Module- 4 - Linear Transformation

Definition and examples, algebra of linear transformations, matrix of a linear transformation. Singular, non-singular linear transformations and invertible linear transformations. Rank and nullity of linear transformations, Rank-Nullity theorem.

Textbook-3: Chapter 5: Sections 5.3- 5.7 **Chapter 6: Sections-6.1-6.2**

LECTURE 1:

Linear Transformations: Definition and examples

Recall:

1. What is a basis of a vector space?
2. What does it mean for a basis to be ordered?
3. How do you check if a set of vectors forms a basis for a vector space?
4. What is the dimension of a vector space?
5. Can a vector space have more than one basis?
6. What are the two properties that define a linear transformation?
7. What is the effect of a linear transformation on the zero vector?

Linear transformations or Linear mappings

Definition: Let U and V be vector space over the same field (hear $\mathbb{R}$).

A function (or mapping) $T: U \longrightarrow V$ is said to be linear transformation or linear mapping if

(i) $T(u+v) = T(u) + T(v)$ for all $u,\ v \in U$.

(ii) $T(cu) = cT(u)$ for all $u \in U$ and $c \in \mathbb{R}$ (scalar field)

(That means T preserves the two basic operations of vector space, that is vector addition and scalar multiplication)

Therefore, (i) $T(\mathbf{0}) = T(0.u) = 0.T(u) = \mathbf{0}$

(ii) $T(au + bv) = aT(u) + bT(v)$, for any $u, v \in U$ and $a, b \in \mathbb{R}$.

Examples

1. Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $T(x, y, z) = (x, y, 0)$, prove that T is a linear transformation.
   Let $u = (x_1,\ y_1,\ z_1), v = (x_2,\ y_2,\ z_2)$. Then $T(u) = (x_1,\ y_1,\ 0)$ and $T(v) = (x_2,\ y_2,\ 0)$.
   i) $T(u+v) = T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$
   $= (x_1 + x_2,\ y_1 + y_2,\ 0) = (x_1,\ y_1,\ 0) + (x_2,\ y_2,\ 0) = T(u) + T(v)$.
   ii) $T(cu) = T(cx_1,\ cy_1,\ cz_1) = (cx_1,\ cy_1,\ 0) = c(x_1,\ y_1,\ 0) = cT(u)$.
   Therefore, T is a linear transformation.
2. (Constant function) Let $T: U \longrightarrow V$, such that $T(u) = \mathbf{0}$ (zero of V) for all $u \in U$. Prove that T is a linear transformation.
   i) $T(u+v) = \mathbf{0} = \mathbf{0} + \mathbf{0} = T(u) + T(v)$. ii) $T(cu) = \mathbf{0} = c\mathbf{0} = cT(u)$.
   Therefore, T is a linear transformation.
3. (Identity function) Let $T: U \longrightarrow U$, defined by $T(u) = u$, for all $u \in U$. Prove that T is a linear transformation.
   i) $T(u+v) = u + v = T(u) + T(v)$. ii) $T(cu) = cu = cT(u)$.
   Therefore, T is a linear transformation.
4. Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that $T(x, y, z) = (x + y,\ y + z)$. Prove that T is a linear transformation.
   Let $u = (x_1,\ y_1,\ z_1), v = (x_2,\ y_2,\ z_2)$.
   Then $T(u) = (x_1 +\ y_1,\ y_1 +\ z_1)$ and $T(v) = (x_2 +\ y_2,\ y_2 +\ z_2)$.
   i) $T(u+v) = T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$
   $= (x_1 + x_2 + y_1 + y_2,\ y_1 + y_2 + z_1 + z_2) = (x_1 + y_1 + x_2 + y_2,\ y_1 + z_1 + y_2 + z_2)$
   $= (x_1 +\ y_1,\ y_1 +\ z_1) + (x_2 +\ y_2,\ y_2 +\ z_2) = T(u) + T(v)$.
   ii) $T(cu) = T\ (cx_1,\ cy_1,\ cz_1) = (cx_1 +\ cy_1,\ cy_1 + cz_1) = c(x_1 +\ y_1,\ y_1 +\ z_1) = cT(u)$.
   Therefore, T is a linear transformation.
5. Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that $T(x, y, z) = (x - y,\ x + z)$. Prove that T is a linear transformation.
   Let $u = (x_1,\ y_1,\ z_1), v = (x_2,\ y_2,\ z_2)$.
   Then $T(u) = (x_1 -\ y_1,\ x_1 +\ z_1)$ and $T(v) = (x_2 -\ y_2,\ x_2 +\ z_2)$.

i) $T(u+v) = T(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$
$= (x_1+x_2-y_1-y_2,\ x_1+x_2+z_1+z_2) = (x_1-y_1+x_2-y_2,\ x_1+z_1+x_2+z_2)$
$= (x_1-y_1,\ x_1+z_1)+(x_2-y_2,\ x_2+z_2) = T(u)+T(v)$.

ii) $T(cu) = T\ (cx_1,\ cy_1,\ cz_1) = (cx_1-cy_1,\ cx_1+cz_1) = c(x_1-y_1,\ x_1+z_1) = cT(u)$.

Therefore, T is a linear transformation.

6. Let $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $T(x,y,z) = (2x-3y,\ x+4,\ 5z)$. Prove that T is not a linear transformation.

Let $u = (x_1,\ y_1,\ z_1), v = (x_2,\ y_2,\ z_2)$.
Then $T(u) = (2x_1-3y_1,\ x_1+4,\ 5z_1)$ and $T(v) = (2x_2-3y_2,\ x_2+4,\ 5z_2)$.
$\Longrightarrow\ T(u)+T(v) = (2x_1+2x_2-3y_1-3y_2,\ x_1+x_2+8,\ 5z_1+5z_2)$
But, $T(u+v) = T(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$
$= (2(x_1+x_2)-3(y_1+y_2),\ (x_1+x_2)+4,\ 5(z_1+z_2))$
Since $x_1+x_2+8 \neq (x_1+x_2)+4,\ T(u+v) \neq T(u)+T(v)$, T is not a linear transformation.

Review:

1. What is a linear transformation?
2. What are the two main properties that define a linear transformation T: V → W?
3. How does a linear transformation differ from a general function?
4. How can a linear transformation be represented using a matrix?
5. If a linear transformation maps $R^3 \to R^2$, what can be said about its matrix representation?
6. What is the effect of a linear transformation on the zero vector?
7. What does it mean for a linear transformation to be one-to-one (injective)?
8. What does it mean for a linear transformation to be onto (surjective)?

LECTURE 2:

Algebra of transformations, matrix of a linear transformation

Recall:

1. What is a linear transformation? Explain with an example.
2. How do you verify whether a mapping $T: V \longrightarrow W$ is linear?
3. How does a linear transformation differ from a general function?
4. Why is the zero vector always mapped to the zero vector under a linear map?
5. If $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ what can be said about its matrix representation?
6. Explain how linearity of (T) is checked for polynomial mappings.

Standard matrix of a linear transformation:

Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, then there exists a unique matrix A such that

$$T(X) = AX, \text{for all } X \in A.$$

A is the $m \times n$ matrix with the columns $T(e_1),\ T(e_2),\ \cdots T(e_n)$,
where $\{e_1, e_2, \cdots e_n\}$ is the standard basis of $\mathbb{R}^n$.

Example: 1. Let $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $T(x,y,z) = (x-y,\ x+z)$.

Then the matrix of transformation is $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

2. Let $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation such that $T(x,y,z) = (3x+4y+z,\ x-y+2z,\ 2x-z)$.

Then the matrix of transformation is $A = \begin{bmatrix} 3 & 4 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$.

3. Find the matrix of transformation $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$, given that $T(-1,\ 1) = (-1,\ 0,\ 2)$ and $T(2,\ 1) = (1,\ 2,\ 1)$.

Solution: Let $a(-1,\ 1)+b(2,\ 1) = (1,\ 0)\ \Longrightarrow a = -\frac{1}{3},\ b = \frac{1}{3}$.

$$T(1,\ 0) = T\left[-\tfrac{1}{3}(-1,\ 1)+\tfrac{1}{3}(2,\ 1)\right] = -\tfrac{1}{3}T(-1,\ 1)+\tfrac{1}{3}T(2,\ 1)$$
$$= -\tfrac{1}{3}(-1,\ 0,\ 2)+\tfrac{1}{3}(1,\ 2,\ 1) = \left(\tfrac{2}{3},\ \tfrac{2}{3},\ -\tfrac{1}{3}\right)$$

Let $a(-1,\ 1)+b(2,\ 1) = (0,\ 1)\ \Longrightarrow a = \frac{2}{3},\ b = \frac{1}{3}$.

$$T(0,\ 1) = T\left[\tfrac{2}{3}(-1,\ 1) + \tfrac{1}{3}(2,\ 1)\right] = \tfrac{2}{3}T(-1,\ 1) + \tfrac{1}{3}T(2,\ 1)$$
$$= \tfrac{2}{3}(-1,\ 0,\ 2) + \tfrac{1}{3}(1,\ 2,\ 1) = \left(-\tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{5}{3}\right).$$

Therefore the matrix of transformation is $A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{5}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ -1 & 5 \end{bmatrix}$.

Matrix of a linear transformation with any basis:

$T: U \longrightarrow V$ be a linear transformation, and basis of U is $\mathcal{B} = \{u_1,\ u_2,\ u_3, \cdots, u_n\}$, and basis of V is $\mathcal{B}' = \{v_1,\ v_2,\ v_3, \cdots, v_m\}$.

$$T(u_1) = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1m}v_m$$
$$T(u_2) = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2m}v_m$$
$$\vdots$$
$$T(u_n) = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nm}v_m$$

Therefore matrix of transformation relative to basis $\mathcal{B}, \mathcal{B}' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$

Example: Find the linear transformation, $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, whose matrix relative to basis $\mathcal{B}, \mathcal{B}' = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$.

Where $\mathcal{B} = \{(1,\ 1),\ (0,\ 2)\}$, $\mathcal{B}' = \{(0,\ 1,\ 1),\ (1,\ 0,\ 1),\ (1,\ 1,\ 0)\}$

Clearly, $T\big((1,\ 1)\big) = 1(0,\ 1,\ 1) - 2(1,\ 0,\ 1) + 0(1,\ 1,\ 0) = (-2,\ 1, -1)$

$T\big((0,\ 2)\big) = -1(0,\ 1,\ 1) + 3(1,\ 0,\ 1) + 1(1,\ 1,\ 0) = (4,\ 0,\ 2)$

For any $(x,\ y) \in \mathbb{R}^2$, $(x,\ y) = a(1,\ 1) + b(0,\ 2) \Longrightarrow a = x,\ a + 2b = y \Longrightarrow b = \frac{1}{2}(y - x)$

$$\therefore (x,\ y) = x(1,\ 1) + \tfrac{1}{2}(y - x)(0,\ 2)$$
$$\Longrightarrow T(x,\ y) = xT(1,\ 1) + \tfrac{1}{2}(y - x)T(0,\ 2)$$
$$= x(-2,\ 1, -1) + \tfrac{1}{2}(y - x)(4,\ 0,\ 2) = (-2x,\ x, -x) + (y - x)(2,\ 0,\ 1)$$
$$= (2y - 4x,\ \ x,\ y - 2x)$$

Review:

1. What is meant by the algebra of transformations?
2. How do you define the sum of two linear transformations?
3. How do you define the product (composition) of two linear transformations?
4. What is the identity transformation?
5. What is the inverse of a linear transformation?
6. How do you represent a linear transformation as a matrix?

LECTURE 3:

Singular, non-singular linear transformations

Recall:

1. What is meant by the linear span of a set of vectors?
2. What is the difference between a span and a subspace?
3. How do we find the span of a given set of vectors?
4. What does it mean for a linear transformation to be singular?
5. What does it mean for a transformation to be non-singular?
6. What is the kernel (null space) of a linear transformation?

Singular and Non-singular Linear Mappings:

A linear transformation $T: U \longrightarrow V$ is called **singular** if there exists a non-zero vector $u \in U$ such that $T(u) = \mathbf{0}$ (the zero vector in V).
In matrix terms, this means its corresponding matrix $\boldsymbol{A}$ has a determinant of zero.

A linear transformation $T: U \longrightarrow V$ is called **non-singular** if the zero vector $\mathbf{0}$ is the only vector whose image under T is 0 i.e, if $T(\mathrm{u}) = \mathbf{0}$ then u must be $\mathbf{0}$. (the zero vector in V). In other words, if $\boldsymbol{Ker\ T} = \{\mathbf{0}\}$.
In matrix terms, this means its corresponding matrix $\boldsymbol{A}$ has a non-zero determinant.

Note: i) Let $T: U \longrightarrow V$ be a non-singular linear mapping. Then the image of any linearly independent set is linearly independent.
ii) T is non-singular iff T is one-to-one .

Problems:

1. Show that the transformation $F: \mathrm{R}^2 \longrightarrow \mathrm{R}^2$ defined by $\mathrm{F(x,y)} = (2\mathrm{x} - 4\mathrm{y}, 3\mathrm{x} - 6\mathrm{y})$ is singular and find its Kernel.
Solution: Set $F(x,y) = (2x - 4y, 3x - 6y) = (0,0)$ to find Ker G
$$\Longrightarrow 2x - 4y = 0, 3x - 6y = 0$$
$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}$$
$\Longrightarrow 2x - 4y = 0, \Longrightarrow x = 2y$ (only first column is pivot column)
So, y is a free variable. Let $y = k \Longrightarrow x = 2k$
$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Longrightarrow Ker\ F = span\{(2,1)\}$$
$\therefore$The solution is $x = 2$ and $y = 1$; hence, G is singular.

2. The linear operator $\mathrm{G}: \mathrm{R}^2 \longrightarrow \mathrm{R}^3$ defined by $\mathrm{G(x,y)} = (\mathrm{x} + \mathrm{y}, \mathrm{x} - 2\mathrm{y}, 3\mathrm{x} + \mathrm{y})$. Show that G is non-singular.
Solution: i) Set $G(x,y) = (x + y, x - 2y, 3x + y) = (0,0,0)$ to find Ker G
$$\Longrightarrow x + y = 0, x - 2y = 0, 3x + y = 0.$$
$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$
$\Longrightarrow x + y = 0, 3y = 0 \Longrightarrow x = -y, \ \ y = 0$ (both columns are pivot columns)
So, $Ker\ G = \{0\}$
$\therefore$The only solution is $x = 0$ and $y = 0$; hence, G is nonsingular.

3. Identify the non-singular /singular linear maps from the following linear maps:
$i)$ $F: R^2 \longrightarrow R^3$ defined by $F(x,y) = (x - y, x - 2y)$.
$ii)$ $\mathrm{G}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\mathrm{G(x,y,z)} = (\mathrm{x} + \mathrm{z}, \mathrm{x} - \mathrm{y}, \mathrm{y})$.
iii) $H: R^2 \longrightarrow R^2$ defined by $H(x,y) = (2x - y, 8x - 4y)$.
Solution: i) Set $F(x,y) = (x - y, x - 2y) = (0,0)$ to find Ker G
$$\Longrightarrow x - y = 0, x - 2y = 0$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$\Longrightarrow x - y = 0, y = 0 \Longrightarrow x = y = 0$ (both column are pivot columns)
So, $Ker\ G = \{0\}$
$\therefore$The only solution is $x = 0$ and $y = 0$; hence, G is nonsingular.
ii) Set $G(x,y,\mathrm{z}) = (x + \mathrm{z}, x - y, y) = (0,0,0)$ to find Ker G
$\Longrightarrow x + \mathrm{z} = 0, x - y = 0, y = 0.$(Only trivial solution: $y = 0, \mathrm{x} = 0$ and $\mathrm{z} = 0$)
$$\text{Or, } \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$\Longrightarrow x + \mathrm{z} = 0, y + \mathrm{z} = 0, \mathrm{z} = 0 \Longrightarrow \mathrm{z} = y = \mathrm{x} = 0$ (all columns are pivot columns)
So, $Ker\ G = \{0\}$

$\therefore$ The only solution is $x = y = z = 0$; hence, G is nonsingular.

iii) Set $H(x,y) = (2x - y, 8x - 4y) = (0,0)$ to find Ker G

$\Longrightarrow 2x - y = 0, 8x - 4y = 0$

$\begin{bmatrix} 2 & -1 \\ 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$\Longrightarrow 2x - y = 0, \Longrightarrow y = 2x$ (only first column is pivot column)

So, y is a free variable. Let $y = k \Longrightarrow x = \frac{k}{2}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{k}{2} \\ k \end{bmatrix} = k \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \Longrightarrow Ker\ F = span\{(1,2)\}$

$\therefore$The solution is $x = 1$ and $y = 2$; hence, G is singular.

Review:

1. How do you determine whether a linear map is singular using its kernel?
2. Explain how determinant of the matrix of a transformation indicates singularity.
3. If a mapping is non-singular, what can be said about its one-to-one nature?
4. How does the pivot column structure of a matrix help determine singularity?
5. Why is a singular linear map not invertible?
6. Identify conditions when a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ cannot be invertible.

LECTURE 4:
Invertible linear transformations

Recall:

1. How does determinant help determine linear dependence?
2. Can two non-zero vectors in $\mathbb{R}^2$ be linearly dependent? When?
3. If a set spans a space, is it necessarily linearly independent?
4. When removing a vector from a linearly dependent set, can the remaining set become independent?
5. What is an invertible transformation?
6. What is the identity mapping?

Operations with Linear Mappings:

Let $F: V \longrightarrow$ U and G: $V \longrightarrow$ U be linear mappings over a field R. The sum $F + G$ and the scalar product $F + G$,
where $k \in R$, are defined to be the following mappings from V into U:
$F + G(v) = F(v) + G(v)$, $(kF)(v) = kF(v)$.

Note: If $F: V \longrightarrow$ U and G: $V \longrightarrow$ U are linear then $F + G$ and kF are also linear.

Composition of Linear Mappings:

Suppose V, U and W are vector spaces over the same field R, and suppose $F: V \longrightarrow$ U and F: U $\longrightarrow$ W are linear mappings. We picture these mappings as follows: $V \xrightarrow{\text{F}} \text{U} \xrightarrow{G} \text{W}$

Then the composition function $(\text{G} \circ \text{F})$ is the mapping from V into W defined by $(\text{G} \circ \text{F})(\text{v}) = \text{G}(\text{F}(\text{v}))$.

Note: G $\circ$ F is linear whenever F and G are linear.

Algebra $A(V)$ of Linear Operators:

Linear mappings of the form $F: V \longrightarrow$ V are called linear operators or linear transformations on V,
we write $\boldsymbol{A(V)}$ for the space of all such mappings.

Note: i) $\boldsymbol{A(V)}$ is a vector space over R,and ,if $\dim(V) = n$ then dim $\boldsymbol{c} = \boldsymbol{n^2}$

ii) for any mappings $\boldsymbol{F, G} \in$ A(V) the G $\circ$ F exists and belongs to A(V).

Invertible linear transformations:

A linear transformation $F: V \longrightarrow V$ is said to be invertible if it has an inverse- i.e, if there exists F^{-1} in A(V) such that $FF^{-1} = F^{-1}F = I$. On the other hand, F is invertible as a mapping if F is both **one-to-one** and **onto**. In such a case, F^{-1} is also linear and $F^{-1}$is the inverse of F as a linear operator.

Note: Suppose F is invertible. Then only $0 \in V$ can map into itself, and so F is non-singular. The converse is not true.

Problems:

1. Consider the linear operator T on R^3 defined by $T(x,y,z,) = (2x,\ 4x - y,\ 2x + 3y - z)$. Show that T is invertible and also find formula for T^{-1}

   Solution: Let W = Ker T, we need to show that T is nonsingular (i.e, W = {0})

   Set $T(x,y,z) = (2x,\ 4x - y,\ 2x + 3y - z) = (0,0,0)$ to find Ker G

   $\Longrightarrow 2x = 0, 4x - y = 0, 2x + +3y - z = 0$

   $$\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (All three column are pivot column)

   $\Longrightarrow 2x = 0, y = 0, z = 0$ Which has only the trivial solution

   Thus, W = {0}. Hence, T is non-singular, and so T is invertible.

   To find T^{-1}: Set $T(x,y,z) = (r,s,t)$ and so $T^{-1}(r,s,t) = (x,y,z)$

   $(2x, 4x - y, 2x + 3y - z) = (r,s,t)$

   Solve for x,y,z in terms of r, s,t : we get $x = \frac{r}{2}$, $y = 2r - s, z = 7r - 3s - t$

   so $T^{-1}(r,s,t) = (\frac{r}{2}, 2r - s, 7r - 3s - t)$ or $T^{-1}(x,y,z) = (\frac{x}{2}, 2x - y, 7x - 3y - z)$.

2. Show that the linear operator T: $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by is invertible and also find formula for T^{-1}.

   $T(x,y,z) = (x + z, x - y, y)$

   Solution: Let W = Ker T, we need to show that T is nonsingular (i.e, W = {0})

   Set $T(x,y,z) = (x + z,\ x - y,\ y) = (0,0,0)$ to find Ker G

   $\Longrightarrow x + z = 0, x - y = 0, y = 0$

   $$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ (All three column are pivot column)

   $\Longrightarrow x = 0, y = 0, z = 0$ Which has only the trivial solution

   Thus, W = {0}. Hence, T is non-singular, and so T is invertible.

   To find T^{-1}: Set $T(x,y,z) = (r,s,t)$ and so $T^{-1}(r,s,t) = (x,y,z)$

   $\Longrightarrow (x + z, x - y, y) = (r,s,t)$

   Solve for x,y,z in terms of r,s,t : we get $y = t$, $x = s + t, z = r - s - t$

   so $T^{-1}(r,s,t) = (s + t, t, r - s - t)$ or $T^{-1}(x,y,z) = (y + z, z, x - y - z)$.

3. The linear operator G: $R^2 \longrightarrow R^3$ defined by $G(x,y,) = (x + y, x - 2y, 3x + y)$. Show that G is invertible and also find formula for G^{-1}

   Solution: Set $G(x,y) = (x + y, x - 2y, 3x + y) = (0,0,0)$ to find Ker G

   $\Longrightarrow x + y = 0, x - 2y = 0, 3x + y = 0.$

   $$\begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

   $\Longrightarrow x + y = 0, 3y = 0 \Longrightarrow x = -y,\ \ y = 0$ (both columns are pivot columns)

   $\therefore$The only solution is $x = 0$ and $y = 0$; hence, G is nonsingular.

   Although G is non-singular, G is not invertible as $R^2$and R^3 are of different dimensions.

Review:

1. What conditions must a linear transformation satisfy to be invertible?
2. Explain the relation between nonsingularity of a matrix and invertibility of the corresponding mapping.
3. If (T) is invertible, what properties does its inverse T^{-1} satisfy?

4. How do you compute the inverse mapping from the transformation rule?
5. Why is a mapping between spaces of different dimensions never invertible?
6. Explain how to check invertibility using the kernel and range of a linear operator.

TUTORIAL 1:
Problems on Linear Transformations, Singular, non-singular linear transformations

1. Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $T(x, y) = (x + y, x)$. Prove that T is a linear transformation.
2. Show that the following mappings are not linear:
   i). $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $F(x, y) = (x^2, y^2)$
   ii). $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (x + 1, y + z)$
3. Determine whether or not each of the following linear maps is non-singular
   i).$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $\mathrm{F}(x, y, z) = (x + y + z, 2x + 3y + 5z, x + 3y + 7z)$
   ii).$G: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $G(x, y) = (y, x + y)$
4. Show that the linear operator $\mathrm{T}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by is invertible and also find formula for T^{-1}.
   i).$\mathrm{T(x,y) = (x + 2y, 2x - 3y)}$
   ii).$\mathrm{T(x,y) = (2x - 3y, 3x - 4y)}$

TUTORIAL 2:
Lab Activity 9: Linear transformation-range space and null space

Objectives:
Use python

- Find the range space and null space of linear transformation

1. Find the rank and nullity of linear transformation $T: R^3 \longrightarrow R^2$ defined by
   $T(x, y, z) = (y - x, y - z)$.

```
from numpy import *
from scipy.linalg import null_space
A = array([[-1 , 1 , 0], [0 , 1 , -1]])
rank = linalg.matrix_rank( A )
print(" Rank of the matrix: ", rank)
ns = null_space( A )
print("\n Null space of the matrix \n", ns)
nullity = ns.shape [1]
print("\n Nullity of T:", nullity)
```

2. Find the rank and nullity of linear transformation $T: R^3 \longrightarrow R^3$ defined by $T(x, y, z) = (x + y, x - y, 2x + z)$.

LECTURE 5:
Linear Transformations - Problems

Recall:
1. How can the determinant of a matrix help in determining linear dependence?
2. Can two non-zero vectors be linearly dependent? Under what condition?
3. If a set spans a vector space, is it always linearly independent?
4. What happens when a vector is removed from a linearly dependent set?
5. What is meant by image of a vector under a transformation?

Problems:
1. Let P_n be the vector space of real polynomial functions of degree $\leq n$. Show that the transformation $T : P_2 \rightarrow P_1$, defined by $T(ax^2 + bx + c) = (a + b)x + c$ is linear.

Solution: Let $u = a_1x^2 + b_1x + c_1, v = a_2x^2 + b_2x + c_2$.

Then $T(u) = (a_1 + b_1)x + c_1$ and $T(v) = (a_2 + b_2)x + c_2$.

i) $T(u+v) = T((a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_1))$
$= (a_1 + a_2 + b_1 + b_2)x + c_1 + c_2$
$= (a_1 + b_1)x + c_1 + (a_2 + b_2)x + c_2 = T(u) + T(v)$.

ii) $T(cu) = T(ca_1x^2 + cb_1x + cc_1) = (ca_1 + cb_1)x + cc_1 = c((a_1 + b_1)x + c_1) = cT(u)$.

Therefore, T is a linear transformation.

2. Prove that the transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x,y) = (3x, x + y)$ is linear. Find the images of the vectors $(1,3)$ and $(-1,2)$ under this transformation.

Let $u = (x_1, y_1), v = (x_2, y_2)$.

Then $T(u) = (3x_1, x_1 + y_1)$ and $T(v) = (3x_2, x_2 + y_2)$.

i) $T(u+v) = T(x_1 + x_2, y_1 + y_2)$
$= (3x_1 + 3x_2, x_1 + x_2 + y_1 + y_2) = (3x_1 + 3x_2, x_1 + y_1 + x_2 + y_2)$
$= (3x_1, x_1 + y_1) + (3x_2, x_2 + y_2) = T(u) + T(v)$.

ii) $T(cu) = T(cx_1, cy_1) = (3cx_1, cx_1 + cy_1) = c(3x_1, x_1 + y_1) = cT(u)$.

Therefore, T is a linear transformation.

Images of the vectors $(1,3)$ and $(-1,2)$ are $T(1, 3) = (3, 4)$ and $T(-1, 2) = (-3, 1)$.

3. Prove that $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)$ is a linear transformation.

Solution: Let $u = (a_1, b_1, c_1), v = (a_2, b_2, c_2)$.

Then $T(u) = (3a_1, a_1 - b_1, 2a_1 + b_1 + c_1)$ and $T(v) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$.

$\Longrightarrow T(u) + T(v) = (3a_1 + 3a_2, a_1 - b_1 + a_2 - b_2, 2a_1 + b_1 + c_1 + 2a_2 + b_2 + c_2)$
$= (3(a_1 + a_2), (a_1 + a_2) - (b_1 + b_2), 2(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2))$
$= T(u+v)$

$T(cu) = T(ca_1, cb_1, cc_1) = (3ca_1, ca_1 - cb_1, 2ca_1 + cb_1 + cc_1)$
$= c(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = cT(u)$.

Therefore, T is a linear transformation.

Review:

1. How do you verify whether a polynomial mapping defines a linear transformation?
2. Explain how to determine the image of a vector under a given transformation.
3. Why do polynomial transformations make useful examples in linear algebra?
4. How do you prove linearity of a map defined by algebraic formulas?
5. Discuss the role of vector addition and scalar multiplication in proving linearity.

LECTURE 6:
Rank and Nullity of a linear operator

Recall:

1. What is the row space of a matrix?
2. What is the column space of a matrix?
3. What is the null space of a matrix?
4. What is the rank of a matrix?
5. What is the nullity of a matrix?
6. What is meant by pivot columns?

Row space and column space of a matrix:

Let A be a $m \times n$ matrix.

The row space of A denoted by Row$A$ is the set of all linear combination of the rows of A .

If $R_1, R_2, R_3, \cdots R_m$ are the rows of A then Row$A$ =Span$\{R_1, R_2, R_3, \cdots R_m\}$

The column space of A denoted by Col$A$ is the set of all linear combination of the columns of A .

If $C_1, C_2, C_3, \cdots C_n$ are the columns of A then Col$A$ =Span$\{C_1, C_2, C_3, \cdots C_n\}$

Since the spanning set is subspace, Row$A$ is subspace of R^m and Col$A$ is subspace of R^n.

Basis of Row$A$ is the set of rows A, which are nonzero rows of echelon form of A .

Basis of col$A$ is the set of pivot columns of A.
Null space of matrix A is the set of all solutions of the homogeneous equation $Ax = \mathbf{0}$.

$\therefore nulA = \{x | x \in R^n \textbf{ and } Ax = 0 \}$.

Range space & null space of a linear transformation:

Let $T: U \longrightarrow V$ be a linear transformation.
Range or image space of T is $Range(T) = \{v \in V | v = T(u) for\ some\ u \in U\}$
Null space or Kernel of T is $nul(T) = \{u \in U | T(u) = 0 \}$.
$T: U \longrightarrow V$ be a linear transformation then $Range(T)$ is a subspace of V and $nul(T)$ is a subspace of U.
Dimension of $Range(T)$ is called $Rank(T)$, and dimension of $nul(T)$ is called $nullity\ of\ T$.

Note: $T: \mathrm{R}^n \longrightarrow \mathrm{R}^m$ be a linear transformation and $A_{m\times n}$ is a matrix of transformation then $Range(T) = Col(A)$, $nul(T) = nul(A)$.

Problems:

1. Find the kernel and range of the linear operator $T(x, y, z) = (x + y, z)$ of $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

   Solution: Clearly matrix of transformation is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and is in echelon form.

   $Range(T) = Col(A) = Span[(1,\ 0), (0,\ 1)]$ (Pivot columns of A).
   $\therefore Rank\ of\ T = 2$.
   $Nul(T) = \{(x,\ y,\ z) \in \mathbb{R}^3 |\ T(x,\ y,\ z) = (0,\ 0)\}$
   $\Longrightarrow x + y = 0, z = 0 \quad \Longrightarrow y = -x,\ \ z = 0$.
   $\therefore$ kernel(T) $= Nul(T) = Span\{(1, -1,\ 0)\}$, and $Nullity\ of\ T = 1$.

2. Let $T: V_3(R) \longrightarrow V_2(R)$ Find dimensions of range space and null space of the transformation $T: R^3 \longrightarrow R^2$ defined by $T(x, y, z) = (y - x\ , y - z)$.

   Solution: Clearly matrix of transformation is $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, and is in echelon form.

   $Range(T) = Col(A) = Span[(-1,\ 0), (1,\ 1)]$ (Pivot columns of A).
   $\therefore Rank\ of\ T = 2$.
   $Nul(T) = \{(x,\ y,\ z) \in \mathbb{R}^3 |\ T(x,\ y,\ z) = (0,\ 0)\}$
   $\Longrightarrow y - x = 0, y - z = 0 \quad \Longrightarrow x = y = z$.
   $\therefore$ kernel(T) $= Nul(T) = Span\{(1, 1,\ 1)\}$, and $Nullity\ of\ T = 1$.

3. Apply rank-nullity to find the dimension of range space and null space of transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + 2y - z, y + z\ , x + y - 2z)$.

   Solution: Matrix of transformation A= $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

   $R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} R_2 = R_1 - R_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_3 = R_2 - R_3 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   $\mathrm{Range(T) = Col(A) = Span[(1, 0,1), (2, 1,1)]}$ (Pivot columns of A).
   $\therefore$ Dimension of Range space= Rank of T = 2.
   $\mathrm{Nul(T)} = \{(x,\ y,\ z) \in \mathbb{R}^3 |\ T(x,\ y,\ z) = (0,\ 0)\} \Longrightarrow x + 2y - z = 0, y + z = 0$.
   third column is non-pivot, Let $z = k \Longrightarrow y = -k$ and $x = 3k \Longrightarrow$ Nul (T) = Span$\{(3, -1, 1)\}$
   $\therefore$ dimension of Null space or kernel space = Nullity of T = 1.

4. Find dimensions of range space and null space of the transformation $T: R^3 \longrightarrow R^3$ defined by $T(x, y, z) = (x\ , y, 0)$.

   Solution: Matrix of transformation A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and clearly is in echelon form

   $\mathrm{Range(T) = Col(A) = Span[(1, 0,0), (0, 1,0)]}$ (Pivot columns of A).
   $\therefore$ Dimension of Range space= Rank of T = 2.
   $\mathrm{Nul(T)} = \{(x,\ y,\ z) \in \mathbb{R}^3 |\ T(x,\ y,\ z) = (0, 0,0)\} \Longrightarrow x = 0, y = 0$.
   third column is non-pivot, Let $z = k \Longrightarrow$ Nul (T) = Span$\{(0, 0, 1)\}$
   $\therefore$ dimension of Null space or kernel space = Nullity of T = 1.

Review:

1. How do you find the kernel and range of a linear transformation?
2. Explain how echelon form is used to determine rank and nullity.
3. Why is the null space a subspace of the domain?
4. How do pivot columns correspond to basis of the range space?
5. How do you interpret rank and nullity in practical applications?
6. Apply rank-nullity to a sample 2×3 matrix and interpret results.

LECTURE 7:
Rank – Nullity Theorem

Recall:

1. What is the sum of two linear transformations?
2. What is the composition of linear transformations?
3. What is the identity transformation?
4. How do you represent a linear operator as a matrix?
5. What does the rank–nullity theorem state?
6. What is the relationship between dimension of domain, rank, and nullity?

Rank nullity theorem: Let U and V are the finite dimensional vector space over $\mathbb{R}$, and $T: U \longrightarrow V$ be a linear transformation, then $\dim\big(Range(T)\big) + \dim\big(Nul(T)\big) = \dim(U)$.Or $Rank\ of\ T + Nullity\ of\ T = \dim(U)$.
(Proof not required)

Proof: Let $\dim(U) = n$, and $\dim\big(Nul(T)\big) = r$.

Since $Nul(T)$ is subspace of U, $r \leq n$.

Let $\{u_1,\ u_2, \cdots\ u_r\}$ be the basis of (T) . Thus, the set $\{u_1,\ u_2, \cdots\ u_r\}$ is linearly independent in $Nul(T)$, and hence linearly independent in . So, we can extend it to form a basis of U.

Let $\{u_1,\ u_2, \cdots\ u_r, v_1,\ v_2, \cdots\ v_{n-r}\}$ be the extended basis for U.

Let v be an arbitrary vector in (T) . Then $T(u) = v$ for some $u \in U$.

$u = c_1u_1 + c_2u_2 + \cdots + c_ru_r + c_{r+1}v_1 + c_{r+2}v_2 + \cdots + c_nv_{n-r}$ for some scalars $c_1, c_2, \cdots c_n$.

$$\begin{aligned}T(u) &= T(c_1u_1 + c_2u_2 + \cdots + c_ru_r + c_{r+1}v_1 + c_{r+2}v_2 + \cdots + c_nv_{n-r})\\ &= T(c_1u_1) + T(c_2u_2) + \cdots + T(c_ru_r) + T(c_{r+1}v_1) + T(c_{r+2}v_2) + \cdots + T(\ c_nv_{n-r})\\ &= c_1T(u_1) + c_2T(u_2) + \cdots + c_rT(u_r) + c_{r+1}T(v_1) + c_{r+2}T(v_2) + \cdots + c_nT(\ v_{n-r})\\ &= c_{r+1}T(v_1) + c_{r+2}T(v_2) + \cdots + c_nT(\ v_{n-r})\end{aligned}$$

(Because T is a linear transformation, and $u_1,\ u_2, \cdots\ u_r \in Nul(T)$, $T(u_1) = T(u_2) = \cdots = T(\ u_r) = 0$)

$v = T(u) = c_{r+1}T(v_1) + c_{r+2}T(v_2) + \cdots + c_nT(\ v_{n-r})$.

Therefore v is the linear combination of the set vectors$\{\ T(v_1),\ T(v_2), \cdots, T(\ v_{n-r})\}$.

Hence $Range(T) = Span\{\ T(v_1),\ T(v_2), \cdots, T(\ v_{n-r})\}$.

Now consider $d_1T(v_1) + d_2\ T(v_2) + \cdots + d_{n-r}T(\ v_{n-r}) = 0$, where$d_1,\ d_2, \cdots d_{n-r}$ are scalars.

$\Longrightarrow\ T(d_1v_1) +\ T(d_2v_2) + \cdots + T(d_{n-r}\ v_{n-r}) = 0$.

$\Longrightarrow\ T(d_1v_1 + d_2v_2 + \cdots + d_{n-r}\ v_{n-r}) = 0 \Longrightarrow d_1v_1 + d_2v_2 + \cdots + d_{n-r}\ v_{n-r} \in Nul(T)$.

Hence $d_1v_1 + d_2v_2 + \cdots + d_{n-r}\ v_{n-r} = k_1u_1 + k_2u_2 + \cdots + k_ru_r$, for some scalars $k_1,\ k_1,\ \cdots, k_r$.

$\Longrightarrow -k_1u_1 - k_2u_2 - \cdots - k_ru_r + d_1v_1 + d_2v_2 + \cdots + d_{n-r}\ v_{n-r} = 0$.

Since $\{u_1,\ u_2, \cdots\ u_r, v_1,\ v_2, \cdots\ v_{n-r}\}$ is a basis of U and hence linearly independent,

$\Longrightarrow k_1, k_2,\ \cdots, k_r, d_1, d_2, \cdots, d_{n-r}$ all are zero.

$\therefore\ d_1 = d_2 = \cdots = d_{n-r} = 0$.

Hence, the set of vectors $\{\ T(v_1),\ T(v_2), \cdots, T(\ v_{n-r})\}$ is linearly independent and form a basis of $Range(T)$.

$\Longrightarrow \dim\big(Range(T)\big) = n - r = \dim(U) - \dim(Nul(T))$

$\therefore\ Rank\ of\ T + Nullity\ of\ T = \dim(U)$.

Example:

1. Verify the rank nullity theorem for the transformation $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by
$$T(x,y,z,t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t).$$

Solution: Clearly matrix of transformation is $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$Range(T) = Col(A) = Span[(1,\ 1,\ 1), (-1,\ 1,\ 0)]$ (**Pivot** columns of A).

$\therefore\ Rank\ of\ T = 2.$

$Nul(T) = \{(x,\ y,\ z,\ t) \in \mathbb{R}^4 \mid T(x,\ y,\ z,\ t) = (0,\ 0, 0, 0)\}.$

$x - y + z + t = 0,\ y + z - 2t = 0$. Since 3rd and 4th columns are non-pivot,

Let $z = k_1$ and $t = k_2$,then

$y = -k_1 + 2k_2,$

$x = y - z - t = -k_1 + 2k_2 - k_1 - k_2 = -2k_1 + k_2.$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2k_1 + k_2 \\ -k_1 + 2k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore\ Nul(T) = Span\{(-2, -1,\ 1,\ 0),\ (1,\ 2,\ 0,\ 1\}$, and $Nullity\ of\ T = 2.$

$$Rank\ of\ T + Nullity\ of\ T = 2 + 2 = 4 = \dim(U) = \dim(\mathbb{R}^4).$$

Review:

1. Explain the meaning of the rank–nullity theorem in your own words.
2. How does the theorem help in determining the solvability of systems?
3. Apply the theorem to verify dimensions of kernel and image in an example.
4. Why must rank + nullity = dimension of domain?
5. Explain how the basis-extension argument is used in the theorem.

LECTURE 8:

Rank – Nullity Theorem-Problems

Recall:

1. What is meant by change of basis?
2. How do you find coordinate vectors relative to a basis?
3. What is a transition matrix?
4. How do you compute the dimension of a subspace?
5. Can a vector space have more than one basis?
6. What does nullity represent?

Problems:

1. Apply the rank-nullity theorem to verify it for the transformation $T: R^3 \longrightarrow R^3$ defined by $T(x,y,z) = (x + 2y - z,\ y + z,\ x + y - 2z).$

Solution: Matrix of transformation A= $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} R_2 = R_1 - R_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_3 = R_2 - R_3 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Range}(T) = \text{Col}(A) = \text{Span}[(1, 0,1), (2, 1,1)]$ (Pivot columns of A).

$\therefore$ Dimension of Range space= Rank of T = 2.

$\text{Nul}(T) = \{(x,\ y,\ z) \in \mathbb{R}^3 \mid T(x,\ y,\ z) = (0,\ 0)\} \Longrightarrow x + 2y - z = 0, y + z = 0$.

third column is non-pivot, Let $z = k \Longrightarrow y = -k$ and $x = 3k \Longrightarrow \text{Nul}\ (T) = \text{Span}\{(3, -1, 1)\}$

$\therefore$ dimension of Null space or kernel space = Nullity of T = 1.

$Rank\ of\ T + Nullity\ of\ T = 2 + 1 = 3 = \dim(U) = \dim(\mathbb{R}^3)$.

2. Let $T: V \longrightarrow W$ be a linear transformation defined by $T(x,y,z) = (x+y, x-y, 2x+z)$.Find the range, null space, rank, nullity and hence verify the rank-nullity theorem.

Solution: Matrix of transformation A= $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Range(T) = Col(A) = Span[(1, 1,2), (1,−1,0), (0,0,1)] (Pivot columns of A).

$\therefore$ Dimension of Range space= Rank of T = 3.

$\mathrm{Nul(T)} = \{(x, y, z) \in \mathbb{R}^3 | T(x, y, z) = (0, 0, 0)\} \Longrightarrow x + y = 0, 2y + z = 0, 2z = 0.$

$\Longrightarrow x = y = z = 0 \Longrightarrow \mathrm{Nul}(T) = \{0\}$

$\therefore$ dimension of Null space or kernel space = Nullity of T = 0.

$Rank\ of\ T + Nullity\ of\ T = 3 + 0 = 3 = \dim(U) = \dim(\mathbb{R}^3)$.

Review:

1. How do you construct matrix representation of a mapping using basis vectors?
2. Explain step-by-step verification of rank–nullity for a sample 3×4 matrix.
3. Why do some transformations have non-pivot columns and what do they represent?
4. Discuss how nullity indicates the number of free parameters in the system.
5. Explain how rank–nullity is validated in the examples from the PDF.

TUTORIAL 3:

Problem solving on Rank – Nullity Theorem

1. Make use of rank-nullity theorem to verify $\dim(R^2) = rank(T) + nullity(T)$ for the linear transformation $T: R^2 \longrightarrow R^3$ such that $T(x, y) = (x-y, y-x, -x)$
2. Apply the rank-nullity theorem to verify it for the linear transformation $T: V_3(R) \longrightarrow V_2(R)$ defined by $T(x,y,z) = (y-x, y-z)$.
3. Construct the 2×2 matrix of transformation A that maps $(1, 3)^T$ and $(1, 4)^T$ into $(-2, 5)^T$ and $(3, -1)^T$ respectively.
4. Find the range space and null space of linear transformation $T: R^3 \longrightarrow R^2$ defined $T(x,y,z) = (x+y, y+z)$.

TUTORIAL 4:

Lab Activity 10: Verification of the rank nullity theorem space

Objectives:

Use python

- To verify the Rank nullity theorem of given linear transformation.

1. Verify the rank nullity theorem for the transformation $T: R^3 \longrightarrow R^3$ defined by

i) $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$.

ii) $T(x,y,z) = (x, y, 0)$

```
from numpy import *
from scipy.linalg import null_space
A = array([[1 , 2 , -1], [0 , 1 , 1], [1 , 1 , -2]])
rank = linalg.matrix_rank(A)
print(" Rank of the matrix:", rank)
ns = null_space( A )
print("\n Null space of the matrix \n", ns)
nullity = ns.shape [1]
print("\n Nullity of T:", nullity)
if rank + nullity == A.shape[1]:
    print("\n Rank - nullity theorem holds .")
else :
    print("\n Rank - nullity theorem does not hold .")
```

Course outcome

- Apply the principles of linear transformations and verify properties such as rank, nullity, and invertibility to solve complex mathematical problems and interpret their significance

PRACTICE QUESTION BANK

Linear Transformation,singular,non-singular transformation ,inverible:

1. Let $T: R^3 \longrightarrow R^3$ such that $T(x,y,z) = (2x-3y,\ x+4,\ 5z)$. Prove that T is not a linear transformation.
2. Let $T: R^3 \longrightarrow R^2$ such that $T(x,y,z) = (x+y,\ y+z)$. Prove that T is a linear transformation.
3. Prove that the transformation $T: R^3 \longrightarrow R^3$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)$ is linear.
4. Let $T: R^2 \longrightarrow R^2$ such that $T(x,\ y) = (x+y,\ x)$. Prove that T is a linear transformation
5. Let P_n be the vector space of real polynomial functions of degree $\leq n$. Show that the transformation $T : P_2 \rightarrow P_1$, defined by $T(ax^2 + bx + c) = (a+b)x + c$ is linear.
6. Prove that the transformation $T: R^2 \longrightarrow R^2$ defined by $T(x,y) = (3x, x+y)$ is linear. Find the images of the vectors $(1,3)$ and $(-1,2)$ under this transformation
7. Which of the following functions are linear transformations:
   i) $T: R^3 \longrightarrow R^3$ defined by $T(x,y,z) = (y, -x, -z)$
   ii) $T: R^3 \longrightarrow R^2$ defined by $T(x,y,z) = (2x-3y, 7y+2z)$
   iii) $f: V_2(R) \longrightarrow V_2(R)$ defined by $f(x,y) = (x+6, y+2)$
   iv) $I: V(R) \longrightarrow V_2(R)$ defined by $I(x) = (3x, 5x)$.
8. Identify the non-singular /singular linear maps from the following linear maps:
   i) $F: R^2 \longrightarrow R^3$ defined by $G(x,y) = (x+y, x-2y, 3x+y)$.
   ii) $G: R^2 \longrightarrow R^2$ defined by $G(x,y) = (2x-4y, 3x-6y)$.
   iii) $F: R^2 \longrightarrow R^3$ defined by $G(x,y) = (x-y, x-2y)$.
   iv) $G: R^2 \longrightarrow R^2$ defined by $G(x,y) = (2x-4y, 3x-6y)$.
9. Show that the transformation $F: R^2 \longrightarrow R^2$ defined by $F(a,b) = (2a-4b, 3a-6b)$ is singular and find its Kernel
10. The linear operator $G: R^2 \longrightarrow R^3$ defined by $G(x,y) = (x+y, x-2y, 3x+y)$. Show that G is non-singular.
11. Consider the linear operator T on R^3 defined by $T(x,y,z,) = (2x,\ 4x-y,\ 2x+3y-z)$. Show that T is invertible and also find formula for T^{-1}.
12. Show that the linear operator $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (x+z, x-y, y)$ is invertible and also find T^{-1}.
13. Show that the linear operator T on R^3 is invertible and find a formula for T^{-1} where $T(x,y,z) = (x-3y-2z, y-4z, z)$.
14. Construct the matrix of transformation $T: R^2 \longrightarrow R^3$, given that $T(-1,\ 1) = (-1,\ 0,\ 2)$ and $T(2,\ 1) = (1,\ 2,\ 1)$.
15. Construct the linear transformation, $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, whose matrix relative to basis $\mathcal{B}, \mathcal{B}' = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$.
    Where $\mathcal{B} = \{(1,\ 1),\ (0,\ 2)\}$, $\mathcal{B}' = \{(0,\ 1,\ 1),\ (1,\ 0,\ 1),\ (1,\ 1,\ 0)\}$.
16. Construct the 2×2 matrix of transformation A that maps $(1,\ 3)^T$ and $(1,\ 4)^T$ into $(-2,\ 5)^T$ and $(3,-1)^T$ respectively.

Range and Null Spaces – Rank-Nullity theorem

1. Find the range space and null space of linear transformation $T: R^3 \longrightarrow R^2$ defined $T(x,y,z) = (y-x,\ y-z)$.
2. Apply rank-nullity to find the dimension of range space and null space of transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$.
3. Let $T: R^4 \longrightarrow R^3$ be the linear mapping defined by $T(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$. Make use of Rank-nullity to find a basis and the dimension of (a) the image of T (b) the kernel of T.
4. Identify the kernel and range of the of the transformation $T: R^3 \longrightarrow R^3$ defined by $T(x,y,z) = (x, y, 0)$.
5. Find the kernel and range of the linear operator $T(x,y,z) = (x+y, z)$ of $R^3 \longrightarrow R^2$.
6. Apply the rank-nullity theorem to verify it for the transformation $T: V \longrightarrow W$ defined by $T(x,y,z) = (x+y, x-y, 2x+z)$
7. Make use of rank-nullity theorem to verify $\dim(R^2) = rank(T) + nullity(T)$ for the linear transformation

$T: R^2 \longrightarrow R^3$ such that $T(x, y) = (x - y, y - x, -x)$

8. Apply the rank-nullity theorem to verify it for the linear transformation $T: V_3(R) \longrightarrow V_2(R)$ defined by $T(x, y, z) = (y - x, y - z)$.
9. Verify the rank nullity theorem for the transformation $T: R^4 \longrightarrow R^3$ defined by $T(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$.
10. Verify the rank nullity theorem for for the linear transformation $T: V_3(R) \longrightarrow V_2(R)$ defined by $T(x, y, z) = (y - x, y + z)$
11. Let $T: V_3(R) \longrightarrow V_2(R)$ Find dimensions of range space and null space of the transformation $T: R^3 \longrightarrow R^2$ defined by $T(x, y, z) = (y - x, y - z)$.
12. Let $T: V \longrightarrow W$ be a linear transformation defined by $T(x, y, z) = (x + y, x - y, 2x + z)$. Find the range, null space, rank, nullity and hence verify the rank-nullity theorem.