Module-2: Vector Calculus
Introduction to Vector Calculus in Computer Science & Engineering. Scalar and vector fields. Gradient, directional derivative, curl and divergence - physical interpretation, solenoidal and irrotational vector fields and scalar potential. Introduction to polar coordinates and polar curves. **Curvilinear coordinates**: Scale factors, base vectors, Cylindrical polar coordinates, Spherical polar coordinates, transformation between Cartesian and curvilinear systems, Orthogonality. **(RBT Levels: L1, L2 and L3)**

LECTURE 1:
Vector Differentiation: Scalar and vector fields:

Recall:

1. What is a Vector?
2. If two vectors are collinear, what can be said about them?
3. What is the result of adding two vectors?
4. What is the result of the dot product of two vectors?
5. What is the magnitude of a vector $A = 3i + 4j$?
6. What is a conitial vector?
7. What type of vector has a magnitude of one that indicates direction?
8. What is zero vectors?
9. What is the characteristic of a position vector?

Prerequisite:

Vectors: Let $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$ and $\vec{c} = c_1 i + c_2 j + c_3 k$.
Then magnitude of $\vec{a}$ is $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
Unit vector in the direction of $\vec{a} = \frac{\vec{a}}{|\vec{a}|}$.
Let A and B are two points, then vector $\overrightarrow{AB}$ =Position vector of $B -$ Position vector of A .

Scalar product (dot product):

$\vec{a} \circ \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{a}|\,|\vec{b}| \cos\theta$, where θ is the angle between $\vec{a}$ & $\vec{b}$.
Note:
1. $\vec{a} \circ \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ i.e. $\vec{a}$ & $\vec{b}$ are orthogonal.
2. Acute angle between the vectors $\vec{a}$ & $\vec{b}$ is $\theta = \cos^{-1}\left|\frac{\vec{a}\circ\vec{b}}{|\vec{a}|\,|\vec{b}|}\right|$
3. Component of $\vec{a}$ in the direction of $\vec{b}$ is $\frac{\vec{a}\circ\vec{b}}{|\vec{b}|}$

Vector product (cross product):

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$$, where $\hat{n}$ is the unit vector $\perp$ to both $\vec{a}$ & $\vec{b}$.

Note:
1. Unit vector $\perp$ to both $\vec{a}$ & $\vec{b}$ is $\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.
2. Unit vector $\perp$ to the triangle ABC is $\hat{n} = \frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}$.
3. Sine of angle between $\vec{a}$ & $\vec{b}$ is $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$.

Scalar triple product (Box product):

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \circ (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}].$$

Note:
1. Vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.
2. Four points A, B, C & D are coplanar $\Leftrightarrow [\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}] = 0$.

Vector triple product:
1. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \circ \vec{c})\, \vec{b} - (\vec{a} \circ \vec{b})\vec{c}$.

2. $(\vec{a}\times\vec{b})\times\vec{c}=\left(\vec{a}\circ\vec{c}\right)\vec{b}-(\vec{b}\circ\vec{c})\vec{a}$.

Scalar point function: For every vector $R = xi + yj + zk$ there exists a unique scalar $f = f(x,\ y,\ z)$, and then f is called scalar point function.

i.e, A **scalar point function** is a physical quantity that is defined **at every point in space** and has **magnitude only**, but **no direction**.

Examples: $f(x,\ y,\ z) = xyz - x^2y$

Real-Life Examples:

Temperature field: In a room, each point has a specific temperature value (e.g., 25 °C). The temperature varies from point to point but does not have direction.

Pressure field: The atmospheric pressure at different points on Earth's surface or inside a fluid varies spatially but has no direction.

Electric potential: At each point in an electric field, the potential has a scalar value depending on location.

Vector point function: For every vector $R = xi + yj + zk$ there exists a unique vector $F = f_1(x,\ y,\ z)\boldsymbol{i} + f_2(x,\ y,\ z)\boldsymbol{j} + f_3(x,\ y,\ z)\boldsymbol{k}$, then F is called vector point function.

i.e, A **vector point function** is defined **at every point in space** and has **both magnitude and direction**.

Examples: $F = \boldsymbol{xyi} + yz^2\boldsymbol{j} + x^2z\boldsymbol{k}$

Real-Life Examples:

Velocity field of fluid: In a flowing river or air flow in a duct, every point in the fluid has a specific velocity vector (magnitude = speed, direction = flow direction).

Gravitational field: At every point around Earth, the gravitational force vector points toward the centre of the Earth. |

Electric field: At any point near a charge, the electric field has both magnitude and direction.

Magnetic field: Around a current-carrying conductor, the magnetic field vectors form loops with direction determined by the right-hand rule.

Vector operator: del: $\boldsymbol{\nabla} = \sum \boldsymbol{i}\frac{\partial}{\partial x} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$.

Review:

1. What is Scalar point function?
2. What is Vector point function?
3. What is Vector differential operator?
4. Give two real-life examples of scalar point functions.
5. Mention two real-life examples of vector point functions.
6. What is the main difference between scalar and vector point functions?

LECTURE 2:
Gradient, Directional Derivative:

Recall:

1. Does a scalar point function have direction? Explain.
2. Does a vector point function depend on position?
3. Write the general mathematical form of a scalar point function.
4. Write the general mathematical form of a vector point function.
5. Give some physical example for scalar and vector point function.
6. Why is it important to distinguish between scalar and vector point functions in fluid mechanics?

Gradient of scalar point function: $\boldsymbol{grad(f)} = \boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$.

Geometrical Interpretation:

The Gradient is vector normal (perpendicular) to the surface of constant scalar value at that point.

Note: 1. $\boldsymbol{\nabla} f$ is normal to the surface $f(x,y,z) = c$. ∴ unit normal vector to the surface $f = c$ is $\frac{1}{|\nabla f|}(\boldsymbol{\nabla} f)$

2. Angle between the two surfaces $f = 0$ & $g = 0$ is $\theta = \cos^{-1}\left|\frac{\nabla f\circ\nabla g}{|\nabla f|\ |\nabla g|}\right|$.

3. Directional derivative of f along $\vec{a}$ is $\frac{\nabla f\circ\vec{a}}{|\vec{a}|}$.

Maximum Directional derivative is $|\boldsymbol{\nabla} f|$ and is along $\boldsymbol{\nabla} f$.

Examples:

1. Find the unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1,\ 2, -1)$.

Solution: Since $\boldsymbol{\nabla f}$ is normal to the surface $f(x,y,z) = 0$.

Let $f = x^3 + y^3 + 3xyz - 3$, $\boldsymbol{\nabla} f = (3x^2 + 3yz)\boldsymbol{i} + (3y^2 + 3xz)\boldsymbol{j} + 3xy\boldsymbol{k}$

At the point $(1,\ 2, -1)$, $\boldsymbol{\nabla} f = -3i + 9j + 6k$ **.**

Unit normal vector$=\frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{14}}(-i + 3j + 2k)$.

2. Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of $2i - j - 2k$.

Solution: Clearly $\nabla\emptyset = (2xyz + 4z^2)i + x^2zj + (x^2y + 8xz)k$.

At the point$(1, -2, -1)$, $\nabla\emptyset = 8i - j - 10k$. And let $\vec{a} = 2i - j - 2k$.

Directional derivative of $\emptyset$ along $\vec{a}$ is $\frac{\nabla\emptyset \circ \vec{a}}{|\vec{a}|} = \frac{16+1+20}{\sqrt{4+1+4}} = \frac{37}{3}$

3. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4,\ 1,\ 2)$ and $(3,\ 3, -3)$.

Solution: Since ∇f is normal to the surface $f = 0$, let $f = xy - z^2$, $\nabla f = yi + xj - 2zk$.

Normal at $(4,\ 1,\ 2) = N_1 = i + 4j - 4k$.

And normal at $(3,\ 3, -3) = N_2 = 3i + 3j + 6k$

Angle between the normals $= \cos^{-1}\left|\frac{N_1 \circ N_2}{|N_1|\,|N_2|}\right| = \cos^{-1}\left|\frac{3+12-24}{\sqrt{33}\sqrt{54}}\right| = \cos^{-1}\frac{1}{\sqrt{22}}$.

4. Find the angle between the two surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1,\ 2)$.

Solution: Let $f = x^2 + y^2 + z^2$ and $g = x^2 + y^2 - z$

Then $\nabla f = 2xi + 2yj + 2zk,$ and $\nabla g = 2xi + 2yj - k,$

At $(2, -1,\ 2)$, $\nabla f = 4i - 2j + 4k$ and $\nabla g = 4i - 2j - k,$

Angle between the two surfaces $f = 9$ & $g = 3$ is

$$\theta = \cos^{-1}\left|\frac{\nabla f \circ \nabla g}{|\nabla f|\ |\nabla g|}\right| = \cos^{-1}\left(\frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}}\right) = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Review:

1. What does the gradient of a scalar function represent?
2. Find the gradient of a function V if V= xyz.
3. What is the unit normal vector to the surface $f(x,y,z) = c$?
4. What is the angle between the two surfaces $f = 0$ & $g = 0$?
5. When does the maximum value of the directional derivative occur?
6. For the function $f = \mathrm{x}^2\mathrm{y} + 2\mathrm{y}^2\mathrm{x}$, at the point $P(1,3)$, what is the direction in which the directional derivative is zero?

LECTURE 3:
Curl and Divergence- physical interpretation:

Recall:

1. What is the geometrical significance of the gradient vector at a point on a surface?
2. What does the magnitude of the gradient represent at a point?
3. Find the unit vector normal to the surface V if $V = x^2 + y^2 + z^2$ at $(1,1,1)$?
4. Explain the physical meaning of the directional derivative.
5. What is the condition for the two surfaces $f = 0$ & $g = 0$ to orthogonal?
6. When does the maximum value of the directional derivative occur?

Divergence of a vector field:

$\boldsymbol{div(F)} = \boldsymbol{\nabla} \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$. where $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$**.**

Geometrical Interpretation:

Divergence at a point measures the net rate of **flux** (outflow or inflow) of the vector field per unit volume surrounding that point.

Note: F is **Solenoidal** $\Leftrightarrow div(F) = 0$.

Curl of a vector field: $\boldsymbol{Curl(F)} = \boldsymbol{\nabla} \times \mathrm{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$, where $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$**.**

Geometrical Interpretation:

Curl at a point measures the tendency of the vector field to rotate (circulate) about that point.

Note: F is **irrotational** $\Leftrightarrow \boldsymbol{Curl}(F) = 0$.

If F is irrotational then there exists a scalar potential $\emptyset$ such that $F = \nabla\emptyset$ and
$\emptyset = \int f_1\, dx\ (y,z\ constant)$ $+ \int$ (terms of f_2 not containing x) $dy(z\ constant)$
$+ \int$ (terms of f_3 not containing x and y) $dz + c$.

<u>Theorems</u>: (Proofs are not required)

1. Prove that $curlgrad\emptyset = 0$.

Proof: $$L.H.S = \nabla \times (\nabla\emptyset) = \nabla \times \left[\frac{\partial\emptyset}{\partial x} i + \frac{\partial\emptyset}{\partial y} j + \frac{\partial\emptyset}{\partial z} k\right] = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix}$$

$$= \sum \left[\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial z\partial y}\right] i = 0 = R.H.S\ .$$

2. Prove that $divcurl\ F = 0$.

Proof: $$L.H.S = \nabla \circ (\nabla \times \mathrm{F}) = \sum \mathrm{i}\frac{\partial}{\partial x} \circ \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \sum \frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] = \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} + \frac{\partial^2 f_1}{\partial y\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} = 0\ .$$

3. Prove that $div(\emptyset F) = grad\emptyset \circ F + \emptyset divF$.

Proof: $$L.H.S = \nabla \circ (\emptyset F) = \sum i\frac{\partial}{\partial x} \circ (\emptyset F) = \sum i \circ \frac{\partial}{\partial x}(\emptyset F)$$

$$= \sum i \circ [\frac{\partial\emptyset}{\partial x} F + \emptyset\frac{\partial F}{\partial x}] = \sum i\frac{\partial\emptyset}{\partial x} \circ F + \emptyset \sum i \circ \frac{\partial F}{\partial x}$$

$$= \nabla\emptyset \circ F + \emptyset(\nabla \circ F) = R.H.S\ .$$

4. Prove that $curl(\emptyset F) = grad\emptyset \times F + \emptyset curlF$.

Proof: $$L.H.S = \nabla \times (\emptyset F) = \sum i\frac{\partial}{\partial x} \times (\emptyset F) = \sum i \times \frac{\partial}{\partial x}(\emptyset F)$$

$$= \sum i \times [\frac{\partial\emptyset}{\partial x} F + \emptyset\frac{\partial F}{\partial x}] = \sum i\frac{\partial\emptyset}{\partial x} \times F + \emptyset \sum i \times \frac{\partial F}{\partial x}$$

$$= \nabla\emptyset \times F + \emptyset(\nabla \times F) = R.H.S\ .$$

Note: 1. $grad[f(r)] = \nabla f(r) = \frac{f'(r)}{r} R$ where $\vec{r} = xi + yj + zk,\ \ r = \sqrt{x^2 + y^2 + z^2}$

2. $div(grad[f(r)]) = \nabla \circ \nabla f(r) = \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

3. $div(gradf) = \nabla^2 2x^2yzf = f_{xx} + f_{yy} + f_{zz}$.

<u>Examples:</u>

1. If $F = xy^2 i + 2x^2yzj - 3yz^2 k$, find $curl(F)$ and $div(F)$

Solution: Given that $F = xy^2 i + 2x^2yzj - 3yz^2 k$

Here, $f_1 = xy^2,\ \ f_2 = 2x^2yz,\ \ f_3 = -3yz^2$.

$$Div(F) = \nabla \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = y^2 + 2x^2z, -6yz.$$

And $$Curl(F) = \nabla \times \mathrm{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= (-3z^2 - 2x^2y)i - (0)j + (4xyz - 2xy)k.$$

2. Find $\nabla\log(x^2 + y^2 + z^2)$ and grad$\left(\frac{1}{r}\right)$.

Solution: $$\nabla\log(x^2 + y^2 + z^2) = \nabla\log(r^2)$$

$$= \nabla 2\log(r) = \frac{f'(r)}{r} R = \frac{2}{r^2} R$$

$$= \frac{2}{x^2 + y^2 + z^2}(xi + yj + zk)$$

And $$\mathrm{grad}\left(\frac{1}{r}\right) = \frac{f'(r)}{r} R = -\frac{1}{r^3} R.$$

3. Show that $\nabla^2 (r^n) = n(n+1)r^{n-2}$.

Solution: If $f(r) = r^n$ then $f'(r) = nr^{n-1}$ and $f''(r) = n(n-1)r^{n-2}$.

Since $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$,

$$\nabla^2 (r^n) = n(n-1)r^{n-2} + \frac{2}{r}nr^{n-1} = [n(n-1)+2n]r^{n-2} = n(n+1)r^{n-2}.$$

In particular $\nabla^2 \left(\frac{1}{r}\right) = 0$.

4. Prove that $\nabla(r^n) = nr^{n-2}R$, $R = xi + yj + zk$.

Proof: Since $\nabla f(r) = \frac{f'(r)}{r}R$, $f(r) = r^n$, $f'^{(r)} = nr^{n-1}$.

$$\nabla(r^n) = \frac{nr^{n-1}}{r}R = nr^{n-2}R.$$

5. If $F = grad\,[x^3 + y^3 + z^3 - 3xyz]$, find $divF$ and $curlF$.

Solution: Given that $F = \nabla f$, where $f = x^3 + y^3 + z^3 - 3xyz$.

$$F = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} = (3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k.$$

$\therefore F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$, where $f_1 = (3x^2 - 3yz)$, $f_2 = (3y^2 - 3xz)$, $f_3 = (3z^2 - 3xy)$.

$$Div(F) = \nabla \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 6x + 6y + 6z.$$

And
$$Curl(F) = \nabla \times \mathrm{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$
$$= (-3x + 3x)i - (-3y + 3y)j + (-3z + 3z)k = \mathbf{0}.$$

Or $Curl(F) = \mathrm{Curl}(\mathrm{grad}f) = 0$.

Review:

1. Find the curl of vector field $F = x^2\boldsymbol{i} + 2z\boldsymbol{j} - y\boldsymbol{k}$.
2. What is the divergence of the vector field $F = 3x^2\boldsymbol{i} + 5xy^2\boldsymbol{j} + xyz^3\boldsymbol{k}$ at the point (1, 2, 3)?
3. If the curl of a vector field F is zero, what can be inferred about F?
4. What is the mathematical expression for divergence in Cartesian coordinates?
5. In fluid dynamics, what does the divergence of a velocity field represent?
6. What is the physical interpretation of curl in a vector field?
7. If the divergence of a vector field is positive at a point, what it indicates?

LECTURE 4:

Solenoidal and irrotational vector fields. Problems:

Recall:

1. Define divergence and curl of a vector field.
2. What is the physical significance of divergence in fluid flow?
3. For a fluid flow with velocity field $V = xi + yj + zk$ calculate the divergence and curl.
4. Why divergence is considered a scalar quantity and curl a vector quantity?
5. How can you determine if a vector field is irrotational or solenoidal?
6. In electromagnetism, what does the divergence of the electric field relate to?

Examples:

1. Find the value of a if $F = (ax^2y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)k$ is solenoidal.

Solution: Since F is solenoidal, $div(F) = 0$, that is $\nabla \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0$.

$$\therefore\ 2axy + 2xy + 2xy = 0 \quad \Rightarrow a = -2.$$

2. Find $a,\ b,\ c,$ if $F = (x + by - z)i + (2x - y + cz)j + (ax + y - z)k$ is irrotational. And also find scalar potential $\emptyset$ such that $F = \nabla\emptyset$.

Solution: F is irrotational $\Rightarrow Curl(F) = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + by - z & 2x - y + cz & ax + y - z \end{vmatrix} = 0$$

$$\Rightarrow (1-c)i - (a+1)j + (2-b)k = 0 \quad \Rightarrow a + 1 = 0,\ 2 - b = 0 \text{ and } 1 - c = 0.$$

$$\therefore\ a = -1,\ b = 2,\ c = 1.$$

If F is irrotational then there exists a scalar potential $\emptyset$ such that $F = \nabla\emptyset$ and

$\emptyset = \int f_1\, dx\ (y,z\ constant)$ $+$ $\int$ (terms of f_2 not containing x) $dy(z\ constant)$

$+$ $\int$ (terms of f_3 not containing x and y) $dz + c$.

$$= \int (x+2y-z)\, dx + \int(-y+z)\, dy + \int(-z)\, dz = \frac{x^2}{2} + 2xy - xz - \frac{y^2}{2} + yz - \frac{z^2}{2} + c\ .$$

3. Show that $\frac{xi+yj}{x^2+y^2}$ is both solenoidal and irrotational

Solution: Given that $F = \frac{x}{x^2+y^2}i + \frac{y}{x^2+y^2}j + 0k$

$$Div(F) = \nabla \circ F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2} = 0$$

Therefore F is solenoidal

$$Curl(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= 0i - 0j + \left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]k$$

$$= \left[\frac{0-2xy}{(x^2+y^2)^2} - \frac{0-2xy}{(x^2+y^2)^2}\right]k = 0.$$

Hence F is irrotational.

4. If $F = (x+y+1)i + j - (x+y)k$, Show that $F \circ curlF = 0$.

Solution: $C\ url(F) = \nabla \times \mathrm{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$

$$= -i + j - k$$

Now, $F \circ curlF = (x+y+1)i + j - (x+y)k \circ -i + j - k = -x - y - 1 + 1 + x + y = 0.$

Recall:

1. What is the mathematical expression for a solenoidal vector field A?
2. Define solenoidal vector.
3. Define irrotational vector.
4. Give the vector identities related to solenoidal and irrotational properties of vector field.
5. Give one real – life example of a solenoidal vector field.
6. Give one real – life example of an irrotational vector field.

TUTORIAL 1:

Problems on Gradient, Divergence and Curl:

1. Find the gradient of $f = xlogz - y^2z + yz^3$ at $(1,\ -2,\ 1)$.
2. Find the directional derivative of $\emptyset = xy^2 + yz^3$ at the point $(1,-2,-1)$ along $\vec{a} = -4j - k$.
3. Find divF and curlF, where $\mathrm{F} = \mathrm{grad}\ [\mathrm{xy^3z^2}]\ at\ (1,-1,\ 1)$
4. Show that $F = x(y-z)i + y(z-x)j + z(x-y)k$ is solenoidal.
5. Find the constants a and b if $F = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational, and also find the scalar potential.
6. Find the constants a, b and c if $F = (\sin y + az)i + (bx\cos y + z)j + (x + cy)k$ is irrotational.
7. Find the directional derivative of $div[grad(\emptyset)]$ along $i + 2j - 2k$, where $\emptyset = x^2yz + y^3 - xz^2$ at the point $(1,\ 1,\ 1)$.

TUTORIAL 2:
Lab Activity 3: Finding Gradient, divergence and curl

Objectives:

Use python

- To find partial derivatives of functions of several variables.
- To find Jacobian of function of two and three variables.

1. Find gradient of $\emptyset = x^2yz$

```
from sympy . vector import *
from sympy import symbols
N= CoordSys3D ('N')
x,y,z=  symbols ('x y z')
A=N.x **2* N.y*N.z
print (f"\n Gradient of {A} is \n")
display (gradient(A))
```

2. Find div(F) and curl(F), if $F = xy^2i + 2x^2yzj - 3yz^2k$

```
from sympy . vector import *
from sympy import symbols
N= CoordSys3D ('N')
x,y,z=  symbols ('x y z')
A=N.x*N.y  **2* N.i+2*N.x **2* N.y*N.z*N.j  -3*N.y*N.z  **2* N.k
print (f"\n Divergence of {A} is \n")
display (divergence (A))
print (f"\n Curl of {A} is \n")
display (curl (A))
```

3. Prove that $F = (yz - 2x^2y)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)k$ is solenoidal.
4. Show that $F = (y + z)i + (z + x)j + (x + y)k$ is irrotational.

LECTURE 5:
Introduction to Polar coordinates. Polar curves

Recall:

1. What is a Cartesin Co-ordinate System?
2. How are Cartesin Co-ordinates used in real life?
3. How is the point represented in the Polar Co-ordinates?
4. What is the relationship between Cartesian and Polar Co-ordinates.

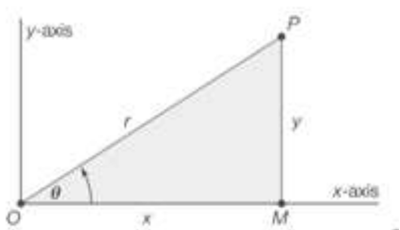
Polar coordinates:

Initial reference is chosen by spotting a point O in the plane called a pole. A line OM drawn through O is called the initial line. If P is any given point in the plane, join the points O and P with the results an angle is formed at O.

The length of OP denoted by r is called the radius vector of the point P and

the angle MOP denoted by θ measured in the anticlockwise direction is called

the vectorial angle. The pair r and θ represented by P=(r, θ) or P(r, θ) are called as the polar co-ordinates of the point P.



Relationship between the Cartesian Co-ordinates (x, y) and the Polar Co-ordinates (r, θ):

Let (x, y) and (r, θ) respectively represent the Cartesian and Polar Co-ordinates of any point P in the plane where the origin O is taken as the pole and the x- axis is taken as the initial line. From the figure we have OM=x, PM=y. Also from the right angled triangle OMP we have

$$\cos\theta = \frac{OM}{OP} = \frac{x}{r} \Rightarrow x = r\cos\theta \qquad \text{...............(1)}$$

$$\sin\theta = \frac{PM}{OP} = \frac{y}{r} \Rightarrow y = r\sin\theta \qquad \text{................(2)}$$

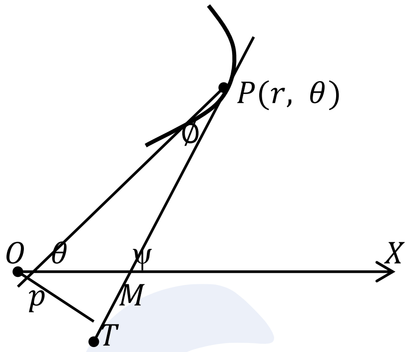
Further squaring and adding (1) and (2) we get $x^2 + y^2 = r^2(cos^2\theta + sin^2\theta)$

$\therefore r = \sqrt{x^2 + y^2}$ $\quad\quad \because (cos^2\theta + sin^2\theta = 1)$ …………….(3)

Also dividing (2) by (1) we get $\frac{rsin\theta}{rcos\theta} = \frac{y}{x} \Rightarrow \tan\theta = \frac{y}{x}$ $\quad\quad \therefore \theta = \tan^{-1}(\frac{y}{x})$ …………….(4)

The relations (1) and (2) determine the Cartesian co-ordinates in terms of polar co-ordinates whereas relations (3) and (4) determine the Polar co-ordinates in terms of Cartesian Co-ordinates.

Polar curves:



O is the pole, OX is the initial line, OP the radius vector , PT is the tangent to the curve at P .
And $OT = p$. In ΔOPM, $\psi = \theta + \phi$.

Review:

1. What is the relationship between the angle ψ and the angle θ and ϕ in polar coordinates.
2. If the radius vector OP makes an angle θ with the positive x-axis, what is the angle ϕ between the radius vector and the tangent PM.
3. Express the cartesian coordinate $(1,1)$ interms of polar coordinate (r,ϑ).
4. Express the cartesian coordinate $(1,-1)$ interms of polar coordinate (r,ϑ).
5. Express the cartesian coordinate $(-1,1)$ interms of polar coordinate (r,ϑ).
6. Express the cartesian coordinate $(-1,-1)$ interms of polar coordinate (r,ϑ).

LECTURE 6:

Curvilinear coordinates: Scale factors, base vectors:

Recall:

1. How the points are represented in plane and space?
2. Why we need polar coordinate system?
3. Give the relationship between Cartesian and polar coordinate system?
4. What do the symbols r and θ represent in polar coordinates?
5. How is straight line represented in polar coordinates?
6. Name the three commonly used orthogonal curvilinear co-ordinate system?

Orthogonal curvilinear co-ordinates:

Let $x = x(u,\ v,\ w)$, $\quad y = y(u,\ v,\ w)$, $\quad z = z(u,\ v,\ w)$ $\quad \cdots\cdots\cdots (1)$

And $u = u(x,\ y,\ z)$, $\quad v = v(x,\ y,\ z)$, $\quad w = w(x,\ y,\ z)$ $\quad \cdots\cdots\cdots (2)$

If the functions in (1) and (2) are single-valued and have continuous partial derivatives, then $(u,\ v,\ w)$ are called curvilinear co-ordinates of $(x,\ y,\ z)$.

Since, $R = x(u,\ v,\ w)\boldsymbol{i} + y(u,\ v,\ w)\boldsymbol{j} + z(u,\ v,\ w)\boldsymbol{k}$,

$dR = \frac{\partial R}{\partial u}du + \frac{\partial R}{\partial v}dv + \frac{\partial R}{\partial w}dw.$

Then $\frac{\partial R}{\partial u} = h_1 T_u$, $\frac{\partial R}{\partial v} = h_2 T_v$, $\frac{\partial R}{\partial w} = h_3 T_w$.

Where T_u , T_v and T_w are unit tangent vectors to u-, v- and w- curves respectively.

Therefore $dR = h_1 du\ T_u + h_2 dv\ T_v + h_3\ dw\ T_w$.

Similarly, Unit normal vectors to the surfaces $u = u_0$, $v = v_0$, $w = w_0$ are

$N_u = \frac{\nabla u}{|\nabla u|}$, $N_v = \frac{\nabla v}{|\nabla v|}$ and $N_w = \frac{\nabla w}{|\nabla w|}$ respectively.

Therefore, at each point P of a curvilinear co-ordinate system there exist two triads of unit vectors
T_u , T_v , T_w tangents to u , v and w- curves and
N_u , N_v , N_w normals to the surfaces $u = u_0$, $v = v_0$, $w = w_0$.

And $T_u = h_1 \nabla u = h_2 h_3 (\nabla v \times \nabla w)$

$T_v = h_2 \nabla v = h_3 h_1 (\nabla w \times \nabla u)$

$$T_w = h_3 \nabla w = h_1 h_2 (\nabla u \times \nabla v)$$

Expressions for gradient, divergence and curl: (For reference only)

1) $\nabla f = \frac{T_u}{h_1}\frac{\partial f}{\partial u} + \frac{T_v}{h_2}\frac{\partial f}{\partial v} + \frac{T_w}{h_3}\frac{\partial f}{\partial w}$.

Let $f(u,\ v,\ w)$ be any scalar point function in terms of $u,\ v,\ w$.

Taking $u,\ v,\ w$ as functions of $x,\ y,\ z$, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \qquad \cdots \text{(i)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \qquad \cdots \text{(ii)}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} \qquad \cdots \text{(iii)}$$

Multiplying (i) by i , (ii) by j , (iii) by k and adding, we have

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w = \frac{T_u}{h_1}\frac{\partial f}{\partial u} + \frac{T_v}{h_2}\frac{\partial f}{\partial v} + \frac{T_w}{h_3}\frac{\partial f}{\partial w}.$$

2) $\nabla . F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 f_1) + \frac{\partial}{\partial v}(h_3 h_1 f_2) + \frac{\partial}{\partial w}(h_1 h_2 f_3)\right]$

Proof: Let $F(u,\ v,\ w)$ be a vector point function such that

$$F = f_1\, T_u + f_2\, T_v + f_3\, T_w = \sum f_1\, h_2 h_3 \nabla v \times \nabla w.$$

$$\begin{aligned}\nabla . F &= \sum \nabla . [(f_1 h_2 h_3)(\nabla v \times \nabla w)]\\ &= \sum [(f_1 h_2 h_3)\nabla . (\nabla v \times \nabla w) + (\nabla v \times \nabla w). \nabla (f_1 h_2 h_3)]\\ &= \sum (\nabla v \times \nabla w). \left\{\frac{\partial (f_1 h_2 h_3)}{\partial \mathrm{u}}\nabla u + \frac{\partial (f_1 h_2 h_3)}{\partial \mathrm{v}}\nabla v + \frac{\partial (f_1 h_2 h_3)}{\partial \mathrm{w}}\nabla w\right\}\\ &= [\nabla u,\ \ \nabla v,\ \nabla w\,]\sum \frac{\partial (f_1 h_2 h_3)}{\partial \mathrm{u}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 f_1) + \frac{\partial}{\partial v}(h_3 h_1 f_2) + \frac{\partial}{\partial w}(h_1 h_2 f_3)\right]\end{aligned}$$

$$\therefore\ \nabla^2 f = \nabla . \nabla f = \nabla . \left(\frac{T_u}{h_1}\frac{\partial f}{\partial u} + \frac{T_v}{h_2}\frac{\partial f}{\partial v} + \frac{T_w}{h_3}\frac{\partial f}{\partial w}\right) = \frac{1}{h_1 h_2 h_3}\sum \frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u} h_2 h_3\right)$$

3) $\nabla \times \mathrm{F} = \begin{vmatrix} \frac{T_u}{h_2 h_3} & \frac{T_v}{h_3 h_1} & \frac{T_w}{h_1 h_2} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$ where $F = f_1\, T_u + f_2\, T_v + f_3\, T_w$.

Let $F(u,\ v,\ w)$ be a vector point function such that

$$F = f_1\, T_u + f_2\, T_v + f_3\, T_w = f_1 h_1 \nabla u + f_2 h_2 \nabla v + f_3 h_3 \nabla w.$$

Then $\nabla \times \mathrm{F} = \sum \nabla \times (f_1 h_1 \nabla u)$

$$\begin{aligned}&= \sum \nabla (f_1 h_1) \times \nabla u + (f_1 h_1). (\nabla \times \nabla u)\\ &= \sum \left(\frac{\partial (f_1 h_1)}{\partial u}\nabla u + \frac{\partial (f_1 h_1)}{\partial v}\nabla v + \frac{\partial (f_1 h_1)}{\partial w}\nabla w\right) \times \nabla u\\ &= \sum \left(\frac{\partial (f_1 h_1)}{\partial v}\nabla v \times \nabla u + \frac{\partial (f_1 h_1)}{\partial w}\nabla w \times \nabla u\right)\\ &= \sum \left(\frac{\partial (f_1 h_1)}{\partial v}\left(-\frac{T_u \times T_v}{h_1 h_2}\right) + \frac{\partial (f_1 h_1)}{\partial w}\left(\frac{T_w \times T_u}{h_3 h_1}\right)\right)\end{aligned}$$

$$= -\frac{\partial (f_1 h_1)}{\partial v}\left(\frac{T_w}{h_1 h_2}\right) + \frac{\partial (f_1 h_1)}{\partial w}\left(\frac{T_v}{h_3 h_1}\right) - \frac{\partial (f_2 h_2)}{\partial w}\left(\frac{T_u}{h_2 h_3}\right) + \frac{\partial (f_2 h_2)}{\partial u}\left(\frac{T_w}{h_1 h_2}\right) - \frac{\partial (f_3 h_3)}{\partial u}\left(\frac{T_v}{h_3 h_1}\right) + \frac{\partial (f_3 h_3)}{\partial v}\left(\frac{T_u}{h_2 h_3}\right)$$

$$= \left(\frac{T_u}{h_2 h_3}\right)\left(\frac{\partial (f_3 h_3)}{\partial v} - \frac{\partial (f_2 h_2)}{\partial w}\right) - \left(\frac{T_v}{h_3 h_1}\right)\left(\frac{\partial (f_3 h_3)}{\partial u} - \frac{\partial (f_1 h_1)}{\partial w}\right) + \left(\frac{T_w}{h_1 h_2}\right)\left(\frac{\partial (f_2 h_2)}{\partial u} - \frac{\partial (f_1 h_1)}{\partial v}\right)$$

$$= \begin{vmatrix} \frac{T_u}{h_2 h_3} & \frac{T_v}{h_3 h_1} & \frac{T_w}{h_1 h_2} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$$

Review:

1. What are curvilinear coordinates?
2. What are scale factors in a curvilinear coordinate system?
3. Define Base vectors in curvilinear coordinate system?
4. What are scale factors for rectangular coordinate system?

Lecture 7:
Cylindrical polar coordinates, Spherical polar coordinates:

Recall:

1. How are orthogonal curvilinear coordinates defined?
2. In orthogonal curvilinear coordinates, how the scale factors are defined?
3. How are the scale factors denoted in orthogonal curvilinear coordinates?
4. What are the unit base vectors in curvilinear coordinates?
5. When we say that base vectors are orthogonal?

Two special curvilinear systems:

1) **Cylindrical coordinates**: $x = \rho\cos\emptyset$, $y = \rho\sin\emptyset$, $z = z$.

Prove that cylindrical co-ordinate system is orthogonal.

Proof: At any point P, we have $x = \rho\cos\emptyset$, $y = \rho\sin\emptyset$, $z = z$.

So that $R = \rho\cos\emptyset\ i + \rho\sin\emptyset\ j + z\ k$.

By definition of scale factors,

$h_1 = |\partial R/\partial\rho| = |\cos\emptyset\ i + \sin\emptyset\ j + 0\ k| = \sqrt{cos^2\emptyset + sin^2\emptyset} = 1$

$h_2 = |\partial R/\partial\emptyset| = |-\rho\sin\emptyset\ i + \rho\cos\emptyset\ j + 0\ k| = \sqrt{\rho^2 sin^2\emptyset + \rho^2 cos^2\emptyset} = \rho$

$h_3 = |\partial R/\partial z| = |1.k| = 1$

$\therefore h_1 = 1, h_2 = \rho, h_3 = 1$ are the scale factors for the cylindrical system.

If T_ρ, $T_\emptyset$, T_z be the unit vectors at P in the directions of the tangents to the ρ, $\emptyset$, z-curves respectively.

Then $T_\rho = \frac{\partial R/\partial\rho}{|\partial R/\partial\rho|} = \cos\emptyset\ i + \sin\emptyset\ j$.

$T_\emptyset = \frac{\partial R/\partial\emptyset}{|\partial R/\partial\emptyset|} = -\sin\emptyset\ i + \cos\emptyset\ j$.

$T_z = \frac{\partial R/\partial z}{|\partial R/\partial z|} = k$.

Now $T_\rho.T_\emptyset = -\cos\emptyset\sin\emptyset + \sin\emptyset\cos\emptyset = 0$.

$T_\emptyset.T_z = 0$ and $T_z.T_\rho = 0$.

And hence cylindrical co-ordinate system is orthogonal.

More over $T_\rho \times T_\emptyset = T_z$, $T_\emptyset \times T_z = T_\rho$ *and* $T_z \times T_\rho = T_\emptyset$. So that cylindrical co-ordinate system is a right handed orthogonal co-ordinate system.

2) **Spherical polar coordinates**: $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$, $z = r\cos\theta$.

Prove that spherical polar co-ordinate system is orthogonal.

Proof: At any point P, we have $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$, $z = r\cos\theta$.

So that $R = r\sin\theta\cos\emptyset\ i + r\sin\theta\sin\emptyset\ j + r\cos\theta\ k$.

By definition of scale factors,

$h_1 = |\partial R/\partial r| = |\sin\theta\cos\emptyset\ i + \sin\theta\sin\emptyset\ j + \cos\theta\ k|$

$= \sqrt{cos^2\theta + sin^2\theta(cos^2\emptyset + sin^2\emptyset)} = 1$

$h_2 = |\partial R/\partial\theta| = |r\cos\theta\cos\emptyset\ i + r\cos\theta\sin\emptyset\ j + r\sin\theta\ k|$

$= \sqrt{r^2 sin^2\theta + r^2 cos^2\theta(cos^2\emptyset + sin^2\emptyset)} = r$

$h_3 = |\partial R/\partial\emptyset| = |-r\sin\theta\sin\emptyset\ i + r\sin\theta\cos\emptyset\ j + 0.k|$

$= \sqrt{r^2 sin^2\theta + (cos^2\emptyset + sin^2\emptyset)} = r\sin\theta$

$\therefore h_1 = 1, h_2 = r, h_3 = r\sin\theta$ are the scale factors for the spherical system.

If T_r, T_θ, $T_\emptyset$ be the unit vectors at P in the directions of the tangents to the r, θ, $\emptyset$-curves respectively.

Then $T_r = \frac{\partial R/\partial r}{|\partial R/\partial r|} = \sin\theta\cos\emptyset\ i + \sin\theta\sin\emptyset\ j + \cos\theta\ k$.

$T_\theta = \frac{\partial R/\partial\theta}{|\partial R/\partial\theta|} = \cos\theta\cos\emptyset\ i + \cos\theta\sin\emptyset\ j - \sin\theta\ k$.

$T_\emptyset = \frac{\partial R/\partial\emptyset}{|\partial R/\partial\emptyset|} = -\sin\emptyset\ i + \cos\emptyset\ j$.

Now $T_\rho.T_\emptyset = \sin\theta\ \cos\theta\cos^2\emptyset + \sin\theta\ \cos\theta\sin^2\emptyset - \sin\theta\ \cos\theta = 0$.

$T_\theta.T_\emptyset = -\cos\theta\sin\emptyset\ \cos\emptyset + \cos\theta\sin\emptyset\ \cos\emptyset = 0$

and $T_{\emptyset}.T_r = -\sin\theta\sin\emptyset\cos\emptyset + \sin\theta\sin\emptyset\cos\emptyset = 0$.

And hence spherical polar co-ordinate system is orthogonal.

More over $T_\rho \times T_\emptyset = T_z$, $T_\emptyset \times T_z = T_\rho$ *and* $T_z \times T_\rho = T_\emptyset$. So that spherical polar co-ordinate system is a right handed orthogonal co-ordinate system.

Review:

1. What are the scale factors for spherical coordinates?
2. What are the scale factors in cylindrical coordinates?
3. How are base vectors in cylindrical coordinates expressed in terms of Cartesian base vectors?
4. How are base vectors in spherical coordinates expressed in terms of Cartesian base vectors?
5. Why are base vectors in curvilinear coordinates not always constant in space?

LECTURE 8:

Transformation between Cartesian and curvilinear systems:

Recall:

1. What are the scale factors in spherical coordinates?
2. What are the scale factors in cylindrical coordinates?
3. How do base vectors in curvilinear coordinates differ from Cartesian base vectors?
4. Explain why unit base vectors in cylindrical and spherical coordinates vary with position

Let $u = x, v = y, w = z$ then $R = xi + yj + zk \Rightarrow R = ui + yj + wk$.

1.Tangent vectors:

$$\frac{\partial R}{\partial u} = \frac{\partial R}{\partial x} = i, \qquad \frac{\partial R}{\partial v} = \frac{\partial R}{\partial y} = j, \qquad \frac{\partial R}{\partial w} = \frac{\partial R}{\partial z} = k.$$

2. The scale factors:

$$h_1 = |\partial R/\partial u| = |i| = 1, \qquad h_2 = |\partial R/\partial v| = |j| = 1, \quad h_3 = |\partial R/\partial w| = |k| = 1$$

3. The base (unit-tangent) vectors:

$$T_u = \frac{\partial R/\partial u}{|\partial R/\partial u|} = i, \qquad T_v = \frac{\partial R/\partial v}{|\partial R/\partial v|} = j, \quad T_w = \frac{\partial R/\partial w}{|\partial R/\partial w|} = k.$$

4. For Cartesian system: $h_1 = h_2 = h_3 = 1$ and $T_u = i, T_v = j, T_w = k$.

5. Also we have $T_u.T_v = 0$, $T_v.T_w = 0$ and $T_u.T_w = 0$.

$$T_u \times T_v = T_w, \quad T_u \times T_w = T_v \quad and \quad T_v \times T_w = T_u.$$

Thus the Cartesian system is orthogonal.

<u>Del applied to the functions in cylindrical co-ordinates</u>: (For reference)

Since $u = \rho,\ v = \emptyset,\ w = z \quad and \quad h_1 = 1,\ h_2 = \rho,\ h_3 = 1.$

$$T_\rho = \cos\emptyset\ i + \sin\emptyset\ j, \quad T_\emptyset = -\sin\emptyset\ i + \cos\emptyset\ j \quad \text{and } T_z = k\ .$$

1. $\nabla f = \frac{T_u}{h_1}\frac{\partial f}{\partial u} + \frac{T_v}{h_2}\frac{\partial f}{\partial v} + \frac{T_w}{h_3}\frac{\partial f}{\partial w} = \frac{T_\rho}{h_1}\frac{\partial f}{\partial \rho} + \frac{T_\emptyset}{h_2}\frac{\partial f}{\partial \emptyset} + \frac{T_z}{h_3}\frac{\partial f}{\partial z}$

$$= \frac{\partial f}{\partial \rho}(\cos\emptyset\ i + \sin\emptyset\ j) + \frac{1}{\rho}\frac{\partial f}{\partial \emptyset}(-\sin\emptyset\ i + \cos\emptyset\ j) + \frac{\partial f}{\partial z}\ (k)\ .$$

2. $\nabla.F = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1) + \frac{\partial}{\partial v}(h_3h_1f_2) + \frac{\partial}{\partial w}(h_1h_2f_3)\right]$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho f_1) + \frac{\partial}{\partial \emptyset}(f_2) + \frac{\partial}{\partial z}(\rho f_3)\right]$$

3. $\nabla \times \mathrm{F} = \begin{vmatrix} \frac{T_u}{h_2h_3} & \frac{T_v}{h_3h_1} & \frac{T_w}{h_1h_2} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix} = \begin{vmatrix} \frac{\cos\emptyset\ i+\sin\emptyset\ j}{\rho} & -\sin\emptyset\ i + \cos\emptyset\ j & \frac{k}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$

<u>Del applied to the functions in spherical polar co-ordinates</u>: (For reference)

Since $u = r,\ v = \theta,\ w = \emptyset \quad and \quad h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta.$

$$T_r = \sin\theta\cos\emptyset\ i + \sin\theta\sin\emptyset\ j + \cos\theta\ k$$

$$T_\theta = \cos\theta\cos\emptyset\ i + \cos\theta\sin\emptyset\ j - \sin\theta\ k$$

and $T_\emptyset = -\sin\emptyset\ i + \cos\emptyset\ j$.

1. $\nabla f = \frac{T_u}{h_1}\frac{\partial f}{\partial u} + \frac{T_v}{h_2}\frac{\partial f}{\partial v} + \frac{T_w}{h_3}\frac{\partial f}{\partial w} = \frac{T_r}{h_1}\frac{\partial f}{\partial r} + \frac{T_\theta}{h_2}\frac{\partial f}{\partial \theta} + \frac{T_\emptyset}{h_3}\frac{\partial f}{\partial \emptyset}$

$$= \frac{\partial f}{\partial r}(\sin\theta\cos\emptyset\ i + \sin\theta\sin\emptyset\ j + \cos\theta\ k) + \frac{1}{r}\frac{\partial f}{\partial \theta}(\cos\theta\cos\emptyset\ i + \cos\theta\sin\emptyset\ j - \sin\theta\ k)$$

$+\frac{1}{r\sin\theta}\frac{\partial f}{\partial \emptyset}\,(-\sin\emptyset\ i+\cos\emptyset\ j)$.

2. $\nabla.F=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1)+\frac{\partial}{\partial v}(h_3h_1f_2)+\frac{\partial}{\partial w}(h_1h_2f_3)\right]$

$=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, f_1)+\frac{\partial}{\partial\theta}(r\sin\theta\, f_2)+\frac{\partial}{\partial\emptyset}(rf_3)\right]$

3. $\nabla\times \mathrm{F}=\begin{vmatrix}\frac{T_u}{h_2h_3} & \frac{T_v}{h_3h_1} & \frac{T_w}{h_1h_2}\\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w}\\ h_1f_1 & h_2f_2 & h_3f_3\end{vmatrix}=\begin{vmatrix}\frac{T_r}{r^2\sin\theta} & \frac{T_\theta}{r\sin\theta} & \frac{T_\emptyset}{r}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\emptyset}\\ f_1 & rf_2 & r\sin\theta\, f_3\end{vmatrix}$

Problems:

1. Express the vector $z\,\boldsymbol{i}-2x\,\boldsymbol{j}+y\,\boldsymbol{k}$ in cylindrical coordinates.

Solution: Let the expression for $F=z\,\boldsymbol{i}-2x\,\boldsymbol{j}+y\,\boldsymbol{k}$ in cylindrical coordinates be $F=f_1\,T_\rho+f_2\,T_\emptyset+f_3\,T_z$

$F=z\,\boldsymbol{i}-2x\,\boldsymbol{j}+y\,\boldsymbol{k}$

$T_\rho=\cos\emptyset\ i+\sin\emptyset\ j+0k,$

$T_\emptyset=-\sin\emptyset\ i+\cos\emptyset\ j+0k$

$T_z=0i+0j+k$.

$f_1=F.\,T_\rho=z\cos\emptyset-2x\sin\emptyset$

$f_2=F.\,T_\emptyset=-z\sin\emptyset-2x\cos\emptyset$

$f_3=F.\,T_z=y.$

Since , $x=\rho\cos\emptyset$, $y=\rho\sin\emptyset$, $z=z$

$f_1=z\cos\emptyset-\rho\sin2\emptyset$, $f_2=z\sin\emptyset+2\rho\cos^2\emptyset$, $f_3=\rho\sin\emptyset.$

∴ In cylindrical coordinates

$F=(z\cos\emptyset-\rho\sin2\emptyset)\ T_\rho-(z\sin\emptyset+2\rho\cos^2\emptyset)\,T_\emptyset+\rho\sin\emptyset\ T_z.$

2. Express the vector $2y\,\boldsymbol{i}-z\,\boldsymbol{j}+3x\,\boldsymbol{k}$ in spherical polar coordinates.

Solution: Let the expression for $F=2y\,\boldsymbol{i}-z\,\boldsymbol{j}+3x\,\boldsymbol{k}$ in spherical polar coordinates be $F=f_1\,T_r+f_2\,T_\theta+f_3\,T_\emptyset$

$T_r=\sin\theta\cos\emptyset\ i+\sin\theta\sin\emptyset\ j+\cos\theta\ k$.

$T_\theta=\cos\theta\cos\emptyset\ i+\cos\theta\sin\emptyset\ j-\sin\theta\ k$

$T_\emptyset=-\sin\emptyset\ i+\cos\emptyset\ j.$

$f_1=F.\,T_r=2y\sin\theta\cos\emptyset-z\sin\theta\sin\emptyset+3x\cos\theta$,

$f_2=F.\,T_\theta=2y\cos\theta\cos\emptyset-z\cos\theta\sin\emptyset-3x\sin\theta$,

$f_3=F.\,T_\emptyset=-2y\sin\emptyset-z\cos\emptyset$.

$x=r\sin\theta\cos\emptyset$, $yr\sin\theta\sin\emptyset$, $zr\cos\theta$

$$F=\left(2r\sin^2\theta\sin\emptyset\ \cos\emptyset-r\sin\theta\cos\theta\sin\emptyset+3r\sin\theta\cos\theta\cos\emptyset\right)\,T_r$$
$$+(2r\sin\theta\cos\theta\sin\emptyset\cos\emptyset-r\cos^2\theta\sin\emptyset-3r\sin^2\theta\cos\emptyset)\,T_\theta$$
$$-(2r\sin\theta\sin^2\emptyset+r\cos\theta\cos\emptyset)\,T_\emptyset\,.$$

Review:

1. What is the relationship between Cartesian coordinates and cylindrical polar coordinates?
2. What is the relationship between Cartesian coordinates and spherical polar coordinates?
3. Discuss the role of scale factors in expressing differential volume elements.
4. Show how the gradient of a scalar function is expressed in terms of scale factors in orthogonal curvilinear coordinates.
5. Why is it essential to use the correct base vectors and scale factors when dealing with vector calculus in cylindrical or spherical system?
6. Compare the simplicity of differential operators in Cartesian vs. curvilinear coordinate systems

TUTORIAL 3:
Problems Solving on Orthogonality

1. Express the vector $yz\,\boldsymbol{i}-y\,\boldsymbol{j}+xz^2\,\boldsymbol{k}$ in cylindrical coordinates.
2. Express the vector $y\,\boldsymbol{i}-z\,\boldsymbol{j}+x\,\boldsymbol{k}$ in spherical polar coordinates.
3. Express the vector $2x\,\boldsymbol{i}-3y^2\,\boldsymbol{j}+xz\,\boldsymbol{k}$ in cylindrical polar coordinates.
4. Express the vector $\mathrm{x}\,\boldsymbol{i}+2y\,\boldsymbol{j}+yz\,\boldsymbol{k}$ in spherical coordinates
5. Find the Scale factors and base vectors for cylindrical system.
6. Find the Scale factors and base vectors for spherical system.

TUTORIAL 4:
Problems Solving on Vecor calculus

1. Find the unit vector normal to the surface $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.
2. Find the directional derivative of $\emptyset = \frac{xz}{x^2+y^2}$ at the point $(1, -1, 1)$ in the direction of $i - 2j + k$.
3. Find $\nabla^2 f$ at $(1, 1, 0)$ if $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$.
4. Evaluate $Curl(CurlF)$ and $Div(CurlF)$, if $F = x^2yi + y^2zj + z^2xk$.
5. Show that $F = (y^2 - z^2 + 3yz + 2xy)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational.
6. Find the constant c if $F = (cxy - z^3)i + (c-2)y^2j + c(1-c)yzk$ is irrotational.
7. If $\vec{v} = \vec{w} \times \vec{r}$, then prove that curl$\vec{v} = 2\vec{w}$.

MODULE 2: VECTOR CALCULUS – PRACTICE QUESTION BANK

VECTOR DIFFERENTIATION – Gradient, Divergence and Curl:

1. Find the gradient of $f = xlogz - y^2z + yz^3$ at $(1, -2, 1)$.
2. Find the unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.
3. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.
4. Find the angle between the two surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.
5. Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4xy + z = 4$ at the point $(1, -1, 2)$?
6. Find the value of p and q so that surfaces $px^2 - qyz = (p+2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1, -1, 2)$.
7. Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of $2i - j - 2k$.
8. Find the directional derivative of $\emptyset = \frac{xz}{x^2+y^2}$ at the point $(1, -1, 1)$ in the direction of $i - 2j + k$.
9. In what direction from $(3, 1, -2)$ is the directional derivative of $\emptyset = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.
10. Find the directional derivative of $\emptyset = xyz$ in the direction of the normal to the surface $x^2z + y^2x + z^2y = 3$ at $(1, 1, 1)$.
11. Find the directional derivative of $\emptyset = xy^2 + yz^3$ at the point $(1, -2, -1)$ in the direction of the normal to the surface $xlogz - y^2 = -4$ at $(-1, 2, 1)$.
12. Find $\nabla^2 f$ at $(1, 1, 0)$ if $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$.
13. If $F = xy^2i + 2x^2yzj - 3yz^2k$, find $curl(F)$ and $div(F)$.
14. If $= e^{xyz}(i + j + k)$, find $curlF$ and $divF$.
15. Find divF and curlF, where F = grad $[xy^3z^2]$ at $(1, -1, 1)$.
16. If $F = grad\ [x^3 + y^3 + z^3 - 3xyz]$, find $divF$ and $curlF$.
17. Evaluate $Curl(CurlF)$ and $Div(CurlF)$, if $F = x^2yi + y^2zj + z^2xk$.
18. If $F = (x + y + 1)i + j - (x + y)k$, Show that $F \circ curlF = 0$.
19. If $A = 2x^2i - 3yzj + xz^2k$ and $\emptyset = 2z - x^3y$,Compute $A \circ \nabla\emptyset\ and\ A \times \nabla\emptyset\ at\ (1, -1, 1)$.
20. Show that $F = x(y - z)i + y(z - x)j + z(x - y)k$ is solenoidal.
21. Show that $F = (y + z)i + (z + x)j + (x + y)k$ is irrotational and also find scalar potential $\emptyset$ such that $F = \nabla\emptyset$.
22. Show that $F = (y^2 - z^2 + 3yz + 2xy)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational.
23. Show that $\frac{xi+yj}{x^2+y^2}$ is both solenoidal and irrotational.
24. Find the directional derivative of $div[grad(\emptyset)]$ along $i + 2j - 2k$, where $\emptyset = x^2yz + y^3 - xz^2$ at the point $(1, 1, 1)$.
25. Find the value of a if $F = (ax^2y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)k$ is solenoidal,
26. Define an irrotational vector. Find the constant a, b and c such that $F = (axy - z^3)i + (bx^2 + z)j + (bxz^2 + cy)k$ is irrotational.

27. Find $a, b, c,$ if $F = (x + by - z)i + (2x - y + cz)j + (ax + y - z)k$ is irrotational. And also find scalar potential $\emptyset$ such that $F = \nabla\emptyset$.
28. Find the constants a and b if $F = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational, and also find the scalar potential.
29. Find the constants a, b and c if $F = (\sin y + az)i + (bx\cos y + z)j + (x + cy)k$ is irrotational.
30. If $F = (x + y + az)i + (bx + 2y - z)j + (x + cy + 2z)k$, Find the constant $a,$ b and c such that F is irrotational.
31. If $R = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ **,** show that i) $\nabla\frac{1}{r^2} = -\frac{2}{r^4}R$ ii) $\nabla \circ \frac{R}{r^2} = \frac{1}{r^2}$ iii) $div(r^n R) = (n + 3)r^n$ iv) $curl(r^n R) = 0$ v) $\nabla\left(\nabla \circ \frac{R}{r}\right) = -\frac{2}{r^3}R$
32. If $R = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then for what value of n, $r^n R$ is a) solenoidal b) irrotational c) both solenoidal and irrotational.
33. Find $\nabla\log(x^2 + y^2 + z^2)$ and grad$\left(\frac{1}{r}\right)$.
34. Show that i) $\nabla(r^n) = nr^{n-2}R,\ R = xi + yj + zk$ ii) $\nabla^2\,(r^n) = n(n + 1)r^{n-2}$.
35. If $f = (x^2 + y^2 + z^2)^{-n}$, Find $\mathrm{div}(\mathrm{grad}f)$ and determine n if $\mathrm{div}(\mathrm{grad}f) = 0$.
36. If $\vec{v} = \vec{w} \times \vec{r}$, then prove that $\mathrm{curl}\vec{v} = 2\vec{w}$.

ORTHOGONAL CURVILINEAR COORDINATES- Cylindrical and Spherical polar coordinates

37. Find the Scale factors for cylindrical system.
38. Find the Scale factors for spherical system.
39. Prove that cylindrical co-ordinate system is orthogonal.
40. Prove that spherical co-ordinate system is orthogonal.
41. Express the vector $z\,\boldsymbol{i} - 2x\,\boldsymbol{j} + y\,\boldsymbol{k}$ in cylindrical coordinates.
42. Express the vector $2x\,\boldsymbol{i} - 3y^2\,\boldsymbol{j} + xz\,\boldsymbol{k}$ in cylindrical coordinates.
43. Express the vector $yz\,\boldsymbol{i} - y\,\boldsymbol{j} + xz^2\,\boldsymbol{k}$ in cylindrical coordinates.
44. Express the vector $2y\,\boldsymbol{i} - z\,\boldsymbol{j} + 3x\,\boldsymbol{k}$ in spherical polar coordinates.
45. Express the vector $y\,\boldsymbol{i} - z\,\boldsymbol{j} + x\,\boldsymbol{k}$ in spherical coordinates.
46. Express the vector $x\,\boldsymbol{i} + 2y\,\boldsymbol{j} + yz\,\boldsymbol{k}$ in spherical coordinates.