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Final Project Report

Portfolio Optimisation Using Markowitz Mean Variance Model

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Declaration

We, Utkarsh Agarwal, Ayush Bhatnagar, Dhirain Vij, and Arya Lamba, certify that this project is our own work, based on our personal study and/or research and that we have acknowledged all material and sources used in its preparation, whether they be books, articles, reports, lecture notes, and any other kind of document, electronic or personal communication. We also certify that this project has not previously been submitted for assessment in any academic capacity, and that we have not copied in part or whole or otherwise plagiarised the work of other persons. We confirm that we have identified and declared all possible conflicts that we may have.

Signed and submitted

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Abstract

This project delves into the practical application and limitations of the Mean-Variance optimization model in portfolio management, exploring both conventional financial contexts and innovative non-financial applications. The study begins with a detailed analysis of the Mean-Variance model, emphasizing its reliance on historical data and the assumption of a stable market environment. Mathematical insights into quadratic programming, analytical solutions, Lagrange multipliers, and constraint formulation are provided, highlighting the elegance and challenges of these methodologies.

The empirical investigation utilizes real-world data from Yahoo Finance, incorporating ten portfolios representing various sectors. Findings include a comparative analysis of portfolios optimized for minimum risk, equal-weighted portfolios, and portfolios maximizing quadratic utility. Volatility and expected vs. actual returns are scrutinized, revealing the model's strengths in delivering low-volatility portfolios but raising questions about its predictive power during economic uncertainties.

The study extends to propose future enhancements, emphasizing the adoption of a multiple factor model to overcome the limitations of historical data reliance. This model incorporates macroeconomic, fundamental, and statistical factors, offering improved diversification, risk assessment, and adaptability to changing market conditions. The integration of dynamic model calibration and continuous learning is suggested to enhance the model's robustness.

Non-conventional applications of the Mean-Variance model in robotics and supply chain management showcase its versatility. In robotics, the model optimizes motion planning by balancing the risk of collisions with the reward of task efficiency. In supply chain management, it identifies configurations offering optimal trade-offs between efficiency and risk.

The key takeaways emphasize the limitations of the Mean-Variance model and the importance of dynamic adaptation, factor-based optimization, and holistic decision-making. The journey from traditional optimization to advanced models reflects the evolution of quantitative techniques in finance, urging practitioners to strike a nuanced balance between mathematical rigor and real-world adaptability.

In conclusion, while the Mean-Variance model remains foundational, this study advocates for a comprehensive and adaptive approach to portfolio optimization. Continuous learning, factor-based models, and non-conventional applications contribute to the ongoing dialogue on the refinement and evolution of quantitative methodologies in financial decision-making.

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1. Introduction

1.1 Project Description

In the ever-evolving landscape of financial markets, effective portfolio management is crucial for investors seeking to optimize returns while managing risk. This project delves into the fundamentals of portfolio construction and optimization, exploring key concepts such as portfolio return, Markowitz Mean–Variance model, and analytical solutions for minimum risk.

By examining scenarios with two, three, and N risky assets, the project provides insights into the complexities of constructing efficient portfolios.

The Markowitz Mean–Variance model, a cornerstone of modern portfolio theory, forms the basis for the analysis. It considers expected returns, standard deviation, and covariance of assets to create portfolios that balance risk and reward. The report progresses from the two-asset case to more generalized scenarios, demonstrating the application of quadratic programming models to find optimal portfolios.

The exploration of efficient portfolios and the efficient frontier emerges as a crucial aspect of our model. By optimizing portfolios for varying levels of risk or return, we unveil the trade-offs that define the frontier. The introduction of quadratic utility, allowing optimization based on varying levels of risk aversion allows investors to make decisions aligned with their risk preferences.

Additionally, the project discusses common constraints in portfolio optimization, emphasizing the importance of factors like budget constraints, upper/lower bounds on positions, exposure limitations, and leverage constraints. The inclusion of these constraints refines the portfolio optimization process, aligning it with specific investment goals and risk preferences.

The goal of this project is to find and analyse an analytical solution to the simplified version of the Markowitz basic mean variance model taking maximum risk aversion, representing the search for the minimum-risk fully invested portfolio.

1.2 Assumptions

1) Normal Distribution of Returns

This assumption is often made for mathematical tractability and simplicity. The normal distribution is well-understood and characterized by its mean and standard deviation, making it convenient for financial modeling. However, financial markets frequently exhibit fat tails and skewness, suggesting deviations from a perfect normal distribution. The assumption is maintained due to its historical adoption in financial theory.

2) Stationarity of Returns

The assumption of stationarity simplifies the modeling process by assuming that statistical properties remain constant over time. This is a common assumption in financial modeling, but it might not hold in dynamic markets where economic conditions, regulations, and investor behaviors change. Despite its limitations, stationarity provides a convenient starting point for modeling.

3) Mean-Variance Framework

The mean-variance framework, introduced by Markowitz, simplifies decision-making by focusing on expected returns and variances only. Higher moments of the return distribution (skewness, kurtosis) are ignored for computational ease. The assumption is maintained because it facilitates the application of mathematical optimization techniques and has been a foundational concept in portfolio theory.

4) Homogeneous Investor Expectations

The assumption of homogeneous expectations simplifies the model by treating all investors as having the same views about future returns and risk. While it eases computational complexity, it oversimplifies reality, as investors in financial markets often have diverse opinions, strategies, and risk tolerances. This simplification is made for the sake of modeling feasibility.

5) Risk Aversion Coefficient

The introduction of a linear risk aversion coefficient in the quadratic utility function is a simplification of investor behavior. It assumes that investors' risk preferences can be captured by a single parameter, making the model more tractable. While this might not fully represent the complexity of individual risk attitudes, it provides a practical approach for optimization.

6) Precision Matrix Properties

The positive semidefinite nature of the covariance matrix and its inverse (precision matrix) is essential for the convexity of optimization models. Convexity ensures the existence of a unique minimum or maximum in the optimization process. The assumption of positive semi definiteness is maintained for theoretical robustness and to guarantee the solvability of the optimization problems in portfolio construction.

In summary, these assumptions are often trade-offs between model simplicity and real-world complexity. While they might not perfectly represent the intricacies of financial markets, they enable the development of models that are computationally feasible and provide valuable insights within the limitations of the assumptions.

2. Portfolio Return

Assume a portfolio must be selected at some initial time t_0 and held until time t. Let $\mathrm{v}_0 = [v_{1,0} \quad \cdots \quad v_{n,0}]^\top$ and $\mathrm{v} = [v_1 \quad \cdots \quad v_n]^\top$ denote the vectors of asset prices at times t_0 and t respectively. The vector v_0 is known whereas v is a vector of random variables. A vector $\mathrm{h} \in \mathbb{R}^n$ of share holdings in each of the assets defines a portfolio whose values at time t_0 and t are $W_0 := \mathrm{v}_0^\top \mathrm{h}$ and $W := \mathrm{v}^\top \mathrm{h}$ respectively.

The value W_0 is known at time t_0 whereas W is a random variable.
The gist of portfolio construction is to choose h to optimize some measure of satisfaction on the random variable W.

$$\text{portfolio return, } r_P = \frac{W - W_0}{W_0}.$$

The return of asset i, which is the same as that of a portfolio entirely invested in asset i, is similarly defined as

$$r_i = \frac{v_i - v_{i,0}}{v_{i,0}}.$$

Instead of the vector of holdings $\mathrm{h} \in \mathbb{R}^n$, the portfolio construction problem is often stated in terms of percentage holdings $\mathrm{x} \in \mathbb{R}^n$ where

$$x_i = \frac{h_i v_{i,0}}{W_0} = \frac{h_i v_{i,0}}{\sum_{j=1}^{n} h_j v_{j,0}}$$

Observe that $W = \mathrm{v}^\top \mathrm{h}$ can be equivalently written as

$$r_P = \sum_{i=1}^{n} r_i x_i = \mathrm{r}^\top \mathrm{x}.$$

This convention runs into difficulties in some cases. For example, the above quantity r_P does not make sense for a long-short portfolio associated with a pairs trading strategy. More broadly, the quantity r_P does not make sense for a situation where the initial value of a portfolio W_0 is zero as when one enters a futures contract or constructs a long-short portfolio with equal long and short cash positions.

This difficulty can be amended by assuming that returns are measured relative to some predefined basis value b as opposed to the initial portfolio value W_0. To make this idea more precise, we associate with each asset and portfolio a basis b that satisfies the following four properties:

- The basis b for a long position of an asset is positive.
- The basis b is measured in the same unit as the asset values.
- The basis is homogeneous: the basis of k shares of an asset is k times the basis of one share.
- The basis is known at time t_0.

New definition of asset and portfolio returns:

$$r_i = \frac{v_i - v_{i,0}}{b_i}, r_P = \frac{W - W_0}{b_P}.$$

Percentage holdings:

$$x_i = \frac{h_i b_i}{b_P}.$$

$W = \mathrm{v}^{\mathsf{T}}\mathrm{h}$ can be equivalently written as $r_P = \mathrm{r}^{\mathsf{T}}\mathrm{x}$.

Throughout this report x will denote the vector of percentage holdings of a portfolio in a universe of n risky assets. When it is applicable and evident from the context, we shall assume the usual basis values $b_i = v_{i,0}$ and $b_P = W_0$ respectively.

3. Markowitz Mean–Variance (Basic Model)

Markowitz's key insight into the above one-period investment problem was to consider the expected value and standard deviation of the return as measures of performance and risk respectively. The portfolio selection problem can then be formally stated as a quadratic programming model. To simplify our discussion of this model, we will proceed in three incremental steps. First, we will look at the case when there are only two assets; second, we will look at the case when there are three risky assets; and finally, we will see the general case with any number of risky assets.

3.1 Two Risky Assets

Suppose we are combining two assets whose random returns are r_1 and r_2. Let

$$\mu_1 := \mathbb{E}(r_1),\ \mu_2 := \mathbb{E}(r_2),$$

and

$$\sigma_1^2 := \text{var}\ (r_1),\ \sigma_2^2 = \text{var}\ (r_2),\ \sigma_{12} = \text{cov}\ (r_1, r_2) = \rho \cdot \sigma_1 \cdot \sigma_2.$$

In this case a portfolio of these two assets is determined by the proportion invested in one of the two assets. Let x denote the proportion in asset 1. Thus, the portfolio return is,

$$r_P = x \cdot r_1 + (1 - x) \cdot r_2$$

the portfolio expected return is,

$$\begin{aligned} \mu_P := \mathbb{E}(r_P) \quad &= x \cdot \mathbb{E}(r_1) + (1 - x) \cdot \mathbb{E}(r_2) \\ &= x \cdot \mu_1 + (1 - x) \cdot \mu_2, \end{aligned}$$

and the portfolio variance is,

$$\sigma_P^2 = x^2 \sigma_1^2 + (1 - x)^2 \sigma_2^2 + 2 \cdot x(1 - x) \cdot \rho \cdot \sigma_1 \cdot \sigma_2.$$

3.2 Three Risky Assets

Suppose now that there are three assets with random returns r_1, r_2, and r_3. As before, let,

$$\mu_j = \mathbb{E}(r_j),\ \sigma_j^2 := \text{var}\ (r_j)\ \text{ for } j = 1,2,3,$$

and

$$\sigma_{ij} := \text{cov}\ (r_i, r_j) = \rho_{ij} \cdot \sigma_i \cdot \sigma_j \text{ for } i, j = 1,2,3.$$

Now a portfolio determines the holdings in the three assets. Let x_j denote the proportion (weight) invested in asset j, for $j = 1,2,3$. Notice that these proportions should add up to one if the portfolio is fully invested in the three assets:

$$x_1 + x_2 + x_3 = 1.$$

Like what we did before, the portfolio return is,

$$r_P = r_1 x_1 + r_2 x_2 + r_3 x_3.$$

portfolio expected return is,

$$\mu_P = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3,$$

portfolio variance is,

$$\sigma_P^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 + 2(\sigma_{12} x_1 x_2 + \sigma_{23} x_2 x_3 + \sigma_{13} x_1 x_3).$$

A portfolio is efficient if it has minimum risk for a given target return, or equivalently, if it has the maximum expected return for a given target risk. This naturally leads to the following quadratic programming formulation.

To find a portfolio of minimum risk (variance) with expected return at least $\bar{\mu}$ we need to solve the following mean-variance optimization model:

$$\begin{aligned} \min_{\mathrm{x}} \quad & \sum_{i=1}^{3} \sigma_{ii} x_i^2 + 2 \sum_{i=1}^{3} \sum_{j=i+1}^{3} \sigma_{ij} x_i x_j \\ \text{s.t.} \quad & \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 \geq \bar{\mu} \\ & x_1 + x_2 + x_3 = 1. \end{aligned}$$

The efficient frontier is the set of efficient portfolios. The efficient frontier is often "visualized" by plotting the expected return against the standard deviation of the efficient portfolios. To generate portfolios on the efficient frontier, we can minimize variance, for varying target return $\bar{\mu}$:

$$\begin{aligned} \min_{\mathrm{x}} \quad & \sum_{i=1}^{3} \sigma_{ii} x_i^2 + 2 \sum_{i=1}^{3} \sum_{j=i+1}^{3} \sigma_{ij} x_i x_j \\ \text{s.t.} \quad & \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 \geq \bar{\mu} \\ & x_1 + x_2 + x_3 = 1. \end{aligned}$$

We can also maximize return, for varying target variance $\bar{\sigma}^2 > 0$:

$$\begin{aligned} \max_{\mathrm{x}} \quad & \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 \\ \text{s.t.} \quad & \sum_{i=1}^{3} \sigma_{ii} x_i^2 + 2 \sum_{i=1}^{3} \sum_{j=i+1}^{3} \sigma_{ij} x_i x_j \leq \bar{\sigma}^2 \\ & x_1 + x_2 + x_3 = 1. \end{aligned}$$

Or we can maximize quadratic utility, for varying risk aversion $\gamma > 0$:

$$\begin{aligned} \max_{\mathrm{x}} \quad & \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 - \frac{\gamma}{2} \left(\sum_{i=1}^{3} \sigma_{ii} x_i^2 + 2 \sum_{i=1}^{3} \sum_{j=i+1}^{3} \sigma_{ij} x_i x_j \right) \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1. \end{aligned}$$

3.3 N Risky Assets

Assume we have n risky assets. Let r ∈ Rn be the n-dimensional random vector of returns, i.e., r denotes the return of asset i between times t_0 and t. Let $\mu \in \mathbb{R}^n$ denote the vector of expected returns, and V ∈ $\mathbb{R}^{n\times n}$ denote the return covariance matrix. More precisely,

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}, \mathrm{V} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$$

where $\mu_i := \mathbb{E}(r_i), \sigma_{ij} := \mathrm{co\,v}(r_i, r_j), i, j = 1, \dots, n$.

The expected return and variance of a given portfolio $\mathrm{x} = [x_1 \quad \cdots \quad x_n]^\top$ of the risky assets are respectively,

$$\mu^\top \mathrm{x} = \sum_{j=1}^{n} \mu_j x_j$$

and

$$\mathrm{x}^\top \mathrm{Vx} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j = \sum_{i=1}^{n} \sigma_{ii} x_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sigma_{ij} x_i x_j$$

A fully invested portfolio is efficient if it has minimum risk for a given level of return, or equivalently if it has maximum expected return for a given level of risk.

A fully invested efficient portfolio can then be characterized as the solution to the following quadratic program:

$$\begin{aligned} \max_{\mathrm{x}} \quad & \mu^\top \mathrm{x} - \tfrac{1}{2}\gamma \cdot \mathrm{x}^\top \mathrm{Vx} \\ & 1^\top \mathrm{x} = 1 \end{aligned} \tag{1}$$

for some risk-aversion coefficient $\gamma > 0$.

The set of efficient portfolios can also be obtained as the set of solutions to the quadratic program:

$$\begin{aligned} \min_{\mathrm{x}} \quad & \mathrm{x}^\top \mathrm{Vx} \\ \text{s.t.} \quad & \mu^\top \mathrm{x} \geq \bar{\mu} \\ & 1^\top \mathrm{x} = 1, \end{aligned} \tag{2}$$

and as the set of solutions to:

$$\begin{aligned} \max_{\mathrm{x}} \quad & \mu^\top \mathrm{x} \\ \text{s.t.} \quad & \mathrm{x}^\top \mathrm{Vx} \leq \bar{\sigma}^2 \\ & 1^\top \mathrm{x} = 1 \end{aligned} \tag{3}$$

by varying $\bar{\mu}$ and $\bar{\sigma}$ respectively.

4. Analytical Solution To Basic Mean–Variance Model (Minimum Risk)

Throughout, we assume that the covariance matrix of asset returns V is positive definite. In particular, V^{-1} exists.

First, let us look at a special case of a quadratic program with equality constraints only. Consider the problem -

$$\min \quad \frac{1}{2}x^T Qx + c^T x$$

$$\text{s.t.} \quad Ax = b,$$

where $(c \in R^n), (Q \in R^{n\times n}), (A \in R^{m\times n}), (b \in R^m)$, and Q is symmetric and positive semidefinite. In this case, the optimality conditions simplify to

$$Qx + c - A^T y = 0 \quad \text{and} \quad Ax - b = 0.$$

The optimality conditions in turn can be stated in terms of the LaGrange function -

$$L(x, y) = \frac{1}{2}x^T Qx + c^T x + y^T(b - Ax).$$

When Q is positive definite and A has full row rank, problem has a unique minimizer x and a unique Lagrange multiplier y given by

$$x = Q^{-1}A^T(AQ^{-1}A^T)^{-1}b$$

$$y = (AQ^{-1}A^T)^{-1}b$$

Where y is the LaGrange multiplier.

Now, coming back to our quadratic function, consider the simplified version of Eq. (1) that is obtained in the limit when $\gamma \to \infty$:

$$\min_{\mathrm{x}} \quad \mathrm{x}^\mathrm{T}\mathrm{V}\mathrm{x} \qquad 1^\mathrm{T}\mathrm{x} = 1 \tag{4}$$

From the above discussion it readily follows that the optimal solution is:

$$\mathrm{x}^* = \frac{1}{1^\mathrm{T}\mathrm{V}^{-1}1}\mathrm{V}^{-1}1 \tag{5}$$

The solution is obtained by utilizing Lagrange multipliers and setting up the Lagrangian for the optimization problem. Solving the resulting system of equations, which includes partial derivatives with respect to the decision variables (portfolio weights) and the Lagrange multiplier.

5. More Common Constraints

Aside from a target expected return or a target variance, the only portfolio constraint in the basic mean-variance model is the full investment constraint.

$$1^{\mathsf{T}}x = 1$$

Furthermore, this constraint disappears if the portfolio is allowed to include holdings in a risk-free asset. In both cases the individual portfolio holdings could in principle take arbitrary positive and negative values as there is no explicit restriction on them.

This motivates the following types of constraints that are often included in a mean-variance model:

- Budget constraints, such as fully invested portfolios.
- Upper and/or lower bounds on the size of individual positions.
- Upper and/or lower bounds on exposure to industries or sectors.
- Leverage constraints such as long-only, or 130/30 constraints.
- Turnover constraints.

The above types of constraints replace the single portfolio constraint,

$$1^{\mathsf{T}}x = 1$$

by a more elaborate set of constraints of the form

$$\begin{aligned} Ax &= b \\ Dx &\geq d \end{aligned}$$

The linear equality constraints impose restrictions on the portfolio weights. These constraints could represent, for example, budget constraints or exposure constraints to certain assets. The linear inequality constraints introduce additional constraints on the portfolio weights. These constraints could represent limits on sector exposures, risk factor exposures, or other relevant constraints.
Consequently, we get the following general version of the basic mean-variance model:

$$\begin{aligned} \max_{x} \quad & \mu^{\mathsf{T}}x - \tfrac{1}{2}\gamma \cdot x^{\mathsf{T}}Vx \\ & Ax = b \\ & Dx \geq d \end{aligned} \tag{6}$$

The set of portfolios obtained via the model can also be obtained via the following two equivalent models. The first one enforces a target expected return:

$$\begin{aligned} \min_{x} \quad & x^{\mathsf{T}}Vx \\ \text{s.t.} \quad & \mu^{\mathsf{T}}x \geq \bar{\mu} \\ & Ax = b \\ & Dx \geq d \end{aligned} \tag{7}$$

The second one enforces a target variance of return:

$$\begin{array}{ll} \max\limits_{\mathrm{x}} & \mu^\top \mathrm{x} \\ \text{s.t.} & \mathrm{x}^\top \mathrm{V} \mathrm{x} \leq \bar{\sigma}^2 \\ & \mathrm{Ax} = \mathrm{b} \\ & \mathrm{Dx} \geq \mathrm{d}. \end{array} \tag{8}$$

A "130/30" leverage constraint means that the total value of the holdings in short positions must be at most 30% of the portfolio value. In general, suppose that we want the value of the total short positions to be at most L. This means that we want to enforce the following restriction:

$$\sum_{j=1}^{n} \min(x_j, 0) \geq -L \Leftrightarrow \sum_{j=1}^{n} \max(-x_j, 0) \leq L$$

Although this is a correct mathematical formulation of the constraint, it is not ideal for computational purposes because of the non-smooth terms $\max(-x_j, 0)$.

If a constraint were written in this form the resulting mean variance model would not be a quadratic program. To formulate this constraint efficiently in the quadratic optimization model, we trade terms of the form $\max(-x_j, 0)$ for new terms involving possibly new variables and linear inequalities. To that end, add the new vector of variables $\mathrm{y} = [y_1 \quad \cdots \quad y_n]^\top$ and constraints,

$$\begin{gathered} \mathrm{x} \geq -\mathrm{y} \\ \sum_{j=1}^{n} y_j \leq L \\ \mathrm{y} \geq 0 \end{gathered}$$

A turnover constraint is a constraint on the total change in the portfolio positions. This constraint is generally included to limit certain kinds of costs such as taxes and transaction costs. Suppose that we have an initial portfolio $\mathrm{x}^0 = [x_1^0 \quad \cdots \quad x_n^0]^\top$ and we want to ensure that the new portfolio incurs a total turnover no larger than h. This means that we want to enforce the restriction,

$$\sum_{j=1}^{n} |x_j^0 - x_j| \leq h$$

To formulate this constraint efficiently in the quadratic optimization model, add the new vector of variables $\mathrm{y} = [y_1 \quad \cdots \quad y_n]^\top$ and constraints,

$$\begin{gathered} x_j - x_j^0 \leq y_j \\ x_j^0 - x_j \leq y_j \\ \sum_{j=1}^{n} y_j \leq h \end{gathered}$$

6. Case Study

In this case study, we explore the practical application of the model, considering its ability to deliver optimal portfolios with minimum risk in comparison to simple equal-weighted portfolios and those maximizing quadratic utility.

6.1 Aim

The overarching goal is to evaluate the performance of the portfolio optimization model under various portfolios and assess its suitability for real-world investment decisions. By comparing the model-generated portfolios with equal-weighted portfolios and those maximizing quadratic utility, we aim to understand the model's strengths, weaknesses, and its ability to adapt to changing market conditions.

6.2 Methodology

We implemented the portfolio optimization model using historical stock price data obtained from Yahoo Finance. The model utilizes the Efficient Frontier framework, considering expected returns and covariance matrix for the universe of risky assets. The optimization process involves finding the minimum-risk portfolio, an equal-weighted portfolio, and a portfolio maximizing quadratic utility and them performing a comparative analysis between the three using real-world data.

6.3 Data

To provide a comprehensive analysis, we incorporated ten different portfolios representing various sectors, each containing 4 assets and they are as follows -

1 - 'RELIANCE.BO', 'TCS.BO', 'HDFCBANK.BO', 'ICICIBANK.BO'
2 - 'INFY.BO', 'HINDUNILVR.BO', 'KOTAKBANK.BO', 'HCLTECH.BO’
3 - 'ITC.BO', 'BAJAJFINSV.BO', 'LT.BO', 'HDFCLIFE.BO'],
4 - 'AXISBANK.BO', 'POWERGRID.BO', 'WIPRO.BO', 'ULTRACEMCO.BO'
5 - 'BHARTIARTL.BO', 'COALINDIA.BO', 'SUNPHARMA.BO', 'CIPLA.BO'
6 - 'JSWSTEEL.BO', 'ONGC.BO', 'IOC.BO', 'TITAN.BO'
7 - 'MARUTI.BO', 'HEROMOTOCO.BO', 'ADANIPORTS.BO', 'UPL.BO'
8 - 'ASIANPAINT.BO', 'NTPC.BO', 'HINDALCO.BO', 'BAJFINANCE.BO'
9 - 'GRASIM.BO', 'EICHERMOT.BO', 'INDUSINDBK.BO', 'BPCL.BO'
10 - 'M&M.BO', 'ZEEL.BO', 'SBIN.BO', 'CIPLA.BO'

The average of daily returns from March 2010 – March 2019 was used to calculate the expected returns and the covariance matrix.

6.4 Tests Performed

1 - Comparison of the volatility of the various portfolios with optimal weights for minimum risk, equal weights, and optimal weights for maximum quadratic utility.

2 - Comparison of the expected vs actual returns of the various portfolios with optimal weights for minimum risk based on real-world data from March 2019 to March 2021.

6.5 Observations

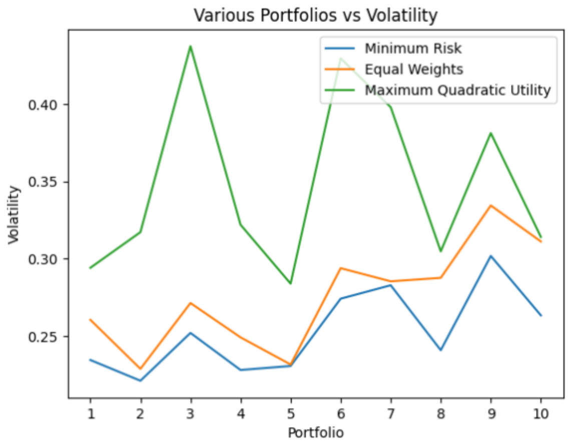

Fig. 3. Volatility Comparison of minimum risk, equal weight and maximum utility portfolios

The graph plotted with portfolio number on the x-axis and volatility on the y-axis clearly depicted the distinctive characteristics of each portfolio. Notably, the portfolio optimized for minimum risk consistently exhibited the lowest volatility among the three. This observation aligns with the foundational principles of portfolio theory, emphasizing the importance of diversification and risk minimization. The optimization process tailored towards minimizing volatility successfully delivered portfolios with the least susceptibility to market fluctuations. In contrast, portfolios with equal weights and those maximizing quadratic utility exhibited comparatively higher volatilities.

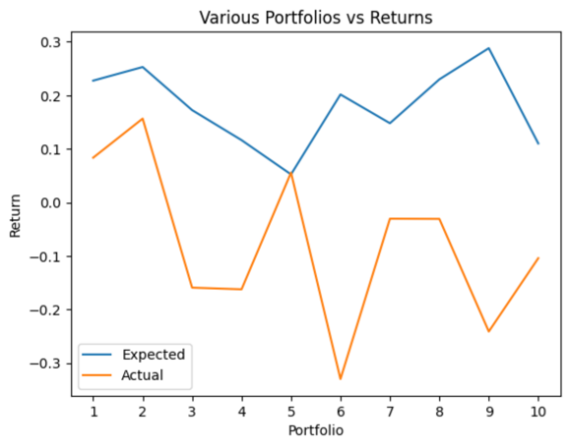

Fig. 4. Expected returns vs Actual returns for minimum risk portfolios

Our examination extended to comparing the expected versus actual returns of portfolios optimized for minimum risk, utilizing real-world data from March 2019 to March 2021. The graph, with portfolio number on the x-axis and returns on the y-axis, offered a revealing snapshot of the performance of these portfolios in a dynamic market environment. Strikingly, the expected returns, derived from historical data and the minimum risk optimization model, consistently outpaced the actual returns observed during the specified period. This stark disparity prompts a critical evaluation of the model's efficacy, suggesting potential limitations in relying solely on historical returns for predictive purposes.

6.6 Inferences

The case study highlights the importance of considering external factors and dynamic market conditions in portfolio optimization. While the model excelled in identifying minimum-risk portfolios under normal circumstances, its performance faltered during periods of economic uncertainty. This emphasizes the need for incorporating additional factors, such as economic indicators or market sentiment, to enhance the model's predictive capabilities.

Furthermore, the deviation between expected and actual returns underscores the inherent limitations of relying solely on historical data. Investors should exercise caution and supplement the model with a comprehensive analysis of current market conditions and economic indicators to make informed investment decisions.

In conclusion, the case study provides valuable insights into the strengths and limitations of the portfolio optimization model. While it excels in identifying optimal portfolios under standard conditions, its predictive power weakens during extraordinary events. Investors should consider a holistic approach, combining historical data with real-time market analysis, to navigate the complexities of the financial landscape effectively. The study contributes to the ongoing dialogue on portfolio optimization methodologies, urging practitioners to adopt a nuanced perspective when applying mathematical models to real-world investment scenarios.

6.7 Limitations

Volatility and Covariance Changes: Mean-variance optimization relies heavily on historical return data, and it assumes that future returns will follow a similar distribution. However, major events like the 2018 and subsequent COVID-19 pandemic can significantly alter the volatility and covariance structure of asset returns. These events can lead to abrupt changes in the relationships between assets, making historical estimates less reliable.

Stability of Expected Returns: The use of average daily returns might not capture changes in the underlying factors influencing stock prices. Economic conditions, industry trends, and company-specific factors can evolve over time, impacting future returns. In times of significant economic disruptions, like the COVID-19 pandemic, historical averages may not be indicative of future performance.

7. Conclusion

While Markowitz's Mean-Variance model has been a pioneering force in portfolio optimization, its limitations necessitate a more nuanced approach to financial modelling and optimization. The model's reliance on historical data assumes a stable market environment, making it susceptible to disruptions such as economic crises or unforeseen events like the COVID-19 pandemic. The inherent assumption of constant expected returns and fixed covariance structures overlooks the dynamic nature of financial markets. Volatility and correlation patterns can shift abruptly, challenging the model's ability to accurately predict risk and return under changing conditions. Additionally, the model's sensitivity to input parameters, such as expected returns and covariance matrices, poses challenges in real-world applications. Minor inaccuracies in these inputs can lead to suboptimal portfolio allocations, potentially exposing investors to higher-than-anticipated risks.

Despite its shortcomings, Markowitz's model serves as a foundational framework, providing valuable insights into the principles of diversification and risk management.

Mathematical Learnings:

a) Quadratic Programming Elegance: Delving into quadratic programming showcased the elegance of mathematical optimization in addressing complex financial decision-making problems. The utilization of quadratic forms allowed for the formulation of objective functions that efficiently captured risk and return trade-offs.

b) Analytical Solution and Lagrange Multipliers: Solving the simplified version of the optimization problem elucidated the importance of Lagrange multipliers in handling equality constraints. The analytical solution underscored the power of mathematical tools in deriving optimal portfolio weights.

c) Constraint Formulation: The inclusion of constraints in the mean-variance model demonstrated the significance of realistic constraints in tailoring mathematical models to reflect practical investment scenarios.

The limitations of the Mean-Variance model underscore the importance of adopting a comprehensive and adaptive approach to portfolio optimization. Modern financial models incorporate advanced optimization techniques, machine learning algorithms, and a broader set of factors, including macroeconomic indicators and sentiment analysis. These models aim to capture the complexities of financial markets more accurately, offering investors a more robust toolset for decision-making.

8. Future Scope

8.1 Adopting A Multiple Factor Model

The traditional mean-variance optimization model relies on historical return data to estimate expected returns and covariance matrices. However, this approach has limitations, especially during periods of economic uncertainty or major market events. The multiple factor model seeks to overcome these limitations by considering a diverse set of factors that influence asset returns. Multiple Factor Model allows for a more comprehensive assessment of risk by accounting for the diverse factors and a more detailed breakdown of risk, leading to better diversification across different risk factors.

The main classes of factors are:

1) Macroeconomic Factors: Incorporating macroeconomic factors such as inflation rates, economic growth, and interest rates allows the model to capture broader economic trends. This is crucial, especially in environments where macroeconomic shifts significantly impact market dynamics.
2) Fundamental Factors: Factors like price-to-earnings (P/E) ratios, dividend yields, and market capitalization provide insights into a company's financial health and growth potential. By including fundamental factors, the model can better assess the intrinsic value of assets.
3) Statistical Factors: Techniques like principal component analysis or hidden factor analysis introduce statistical factors that may not be apparent through traditional analysis. These factors can capture latent patterns and relationships in asset returns.

Out of these three, fundamental factors like PE ratios, Operating profit margins etc help us to get a better idea on the performance of a company which in turn helps us to predict its price rather than predicting on the past price movement.

Implementation

a) Factor Identification and Selection – First we'll identify relevant factors based on economic theories, empirical research and consider factors that exhibit a significant impact on asset returns and are expected to persist across various market conditions.

b) Factor Weighting – In the second step we'll assign weights to each factor based on their perceived importance and relevance. Dynamic factor weighting allows for adjustments based on real-time data and changing market conditions.

c) Utility Function Modification: In the third step we'll modify the utility function within the optimization model to include factor weights and penalize portfolios that deviate from the chosen factors. The modified utility function guides the optimization process towards portfolios aligned with the selected factors.

$$r_i = \sum_{k=1}^{K} B_{ik} f_k + u_i,$$

• ri: excess return of asset i
• Bik: exposure of asset i to factor k
• fk: rate of return of factor k
• ui: specific (or residual) return of asset i.

Benefits

1) Improved Diversification: By considering a diverse set of factors, the model ensures that portfolios are not overly concentrated in specific risk exposures. Improved diversification enhances the stability of the portfolio under different economic scenarios.
2) Enhanced Risk-Return Trade-off: Factor-based optimization allows for a more granular assessment of risk, enabling investors to tailor portfolios to specific risk preferences.
3) Adaptability to Market Changes: The inclusion of dynamic factor weights ensures adaptability to changing market conditions.

The adoption of a multiple factor model represents a forward-looking approach to portfolio optimization, acknowledging the limitations of traditional mean-variance models. By incorporating macroeconomic, fundamental, and statistical factors, investors can construct portfolios that are more resilient, diversified, and aligned with their risk preferences.

8.2 Non-Conventional Areas Of Application

Robotics

In the realm of robotics, the Mean-Variance Markowitz model finds an unconventional yet highly impactful application in motion planning and navigation. The core principles of balancing risk and reward, central to financial portfolio optimization, are adapted to ensure the efficiency and safety of robotic systems.

Risk Factors in Robotics: In robotics, risk translates into the probability of collisions, mechanical failures, or other hazardous events. For example, in the context of an autonomous vehicle, the risk may involve the likelihood of collisions with obstacles or pedestrians.

Reward Factors in Robotics: The reward component in robotics refers to the efficiency and speed of task completion. For instance, in an industrial robot tasked with assembling products on an assembly line, efficiency equates to completing tasks in the least amount of time.

Efficiency in Motion Planning: The Mean-Variance model is adapted to formulate a motion planning strategy that optimally balances risk and reward. This involves finding trajectories and paths for the robotic system that minimize the risk of collisions and mechanical failures while maximizing the efficiency of task completion. The model considers various potential paths and evaluates them based on their risk and reward characteristics, akin to the evaluation of different investment portfolios.

Supply Chain Management

Another non-conventional application of the Mean-Variance Markowitz model is found in the domain of supply chain management. Here, the model is employed to balance efficiency and reliability, addressing the complex challenges inherent in modern supply chains.

Efficiency and Cost Savings: In supply chain management, efficiency is often synonymous with cost savings. The Mean-Variance model, typically associated with financial portfolios, is adapted to identify configurations in the supply chain that offer optimal efficiency in terms of cost and resource utilization.

Risk in Supply Chain: Risks in the supply chain encompass a spectrum of factors, including supplier reliability, transportation issues, demand variability, and inventory management challenges. Analogous to financial risks in a portfolio, supply chain risks represent potential disruptions and cost increases.

Identification of Efficient Frontier: The Mean-Variance model, in this context, is used to identify the efficient frontier in the supply chain. The efficient frontier determines configurations that provide optimal trade-offs between risk and efficiency. For example, in inventory management, the mean (average) might represent the risk of stockouts, and the variance could signify the cost of holding inventory.

The application of the Mean-Variance Markowitz model in robotics and supply chain management showcases the versatility and adaptability of mathematical optimization concepts beyond their traditional domains. By leveraging the principles of balancing risk and reward, these non-conventional applications demonstrate how quantitative models can enhance decision-making in diverse and complex operational contexts.

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