4.

(a) $y^T P y = y^T A^T A y = (Ay)^T (Ay) \geq 0$ —— (i)

$z^T Q z = z^T A A^T z = (A^T z)^T (A^T z) \geq 0$ —— (ii)

Suppose u is an eigenvector of P corresponding to an eigenvalue λ. We have

$$Pu = \lambda u \Rightarrow u^T P u = u^T \lambda u = \lambda (u^T u)$$

Now, $u^T P u \geq 0$ (from (i)), as is $u^T u$. Thus, $\lambda \geq 0$.

Similarly, let v be an eigenvector of Q corresponding to an eigenvalue μ.

$$Qv = \mu v \Rightarrow v^T Q v = v^T \mu v = \mu (v^T v)$$

Again, $v^T Q v \geq 0$ ((ii)), and $v^T v \geq 0$, so $\mu \geq 0$.

Thus, the eigenvalues of P and Q are non-negative.

(b) $Pu = \lambda u \Rightarrow A^T A u = \lambda u$

$$\Rightarrow A A^T A u = A \lambda u = \lambda (Au)$$

Thus, $Q(Au) = \lambda (Au)$, so Au is an eigenvector of Q with eigenvalue λ, if u is an eigenvector of P with eigenvalue λ.

Similarly, let v be an eigenvector of Q with eigenvalue μ, so

$$Qv = \mu v \Rightarrow AA^T v = \mu v$$

$$\Rightarrow A^T A A^T v = A^T \mu v = \mu (A^T v)$$

Thus, $P(A^T v) = \mu (A^T v)$, meaning $A^T v$ is an eigenvector of P with eigenvalue μ.

A is of size $m \times n$, so A^T is of size $n \times m$.

$P = A^T A$ is $n \times n$, and $Q = AA^T$ is $m \times m$.

Thus, u is of size $(n \times 1)$, and v is of size $(m \times 1)$.

(c) Suppose v_i is an eigenvector of Q with eigenvalue λ_i.

Thus, $Qv_i = \lambda_i v_i \Rightarrow AA^T v_i = \lambda_i v_i$

$$u_i = \frac{A^T v_i}{\|A^T v_i\|}, \text{ so } Au_i = A\left(\frac{A^T v_i}{\|A^T v_i\|}\right) = \frac{AA^T v_i}{\|A^T v_i\|}$$

Thus, $$Au_i = \frac{\lambda_i v_i}{\|A^T v_i\|} = \left(\frac{\lambda_i}{\|A^T v_i\|}\right) v_i = \gamma_i v_i,$$

where $\gamma_i = \dfrac{\lambda_i}{\|A^T v_i\|}$ is real and non-negative

(as λ_i being an eigenvalue of Q is ≥ 0 as shown in (a))

(d) $U = [v_1 | v_2 | \cdots | v_m]$, $V = [u_1 | u_2 | \cdots | u_n]$

Case I : $m \geq n$

$$U \Gamma V^T = [v_1 | v_2 | \cdots | v_m] \begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_n \end{bmatrix} [u_1 | u_2 | \cdots | u_n]^T$$

$$= [r_1 v_1 | r_2 v_2 | \cdots | r_n v_n] [u_1 | u_2 | \cdots | u_n]^T$$

$$= [Au_1 | Au_2 | \cdots | Au_n] [u_1 | u_2 | \cdots | u_n]^T \text{ (from (c))}$$

$$= A [u_1 | u_2 | \cdots | u_n] [u_1 | u_2 | \cdots | u_n]^T$$

$$= AI = A$$

where the last result follows from the facts that $u_i^T u_j = 0$, $i \neq j$ and $u_i^T u_i = 1$ (as u_i is a normalized vector), so $VV^T = V^T V = I$.

Case II : $m < n$

$$U \Gamma V^T = [v_1 | v_2 | \cdots | v_m] \begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_m \end{bmatrix} [u_1 | u_2 | \cdots | u_n]^T$$

$$= [v_1 | v_2 | \cdots | v_m] [r_1 u_1 | r_2 u_2 | \cdots | r_m u_m]^T$$

$$= [r_1 v_1 | r_2 v_2 | \cdots | r_m v_m] [u_1 | u_2 | \cdots | u_m]^T$$

$$= [Au_1 | Au_2 | \cdots | Au_m] [u_1 | u_2 | \cdots | u_m]^T$$

$$= A [u_1 | u_2 | \cdots | u_m] [u_1 | u_2 | \cdots | u_m]^T = AI = A.$$