W4

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1. **Temperature field.** Let the **isotherms** (curves of constant temperature) in a body in the upper half-plane $y > 0$ be given by $4x^2 + 9y^2 = c$. Find the orthogonal trajectories (the curves along which heat will flow in regions filled with heat-conducting material and free of heat sources or heat sinks).

2. **Cauchy–Riemann equations.** Show that for a family $u(x, y) = c = \text{const}$ the orthogonal trajectories $v(x, y) = c^* = \text{const}$ can be obtained from the following *Cauchy–Riemann equations* (which are basic in complex analysis in Chap. 13) and use them to find the orthogonal trajectories of $e^x \sin y = \text{const}$. (Here, subscripts denote partial derivatives.)

$$u_x = v_y, \quad u_y = -v_x$$

Q1) a) $4x^2 + 9y^2 = c$

diff. w.r.t x

$8x + 18yy' = 0$

$y' = \frac{-4x}{9y}$ ① $= f(x,y)$

so orthogonal trajectories is

$\bar{y}' = -\frac{1}{f(x,y)}$

$\frac{d\bar{y}}{\bar{y}} = \frac{9}{4} \cdot \frac{dx}{x}$

Integrate both side

$\ln \bar{y} = \frac{9}{4} \cdot \ln(x) + C$

$y = e^{9/4 \ln(x+c)}$

$\boxed{y = Cx^{9/4}}$

Q2) a) $u(x,y) = c$

$u_x\,dx + u_y\,dy = 0$

$y' = \frac{-u_x}{u_y}$ so O.T $\Rightarrow y' = \frac{u_y}{u_x}$

$V(x,y) = c^*$

after differentiating w.r.t x, so $y' = \frac{-V_x}{V_y}$

A.T.Q let $u = e^x \sin y$

$u_x = e^x \sin y$ & $u_y = e^x \cos y$

$[\, u_x = V_y \text{ (given)} \,]$

lets prove this

$\int V_y = \int e^x \sin y \Rightarrow V = -e^x \cos y + f(x)$

$V_x = -e^x \cos y + f'(x)$

$f'(x) = 0$ so $f(x) = C$

$V = -e^x \cos y + c \Rightarrow \boxed{e^x \cos y = C_1}$

Q3) $y' = f(x)$

$\int dy = \int f(x)dx \Rightarrow y = g(x) + C$ here we can see that $g(x)$ is independent of y

y is the shape of family will be same & there $C \to$ represent transformation but get similar curve

for their O.T the case remains the same just $y' = \frac{-1}{f(y)}$

4. **Hanging cable.** It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving $y'' = k\sqrt{1 + y'^2}$, where the constant k depends on the

Answer) $y'' = k\sqrt{1+y'^2}$ $k=1$, $y(1) = y(-1) = 0$

let $y' = z$

Hanging cable. It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving

$y'' = k\sqrt{1 + y'^2}$, where the constant k depends on the weight. This curve is called *catenary* (from Latin *catena* = the chain). Find and graph $y(x)$, assuming that $k = 1$ and those fixed points are $(-1, 0)$ and $(1, 0)$ in a vertical xy-plane.

Answer) $y = k\sqrt{1+y'^2}$ $\quad k = 1$, $y(1) = y(-1)$

let $y' = z$

so $z' = k\sqrt{1+z^2} \Rightarrow \dfrac{dz}{dx} = k\sqrt{1+z^2}$

$z' = k\sqrt{1+z^2}$ [we know $k=1$]

$z' = \sqrt{1+z^2}$

$z = \sinh(x + c_1)$

$\dfrac{dy}{dx} = \sinh(x+c_1)$ [we know $z = y'$]

$\boxed{y = \cosh(x+c_1) + c_2}$

we know $y(-1) = y(1) = 0 \Rightarrow$ $1 = \cosh(x + c_1) + c_2$ (1)

$-1 = \cosh(x+c_1) + c_2$ (2)

solving (1) & (2)

$\boxed{c_1 = 0 \quad \& \quad c_2 = \cosh(1)}$

Q5) $v = \dfrac{1}{a}$ (given)

$a = \dfrac{1}{v} \Rightarrow \dfrac{dv}{dt} = \dfrac{1}{v} \Rightarrow \int v\,dv = \int dt \Rightarrow \dfrac{v^2}{2} = t + c$ $\quad$ [at $t = 0$, $v = v_0$]

$\dfrac{v^2}{2} = c \Rightarrow \dfrac{v^2}{2} = t + \dfrac{v_0^2}{2}$

$\boxed{v = \sqrt{(2t + v_0^2}}$

$\dfrac{dy}{dt} = (2t + v_0^2)^{1/2} \Rightarrow \int dy = \int (2t + v_0^2)^{1/2} \Rightarrow y + c = \dfrac{1}{2}\dfrac{(2t+v_0^2)^{3/2}}{3/2}$

[$y(0) = y_0$]

$y_0 + c = \dfrac{v_0^3}{3}$, $\quad c = \dfrac{v_0^3}{3} - y_0$

$\boxed{y(t) = \dfrac{(v_0^2 + 2t)^{3/2}}{3} - \dfrac{v_0^3}{3} + y_0}$