Game Theory: Exploring Key Concepts Mathematically

Introduction

Game theory, a branch of mathematics, analyzes strategic interactions among rational decision-makers. It has diverse applications in economics, biology, political science, and artificial intelligence. This report explores fundamental concepts, focusing on strategic dominance, equilibrium concepts, sequential games, repeated and stochastic games, extensive form games, and evolutionary dynamics, emphasizing their mathematical foundations.

1 Strategic Dominance

Strictly Dominant Strategies

Definition: In a game, a strategy S_i for player i is strictly dominant if it always yields a higher payoff than any other strategy, regardless of the choices made by other players.

$$u_i(S_i, S_{-i}) > u_i(S_i', S_{-i}) \quad \forall S_i' \neq S_i$$

Weakly Dominant Strategies

Definition: A strategy S_i is weakly dominant if it yields a payoff at least as good as any other strategy, and strictly better when others do not play their strictly dominant strategies.

$$u_i(S_i, S_{-i}) \geq u_i(S_i', S_{-i}) \quad \forall S_i' \neq S_i \text{ or } S_i' = S_i \text{ and } S_{-i} \text{ does not play their strictly dominant str}$$

Maxmin Strategies

Definition: In a game, the maxmin strategy for a player is the strategy that maximizes the minimum payoff the player can receive, assuming rational opponents.

$$\text{Maxmin Strategy: } \max_{S_i} \min_{S_{-i}} u_i(S_i, S_{-i})$$

2 Equilibrium Concepts

Nash Equilibrium

Definition: A strategy profile $(S_1^*, S_2^*, ..., S_n^*)$ is a Nash Equilibrium if no player can unilaterally deviate to achieve a higher payoff.

$$u_i(S_1^*, S_2^*, ..., S_i, ..., S_n^*) \leq u_i(S_1^*, S_2^*, ..., S_i', ..., S_n^*) \quad \forall i, \forall S_i' \in S_i$$

Iterative Removal of Strictly Dominated Strategies

Definition: Iteratively eliminate strictly dominated strategies until no strictly dominated strategies remain. The resulting set of strategy profiles is a subset of Nash Equilibria.

3 Sequential and Repeated Games

Game Trees

Definition: A game tree represents sequential games. At each node, players make decisions based on the actions of previous players.

Repeated Games

Definition: In a repeated game G, strategies are applied over T periods, allowing players to strategize for the long term, considering future interactions.

$$u_i(s_1, s_2, ..., s_n) = \sum_{t=1}^{T} u_i(s_1(t), s_2(t), ..., s_n(t))$$

4 Extensive Form Games

Pure Strategies Example

Consider an extensive form game represented as a tree with nodes N and edges H. Each node $n \in N$ represents a decision point for a player. At each node, a player can choose an action from the set of available actions A_n. The payoffs are represented by the function $u_n(a_1, a_2, ..., a_n)$ where a_i is the action chosen by player i. For example, in the game tree below, player 1 chooses from actions $A_1 = \{A, B, C\}$, and player 2 chooses from actions $A_2 = \{X, Y\}$. The payoffs are represented at the terminal nodes.

		1	
	/	|	\
1		2	
3			
/	\	|	\
/	\		
A	B	C	X
Y	Z		
$(2,1)$	$(3,4)$	$(0,2)$	$(2,3)$
$(1,2)$	$(3,5)$		

Induced Normal Form

The induced normal form of an extensive form game converts the game tree into a matrix representation. Each combination of actions chosen by the players corresponds to a cell in the matrix. For the above game:

				Player 2		
		XA	XB	XC	YA	YB
YC						
3*Player 1	A	$(2,1)$	$(3,4)$	$(0,2)$	$(2,3)$	$(1,2)$
$(3,5)$						
	B	$(\dots)$	$(\dots)$	$(\dots)$	$(\dots)$	$(\dots)$
$(\dots)$						
	C	$(\dots)$	$(\dots)$	$(\dots)$	$(\dots)$	$(\dots)$
$(\dots)$						

Subgame Perfection

Definition: A strategy profile $(\sigma_1, \sigma_2, ..., \sigma_n)$ is subgame perfect if it represents a Nash Equilibrium in every subgame of the original game. Subgame perfection is achieved through backward induction, where each player optimizes their payoff at every information set.

Let S_i represent the set of pure strategies for player i. A strategy profile σ_i for player i in an information set I is subgame perfect if, for all players j and all strategies σ'_j in S_j, the following condition holds for all $I' \subseteq I$:

$$u_i(\sigma_i(I), \sigma'_j, \sigma_{-i}(I')) \geq u_i(\sigma_i(I), \sigma_j, \sigma_{-i}(I'))$$

where σ_{-i} represents strategies of all players except player i, and u_i is the payoff function for player i.

Imperfect Information Extensive Form

Formal Definition: In imperfect information extensive form games, players make decisions based on their private information and their beliefs about the opponents' strategies. Each player i has a set of information sets I_i representing their observations. Player i's strategy σ_i specifies actions for each information set I_i.

$$\sigma_i : I_i \to S_i$$

where S_i represents the set of pure strategies for player i.

5 Mixed and Behavioral Strategies

Mixed Strategies

Definition: In mixed strategies, players randomize over their pure strategies using probabilities. Let S_i be the set of pure strategies for player i. A mixed strategy for player i, denoted as σ_i, is a probability distribution over S_i.

$$\sigma_i = (p_{i1}, p_{i2}, ..., p_{in})$$

where p_{ij} represents the probability of choosing strategy j for player i.

6 Repeated Games

OPEC and Cartel Examples

In the context of repeated games, let's consider two classic examples: OPEC (Organization of the Petroleum Exporting Countries) and a cartel. These are illustrative of how repeated games can impact real-world decision-making.

OPEC Example

Imagine OPEC countries repeatedly deciding whether to cut or maintain their oil production levels. The payoff for each country depends on their collective decisions over time, affecting oil prices and market share.

Cartel Example

In a cartel, firms repeatedly decide whether to collude or compete in setting prices. This decision affects their individual profits and the overall market structure.

Infinitely Repeated Games and Utility

In infinitely repeated games, strategies are chosen in each period, and players aim to maximize their cumulative utility over an infinite horizon. Utility in such games is often expressed using discounted rewards.

Discounted Reward Math

Let $u_i(t)$ be the payoff for player i at time t. The utility of player i in an infinitely repeated game with a discount factor δ is given by:

$$U_i = \sum_{t=0}^{\infty} \delta^t u_i(t)$$

Here, δ represents the discount factor, which determines how much weight players assign to future payoffs. Smaller δ values prioritize immediate rewards, while larger δ values prioritize long-term rewards.

7 Stochastic Games

Stochastic games are a generalization of repeated games where the game's dynamics include randomness. Players make decisions in uncertain environments, making stochastic games suitable for modeling real-world scenarios.

Generalization of Repeated Game

In a stochastic game, players choose actions at each stage, and the resulting payoffs depend not only on their actions but also on a random state transition. The game is described by a tuple (S, A, P, R), where: - S is the set of states. - A is the set of actions for each player. - P represents the state transition probabilities. - R is the reward function.

Fictitious Play

Fictitious play is a dynamic learning process in repeated games where players form beliefs about their opponents' strategies based on past observations. It's a useful concept for studying how strategies evolve over time.

Convergence of Nash Equilibrium Theorem

Fictitious play converges to a Nash equilibrium in many repeated games. Specifically, it is shown that, in a two-player, zero-sum game, if players follow fictitious play, their average strategies converge to a Nash equilibrium.

Mixed Strategy and No Regret Learning

Mixed strategies involve randomizing over pure strategies with specific probabilities. Players may adopt mixed strategies in repeated games to maximize their expected utility. No regret learning is a common way to adapt and refine strategies over time.

Discounted Repeated Games and Folk Theorem

Discounted repeated games introduce the concept of discounting to encourage cooperation in repeated games. The Folk Theorem provides conditions under which a wide range of equilibrium outcomes can be supported in repeated games with discounting.

Folk Theorem

The Folk Theorem states that in infinitely repeated games with discounting, almost any payoff vector that provides players with payoffs exceeding the minmax values of the stage game can be sustained as a subgame perfect Nash equilibrium.

Understanding these concepts and their mathematical foundations is essential for analyzing strategic interactions in repeated games and stochastic environments.

8 Bayesian Games

Expected Utility

Definition: In a Bayesian game, each player has private information represented by a type. The expected utility of a strategy for player i is calculated by integrating over the player's type space Θ_i.

$$u_i(s_i, s_{-i}) = \int_{\Theta_i} u_i(t_i, s_{-i}) \cdot f(t_i)\, dt_i$$

where $u_i(t_i, s_{-i})$ represents the utility of player i given their type t_i and others' strategies s_{-i}, and $f(t_i)$ is the probability density function of player i's type.

Interim Expected Utility

Definition: Interim expected utility considers that one player knows their own type but not the opponent's. It involves averaging the expected utility over the uncertain opponent's types.

$$\bar{u}_i(t_i, s_{-i}) = \int_{\Theta_{-i}} u_i(t_i, t_{-i}, s_{-i}) \cdot f_{-i}(t_{-i})\, dt_{-i}$$

where $u_i(t_i, t_{-i}, s_{-i})$ represents the utility of player i given their own type t_i, opponent's type t_{-i}, and strategies s_{-i}, and $f_{-i}(t_{-i})$ is the probability density function of opponent's types.

Bayesian Equilibrium

Definition: A strategy profile $(\sigma_1, \sigma_2, ..., \sigma_n)$ is a Bayesian Equilibrium if, for each player i, their strategy maximizes their expected utility given their beliefs about others' types and strategies.

$$\sigma_i^* \in \arg\max_{\sigma_i} \int_{\Theta_{-i}} \int_{\Theta_i} u_i(t_i, t_{-i}, \sigma_i, \sigma_{-i}) \cdot f(t_i, t_{-i}) \, dt_i \, dt_{-i}$$

where $u_i(t_i, t_{-i}, \sigma_i, \sigma_{-i})$ represents the utility of player i given their own type t_i, opponent's type t_{-i}, and strategies σ_i, σ_{-i}, and $f(t_i, t_{-i})$ is the joint probability density function of players' types.

9 Coalition Games

Superadditive Games

Definition: In a coalition game, the worth of a coalition is superadditive if the worth of the coalition is greater than or equal to the sum of the individual worth of its members.

$$v(C_1 \cup C_2) \geq v(C_1) + v(C_2) \quad \forall C_1, C_2 \subseteq N$$

where $v(C)$ represents the worth of coalition C and N is the set of all players.

Shapley Value

Definition: The Shapley value assigns a unique payoff to each player in a coalition game, considering all possible orders in which players can join the coalition. It represents the average marginal contribution of a player across all possible coalitions.

$$\phi_i(v) = \sum_{\pi \in \Pi} \frac{1}{|\Pi|} \left(v(\pi^{-1}(i)) - v(\pi^{-1}(i) \setminus \{i\}) \right)$$

where $\phi_i(v)$ is the Shapley value for player i, $v(\pi^{-1}(i))$ is the worth of the coalition in permutation π where player i joins, and $|\Pi|$ is the total number of possible permutations.

Core Existence and Uniqueness

Definition: The core of a coalition game is the set of payoff vectors where no coalition can improve

Conclusion

Game theory provides a rigorous framework for analyzing strategic interactions. Understanding concepts like strategic dominance, equilibrium, sequential and repeated games, extensive form games, imperfect information games, mixed and behavioral strategies, repeated games with discounting, stochastic games, fictitious play, mixed strategy, and no-regret learning, and the Folk Theorem mathematically deepens insights into strategic decision-making. These mathematical foundations empower scholars and practitioners to model and predict complex real-world scenarios, fostering advancements across diverse fields.