vector space of all 2×2 metrices, and $W = \left\{ \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}$.

   Show that W is a subspace of V.

Solution: Let $u = \begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix} \in W$ and $\in \mathbb{R}$,

then $cu + v = c\begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix} + \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix} = \begin{bmatrix} cx_1 + x_2 & -(cx_1 + x_2) \\ cy_1 + y_2 & cz_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_3 & -x_3 \\ y_3 & z_3 \end{bmatrix} \in W$.

Hence by the theorem1, W is a subspace of V.

4. Let $X = (1, 2, -3)$, $Y = (-2, 3, 0)$ and $W = \{aX + bY | a, b \in \mathbb{R}\}$. Show that W is subspace of $\mathbb{R}^3$.

   Solution: Clearly $W \subseteq \mathbb{R}^3$.
   Let $u = a_1X + b_1Y$, $v = a_2X + b_2Y \in W$ and $c \in \mathbb{R}$,
   then $cu + v = c(a_1X + b_1Y) + a_2X + b_2Y$

$$= ca_1X + cb_1Y + a_2X + b_2Y = (ca_1 + a_2)X + (cb_1 + b_2)Y$$
$$= a_3X + b_3Y \in W. \qquad (\because a_3 = ca_1 + a_2 \text{ and } b_3 = cb_1 + b_2 \in \mathbb{R})$$

Hence by the theorem1, W is a subspace of $\mathbb{R}^3$.

5. Let $W = \{(a,\ a^2,\ b) \mid a, b \in \mathbb{R}\}$, is W a subspace of $\mathbb{R}^3$? Give reason.

Solution: No, W is not the subspace of $\mathbb{R}^3$.
Because $(1,\ 1,\ 2),\ (2,\ 4,\ 3) \in W$, but $(1,\ 1,\ 2) + (2,\ 4,\ 3) = (3, 5, 5) \notin W$.
W is not closed under addition.

6. Let $W = \{(x,\ y,\ z) \mid lx + my + nz = 0\}$, then prove that W is a subspace of $\mathbb{R}^3$.

Solution: Let $u = (x_1,\ y_1,\ z_1),\ v = (x_2,\ y_2,\ z_2) \in W$ and $\in \mathbb{R}$,
Then $lx_1 + my_1 + nz_1 = 0$ and $lx_2 + my_2 + nz_2 = 0$.

$cu + v = (cx_1 + x_2,\ cy_1 + y_2,\ cz_1 + xz_2) \in W$

Because $l(cx_1 + x_2) + m(cy_1 + y_2) + n(cz_1 + z_2) = c(lx_1 + my_1 + nz_1) + (lx_2 + my_2 + nz_2) = 0$.

Therefore, W is a subspace of $\mathbb{R}^3$.

7. Prove that $W = \{(x,\ y,\ z) \mid x - 3y + 4z = 0\}$ is a subspace of $\mathbb{R}^3$.

Solution: Let $u = (x_1,\ y_1,\ z_1),\ v = (x_2,\ y_2,\ z_2) \in W$ and $\in \mathbb{R}$,
Then $x_1 - 3y_1 + 4z_1 = 0$ and $x_2 - 3y_2 + 4z_2 = 0$.

$cu + v = (cx_1 + x_2,\ cy_1 + y_2,\ cz_1 + xz_2) \in W$

Because $(cx_1 + x_2) - 3(cy_1 + y_2) + 4(cz_1 + z_2) = c(x_1 - 3y_1 + 4z_1) + (x_2 - 3y_2 + 4z_2) = 0$.

Therefore, W is a subspace of $\mathbb{R}^3$.

Theorem2: If H and K are subspaces of V then $H \cap K$ is also subspace of V.

Proof: Let $u,\ v \in H \cap K \implies u,\ v \in H$ and $u,\ v \in K$
Since H and K are subspace, for any scalar , $cu + v \in H$ and $cu + v \in K$.
$\implies cu + v \in H \cap K$, and hence by the theorem1, $H \cap K$ is also subspace of V.

Note: If H and K are subspaces of , $H \cup K$ need not be a subspace of V.

Example: Let $H = \{(x,\ 0) \mid x \in \mathbb{R}\}$ and $K = \{(0,\ y) \mid y \in \mathbb{R}\}$.

For any $u = (x_1, 0),\ v = (x_2, 0) \in H$ and for any scalar c, $cu + v = (cx_1 + x_2, 0) \in H$.
Similarly, for any $u = (0,\ y_1),\ v = (0,\ y_2) \in K$ and for any scalar c, $cu + v = (0,\ cy_1 + y_2) \in K$.
Therefore, H and K are subspace of $\mathbb{R}^2$.

Elements of $H \cup K$ are of the form either $(x,\ 0)$ or $(0,\ y)$.

$(2,\ 0),\ (0,\ 3) \in H \cup K$, but $(2,\ 0) + (0,\ 3) = (2,\ 3) \notin H \cup K$.

$\implies H \cup K$ is not closed with respect to addition, and hence not a subspace.

Theorem3: If W_1 and W_2 are subspaces of a vector space V, then the set
$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$$ is also a subspace.

Proof: Suppose $u,\ v \in W_1 + W_2$, then $u = w_1 + w_2,\ v = w_1' + w_2'$ for some $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$.

For any scalar c, $cu + v = cw_1 + cw_2 + w_1' + w_2'$
$$= (cw_1 + w_1') + (cw_2 + w_2') \in W_1 + W_2.$$

(Since W_1 and W_2 are subspaces, $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2 \implies cw_1 + w_1' \in W_1$ and $cw_2 + w_2' \in W_2$)
Therefore $W_1 + W_2$ is also a subspace.

Review:

1. What is a subspace of a vector space?
2. Explain the difference between a vector space and a subspace.
3. What conditions must a subset of a vector space satisfy to be a subspace?
4. Give two examples of subspaces in R^3.

LECTURE 3:
Linear span ,Linearly independent and dependent sets

Recall:

1. Is the empty set a subspace of a vector space?
2. Explain what is meant by the linear span of a set of vectors.
3. What is the difference between a subspace and a span?
4. How do we find the span of a given set of vectors?

Linear combination: A Linear combination of a set $\{v_1, v_2, v_3, \cdots, v_n\}$ of vectors in vector space V is a vector v of the form $v = c_1v_1 + c_2v_2 + c_3v_3 \cdots + c_nv_n$ for some scalars $c_1, c_2, c_3, \cdots, c_n$.

Span of set of vectors: Consider any set of vectors $\{v_1, v_2, v_3, \cdots, v_n\}$ of a vector space. Then the set of all linear combinations of $\{v_1, v_2, v_3, \cdots, v_n\}$ is called span of the vectors $v_1, v_2, v_3, \cdots, v_n$.

$\therefore$ **Span**$\{v_1, v_2, v_3, \cdots, v_n\} = \{c_1v_1 + c_2v_2 + c_3v_3 \cdots + c_nv_n | \, c_1, c_2, c_3, \cdots, c_n \in \mathbb{R}\}$.

Theorem: The set $H = Span\{v_1, v_2\}$ is a subspace of a vector space V, for any $v_1, v_2 \in V$.

Proof: Let $u, v \in H$, then there exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $u = c_1v_1 + c_2v_2$ and $v = c_3v_1 + c_4v_2$.

For any scalar c, $cu + v = c(c_1v_1 + c_2v_2) + c_3v_1 + c_4v_2$

$= (cc_1 + c_3)v_1 + (cc_2 + c_4)v_2 \in H$.

Therefore, H is a subspace of .

In general **Span**$\{v_1, v_2, v_3, \cdots, v_n\}$ is a subspace of V for any set of vectors $\{v_1, v_2, \cdots, v_n\}$ of a vector space V.

Problems:

1. Express the vector $u = (1, -2, 5)$ as the linear combination of vectors $u_1 = (1,1,1), u_2 = (1, 2, 3)$ and $u_3 = (2, -1, 1)$.

Solution: Let $c_1u_1 + c_2u_2 + c_3u_3 = u \quad \Longrightarrow \begin{cases} c_1 + c_2 + 2c_3 = 1 \\ c_1 + 2c_2 - c_3 = -2 \\ c_1 + 3c_2 + c_3 = 5 \end{cases} \Longrightarrow c_1 = -6, \; c_2 = 3, \quad c_3 = 2$

$\therefore (1, -2, 5) = -6(1,1,1) + 3(1, 2, 3) + 2(2, -1, 1)$.

2. Can $v = (2, -5, 3)$ in $\mathbb{R}^3$ be represented as a linear combination of the vectors $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$, $u_3 = (1, -5, 7)$.

Solution: Let $c_1u_1 + c_2u_2 + c_3u_3 = v$.

Then the augmented matrix is $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 = 2R_3 + 5R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Clearly $\rho(A) = 2 \neq \rho(A|B) = 3$.

Since there is no solution, v cannot be represented as a linear combination of the vectors u_1 u_2, u_3.

3. For which value of k the vectors $u = (1, -2, k)$ in $\mathbb{R}^3$ be a linear combination of the vectors $u_1 = (3, 0, -2)$ and $u_2 = (2, -1, -5)$.

Solution: Let $c_1u_1 + c_2u_2 = v \Longrightarrow \begin{cases} 3c_1 + 2c_2 = 1 \\ 0 - c_2 = -2 \\ -2c_1 - 5c_2 = k \end{cases} \Longrightarrow c_1 = -1,\ c_2 = 2$. $\quad \therefore k = -2c_1 - 5c_2 = -8$.

4. Write the matrix $E = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as a linear combination of the matrices $= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$.

Solution: Let $c_1A + c_2B + c_3C = E \Longrightarrow \begin{cases} c_1 = 3 \\ c_1 + 2c_3 = 1 \\ c_1 + c_2 = 1 \\ c_2 - c_3 = -1 \end{cases} \Longrightarrow c_1 = 3,\quad c_2 = -2,\quad c_3 = -1$.

Therefore, $E = 3A - 2B - C$. (Solution of first three equations satisfies fourth equation)

5. Let $f\ (x) = 2x^2 - 5\ and\ g(x) = x + 1$. Show that the function $h(x) = 4x^2 + 3x - 7$ lies in the subspace Span$\{f\ , g\}$ of P_2.

Solution: Let $h(x) = c_1f(x) + c_2g(x) \Longrightarrow 4 = 2c_1,\ 3 = c_2$ and $-7 = -5c_1 + c_2$.

First two equations gives, $c_1 = 2$, $c_2 = 3$ and it satisfies third equation.

Therefore, $h(x) = 2f(x) + 3g(x) \Longrightarrow h \in$ Span$\{f\ , g\}$.

Linearly dependent and linearly independent vectors:

Set of vectors $\{v_1,\ v_2, \cdots, v_n\}$ is said to be linearly dependent if there exist scalars $c_1,\ c_2,\ c_3, \cdots, c_n$, not all of them zeros, such that $c_1v_1 + c_2v_2 + c_3v_3 \cdots + c_nv_n = 0$.

If $c_1v_1 + c_2v_2 + c_3v_3 \cdots + c_nv_n = 0$ has only the trivial solution, that is each $c_i = 0$ for $i = 0, 1, 2, 3, \cdots, n$, then set $\{v_1,\ v_2, \cdots, v_n\}$ is linearly independent.

Theorem: Let V be a vector space over $\mathbb{R}$, the set $\{v_1,\ v_2,\ v_3, \cdots, v_n\}$ of vectors in V is said to be linearly dependent if and only if at least one of them is linear combination of others.

Proof: Suppose that the vector v_j is the linear combination of other vectors, then

$$v_j = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_{j-1}v_{j-1} + c_{j+1}v_{j+1} + \cdots + c_nv_n\ .$$

$$\Longrightarrow c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_{j-1}v_{j-1} + (-1)v_j + c_{j+1}v_{j+1} + \cdots + c_nv_n$$

Where the coefficient of v_j is -1 which is not equal to zero.

Therefore, the set $\{v_1,\ v_2,\ v_3, \cdots, v_n\}$ of vectors is linearly dependent.

Conversely, Let the set $\{v_1,\ v_2,\ v_3, \cdots, v_n\}$ of vectors in V is linearly dependent.

Then there exist scalars $c_1,\ c_2,\ \cdots, c_n$, not all of them zeros, such that $c_1v_1 + c_2v_2 + c_3v_3 \cdots + c_nv_n = 0$.

Let it be c_i, that means $c_i \neq 0$, then $c_iv_i = -(c_1v_1 + c_2v_2 + \cdots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \cdots + c_nv_n)$

$$\Longrightarrow v_i = -\frac{c_1}{c_i}v_1 - \frac{c_2}{c_i}v_2 \cdots - \frac{c_{i-1}}{c_i}v_{i-1} - \frac{c_{i+1}}{c_i}v_{i+1} - \cdots - \frac{c_n}{c_i}v_n.$$

Therefore, v_i is linear combination of others.

Note: Any set of vectors containing the **0** vector is linearly dependent.

Consider the set $\{v_1, v_2, v_3, \cdots, v_n\}$ of vectors containing $\mathbf{0}$ vector. Let $v_i = \mathbf{0}$ vector.
Then the equation $c_1v_1 + c_2v_2 + \cdots + c_iv_i \cdots + c_nv_n = 0$ has a nontrivial solution.
One of the solution is $c_i = 1$ and remaining coefficients zero.
Therefore, set of vectors containing the $\mathbf{0}$ vector is linearly dependent.

Let A be a matrix of any order, and E be the echelon form of A.

Pivots are non-zero leading entry of a row in an echelon form of the matrix.

Pivot columns are the columns of A (original matrix) that contains a pivot.

Identify the pivots and pivot columns of A. Given that $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & 6 \end{bmatrix} \sim E = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Pivots are $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and pivot columns are $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & 6 \end{bmatrix}$

To check the given set of vectors for linearly independent or linearly dependent, take all the given vectors as columns of a matrix. Then reduce that matrix to echelon form to find pivot columns and non-pivot columns. If all the columns are pivot columns, then the given vectors are linearly independent. If the non-pivot column (columns) exist, then the vectors are linearly dependent. Vectors of pivot columns are linearly independent. Any non- pivot column can be expressed as linear combinations of the pivot columns on their left.

Problems:

1. Check whether the following vectors are linearly dependent or linearly independent vectors?

i) $(1,\ 4,\ 5), (4,\ 4,\ 8), (3, -3,\ 0)$.

Solution: Let A be the matrix with columns are given vectors.

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 4 & -3 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 4 & 5 \\ 0 & 4 & 5 \end{bmatrix} \quad \begin{pmatrix} R_2 = \frac{R_2 - 4R_1}{-3} \\ R_3 = \frac{R_3 - 5R_1}{-3} \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad (R_3 = R_3 - R_2)$$

Clearly third column is non-pivot column, hence third vector is linear combination of first two vectors.
Therefore given vectors are linearly dependent. $u_3 = -2u_1 + \frac{5}{4}u_2$.

ii) $(1,\ 0, -1,\ 2), (4,\ 2,\ 0, -1), (6,\ 4, -2,\ 3)$.

Solution: Let A be the matrix with columns are given vectors.

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 4 \\ -1 & 0 & -2 \\ 2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{pmatrix} R_3 = \frac{R_3 + R_1}{4} \\ R_4 = \frac{R_4 - 2R_1}{-9} \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{pmatrix} R_4 = R_4 - R_3 \\ R_3 = \frac{R_3 - 0.5R_2}{-1} \end{pmatrix}$$

Clearly all the columns are pivot columns, hence the given vectors are linearly independent.

2. Show that vectors $(1,1,\ 2,\ 4), (2, -1, -5,\ 2), (1, -1, -4,\ 0), (2, 1, 1, 6)$ are linearly dependent.

Solution: Let A be the matrix with columns are given vectors.

$$A=\begin{bmatrix}1&2&1&2\\1&-1&-1&1\\2&-5&-4&1\\4&2&0&6\end{bmatrix}\sim\begin{bmatrix}1&2&1&2\\0&-3&-2&-1\\0&-9&-6&-3\\0&-6&-4&-2\end{bmatrix}\sim\begin{bmatrix}1&2&1&2\\0&-3&-2&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Clearly third and fourth columns are non-pivot column, hence third and fourth vectors are linear combination of first two vectors. Therefore given vectors are linearly dependent. $u_3=-\frac{1}{3}u_1+\frac{2}{3}u_2,\quad u_4=\frac{4}{3}u_1+\frac{1}{3}u_2$.

3. Let V be a vector space of all 2×3 matrices over $\mathbb{R}$. Show that $A=\begin{bmatrix}2&1&-1\\3&-2&4\end{bmatrix}$, $B=\begin{bmatrix}1&1&-3\\-2&0&5\end{bmatrix}$ and $C=\begin{bmatrix}4&-1&2\\1&-2&3\end{bmatrix}$ form a linearly independent set.

Solution: Let $c_1A+c_2B+c_3C=\mathbf{0}\Longrightarrow\begin{cases}2c_1+c_2-c_3=0,\ c_1+c_2-c_3=0,\ -c_1-3c_2+2c_3=0\\3c_1-2c_2+c_3=0,-2\,c_1+0c_2-2c_3=0,\ 4c_1+5c_2+3c_3=0\end{cases}$

. Solving the first three equations $2c_1+c_2-c_3=0,\ c_1+c_2-c_3=0,\ -c_1-3c_2+2c_3=0$ we get, $c_1=0,\ c_2=0,\ c_3=0$. That is only trivial solutions. Hence, A,B,C are linearly independent.

(If the first three equations have nontrivial solutions, and that solutions satisfies remaining equations, then vectors are linearly dependent, If the nontrivial solutions does not satisfies even one of the remaining equations, then A,B,C are linearly independent.)

Review:

1. What does it mean for a set of vectors to be linearly independent?
2. How do you determine if a set of vectors is linearly dependent?
3. If a set of vectors is linearly dependent, what does that imply about one or more of the vectors in the set?
4. What is the minimum number of vectors required for a set to be linearly dependent in an n-dimensional space?

LECTURE 4:
Basis and dimensions

Recall:

1. How can the determinant of a matrix help in determining linear dependence?
2. Can two nonzero vectors in $\mathbb{R}^3$ be linearly dependent? If yes, under what condition?
3. If a set of vectors spans a space, is it necessarily linearly independent?
4. If you remove a vector from a linearly dependent set, can the remaining set become independent? Explain with an example.

Basis for a vector space: A set $\mathcal{B}=\{u_1,\ u_2,\ u_3,\cdots,u_n\}$ of vector space V is said to be basis of V if,

(i) $\mathcal{B}$ is linearly independent.

(ii) $V=span\{\mathcal{B}\}$. That is $\mathcal{B}$ spans V.

Thus, the basis of vector space V is the linearly independent set that spans V.

Note: If a set $\mathcal{B}$ is basis of vector space V, then

(i) $\mathcal{B}$ is the minimal spanning set, that is, there is no smaller set of vectors in $\mathcal{B}$ that span V.

(ii) $\mathcal{B}$ is the maximal linearly independent set of vectors in V , that is, adding any other vector to $\mathcal{B}$ it will become linearly dependent.

Dimension: The number of vectors in a basis of a vector space V is called dimension of V, denoted by $dim(V)$.

Theorem: Let V be a vector space with a basis $\mathcal{B}=\{u_1,\ u_2,\ u_3,\cdots,u_n\}$. Then every vector in V is uniquely written as a linear combination of vectors in $\mathcal{B}$.

Proof: Let $v \in V$. Assume that v can be written as linear combination of vectors in $\mathcal{B}$ in two different ways.

$v = c_1u_1 + c_2u_2 + c_3u_3 \cdots + c_nu_n$ and $v = d_1u_1 + d_2u_2 + d_3u_3 \cdots + d_nu_n$.

Since $v - v = 0$, $(c_1 - d_1)u_1 + (c_2 - d_2)u_2 + (c_3 - d_3)u_3 \cdots + (c_n - d_n)u_n = 0$.

Since the vectors of $\mathcal{B}$ are linearly independent, each $c_i - d_i = 0$ for $i = 1, 2, 3, \cdots n$.

Hence $c_i = d_i$ for $i = 1, 2, 3, \cdots n$. Which shows that the two representations of v are same.

Problems:

1. Check whether the set $\{(2,\ 3), (0,\ 4)\}$ is a basis for $\mathbb{R}^2$. If so express the vectors of $\mathbb{R}^2$ as linear combination of these vectors.

Solution: The set $\mathcal{B} = \{u_1,\ u_2\}$ is a basis of $\mathbb{R}^2$ if and only if, $\mathcal{B}$ is linearly independent spanning set of $\mathbb{R}^2$. Let $u_1 = (2,\ 3)$, $u_2 = (0,\ 4)$. Since, u_1 is not a scalar multiple of u_2, and u_2 is not a scalar multiple of u_1, u_1 and u_2 are linearly independent.

Let $(x, y) \in \mathbb{R}^2$, and $(x, y) = c_1(2,\ 3) + c_2(0,\ 4) \Longrightarrow x = 2c_1,\ y = 3c_1 + 4c_2$

$$\Longrightarrow c_1 = \frac{x}{2}, \quad c_2 = \frac{y - \frac{3}{2}x}{4} = \frac{2y - 3x}{8}.$$

$$\therefore (x, y) = \frac{x}{2}u_1 + \frac{2y - 3x}{8}u_2.$$

Thus we can express any vector of $\mathbb{R}^2$ as linear combination of u_1 and u_2. Hence set $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

2. Check whether the set $\{(1,3,1), (0,2,0), (0,0,7)\}$ is a basis of $\mathbb{R}^3$. If so express the vectors of $\mathbb{R}^3$ as linear combination of these vectors.

Solution: To check whether the set is independent, form the matrix whose columns are given vectors and reduce it in to echelon form:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{pmatrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - R_1 \end{pmatrix}.$$

Since every column of A are pivot columns, set of vectors are linearly independent.

Let $(x,\ y,\ z) = c_1(1,3,1) + c_2(0,2,0) + c_3(0,0,7) \Longrightarrow x = c_1,\ \ y = 3c_1 + 2c_2,\ z = c_1 + 7c_3$.

$\Longrightarrow c_1 = x,\ \ c_2 = \frac{y - 3x}{z},\ \ c_3 = \frac{z - x}{7}$. Therefore, every vector of $\mathbb{R}^3$ can be expressed as linear combinations of the vectors $(1,3,1), (0,2,0), (0,0,7)$. Hence the given set form a basis of $\mathbb{R}^3$.

3. Find the dimension of the subspace W of $\mathbb{R}^3$ spanned by the vectors $\{(3,1,0), (2,1,0), (1,1,-2)\}$.

Solution: Form the matrix A whose columns are the vectors in the given set. Then, reduce A to echelon form.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} (R_2 \leftrightarrow R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \quad (R_2 = R_2 - 3R_1).$$

Every column has a pivot entry; hence, no vector is a linear combination of the previous vectors. Thus, the vectors are linearly independent and hence they form the basis for the subspace W spanned by the vectors $\{(3,1,0), (2,1,0), (1,1,-2)\}$ Therefore, $\dim(W) = 3$.

4. Find the dimension of the subspace W of $\mathbb{R}^3$ spanned by the vectors $\{(2,4,2), (1,-1,0), (1,2,1), (0,3,1)\}$.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & -1 & 2 & 3 \\ 2 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{pmatrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (R_3 = 3R_3 - R_2).$$

Since the first and second columns of A are pivot columns, and are linearly independent. The basis of W is the set $\{(2,4,2), (1,-1,0)\}$ Therefore, $\dim(W) = 2$.

Review:

1. What is a basis of a vector space?
2. What does it mean for a basis to be ordered?
3. How do you check if a set of vectors forms a basis for a vector space?
4. What is the dimension of a vector space?
5. Can a vector space have more than one basis?

TUTORIAL 1:

Problem Solving on Linearly independent and dependent sets

1. Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space M_2 of 2×2 matrices.
2. Write the vector $v = (1,3,9)$ as a linear combination of the vectors $u_1 = (2,1,3)$, $u_2 = (1,-1,1)$, and $u_3 = (3,1,5)$ in the vector space $\mathbb{R}^3$.
3. Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether b can be written as a linear combination of a_1 and a_2.
4. Check whether the following vectors are linearly dependent or linearly independent vectors?
   i) $(1, 4, 5), (4, 2, 3), (3, -1, 0)$ ii) $(1, 1, 1, 1), (1, 2, 3, 4), (1, 2, 1, 2), (3, 5, 5, 7)$
   iii) $(1, 2, 1, 3), (3, 1, 5, -1), (2, -1, 0, 2)$
5. Find the condition on a, b, c so that W = (a, b, c) is a linear combination of u = (1,−3, 2) and v = (2,−1, 1) in R^3 so that W ∈ Span(u, v).

TUTORIAL 2:

Lab Activity 7: Linearly Independence and Dependence sets

Objectives:

Use python

- To find row reduced echelon form (RREF)
- To check independency of given vectors.

1. Check whether the following vectors are linearly dependent or linearly independent vectors?
   (i). $(1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0), (2, 1, 1, 6)$

```python
from sympy import *
A = Matrix([[1, 2,  1, 2],[1, -1, -1, 1],[2, -5, -4, 1],[4, 2,  0, 6]])
rref_matrix, pivots = A.rref()
print("RREF =")
print(rref_matrix)
print("\nPivot columns =", pivots)
rank = len(pivots)
print("\nRank =", rank)
if rank == A.shape[1]:
    print("Vectors are linearly Independent")
else:
    print("Vectors are linearly Dependent")
```

2. Show that vectors $(1, 0, -1, 2)$, $(4, 2, 0, -1)$, $(6, 4, -2, 3)$ are linearly independent.

```python
from sympy import *
A = Matrix([[1, 4, 6],[0, 2, 4],[-1, 0, -2],[2, -1,  3]])
rref_matrix, pivots = A.rref()
print("RREF =")
print(rref_matrix)
print("\nPivot columns =", pivots)
rank = len(pivots)
print("\nRank =", rank)
if rank == A.shape[1]:
    print("Vectors are linearly Independent")
else:
    print("Vectors are linearly Dependent")
```

3. Determine whether the given vectors are linearly dependent or linearly independent in $\mathbb{R}^4$
   i). $v_1 = (1,3,-1,0), v_2 = (2,9,-1,3), v_3 = (4,5,6,11), v_4 = (1,-1,2,5), v_5 = (3,-2,6,7)$.
   ii). $v_1 = (1,4,1,7), v_2 = (3,-5,2,3), v_3 = (2,-1,6,9), v_4 = (-2,3,1,6)$.

LECTURE 5:
Basis and dimensions-Problems

Recall:

1. How can the determinant of a matrix help in determining linear dependence?
2. Can two nonzero vectors in $\mathbb{R}^3$ be linearly dependent? If yes, under what condition?
3. If a set of vectors spans a space, is it necessarily linearly independent?
4. If you remove a vector from a linearly dependent set, can the remaining set become independent? Explain with an example.

Problems:

5. Let W be the subspace spanned by vectors $u_1 = (1,-2,5,-3)$, $u_2 = (3,8,-3,-5)$, $u_3 = (2,3,1,-4)$. Find the basis and dimension of . Extend the basis of W to a basis of $\mathbb{R}^4$.

Solution: Form the matrix A whose columns are the vectors in the given set. Then, reduce A to echelon form.

$$A=\begin{bmatrix}1&3&2\\-2&8&3\\5&-3&1\\-3&-5&-4\end{bmatrix}\sim\begin{bmatrix}1&3&2\\0&2&1\\0&2&1\\0&2&1\end{bmatrix}\begin{pmatrix}R_2=\frac{R_2+2R_1}{7}\\R_3=\frac{R_3-5R_1}{-9}\\R_4=\frac{R_4+3R_1}{4}\end{pmatrix}$$

$$\sim\begin{bmatrix}1&3&2\\0&2&1\\0&0&0\\0&0&0\end{bmatrix}\quad\begin{pmatrix}R_3=R_3-R_2\\R_4=R_4-R_2\end{pmatrix}.$$

Since the first and second columns of A are pivot columns, and are linearly independent. The basis of W is the set $\{u_1, u_2\}$ Therefore, $\dim(W)=2$.

We seek four linearly independent vectors, which include the above two linearly independent vectors.

The four vectors $\{(1,-2,5,8),(3,8,-3,-5),(0,0,1,0),(0,0,0,1)\}$ form the basis of $\mathbb{R}^4$. Which is the extension of the basis of W to a basis of $\mathbb{R}^4$.

6. Determine whether $(1,1,1,1),(1,2,3,2),(2,5,6,4),(2,6,8,5)$ form a basis of $\mathbb{R}^4$. If not find the dimension of subspace spanned by these vectors.

Solution: $$A=\begin{bmatrix}1&1&2&2\\1&2&5&6\\1&3&6&8\\1&2&4&5\end{bmatrix}\sim\begin{bmatrix}1&1&2&2\\0&1&3&4\\0&2&4&6\\0&1&2&3\end{bmatrix}\begin{pmatrix}R_2=\frac{R_2-R_1}{7}\\R_3=\frac{R_3-R_1}{-9}\\R_4=\frac{R_4-R_1}{4}\end{pmatrix}$$

$$\sim\begin{bmatrix}1&1&2&2\\0&1&3&4\\0&0&1&-1\\0&0&1&-1\end{bmatrix}\begin{pmatrix}R_3=\frac{R_3-2R_2}{2}\\R_4=R_4-R_2\end{pmatrix}$$

$$\sim\begin{bmatrix}1&1&2&2\\0&1&3&4\\0&0&1&-1\\0&0&0&0\end{bmatrix}(R_4=R_4-R_3).$$

Since the first three columns of A are pivot columns, and are linearly independent and the fourth column is non-pivot set containing the given vectors are not form a basis of $\mathbb{R}^4$.

Let the W be the subspace spanned by given vectors. Then the $\dim(W)=3$.

7. Let W be the subspace of $\mathbb{R}^5$ spanned by $x_1=(1,2,-1,3,4)$, $x_2=(2,4,-2,6,8)$, $x_3=(1,3,2,2,6)$, $x_4=(1,4,5,1,8)$, $x_5=(2,7,3,3,9)$.
Find a subset of vectors which forms a basis of W.

Solution: Let $$A=\begin{bmatrix}1&2&1&1&2\\2&4&3&4&7\\-1&-2&2&5&3\\3&6&2&1&3\\4&8&6&8&9\end{bmatrix}\sim\begin{bmatrix}1&2&1&1&2\\0&0&1&2&1\\0&0&3&6&5\\0&0&1&2&3\\0&0&2&4&1\end{bmatrix}\begin{pmatrix}R_2=R_2-2R_1\\R_3=R_3+R_1\\R_4=\frac{R_4-3R_1}{-1}\\R_5=R_5-4R_1\end{pmatrix}$$

$$\sim\begin{bmatrix}1&2&1&1&2\\0&0&1&2&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\begin{pmatrix}R_3=\frac{R_3-3R_2}{2}\\R_4=\frac{R_4-R_2}{2}\\R_5=\frac{R_5-2R_2}{-5}\end{pmatrix}$$

$$\sim\begin{bmatrix}1&2&1&1&2\\0&0&1&2&1\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{pmatrix}R_4=R_4-R_3\\R_5=R_5-R_3\end{pmatrix}$$

Since the first, third and fifth columns of A are pivot columns, and are linearly independent, basis of W is the set $\{(1,2,-1,3,4),\ (1,3,2,2,6),\ (2,7,3,3,9)\}$.

8. Let V be a vector space of polynomials over $\mathbb{R}$. Find a basis and dimension of the subspace W of V spanned by the polynomials; $x_1 = t^3 - 2t^2 + 4t + 1$, $x_2 = 2t^3 - 3t^2 + 9t - 1$, $x_3 = t^3 + 6t - 5$, and $x_4 = 2t^3 - 5t^2 + 7t + 5$.

   (Taking coefficients of each polynomial is taken as columns)

Solution: Let $A = \begin{bmatrix}1&2&1&2\\-2&-3&0&-5\\4&9&6&7\\1&-1&-5&5\end{bmatrix} \sim \begin{bmatrix}1&1&2&2\\0&1&2&-1\\0&1&2&-1\\0&1&2&-1\end{bmatrix}\begin{pmatrix}R_2=R_2+2R_1\\R_3=R_3-4R_1\\R_4=\frac{R_4-R_1}{-3}\end{pmatrix}$

$$\sim\begin{bmatrix}1&1&2&2\\0&1&2&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{pmatrix}R_3=R_3-R_2\\R_4=R_4-R_2\end{pmatrix}$$

Since the first and second columns of A are pivot columns, basis of W is $\{x_1,\ x_2\}$, $\dim(W) = 2$.

Review:

1. How do you determine the dimension of a subspace?
2. What is the relationship between the number of vectors in a basis and the dimension of a vector space?
3. If a vector space has dimension n, how many vectors must a basis contain?
4. What does it mean for a basis to be ordered?
5. How do you check if a set of vectors forms a basis for a vector space?
6. What is the dimension of a vector space?

LECTURE 6:
Row space and Column space of a matrix

Recall:

1. How can the determinant of a matrix help in determining linear dependence?
2. Can two nonzero vectors in $\mathbb{R}^{3\,3}$ be linearly dependent? If yes, under what condition?
3. If a set of vectors spans a space, is it necessarily linearly independent?
4. If you remove a vector from a linearly dependent set, can the remaining set become independent? Explain with an example.
5. If a vector space has dimension n, how many vectors must a basis contain?

Row space and column space of a matrix:

Let A be a $m \times n$ matrix.

The row space of A denoted by Row$A$ is the set of all linear combination of the rows of A .

If $R_1, R_2, R_3, \cdots R_m$ are the rows of A then Row$A$ =Span$\{R_1, R_2, R_3, \cdots R_m\}$

The column space of A denoted by Col$A$ is the set of all linear combination of the columns of A .
If $C_1, C_2, C_3, \cdots C_n$ are the columns of A then Col$A$ =Span$\{C_1, C_2, C_3, \cdots C_n\}$

Since the spanning set is subspace, Row$A$ is subspace of R^m and Col$A$ is subspace of R^n.

Basis of Row$A$ is the set of rows A, which are nonzero rows of echelon form of A .
Basis of col$A$ is the set of pivot columns of A.

Null space of matrix A is the set of all solutions of the homogeneous equation $Ax = \mathbf{0}$.

$$\therefore nulA = \{x | x \in R^n \ \boldsymbol{and} \ Ax = 0 \ \}.$$

Example: 1. Find the basis and dimension of Row$A$, Col$A$ and $nulA$ for $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

Solution: Reducing A into echelon form we get,

$$\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Row$A = \{(1, 3, 1, -2, -3), (1, 4, 3, -1, -3\}$, and dim (Row$A$) = 2.

Basis of Col$A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \\ 8 \end{bmatrix} \right\}$, and dim (Col$A$) = 2.

To find the basis of null space, that is solution of the system $Ax = 0$,

Consider $x_1 + 3x_2 + x_3 - 2x_4 - 3x_5 = 0$ and $x_2 + 2x_3 + x_4 - x_5 = 0$ (Coefficients from the nonzero rows of echelon form) Since the last three columns of A are non-pivot columns, x_3, x_4, x_5 are free variables.

Let $x_3 = c_1$, $x_4 = c_2$, $x_5 = c_3$,
then $x_2 = -2c_1 - c_2 + c_3$, $x_1 = -3(-2c_1 - c_2 + c_3) - c_1 + 2c_2 + 3c_3 = 5c_1 + 5c_2$.

Hence, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5c_1 + 5c_2 \\ -2c_1 - c_2 + c_3 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Therefore, basis of $(nulA) = \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and dim $(nulA) = 3$.

2. Find the basis and dimension of row space, column space and null space of the matrix
$\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$

Solution: Reducing A into echelon form we get,

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Row$A = \{(1, -1, 1, 3, 2), (2, -1, 1, 5, 1\}$, and dim (Row$A$) = 2.

Basis of Col$A$ $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$, and dim $(\text{Col}A) = 2$.

To find the basis of null space, that is solution of the system $Ax = 0$,

Consider $x_1 - x_2 + x_3 + 3x_4 + 2x_5 = 0$ and $x_2 - x_3 - x_4 - 3x_5 = 0$ (Coefficients from the nonzero rows of echelon form) Since the last three columns of A are non-pivot columns, $x_3,\ x_4,\ x_5$ are free variables.

Let $x_3 = c_1,\ x_4 = c_2,\ x_5 = c_3$,
then $x_2 = c_1 + c_2 + 3c_3$, $x_1 = (c_1 + c_2 + 3c_3) - c_1 - 3c_2 - 2c_3 = -2c_2 + c_3$.

Hence, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2c_2 + c_3 \\ c_1 + c_2 + 3c_3 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Therefore, basis of $(nulA) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and dim $(nulA) = 3$.

Review:

1. What are the main assumptions made in the traffic flow model?
2. Write the basic condition for traffic balance at an intersection.
3. What mathematical method is used to solve the system of equations in this model?
4. What is meant by a directed network in the context of traffic flow?

LECTURE 7:
Coordinate vectors

Recall:

1. What is meant by the algebra of transformations?
2. How do you define the sum of two linear transformations?
3. How do you define the product (composition) of two linear transformations?
4. What is the identity transformation?
5. What is the inverse of a linear transformation?
6. How do you represent a linear transformation as a matrix?

Coordinates: Let V be a n- dimensional vector space with basis, $\mathcal{B} = \{u_1,\ u_2,\ u_3, \cdots, u_n\}$.

Any vector $v \in V$ can be expressed uniquely as a linear combination of vectors in $\mathcal{B}$.

Let $v = c_1 u_1 + c_2 u_2 + c_3 u_3 \cdots + c_n u_n$.

Then coordinates of v with respect to the basis $\mathcal{B}$ is $[v]_{\mathcal{B}} = (c_1,\ c_2,\ c_3, \cdots, c_n)$.

Examples:

1. Determine the coordinates of vector $(5,\ 3,\ 4)$ relative to the basis $\mathcal{B} = \{(1,-1,0),(1,1,0),(0,1,1)\}$

Solution: Let $(5,\ 3,\ 4) = c_1(1,-1,0) + c_2(1,1,0) + c_3(0,1,1)$
$\Rightarrow c_1 + c_2 = 5,\ -c_1 + c_2 + c_3 = 3,\ c_3 = 4 \quad \Rightarrow c_1 = 3,\ c_2 = 2,\ c_3 = 4.$
Coordinates of $(5,\ 3,\ 4)_{\mathcal{B}} = (3,\ 2,\ 4)$.

Or, $(c_1,\ c_2,\ c_3)\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} = (5,\ 3,\ 4)$

Transition matrix of change of basis is $P = \begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} \Longrightarrow P^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\\ -1 & 1 & 0\\ 1 & -1 & 2\end{bmatrix}$.

$$(c_1,\ c_2,\ c_3) = (5,\ 3,\ 4)P^{-1} = (5,\ 3,\ 4)\frac{1}{2}\begin{bmatrix}1 & 1 & 0\\ -1 & 1 & 0\\ 1 & -1 & 2\end{bmatrix} = (3,\ 2, 4)\ .$$

2. Let coordinate of $[v]_{\mathcal{B}} = (3,\ 2)$, where $\mathcal{B} = \{(1,2),(0,-1)\}$.

Find the coordinate of $[v]_{\mathcal{B}'}$, where $\mathcal{B}' = \{(1,1),(2,-1)\}$.

Solution: Given that $v = 3(1,2) + 2(0,-1) = (3,4)$.

$$(3,4) = a(1,1) + b(2,-1) \quad \Longrightarrow a = \frac{11}{3},\ b = -\frac{1}{3}.$$

$$\therefore [v]_{\mathcal{B}'} = \left(\frac{11}{3}, -\frac{1}{3}\right).$$

Or $[v]_{\mathcal{B}}P = v$ and $[v]_{\mathcal{B}'}P' = v \Longrightarrow [v]_{\mathcal{B}'}P' = [v]_{\mathcal{B}}P \Longrightarrow [v]_{\mathcal{B}'} = [v]_{\mathcal{B}}P(P')^{-1}$

Given that $P = \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix},\ P' = \begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix} \Longrightarrow (P')^{-1} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$

$$P(P')^{-1} = \frac{1}{3}\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}5 & -1\\ -2 & 1\end{bmatrix}.$$

$$[v]_{\mathcal{B}'} = [v]_{\mathcal{B}}P(P')^{-1} = \frac{1}{3}(3,\ 2)\begin{bmatrix}5 & -1\\ -2 & 1\end{bmatrix} = \frac{1}{3}(11,-1)\quad .$$

Review:

1. What is meant by a change of basis in a vector space?
2. How do you find the coordinate vector of a given vector relative to a new basis?
3. What is the transition matrix from one basis to another?
4. Can a vector space have more than one basis?
5. How do you determine the dimension of a subspace?
6. What is the relationship between the number of vectors in a basis and the dimension of a vector space?

LECTURE 8:

Inner products and Orthogonality

Recall:

1. What is meant by a change of basis in a vector space?
2. How do you find the coordinate vector of a given vector relative to a new basis?
3. What is the transition matrix from one basis to another?
4. Can a vector space have more than one basis?
5. How do you determine the dimension of a subspace?
6. What is the relationship between the number of vectors in a basis and the dimension of a vector space?

Inner product: If u and v are two vectors in $\mathbb{R}^n$, and consider u and v as $n \times 1$ matrices. Then the inner product or dot product of u and v is $u^T v$, denoted by $u \cdot v$.

Let $u = \begin{bmatrix}u_1\\ u_2\\ \vdots\\ u_n\end{bmatrix}$ and $v = \begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix}$.

$$u\cdot v=u^T v=[u_1 \quad u_2 \quad \cdots \quad u_n]\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix}=u_1v_1+u_2v_2+\cdots+u_nv_n.$$

Properties: For any three vectors , v and w in $\mathbb{R}^n$, and for any scalar c,

1. $u\cdot v=v\cdot u$.
2. $(u+v)\cdot w=u\cdot w+v\cdot w$.
3. $(cu)\cdot v=c(u\cdot v)=u\cdot(cv)$.
4. $u\cdot u\geq 0$, and $u\cdot u=0\Leftrightarrow u$.

Length of a vector (or norm) $u=(u_1,u_2,\cdots,u_n)$ is $\|u\|=\sqrt{u_1^2+u_2^2+\cdots+u_n^2}$.

Distance between the two vectors u and v is $dist(u,v)=\|u-v\|$.

If $u\cdot v=0$, then u and v are orthogonal.

Problems:

1. Consider vector $u=(1,2,4), v=(2,-3,5)$and $w=(4,2,-3\ )$in $\mathbf{R}^3$.
   Find (i) $<\ u\,.v\ >$ (ii) $<\ u.w\ >$ (iii) $<\ v\,.w>$ (iv) $<(u+v)\,.w\ >$ (v) $\|u\|$

   Solution: (i) $<\ u\,.v\ >\ =u^Tv$=$[1 \quad 2 \quad 4]\begin{bmatrix}2\\ -3\\ 5\end{bmatrix}=2-6+20=16.$

   (iii) $u\cdot w=u^Tw$= $4+4-12=-4$ (iii) $<\ v\,.w>$ =-13

   (iv) $<(u+v).w\ >$ =$(3,-1,9)\cdot(4,2,-3\ )=-17$

   (v) $\|u\|=\sqrt{u_1^2+u_2^2+u_3^2}=\sqrt{21}$
2. Verify the vectors $(1,1,1), v=(1,2,-3)$and $w=(1,-4,3\ )$ in $\mathbf{R}^3$ are orthogonal.
   Solution: (i) Solution: (i) $<\ u\,.v\ >\quad 0$, $<\ v\,.w\ >\ =1-8-9=-16$ and $<\ u\,.w\ >=0$
   Thus u is orthogonal to v and w, But v and w are not orthogonal.

3. Show that the functions $f\ (x)\ =\ 3x-2$ and $g(x)\ =\ x$ are orthogonal in P_n with inner product $<\ f\,.g\ >=\ \int_0^1 f\ (x)\ g(x)\ dx$.

   Solution: Given that $f\cdot g=<\ f\,,g\ >=\ \int_0^1 f\ (x)\ g(x)\ dx$

   $$=\int_0^1\ (3x-2)\ x\,dx=\int_0^1\ (3x^2-2x)\ dx=x^3-x^2\Big|_0^1=1-1=0 \quad .$$

   $f\cdot g=0\Longrightarrow\ f$ and g are orthogonal.

4. Consider $f\ (t)=\ 4t+3$ and $g(x)=\ t^2$ with inner product
   $<\ f\,.g\ >=\ \int_0^1 f\ (t)\ g(t)\ dt$. Find (i) $<\ f\,,g\ >$ (ii) $\|g\|$

   Solution: (i) Given that $f\cdot g=<\ f\,,g\ >=\ \int_0^1 f\ (t)\ g(t)\ dt$

   $$=\ \int_0^1(4t^3+3t^2)dt=t^4+t^3\Big|_0^1=2\ .$$

   (ii) $\|g\|=\sqrt{<\ g\,,g\ >}$ where $<\ g\,,g\ >=\int_0^1 t^4\ dt=\frac{t^5}{5}\Big|_0^1=\frac{1}{5}$

   $$\therefore\ \|g\|=\sqrt{\frac{1}{5}}\,.$$

Review:

1. What is an inner product space? Give an example.
2. Define the dot product and explain its relation to the inner product.
3. How does the inner product define the length (norm) of a vector?
4. Define Orthogonality in an inner product space.
5. If two vectors have zero inner product, what does that imply?

TUTORIAL 3:
Problem Solving on Basis and dimensions, Inner Product

1. Find the basis and dimension of the subspace W of $\mathbb{R}^3$ spanned by the vectors $\{(1,2,3),(2,4,6),(0,1,1)\}$.
2. Show that the sets of vectors $\{(\mathbf{1},\mathbf{2},\mathbf{1}),(\mathbf{3},\mathbf{1},\mathbf{5}),(-\mathbf{1},\mathbf{0},\mathbf{1}),(\mathbf{1},-\mathbf{1},\mathbf{2})\}$ do not form a basis for $\boldsymbol{V}(\boldsymbol{R})$..
3. Let W be the subspace of $\mathbb{R}^5$ spanned by $x_1 = (1,2,1,-1,4),\ x_2 = (2,1,-2,0,1), x_3 = (1,-1,2,2,6)$ $x_4 = (1,2,5,1,8),\ x_5 = (2,7,1,2,-1)$. Find a subset of vectors which forms a basis of W.
4. Determine whether or not each of the following form a basis in ℝ3.
   i){(1,3,1),(0,2,0),(0,0,7)}
   ii){(1,2,9),(2,-3,4),(1,3,7),(2,4,8)}
   iii){(2,2,1),(1,3,7),(1,2,2)}
5. Determine the coordinates of vector $(1,-3,\ 2)$ relative to the basis $S = \{(1,1,1),(1,1,0),(1,0,0)\}$
6. Find the inner products $< u.v >,< v.w >$ and $< w.u >$ where $u = (1,1,1,1), v = (1,2,4,5)$ and $w = (1,-3,-4,-2\ )$.

TUTORIAL 4:
Lab Activity 8: Basis and dimension ,Inner product

Objectives:

Use python

- To compute the dimension of vector space.
- To compute the inner product of two vectors.
- To check whether the given vectors are orthogonal..

1. Find the dimension of subspace spanned by the vectors (1, 2, 3), (2, 3, 1) and (3, 1, 2).

```python
from numpy import *
V = array ([[1 , 2 , 3],[2 , 3 , 1],[3 , 1 , 2]])
db = linalg.matrix_rank( V )
dimension = V.shape [0]
print (" Dimension of Basis of the matrix ", db )
print (" Dimension of the matrix ", dimension )
```

2. Find the inner product of the vectors (2, 1, 5, 4) and (3, 4, 7, 8).

```python
from numpy import *
A = array ([2 , 1 , 5 , 4])
B = array ([3 , 4 , 7 , 8])
output = dot(A , B )
print ( output )
```

3. Verify whether the following vectors (2, 1, 5, 4) and (3, 4, 7, 8) are orthogonal.

```python
from numpy import *
A = array ([2 , 1 , 5 , 4])
B = array ([3 , 4 , 7 , 8])
output = dot(A , B )
print ('Inner product is :', output )
if output ==0:
    print ('given vectors are orthognal ')
else :
    print ('given vectors are not orthognal ')
```

Course outcome

- Demonstrate the idea of linear dependence and independence of sets in the Vector Space and apply the concepts to problems in computer science engineering.

PRACTICE QUESTION BANK
MODULE 3: VECTOR SPACE – QUESTION BANK

Vector Spaces – Subspace, linear span, linearly independent and dependent sets, basis and dimensions:

1. Let $V = \{(x,\ y)|x, y\ \in R\} = R^2$, For $u = (x_1,\ y_1),\ v = (x_2,\ y_2)\ \in V,\quad c \in R$, define
   a. Addition: $u + v = (x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 +\ y_2)$.
   b. Scalar multiplication: $c(u) = c(x_1,\ y_1) = (cx_1, cy_1)$

   then show that V is vector space over R with respect to the addition and scalar multiplication defined above.
2. Check whether the set $V = \{(x,\ y)|x, y \in R\}$with the vector addition defined by $(x_1,\ y_1) + (x_2,\ y_2) = (x_1y_2,\ x_2y_1)$ and the scalar multiplication defined by $c(x, y) = (cx, cy)$ is a vector space.
3. Prove that the set of all (exact) second-degree polynomial functions is not a vector space.
4. Prove that $H = \{(x,\ 0)|\ x \in R\}$ is a subspace of R^2.
5. Let V be the vector space of all 2×2 metrices, and $W = \left\{\begin{bmatrix} x & -x \\ y & z \end{bmatrix}\ \middle|\ x, y, z \in R\right\}$.

   Show that W is a subspace of V.
6. Check whether the subset $U = \{(x,\ y) \in R|\ x^2 + y^2 = 1\}$ is a subspace?
7. Let $X = (1, 2, -3),\ \ Y = (-2, 3, 0)$ and $W = \{aX + bY|a, b \in R\}$. Show that W is subspace of R^3.
8. Let $W\ = \{(a,\ a^2,\ b)|\ a, b \in R\}$, is W a subspace of R^3? Give reason.
9. Let $W\ = \{(x,\ y,\ z)|\ lx + my + nz = 0\}$, then prove that W is a subspace of R^3.
10. Prove that $W\ = \{(x,\ y,\ z)|\ x - 3y + 4z = 0\}$ is a subspace of R^3.
11. If H and K are subspaces of V then prove that $H \cap K$ is also subspace of V.
12. Let H and K are subspaces of V ,Is $H \cup K$ is also subspace of V?
13. Let $U = \{(a, 0, 0) \in \mathbf{R}^3\ \}$ be the subset of $\mathbf{R}^3$,Show that U is a subspace of $\mathbf{R}^3$.
14. Prove that the set W of 2×2 diagonal matrices is a subspace of the vector space M22 of 2×2 matrices.

15. Let V be the vector space of all square matrices over R. Determine which of the following are subspaces of V.
    i) $W = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} \middle| \; x, y, z \in R \right\}$
    ii) $W = \{A|A \in V$ and A is singular$\}$
    iii) $W = \{A|A \in V, A2 = A\}$
16. Express the vector $u = (1, -2, 5)$ as a linear combination of the vectors $u_1 = (1, 1, 1), u_2 = (1, 2, 3)$, $u_3 = (2, -1, 1$ in the vector space R^3.
17. Can v = (2,−5, 3) in R^3 be represented as a linear combination of the vectors u_1 = (1,−3, 2), u_2 = (2,−4,−1), u_3= (1,−5, 7).
18. For which value of k the vectors u = (1,−2, k) in R^3 be a linear combination of the vectors u_1 = (3, 0,−2) and u_2 = (2,−1,−5).
19. Write the matrix $E = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $and\ C = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$.
20. Let a_1 = (1,−2,−5), a_2 = (2, 5, 6) and b = (7, 4,−3). Determine whether b can be written as a linear combination of a_1 and a_2.
21. Let u_1 = (1,2, 3), u_2 = (0,1,2), u_3= (-1,0, 1).Prove
    a) $w = (1,1,1)$ is not a linear combination of u_1, u_2, u_3.
    b) $w = (1, -2, 2)$ is a linear combination of u_1, u_2, u_3.
22. Write the vector $u = (1,3,\ 9)$ as a linear combination of the vectors $u_1 = (2,\ 1,\ 3), u_2 = (1,\ -1,\ 1), u_3 = (3,1,\ 5)$ in the vector space R^3.
23. Check whether the vector v = (4, 2, 1) can be written as a linear combination of the vectors u_1 = (1,−3, 1), u_2 = (0, 1, 2), u_3 = (5, 1, 37) in the vector space R^3.
24. Is $E = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$ in vector space of 2×2 matrices , a linear combination of B= $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ $and\ D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$?
25. Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$in the vector space M_{22} of 2×2 matrices .
26. Find the condition on a, b, c so that W = (a, b, c) is a linear combination of u = (1,−3, 2) and v = (2,−1, 1) in R^3 so that W ∈ Span(u, v).
27. Show that the matrix $E = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ lies in the subspace span $\{A, B, C\}$ of vector space M_{22} of 2×2 matrices.where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$.
28. Let $f\ (x) = 2x^2 - 5\ and\ g(x) = x + 1$. Show that the function $h(x) = 4x^2 + 3x - 7$ lies in the subspace Span$\{f\ , g\}$ of P_2.
29. Check whether the following vectors are linearly dependent or linearly independent vectors?
    ii) $(1,\ 4,\ 5)\ ,(4,\ 4,\ 8),\ (3, -3,\ 0)$. ii) $(1,\ 0, -1,\ 2)\ ,(4,\ 2,\ 0, -1),\ (6,\ 4, -2,\ 3)$.
6. $(1,1,\ 2,\ 4)\ ,(2, -1, -5,\ 2),\ (1, -1, -4,\ 0), (2,1,1,6)$

30. Let V be a vector space of all 2×3 matrices over R. Show that $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ form a linearly independent set.
31. Check whether the set $\{(2,\ 3), (0,\ 4)\}$ is a basis for R^2. If so express the vectors of $\mathbb{R}^2$ as linear combination of these vectors.
32. Check whether the set $\{(1,3,1), (0,2,0), (0,0,7)\}$ is a basis of R^3. If so express the vectors of $\mathbb{R}^3$ as linear combination of these vectors.
33. Find the dimension of the subspace W of $\mathbb{R}^3$ spanned by the vectors $\{(3,1,0), (2,1,0), (1,1,-2)\}$.
34. Find the dimension of the subspace W of R^3 spanned by the vectors $\{(2,4,2), (1,-1,0), (1,2,1), (0,3,1)\}$.
35. Let W be the subspace spanned by vectors $u_1 = (1,-2,5,-3)$, $u_2 = (3,8,-3,-5)$, $u_3 = (2,3,1,-4)$. Find the basis and dimension of . Extend the basis of W to a basis of R^4 .
36. Determine whether $(1,1,1,1), (1,2,3,2), (2,5,6,4), (2,6,8,5)$ form a basis of $\mathbb{R}^4$. If not find the dimension of subspace spanned by these vectors.
37. Let W be the subspace of R^5 spanned by $x_1 = (1,2,-1,3,4)$, $x_2 = (2,4,-2,6,8)$, $x_3 = (1,3,2,2,6)$, $x_4 = (1,4,5,1,8)$, $x_5 = (2,7,3,3,9)$. Find a subset of vectors which forms a basis of W.
38. Let V be a vector space of polynomials over R. Find a basis and dimension of the subspace W of V spanned by the polynomials; $x_1 = t^3 - 2t^2 + 4t + 1$, $x_2 = 2t^3 - 3t^2 + 9t - 1$, $x_3 = t^3 + 6t - 5$, and $x_4 = 2t^3 - 5t^2 + 7t + 5$.(Taking coefficients of each polynomial is taken as columns)
39. Determine the coordinates of vector $(5,\ 3,\ 4)$ relative to the basis $\mathcal{B} = \{(1,-1,0), (1,1,0), (0,1,1)\}$
40. Determine whether or not each of the following form a basis in R^3. i){(1,3,1),(0,2,0),(0,0,7)} ii){(1,2,9),(2,-3,4),(1,3,7),(2,4,8)} iii){(2,2,1),(1,3,7),(1,2,2)}
41. Let coordinate of $[v]_{\mathcal{B}} = (3,\ 2)$, where $\mathcal{B} = \{(1,2), (0,-1)\}$.Find the coordinate of$[v]_{\mathcal{B}'}$, where $\mathcal{B}' = \{(1,1), (2,-1)\}$.

Range space, Column space, null space ,Coordinates, Inner product and Orthogonality:

42. Find the basis and dimension of Row$A$, Col$A$ and $nulA$ for $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$
43. Find the basis and dimension of row space, column space and null space of the matrix $\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$
44. Determine the coordinates of vector $(5,\ 3,\ 4)$ relative to the basis $\mathcal{B} = \{(1,-1,0), (1,1,0), (0,1,1)\}$
45. Determine the coordinates of vector $(1,-3,\ 2)$ relative to the basis $S = \{(1,1,1), (1,1,0), (1,0,0)\}$
46. Let coordinate of $[v]_{\mathcal{B}} = (3,\ 2)$, where $\mathcal{B} = \{(1,2), (0,-1)\}$. Find the coordinate of$[v]_{\mathcal{B}'}$, where $\mathcal{B}' = \{(1,1), (2,-1)\}$
47. Consider vector $u = (1,2,4), v = (2,-3,5)$and $w = (4,2,-3\ )$in $\mathbf{R}^3$. Find (i) $< u . v >$ (ii) $< u . w >$ (iii) $< v . w >$ (iv) $< (u + v) . w >$ (v) $\|u\|$
48. Find the inner products $< u . v >$,$< v . w >$ and $< w . u >$ where $u = (1,1,1,1), v = (1,2,4,5)$ and $w = (1,-3,-4,-2\ )$.
49. Show that the functions $f(x) = 3x - 2$ and $g(x) = x$ are orthogonal in P_n with inner product $< f . g > = \int_0^1 f(x)\, g(x)\, dx$.
50. Consider $f(t) = 4t + 3, g(t) = t^2$,the inner product $\langle f, g \rangle = \int_0^1 f(t) g(t) dt$.Find $\langle f, g \rangle$ *and* $\|g\|$.