Module-3: System of linear equations, Eigenvalues and Eigenvectors
Elementary row transformation of a matrix, Echelon form, rank of a matrix. Consistency and solution of system of linear equations: Gauss elimination method, Gauss Jordan method. Applications: Traffic flow. Eigenvalues and Eigenvectors, diagonalization of the matrix, modal matrix. **(RBT Levels: L1, L2 and L3)** **Textbook-1:** Chapter 2: Sections 2.7,2.10,2.13,2.14,2.16 Chapter 28: Sections 28.6(1,2) Textbook-2: Chapter-7: Section 7.3

LECTURE 1:

Elementary row transformation of a matrix, Echelon form, Rank of a matrix

Recall:

1. What is a Matrix and their types.
2. What are the properties of a Matrix?
3. What is Elementary row transformation of a matrix?
4. What is a Minor and order of a Matrix?

Elementary transformation of a matrix:

1. The interchange of any two rows (columns)
2. The multiplication of any row (column) by a non-zero number.
3. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column)

Two matrices A and B are said to be **equivalent** if one can be obtained from the other by a sequence of Elementary transformation. Equivalent matrices are denoted by $A \sim B$.
A matrix is obtained from the unit matrix by any one of the elementary transformations is called **Elementary matrix.**

Rank: A matrix is said to be of rank $\boldsymbol{r}$, if it has at least one nonzero minor of order r and every minor of order higher then r vanishes. Rank of A is denoted by $\rho(A)$.

Note: 1. If a matrix has nonzero minor of order r , then its rank is $\geq r$.
2. If all the minors of order $r+1$ are zero, then its rank is $\leq r$.
3. Elementary transformations do not change the rank of a matrix.

Echelon Form: A rectangular matrix is in echelon form if,
1. All nonzero rows are above any zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Row reduced Echelon Form: An echelon form is said to be row reduced if, the leading entry in each nonzero row is 1 and each leading 1 is the only nonzero entry in its column.

If a matrix $\boldsymbol{A}$ is equivalent to an echelon matrix $\boldsymbol{E}$, then $\boldsymbol{\rho(A)} =$ Number of nonzero rows in $\boldsymbol{E}$.

Examples: Find the rank of the following matrix:
Solution:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ 2. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ 3. $\begin{bmatrix} 90 & 91 & 92 & 93 & 94 \\ 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \end{bmatrix}$

Solutions: 1. $\begin{bmatrix}1&2&3\\1&4&2\\2&6&5\end{bmatrix}\xrightarrow[R_3=R_3-2R_1]{R_2=R_2-R_1}\begin{bmatrix}1&2&3\\0&2&-1\\0&2&-1\end{bmatrix}\xrightarrow{R_3=R_3-R_2}\begin{bmatrix}1&2&3\\0&2&-1\\0&0&0\end{bmatrix}$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore\ \boldsymbol{\rho(A)=2}$ **.**

2. $\begin{bmatrix}2&3&-1&-1\\1&-1&-2&-4\\3&1&3&-2\\6&3&0&-7\end{bmatrix}\xrightarrow{R_4=R_4-(R_1+R_2+R_3)}\begin{bmatrix}2&3&-1&-1\\1&-1&-2&-4\\3&1&3&-2\\0&0&0&0\end{bmatrix}$

$\xrightarrow[R_3=2R_3-3R_1]{R_2=-(2R_2-R_1)}\begin{bmatrix}2&3&-1&-1\\0&5&3&7\\0&-7&9&-1\\0&0&0&0\end{bmatrix}\xrightarrow{R_3=5R_3+7R_2}\begin{bmatrix}2&3&-1&-1\\0&5&3&7\\0&0&66&44\\0&0&0&0\end{bmatrix}$

Clearly reduced matrix is in echelon form with 3 nonzero rows. $\therefore\ \boldsymbol{\rho(A)=3}$ **.**

3. $\begin{bmatrix}90&91&92&93&94\\91&92&93&94&95\\92&93&94&95&96\\93&94&95&96&97\\94&95&96&97&98\end{bmatrix}\xrightarrow[\substack{R_3=R_3-R_2\\R_2=R_2-R_1}]{\substack{R_5=R_5-R_4\\R_4=R_4-R_3}}\begin{bmatrix}90&91&92&93&94\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{bmatrix}$

$\xrightarrow[R_3=R_3-R_2]{\substack{R_5=R_5-R_2\\R_4=R_4-R_2}}\begin{bmatrix}90&91&92&93&94\\1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{R_2=-(90R_2-R_1)}\begin{bmatrix}90&91&92&93&94\\0&1&2&3&4\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore\ \boldsymbol{\rho(A)=2}$ **.**

Review:

1. What is an Elementary matrix?
2. What are the Elementary transformations of a matrix?
3. What is a Rank of a matrix?
4. What is Echelon and row reduced echelon form.

LECTURE 2:
Rank of a matrix –Problems and Consistency of system of linear equations

Recall:

1. What are the three types of elementary row operations? Provide an example of each.
2. How does applying an elementary row transformation affect the determinant of a matrix?
3. Can the rank of a matrix change when applying row transformations? Justify your answer.
4. When adding a multiple of one row to another, does it alter the rank of the matrix?

Problems: Find the rank of the following matrix:

4. $\begin{bmatrix}0&1&-3&-1\\1&0&1&1\\3&1&0&2\\1&1&-2&0\end{bmatrix}$ 5. $\begin{bmatrix}2&1&-1&3\\1&2&4&3\\3&6&12&9\\3&3&3&6\end{bmatrix}$

Solution:

4. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} R_1 \leftrightarrow R_4 \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

$$\xrightarrow[R_3 = -\frac{1}{2}(R_3 - 3R_1)]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \xrightarrow[R_3 = R_3 + R_2]{R_4 = R_4 + R_2} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore\ \boldsymbol{\rho(A) = 2}$.

5. . $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix} R_1 \leftrightarrow R_4 \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & -1 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$

$$\xrightarrow[R_4 = -\frac{1}{3}(R_4 - 3R_1)]{R_2 = -\frac{1}{3}(R_2 - 2R_1),\ R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 = \ (R_4 - R_2)} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore\ \boldsymbol{\rho(A) = 2}$.

<u>Consistency of Homogeneous linear equations, $AX = 0$</u> :

$X = 0$ is the trivial solution. Thus the homogeneous system is always consistent.

Note: 1. If $\rho(A) =$ number of unknowns, then the system has only trivial solution.

2. If $\rho(A) <$ number of unknowns, then the system has an infinite number of solutions.

<u>Consistency of non-homogeneous linear equations, $AX = B$</u> :

1. If $\rho(A) = \rho(A|B) =$ number of unknowns, then the system has unique solution.
2. If $\rho(A) = \rho(A|B) <$ number of unknowns, then the system has an infinite number of solutions.
3. If $(A) \neq \rho(A|B)$, then system has no solution.

<u>Examples</u>:

1. Test for consistency and solve the system $x + 4y + 3z = 0, \quad x - y + z = 0, \ 2x - y + 3z = 0$.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3=-(R_3-2R_1)]{R_2=-(R_2-R_1)} \left[\begin{array}{ccc|c}1 & 4 & 3 & 0\\ 0 & 5 & 2 & 0\\ 0 & 9 & 3 & 0\end{array}\right]$$

$$\xrightarrow{R_3=5R_3-9R_2} \left[\begin{array}{ccc|c}1 & 4 & 3 & 0\\ 0 & 5 & 2 & 0\\ 0 & 0 & -3 & 0\end{array}\right]$$

Clearly $\rho(A)=\rho(A|B)=3=$ number of unknowns, the system has unique solution that is trivial. $x=y=z=0.$

2. Test for consistency and solve $5x+3y+7z=4,\ 3x+26y+2z=9,\ \ 7x+2y+10z=5$

Solution: Augmented matrix $[A|B]=\left[\begin{array}{ccc|c}5 & 3 & 7 & 4\\ 3 & 26 & 2 & 9\\ 7 & 2 & 10 & 5\end{array}\right]$

$$\xrightarrow[R_3=-(5R_3-7R_1)]{R_2=\frac{1}{11}(5R_2-3R_1)} \left[\begin{array}{ccc|c}5 & 3 & 7 & 4\\ 0 & 11 & -1 & 3\\ 0 & 11 & -1 & 3\end{array}\right]$$

$$\xrightarrow{R_3=R_3-R_2} \left[\begin{array}{ccc|c}5 & 3 & 7 & 4\\ 0 & 11 & -1 & 3\\ 0 & 0 & 0 & 0\end{array}\right]$$

Clearly $\rho(A)=\rho(A|B)=2<$ number of unknowns, the system has infinite solutions.

Let $z=k \implies 11y-z=3 \implies \quad y=\frac{k+3}{11}$ and $5x+3y+7z=4 \implies x=\frac{4-7k-3\frac{k+3}{11}}{5}=\frac{35-74k}{55}$.

Review:

1. What is the trivial solution of the homogeneous system AX=0.
2. If the Rank of a matrix is equal to the number of unknowns, what type of solution does the homogeneous system have?
3. If the Rank of a matrix is less than the number of unknowns, what type of solution does the homogeneous system have?
4. Give the condition for nonhomogeneous system to be consistent.

LECTURE 3:
Consistency and Solution of system of linear equations

Recall:

1. State the rank condition for a unique solution, no solution and infinitely many solutions?
2. What is an augmented matrix?
3. When does a homogeneous system have only the trivial solution?
4. Define the term Echelon form of a matrix?
5. What is the effect of applying elementary row operations on the rank of a matrix?

3. For what values of λ and μ do the system of equations: $x+y+z=6,\quad x+2y+3z=10,$ $x+2y+\lambda z=\mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 10\\ 1 & 2 & \lambda & \mu\end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

(i) If $(A) \neq \rho(A|B)$, then the system has no solution.

If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$ then $\rho(A) = 2 \neq \rho(A|B) = 3$.

Therefore, if $\lambda = 3$ and $\mu \neq 10$ then the system has no solution

(ii) If $\rho(A) = \rho(A|B) =$ number of unknowns, then the system has unique solution.

If $\lambda - 3 \neq 0$ and for any value of μ, $\rho(A) = \rho(A|B) = 3 =$ number of unknowns.

Hence for $\lambda \neq 3$, the system has unique solution.

(iii) If $\rho(A) = \rho(A|B) <$ number of unknowns, then the system has an infinite number of solutions.

If $\lambda - 3 = 0$ and $\mu - 10 = 0$ then $\rho(A) = 2 = \rho(A|B) < 3$.

Therefore if $\lambda = 3$ and $\mu = 10$ then the system has an infinite number of solutions.

4. Show that if $\lambda \neq -5$, the system $3x - y + 4z = 3,\quad x + 2y - 3z = -2,\quad 6x + 5y + \lambda z = -3$ have a unique solution. Find the solution if $\lambda = -5$.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 6R_1]{R_2 = R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda + 18 & 9 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda + 5 & 0 \end{array}\right]$$

Clearly if $\lambda + 5 \neq 0$, $\rho(A) = \rho(A|B) = 3 =$ number of unknowns.

Therefore if $\lambda \neq -5$ then the system has unique solution.

if $\lambda = -5$ then $\rho(A) = 2 = \rho(A|B) < 3$, the system has an infinite number of solutions.

$x + 2y - 3z = -2$ and $-7y + 13z = 9 \Rightarrow y = \frac{13z-9}{7}$, $x = -2 - 2\left(\frac{13z-9}{7}\right) + 3z = \frac{4-5z}{7}$

Therefore solutions are $\begin{pmatrix} \frac{4-5z}{7} \\ \frac{13z-9}{7} \\ z \end{pmatrix}$ for any value of z.

Review:

1. Write a condition for the homogeneous system AX=0 to have a non-trivial solution.
2. What does it mean when the rank of a matrix is equal to the number of unknowns?
3. For what value of k is the system is consistent: $x + 2y + z = 3, 2x + 4y + kz = 6$.
4. Give the condition for nonhomogeneous system to be consistent.

LECTURE 4:
Gauss-elimination method

Recall:

1. When we say that the system of equation $AX = B$ has a unique solution?
2. What is the condition for system of equation $AX = B$ to have infinitely many solutions?

3. If the Rank of a matrix is less then the number of unknowns, what type of solution does the homogeneous system have?
4. What is the maximum rank a m×n matrix can have?
5. What is the condition for the system of equation to have no solution?

Solution of linear simultaneous equations:

1. **Gauss elimination method:**

Consider the equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$.
Reduce augmented matrix into an upper triangular matrix as below

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\xrightarrow[R_3 = a_1R_3 - a_3R_1]{R_2 = a_1R_2 - a_2R_1} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & b'_3 & c'_3 & d'_3 \end{array}\right]$$

$$\xrightarrow{R_3 = b'_2R_3 - b'_3R_2} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

Then $z = \frac{d''_3}{c''_3}$, $y = \frac{d'_2 - zc'_2}{b'_2}$, $x = \frac{d_1 - yb_1 - zc_1}{a_1}$

Example: 1. Solve by Gauss elimination method,
$2x - 3y + z = -1$, $x + 4y + 5z = 25$, $3x - 4y + z = 2$.
Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 1 & 4 & 5 & 25 \\ 3 & -4 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - 3R_1]{R_2 = 2R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 1 & -1 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 = 11R_3 - R_2} \left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 0 & -20 & 26 \end{array}\right]$$

$\therefore$ $z = -\frac{26}{20} = -1.3$, $y = \frac{51 - 9\times(-1.3)}{11} = 5.7$ and $z = \frac{-1+3\times5.7+1.3}{2} = 8.7$.

2. Solve by Gauss elimination method, $2x + 3y + z = -1$, $x - y + z = 6$, $3x + 2y - z = -4$.

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 2 & 3 & 1 & -1 \\ 3 & 2 & -1 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 5 & -1 & -13 \\ 0 & 5 & -4 & -22 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 5 & -1 & -13 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$\therefore$ $-3z = -9$, $5y - z = -13$, $x - y + z = 6 \implies$ $z = 3$, $y = -2$, and $x = 1$.

Review:

1. What is the primary goal of the Gauss Elimination method?
2. In the Gauss Elimination method, what is the first step typically performed on the augmented matrix.
3. Discuss how Gauss- elimination can fail if the pivot element is zero. How is this issue resolved?
4. In Gauss- elimination, does the rank of a matrix change during row operations?

TUTORIAL 1:

Problems on rank of a matrix ,test for consistency and Gauss elimination method

1. Find the rank of the following matrix:

(i) $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 21 & 22 & 23 & 24 \\ 22 & 23 & 24 & 25 \\ 23 & 24 & 25 & 26 \\ 24 & 25 & 26 & 27 \end{bmatrix}$

2. Test for consistency and solve the system

(i) $x + y + z = 3, \quad 2x - y + 3z = 10, \; 4x + y + 5z = 16$.

(ii) $x + y + z = 3, \quad 2x + y + 3z = 5, \; x + 2y = 3$.

3. For what values of λ and μ do the system of equations: $2x + 3y + 5z = 9, \; 7x + 3y - 2z = 8,$ $2x + 3y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.

4. Applying Gauss elimination method solve

1. $2x + 3y - z = 5, \quad 4x + 4y - 3z = 3, \; 2x - 3y + 2z = 2.$

2. $x + y + z = 6, \quad x - 2y + 3z = 8, \; 2x + y - z = 3.$

TUTORIAL 2:

Lab Activity 4: Finding rank, reduced echelon form and Test for Consistency

Objectives:

Use python

- To find rank of a matrix.
- To find solution of system of equations numerically.
- To test for consistency

1. Find the rank of a. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

```
from numpy import *
A= matrix ([[1 ,2 ,3] ,[1 ,4,2] ,[2 ,6 ,5]])
r= linalg . matrix_rank (A)
print("rank of matrix A",r)
```

2. Check whether the following system of homogeneous linear equation has non-trivial solution.

a. $x + 4y + 3z = 0, \quad x - y + z = 0, \; 2x - y + 3z = 0$

b. $x + 2y - z = 0, \quad 2x + y + 4z = 0, \; x - y + 5z = 0$

```
from numpy import *
A= matrix ([[1 ,4 , 3] ,[1 ,-1 ,1] ,[2 ,-1 ,3]])
B= matrix ([[0] ,[0] ,[0]])
r= linalg . matrix_rank (A)
n=A. shape [1]
print(n)
print(”rank of matrix A",r)
if (r==n):
    print (" System has trivial solution ")
 else :
     print (" System has ", n-r, " non - trivial solution (s)")
```

3. Examine the consistency of the following system of equations and solve if consistent
   a. $x+2y-z=1, \quad 2x+y+5z=2, \quad 3x+3y+4z=1$
   b. $x+2y-z=1, \quad 2x+y+4z=2, \quad 3x+3y+4z=2.$

```
from numpy import *
A= matrix ([[1 ,2 , -1] ,[2 ,1 ,5] ,[3 ,3 ,4]])
B= matrix ([[1] ,[2] ,[1]])
AB= concatenate ((A,B), axis =1)
 display (AB)
rA= linalg . matrix_rank (A)
print(”rank of matrix A",rA)
rAB= linalg . matrix_rank (AB)
 print(”rank of matrix A",rAB)
 n=A. shape [1]:
 if (rA==rAB):
        if (rA==n):
                print (" System has unique solution ")
                print (linalg . solve (A,B))
        else :
                 print (" System has infinitely many solutions ")
 else :
                 print (" The System of equation is inconsistent")
```

LECTURE 5:
Gauss-Jordan method

Recall:

1. Define the role of pivot elements in Gaussian elimination.
2. In the Gauss Elimination method, what is the first step typically performed on the augmented matrix.
3. State the condition under which Gauss-elimination can be applied to solve a linear system.
4. What is the main objective of the Gauss-elimination method?

2. **Gauss Jordan method**:

Reduce augmented matrix into a diagonal matrix as below

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\xrightarrow[R_3 = a_1R_3 - a_3R_1]{R_2 = a_1R_2 - a_2R_1} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & b'_3 & c'_3 & d'_3 \end{array}\right]$$

$$\xrightarrow[R_3 = b'_2R_3 - b'_3R_2]{R_1 = b'_2R_1 - b_1R_2} \left[\begin{array}{ccc|c} a'_1 & 0 & c'_1 & d'_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

$$\xrightarrow[R_2 = c''_3R_2 - c'_2R_3]{R_1 = c''_3R_1 - c'_1R_3} \left[\begin{array}{ccc|c} a''_1 & 0 & 0 & d''_1 \\ 0 & b''_2 & 0 & d''_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

Then $x = \frac{d''_1}{a''_1}$, $y = \frac{d''_2}{b''_2}$ and $z = \frac{d''_3}{c''_3}$.

Example:

1. Solve by Gauss Jordan method, $2x - y + 3z = 1, \quad -3x + 4y - 5z = 0, \quad x + 3y - 6z = 0.$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ -3 & 4 & -5 & 0 \\ 1 & 3 & -6 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - R_1]{R_2 = 2R_2 + 3R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 5 & -1 & 3 \\ 0 & 7 & -15 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 = 5R_3 - 7R_2]{R_1 = 5R_1 + R_2} \left[\begin{array}{ccc|c} 10 & 0 & 14 & 8 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & -68 & -26 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 5 & 0 & 7 & 4 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & 34 & 13 \end{array}\right]$$

$$\xrightarrow[R_2 = 34R_2 + R_3]{R_1 = 34R_1 - 7R_3} \left[\begin{array}{ccc|c} 170 & 0 & 0 & 45 \\ 0 & 170 & 0 & 115 \\ 0 & 0 & 34 & 13 \end{array}\right]$$

$\therefore \quad x = \frac{45}{170} = \frac{9}{34} = 0.2647$, $\quad y = \frac{115}{170} = \frac{23}{34} = 0.6765$ and $\quad z = \frac{13}{34} = 0.3824.$

2. Solve by Gauss Jordan method, $2x + y + z = 10, \quad 3x + 2y + 3z = 18, \quad x + 4y + 9z = 16.$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - R_1]{R_2 = 2R_2 - 3R_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{array}\right]$$

$$\xrightarrow[R_3 = -\frac{1}{4}(R_3 - 7R_2)]{R_1 = \frac{1}{2}(R_1 - R_2)} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - 3R_3]{R_1 = R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\therefore \quad x = 7$, $\quad y = -9$ and $\quad z = 5.$

Review:

1. How does the Gauss-Jordan method differ from Gauss- elimination?
2. What is the primary advantage of converting a matrix to reduced row echelon form using the Gauss-Jordan method?

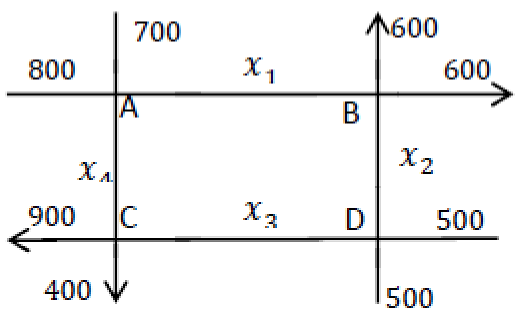
3. What are the additional steps in Gauss-Jordan compared to Gauss-elimination?
4. State one real-world application of the Gauss-Jordan method.

LECTURE 6:
Application: Traffic flow

Recall:

1. Does the Gauss–Jordan method require back substitution? Give a reason.
2. State one difference between the Gauss–Jordan and Gauss–Elimination methods.
3. What is the basic requirement on the rank of a matrix for a unique solution to exist?
4. What happens if the rank of the coefficient matrix is less than the rank of the augmented matrix?
5. What is meant by consistent and inconsistent systems of equations?

Mathematical Model:
A system of linear equations was used to analyze the flow of traffic for a network of four one-way streets.
The pioneering work done by Gareth Williams on Traffic flow has led to greater understanding of this research.
The variables x_1,x_2, x_3, and x_4 represent the flow of the traffic between the four intersections in the network.
The data was obtained by counting the number of vehicles that travelled around the four one-way streets between different time intervals during the mid-week peak traffic hours.
The arrows in the diagram indicate the direction of flow of traffic in and out of the network that is measured in terms of number of vehicles per hour (vph). The diagram in figure below describes the four one-way streets:



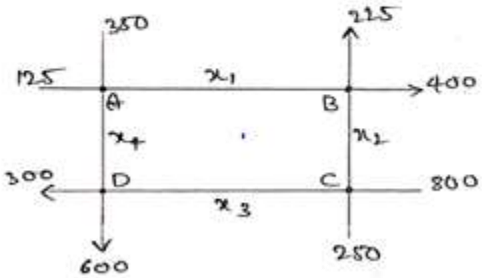
Model Assumptions:

The following assumptions were made in order to ensure the smooth flow of the traffic:
i) Vehicles entering each intersection should always be equal to the number of vehicles leaving the intersection.
ii) The streets must all be one-way with the arrows indicating the direction of traffic flow.
The system of equations for the model was formulated and solved by Gauss–Jordan method:
Note:

1. Traffic network representation: Traffic networks can be described by directed networks.
2. Intersections as nodes: Intersections correspond to the nodes of the network.
3. Roads as directed edges: Roads correspond to the directed edges or branches of the network, with traffic flowing along these branches.
4. Flow conservation at intersections: All traffic entering an intersection must leave that intersection ("At each node, the flow in is equal to the flow out").
5. Total flow balance: The total traffic entering the network equals the total traffic leaving the network.

Problems:

1. The flow of traffic through a network of streets is shown below:



i) Find x_1, x_2, x_3, x_4 to balance the traffic flow.
ii) Find the traffic flow at $x_4 = 0$.
iii) Find the traffic flow at $x_4 = 100$.

Solution:

Nodes	Flow in	Flow out	Equations
A	125+350	$x_1 + x_4$	$x_1 + x_4 = 475$
B	$x_1 + x_2$	400+225	$x_1 + x_2 = 625$
C	800+250	$x_2 + x_3$	$x_2 + x_3 = 1050$
D	$x_3 + x_4$	300+600	$x_3 + x_4 = 900$

The augmented matrix for the linear equations is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 1 & 1 & 0 & 0 & 625 \\ 0 & 1 & 1 & 0 & 1050 \\ 0 & 0 & 1 & 1 & 900 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 1 & 0 & 1050 \\ 0 & 0 & 1 & 1 & 900 \end{array}\right] R_3 = R_3 - R_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 1 & 1 & 900 \end{array}\right]$$

$R_4 = R_4 - R_3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Clearly $\rho(A) = \rho(A|B) = 3 < 4$(number of unknowns), the system has infinite solutions.

Rewriting the equations, we have

$x_1 + x_4 = 475 \Longrightarrow x_1 = 475 - x_4$ ----- (1)

$x_2 - x_4 = 150 \Longrightarrow x_2 = 150 + x_4$ ----- (2)

$x_3 + x_4 = 900 \Longrightarrow x_3 = 900 - x_4$ ----- (3)

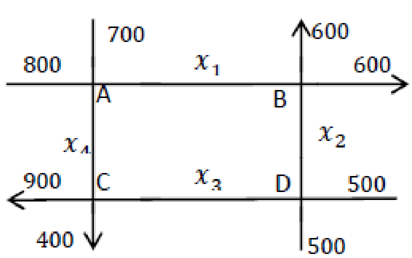
Here, x_4 is a free variable. From (1) we have $x_4 \le 475$, from (2), $x_2 = 150 + x_4 \Longrightarrow x_4 \ge -150 \Longrightarrow x_4 \ge 0$.
Also, from (3), $x_3 = 900 - x_4 \Longrightarrow x_4 \le 900$

$\therefore 0 \le x_4 \le 475$

ii) When $x_4 = 0$.
From (1), (2) and (3) we get $x_1 = 475, x_2 = 150$, $x_3 = 900$.

ii) When $x_4 = 100$.
From (1), (2) and (3) we get $x_1 = 375, x_2 = 250$, $x_3 = 800$.

2. Balance the traffic flow at each intersection:



Solution:

Nodes	Flow in	Flow out	Equations
A	700+800	$x_1 + x_4$	$x_1 + x_4 = 1500$
B	$x_1 + x_2$	600+600	$x_1 + x_2 = 1200$
C	500+500	$x_2 + x_3$	$x_2 + x_3 = 1000$
D	$x_3 + x_4$	400+900	$x_3 + x_4 = 1300$

The augmented matrix for the linear equations is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 1 & 1 & 0 & 0 & 1200 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 1300 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -300 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 1300 \end{array}\right] R_3 = R_3 - R_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -300 \\ 0 & 0 & 1 & 1 & 1300 \\ 0 & 0 & 1 & 1 & 1300 \end{array}\right]$$

$R_4 = R_4 - R_3$ $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -300 \\ 0 & 0 & 1 & 1 & 1300 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Clearly $\rho(A) = \rho(A|B) = 3 < 4$(number of unknowns), the system has infinite solutions.

Rewriting the equations, we have

$x_1 + x_4 = 1500 \Longrightarrow x_1 = 1500 - x_4$ ----- (1)

$x_2 - x_4 = -300 \Longrightarrow x_2 = -300 + x_4$ ----- (2)

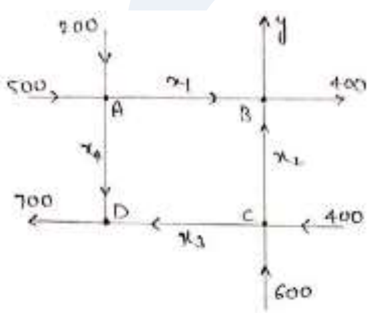
$x_3 + x_4 = 1300 \Longrightarrow x_3 = 1300 - x_4$ ----- (3)

Here, x_4 is a free variable. (1)$\Longrightarrow x_4 \leq 1500$, (2) $\Longrightarrow x_4 \geq 300$ and (3)$\Longrightarrow x_4 \leq 1300$

Also, from (3), $x_3 = 900 - x_4 \Longrightarrow x_4 \leq 900$

$\therefore 300 \leq x_4 \leq 1300$

3.



i. What flow to the north should the traffic light of the intersection B let through to balance the traffic flow in this network?

ii. Assuming that the traffic light at B has been set to balance the total flow in and flow out, what should be the flow is through the inner branches?

Solution: Total flow in: $500 + 200 + 400 + 600$

Total flow out: $y + 400 + 700$

To balance the total flow, we have $500 + 200 + 400 + 600 = y + 400 + 700 \Longrightarrow y = 600.$

Now let's balance the network at each intersection assuming $y = 600$.

Nodes	Flow in	Flow out	Equations
A	200+500	$x_1 + x_4$	$x_1 + x_4 = 700$
B	$x_1 + x_2$	600+400	$x_1 + x_2 = 1000$
C	400+600	$x_2 + x_3$	$x_2 + x_3 = 1000$
D	$x_3 + x_4$	700	$x_3 + x_4 = 700$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 700 \\ 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 700 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \end{array}\right] R_3 = R_3 - R_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 700 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 1 & 1 & 700 \end{array}\right]$$

$$R_4 = R_4 - R_3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 700 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Rewriting into the linear equations, we have

$x_1 + x_4 = 700 \Rightarrow x_1 = 700 - x_4$ ----- (1)

$x_2 - x_4 = 300 \Rightarrow x_2 = 300 + x_4$ ----- (2)

$x_3 + x_4 = 700 \Rightarrow x_3 = 700 - x_4$ ----- (3)

Here, x_4 is a free variable. (1)$\Rightarrow x_4 \le 700$, (2) $\Rightarrow x_4 \ge 0$ and (3)$\Rightarrow x_4 \le 700$

Since, all flow must be non-negative.

$\therefore 0 \le x_4 \le 700$

Review:

1. What are the main assumptions made in the traffic flow model?
2. Write the basic condition for traffic balance at an intersection.
3. What mathematical method is used to solve the system of equations in this model?
4. What is meant by a directed network in the context of traffic flow?

LECTURE 7:
Eigenvalues and Eigenvectors

Recall:

1. Explain how a system of linear equations can represent a network of one-way streets.
2. Describe how the Gauss–Jordan method helps determine unknown traffic flows in the network.
3. Explain why the sum of all inflows and outflows in the traffic network must be equal.
4. Analyze how the Gauss–Jordan method eliminates variables systematically to obtain a reduced form.

Characteristic equation: $|A - \lambda I| = 0$ is the characteristic equation of the square matrix A. Roots are called **Characteristic roots** or **eigenvalues** or **latent roots** of A.

Any vector X satisfying $[A - \lambda I]X = 0$ is called **eigenvector** corresponding to the eigenvalue.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then Characteristic equation is $\lambda^2 - (a+d)\lambda + (ad - cb) = 0$.

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then Characteristic equation is

$$\lambda^3 - (a_1 + b_2 + c_3)\lambda^2 + (sum\ of\ the\ minors\ of\ a_1, b_2 \& c_3)\lambda - |A| = 0 .$$

Properties of eigenvalues:

1) The sum of the eigenvalues of a matrix is the sum of the principal diagonal elements.
2) The product of the eigenvalues of a matrix is equal to its determinant.
3) If λ is the eigenvalue of A , then $1/\lambda$ is eigenvalue of A^{-1}.
4) If λ is the eigenvalue of an orthogonal matrix, then $1/\lambda$ is also its eigenvalue.

5) If λ is the eigenvalue of A, then λ^n is the eigenvalue of A^n. But eigenvectors are same.

Cayley-Hamilton theorem: Every square matrix satisfies its characteristic equation.

1) Find the eigenvalues and eigenvectors of the following matrices.

i) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution:

i) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\sum D = 1 + 5 + 1 = 7.$

$\sum M\ D = 4 - 8 + 4 = 0$

$|A| = -36$

Characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (\ 7\ )\lambda^2 + (\ \ 0\ )\lambda - (\ -36) = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$$

Roots are $-2,\ 3,\ 6$

$\lambda_1 = -2$

$3x + y + 3z = 0$

$x + 7y + z = 0$

$\Rightarrow 20y = 0$, and $z = -x$

$\therefore X_1 = [1, \quad 0, \quad -1]'$

$\lambda_2 = 3$

$-2x + y + 3z = 0$

$x + 2y + z = 0$

$\Rightarrow \quad y = -z$

$X_2 = [1, \quad -1, \quad 1]'$

$\lambda_3 = 6$

$-5x + y + 3z = 0$

$x - y + z = 0$

$\Rightarrow \quad z = x$

$X_3 = [1, \quad 2, \quad 1]'$

Eigenvalues are $-2,\ 3$ and 6, the corresponding eigenvectors are
$[1, \quad 0, \quad -1]'$, $[1, \quad -1, \quad 1]'$ and $[1, \quad 2, \quad 1]'$ respectively.

ii) Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$\sum D = 8 + 7 + 3 = 18.$

$\sum M\ D = 5 + 20 + 20 = 45$

$|A| = 0.$

Characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (\ 18\ )\lambda^2 + (\ \ 45\ )\lambda - (\ 0) = 0$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$$

Roots are $0,\ 3,\ 15$

$\lambda_1 = 0$

$8x - 6y + 2z = 0$

$-6x + 7y - 4z = 0$

$\Rightarrow 10x - 5y = 0$,

$\Rightarrow \quad y = 2x$

$\therefore X_1 = [1, \quad 2, \quad 2]'$

$\lambda_2 = 3$

$5x - 6y + 2z = 0$

$-6x + 4y - 4z = 0$

$\Rightarrow \quad 4x - 8y = 0$

$\Rightarrow \quad x = 2y$

$X_2 = [2, \quad 1, \quad -2]'$

$\lambda_3 = 15$

$-7x - 6y + 2z = 0$

$-6x - 8y - 4z = 0$

$\Rightarrow \quad -20x - 20y = 0$

$\Rightarrow \quad y = -x$

$X_3 = \left[1, \quad -1, \quad \frac{1}{2}\right]'$

Eigenvalues are $0,\ 3$ and 15, the corresponding eigenvectors are
$[1, \quad 2, \quad 2]'$, $[2, \quad 1, \quad -2]'$ and $[2, \quad -2, \quad 1]'$ respectively.

iii) Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$\sum D = 6 + 3 + 3 = 12.$

$\sum M\ D = 8 + 14 + 14 = 36$

$|A| = 32$

Characteristic equation is $|A - \lambda I| = 0$.

$$\Longrightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow \lambda^3 - (\ 12\ )\lambda^2 + (\ 36\ )\lambda - (\ 32) = 0$$

$$\Longrightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Roots are $2,\ 2,\ 8$

(The sum of the eigenvalues of a matrix is the sum of the principal diagonal elements.

$\lambda_1 = 2$	$\lambda_2 = 2$	$\lambda_3 = 8$
$4x - 2y + 2z = 0$	$4x - 2y + 2z = 0$	$-2x - 2y + 2z = 0$
Or $2x - y + z = 0$	Or $2x - y + z = 0$	$-2x - 5y - z = 0$
Let $y = 0$, and $z = -2x$	Let $z = 0$, and $y = 2x$	$\Longrightarrow 3y + 3z = 0$
$\therefore X_1 = [1,\quad 0,\quad -2]'$	$X_2 = [1,\quad 2,\quad 0]'$	$X_3 = [2,\quad -1,\quad 1]'$

Eigenvalues are $2,\ 2$ and 8 , the corresponding eigenvectors are $[1,\quad 0\quad -2]'$, $[1,\quad 2,\quad 0]'$ and $[2\quad -1\quad 1]'$ respectively.

Review:

1. What is the characteristic equation of a square matrix A.?
2. What does the term λ represent in the characteristic equation?
3. What are the characteristic roots of a matrix also known as?
4. What is an eigenvector of a matrix A.?
5. If $\lambda_1,\ \lambda_2, \ldots\ldots\ \lambda_n$ are the eigenvalues of a matrix A, what is the product of the eigenvalues equal to.
6. What does the Cayley-Hamilton theorem state.

LECTURE 8:

Diagonalization of the matrix, modal matrix

Recall:

1. What is the characteristic equation of a square matrix A.?
2. What does the term λ represent in the characteristic equation?
3. What are the characteristic roots of a matrix also known as?
4. What is an eigenvector of a matrix A.?
5. If $\lambda_1,\ \lambda_2, \ldots\ldots\ \lambda_n$ are the eigenvalues of a matrix A, what is the product of the eigenvalues equal to.
6. What does the Cayley-Hamilton theorem state.

Reduction to the diagonal form: If a square matrix A of order n has n linearly independent eigenvectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Let A be a square matrix of order 3. And let $\lambda_1, \lambda_2,\ \lambda_3$ be its eigenvalues and $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ and $\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding eigenvectors. Then $P^{-1}AP = D$, where $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

The matrix P which diagonalises A is called the ***Modal matrix*** of A and D is ***spectral matrix*** of A.

If $B = P^{-1}AP$ or $A = P^{-1}BP$ for some non-singular matrix P, then A and B are ***similar*** matrices.

If $P^{-1}AP = D$ then $A^n = PD^nP^{-1}$.

Problems:

1. Find the matrix $\boldsymbol{P}$ which transforms the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form, and hence calculate A^4.

Solution: Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\sum D = 1 + 5 + 1 = 7.$

$\sum M \;\; D = 4 - 8 + 4 = 0$

$|A| = -36$

Characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (\;7\;)\lambda^2 + (\;0\;)\lambda - (-36) = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$$

Roots are $-2,\; 3,\; 6$

$\lambda_1 = -2$	$\lambda_2 = 3$	$\lambda_3 = 6$
$3x + y + 3z = 0$	$-2x + y + 3z = 0$	$-5x + y + 3z = 0$
$x + 7y + z = 0$	$x + 2y + z = 0$	$x - y + z = 0$
$\Rightarrow 20y = 0$, and $z = -x$	$\Rightarrow \;\; y = -z$	$\Rightarrow \;\; z = x$
$\therefore X_1 = [1,\;\; 0,\;\; -1]'$	$X_2 = [1,\;\; -1,\;\; 1]'$	$X_3 = [1,\;\; 2,\;\; 1]'$

Eigenvalues are $-2,\; 3$ and 6 , the corresponding eigenvectors are $[1,\;\; 0,\;\; -1]'$, $[1,\;\; -1,\;\; 1]'$ and $[1,\;\; 2,\;\; 1]'$ respectively.

$$\therefore \; P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} . \;\; P^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Since $A^n = PD^nP^{-1}$.

$$A^4 = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 48 & 0 & -48 \\ 162 & -162 & 162 \\ 1296 & 2592 & 1296 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 & -8 \\ 27 & -27 & 27 \\ 216 & 432 & 216 \end{bmatrix} = \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}.$$

2. Diagonalise the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, and hence calculate A^6.

Solution: Characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$$

Eigenvalues are $2,$ and 5 , the corresponding eigenvectors are $[1,\;\; -1]'$, and $[1,\;\; 2]'$ respectively.

$$\therefore \; D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \text{ and } \textbf{\textit{Modal matrix }} P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} . \;\; P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Since $A^n = PD^nP^{-1}$

$$A^6 = PD^6P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 64 & 0 \\ 0 & 15625 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 128 & -64 \\ 15625 & 15625 \end{bmatrix}$$

$$= \begin{bmatrix} 5251 & 5187 \\ 10374 & 10438 \end{bmatrix}$$

Or $A^n = PD^nP^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 5^n \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2^{n+1} & -2^n \\ 5^n & 5^n \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 2^{n+1} + 5^n & 5^n - 2^n \\ 2 \times 5^n - 2^{n+1} & 2 \times 5^n + 2^n \end{bmatrix}$$

$$\therefore \ A^6 = \tfrac{1}{3}\begin{bmatrix} 2^7 + 5^6 & 5^6 - 2^6 \\ 2 \times 5^6 - 2^7 & 2 \times 5^6 + 2^6 \end{bmatrix} = \begin{bmatrix} 5251 & 5187 \\ 10374 & 10438 \end{bmatrix}$$

3. Diagonalise the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Solution: i) Let $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$.

$$\Longrightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow \lambda^3 - (\ 1\ )\lambda^2 + (\ -5)\lambda - (\ -5) = 0$$

$$\Longrightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0 \ \Longrightarrow \ (\lambda^2 - 5)(\lambda - 1) = 0$$

$\sum D = -1 + 2 + 0 = 1.$

$\sum M \ \ D = 1 - 2 - 4 = -5$

$|A| = -5$

Eigenvalues are $-\sqrt{5}$, 1, $\sqrt{5}$, and hence required diagonal matrix is $D = \begin{bmatrix} -\sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$

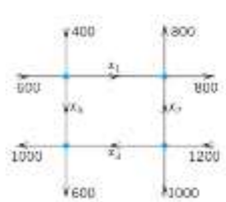
Review:

1. State the condition for a square matrix A to have eigenvalues.
2. What does it mean if an eigenvalue of a matrix is zero?
3. State the eigenvalues of an identity matrix of order n.
4. How many eigenvalues does an n×n matrix have?
5. If λ is an eigenvalue of A, what is an eigenvalue of kA, where k is a scalar?

TUTORIAL 3:

Problems on Gauss-Jordan method ,Traffic flow, eigenvalues, eigenvectors and diagonalization of a matrix

1. Applying Gauss-Jordan method, solve:
   (i). $2x + 3y - z = 5,\quad 4x + 4y - 3z = 3,\ \ 2x - 3y + 2z = 2.$
   (ii). $x + y + z = 6,\quad x - 2y + 3z = 8,\ \ 2x + y - z = 3$
2. Find the Eigen values and Eigen vectors of the following matrices.
   i) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$
3.



   Find the traffic flow (cars per hour) in the net of one-way streets (in the directions indicated by the arrows) shown in the figure. Is the solution unique?
4. Diagonalise the matrix A and hence calculate A^4.
   (i) $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

TUTORIAL 4:

Lab Activity 5:Solving system of linear equations using Gauss elimination method

Objectives:

Use python

- To find the solution of given system, using Gauss-elimination method.

1. Solve by Gauss elimination method,

   a. $2x + 3y + z = -1,\quad x - y + z = 6,\quad 3x + 2y - z = -4.$

   b. $2x - 3y + z = -1,\quad x + 4y + 5z = 25,\quad 3x - 4y + z = 2.$

```
A= matrix ( [[2 , 3 ,1. -1] ,[1 ,-1 ,1,6] ,[3 ,2 ,-1, -4 ]] )
n= 3
for i in range (n):
      p =A[i][i]
     A[i]=[a/p for a in A[i]]
     for k in range (i+1,n):
         f=A[k][i]
         A[k]=[A[k][j] – f * A[i][j] for j in range(n+1)]
print (" Upper Triangular: ")
for r in A:
      print ([ round(x,2) for x in r])
    x=[ 0 ]*n
for i in range (n-1,-1,-1):
     x[I] = A[i][n] - sum(A[i][j] * x[j] for j in range ( i+1,n))
print ("x=%.2f, y=%.2f, z=%.2f" %tuple(X))
```

TUTORIAL 5:

Lab Activity 6: Determine Eigenvalues and Eigenvectors

Objectives:

Use python

- To find eigenvalues and corresponding eigenvectors.

1. Obtain the eigenvalues and eigenvectors for the given matrix.

i) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

```
from numpy import *
A= matrix ([[1 ,1 , 3] ,[1 ,5 ,1] ,[3 ,1 ,1]])
print ( " Given matrix :\n" , A)
u,v= linalg .eig ( A )
print ( "\n Eigenvalues are :\n" , u)
print ("\n Eigenvectors are :\n" , v)
```

Course outcome

- Solve systems of linear equations using direct and iterative methods, and determine eigenvalues and eigenvectors for diagonalization of matrices.

PRACTICE QUESTION BANK

1. Find the Rank of the following Matrices:

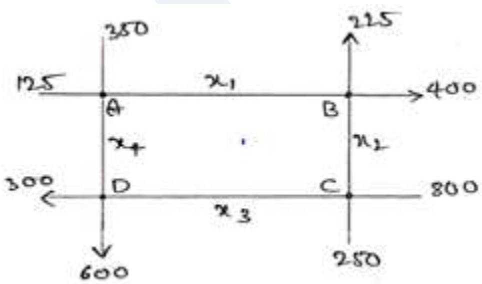
a. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 2 & 5 & 2 \\ 2 & -1 & 1 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$ d. $\begin{bmatrix} 90 & 91 & 92 & 93 & 94 \\ 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \end{bmatrix}$ e. $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$

2. Solve the following system of equations by Gauss elimination method:
   a. $2x + 3y + z = -1$, $x - y + z = 6$, $3x + 2y - z = -4$.
   b. $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$.
   c. $2x + y + 4z = 12$, $4x + 11y - z = 33$, $8x - 3y + 2z = 20$.
   d. $2x - 3y + z = -1$, $x + 4y + 5z = 25$, $3x - 4y + z = 2$.
   e. $2x + 3y - z = 5$, $4x + 4y - 3z = 3$, $2x - 3y + 2z = 2$.
   f. $x + y + z = 6$, $x - 2y + 3z = 8$, $2x + y - z = 3$.

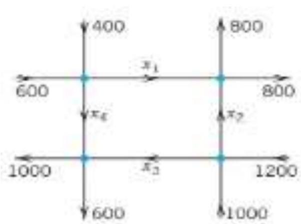
3. Solve the following system of equations by Gauss Jordan method:
   a. $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$.
   b. $x + y + z = 11$, $3x - y + 2z = 12$, $2x + y - z = 3$.
   c. $2x + 5y + 7z = 52$, $2x + y - z = 0$, $x + y + z = 9$.
   d. $2x - y + 3z = 1$, $-3x + 4y - 5z = 0$, $x + 3y - 6z = 0$.
   e. $2x + 3y - z = 5$, $4x + 4y - 3z = 3$, $2x - 3y + 2z = 2$.
   f. $x + y + z = 6$, $x - 2y + 3z = 8$, $2x + y - z = 3$

4. The flow of traffic through a network of streets is shown below:



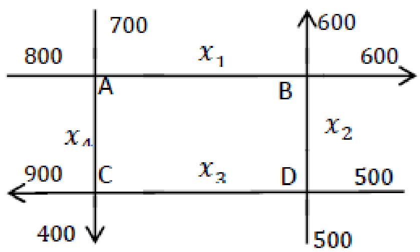
iv) Find x_1, x_2, x_3, x_4 to balance the traffic flow.
v) Find the traffic flow at $x_4 = 0$.
vi) Find the traffic flow at $x_4 = 100$.

5.



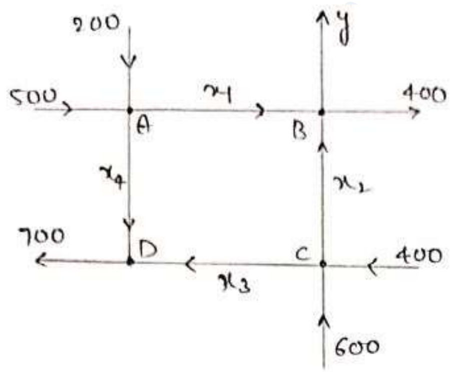
Find the traffic flow (cars per hour) in the net of one-way streets (in the directions indicated by the arrows) shown in the figure. Is the solution unique?

6. Balance the traffic flow at each intersection:



7.

1. What flow to the north should the traffic light of the intersection B let through to balance the traffic flow in this network?
2. Assuming that the traffic light at B has been set to balance the total flow in and flow out, what should be the flow is through the inner branches?



8. Test for consistency and solve the system if consistent:

a. $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$.
b. $x + y + z = 3$, $2x - y + 3z = 10$, $4x + y + 5z = 16$.
c. $x + y + z = 3$, $2x + y + 3z = 5$, $x + 2y = 3$.
d. $x + 2y + 2z = ,$ $2x + y + z = 2$, $3x + 2y + 2z = 3$, $y + z = 0$.

9. Find the eigenvalues and eigenvectors of the following matrices.

i) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

iv) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ v) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ vi) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

9. For what values of λ and μ do the system of equations: $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.

10. For what values of λ and μ do the system of equations: $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.

11. Find the matrix ***P*** which transforms the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form, and hence calculate A^4.

12. Diagonalise the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, and hence calculate A^6.

13. Diagonalise the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

14. Diagonalise the matrix A and hence calculate A^4.

    (i) $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.