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Quiz : 4

Proof by induction on $|X|$.

Base Case : $|X| = 1$, If the poset consists of only 1 element, then the $X = \{x\}$ and assign $f(x) = 1$ hence there is a valid topological sorting.

Induction Hypothesis :→ Assume that for all posets with $|X| = k$ has a valid topological sorting.

Induction Step : For a poset with $|X| = k+1$.

~~I make a small claim that~~

Claim :→ ~~Poset has a maximal element~~. Any finite poset has a maximal element.

Proof :→ Let us assume not, then let's take a element ($|X| = k+1$) $a_1 \in X$. Since a_1 is not maximal, $\exists a_2 \in X$: $a_1 \le a_2$, $a_1 \neq a_2$.

Now continuing the same step $k+1$ times we will have $a_1 \le a_2 \le a_3 \dots \dots \le a_{k+1} \le a_{k+2}$ (all are pairwise distinct) but our set has only $k+1$ elements, Hence this is a contradiction. Hence any poset with finite size must have a maximal element (as k is general).

Now Let a maximal element of $(X, \le)$ be m, Let's take the set $(X - \{m\}, \le)$ here $|X - \{m\}| = k$ and this is a poset so by IH, we have $f_1 : X - \{m\} \to \{1, \dots, k\}$

~~Let us define~~

$$f(x) = \begin{cases} k+1 & \text{if } x = m \\ f_1(x) & \text{otherwise.} \end{cases}$$

This is valid topological sorting, since the f_1 was a valid sorting and given $f(m) = k+1$ means that there will be no x s.t. $m \le x$, hence the topological sort is valid.

Induction Completes.

Any finite set has a valid topo-sort. Hence proved. □