Practical NO: 01

Topic : limits and continuity.

1. $\lim_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

2. $\lim_{x \to 0} \left[\dfrac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a+x}} \right]$

3. $\lim_{x \to \pi/6} \left[\dfrac{\cos x - \sqrt{3}\sin x}{\pi - 6x} \right]$

4. $\lim \left[\dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2-1}} \right]$

Examine the continuity of the following function at given points.

(i)
$$f(x) = \begin{cases} \dfrac{\sin^2 x}{\sqrt{1-\cos x}} & \text{for } 0 < x < \pi/2 \\[2ex] = \dfrac{\cos x}{\pi - 2x} & \text{for } \pi/2 < x < \pi \end{cases} \quad \text{at } x = \pi/2$$

(ii)
$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & 0 < x < 3 \\[2ex] = x+3 & 3 \le x < 6 \\[2ex] = \dfrac{x^2-9}{x+3} & 6 \le x \le 9 \end{cases} \quad \text{at } x = 3,\ x = 6$$

Find value for k, so that the function $f(x)$ is at the indicated point

(i) $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & x < 0 \\ = k & x = 0 \end{cases}$ at $x = 0$

(ii) $f(x) = \begin{cases} (\sec^2 x) & x \neq 0 \\ = k & x = 0 \end{cases}$ at $x = 0$

(iii) $f(x) = \begin{cases} \dfrac{\sqrt{3} - \tan x}{\pi - 3x}, & x \neq \dfrac{\pi}{3} \\ = k, & x = \dfrac{\pi}{3} \end{cases}$ at $x = \dfrac{\pi}{3}$

Discuss the continuity of foll. function which of the function have a removable discontinuity? Redefine the function so as to remove the discontinuity.

(i) $f(x) = \begin{cases} \dfrac{1-\cos 2x}{x \tan x}, & x \neq 0 \\ = 9, & x = 0 \end{cases}$ at $x = 0$

(ii) $f(x) = \begin{cases} (e^{3x} - 1) \sin x^\circ, & x \neq 0 \\ = \dfrac{\pi}{60}, & x = 0 \end{cases}$ at $x = 0$.

(iii) If $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$ for $x \neq 0$ is continuous at $x = 0$. Find $f(0)$

(iv) If $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$ for $x \neq \dfrac{\pi}{2}$ is continuous at $x = \dfrac{\pi}{2}$. find $f\left(\dfrac{\pi}{2}\right)$

1. $\lim\limits_{x\to a}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

$\lim\limits_{x\to a}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\times\dfrac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\times\dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right]$

$\lim\limits_{x\to a}\dfrac{(a+\cancel{x}2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a-3x)(\sqrt{a+2x}+\sqrt{3x})}$

$\lim\limits_{x\to a}\dfrac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(3a-3x)(\sqrt{a+2x}+\sqrt{3x})}$

$\dfrac{1}{3}\lim\limits_{x\to a}\dfrac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(a-x)(\sqrt{a+2x}+3\sqrt{x})}$

$\dfrac{1}{3}\dfrac{\sqrt{3a+x}+2\sqrt{a}}{\sqrt{a+2a}+\sqrt{3a}}$

$\dfrac{1}{3}\times\dfrac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$\dfrac{1}{3}\times\dfrac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$\dfrac{1}{3}\times\dfrac{4^2\sqrt{a}}{2\sqrt{3a}} = \dfrac{2}{3\sqrt{3}}$ //

2. $\lim\limits_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$\lim\limits_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\times\dfrac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$

$\lim\limits_{y\to 0}\dfrac{a+y-a}{y\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$

$\dfrac{1}{\sqrt{a+0}\,(\sqrt{a+0}+\sqrt{a})}$

$\dfrac{1}{\sqrt{a}\,(\sqrt{a}+\sqrt{a})}$

$\dfrac{1}{\sqrt{a}\,(2\sqrt{a})} = \dfrac{1}{2a}$ //

3. $\lim\limits_{x\to \pi/6}\dfrac{\cos x-\sqrt{3}\sin x}{\pi-6x}$

By substituting $x-\frac{\pi}{6}=h$

$x = h+\frac{\pi}{6}$

where $h\to 0$

$$\lim_{h\to 0} \frac{\cos(h+\pi/6) - \sqrt{3}\sin(h+\pi/6)}{\pi - 6\left(h + \frac{\pi}{6}\right)}$$

using
$\cos(A+B) = \cos A\cos B - \sin A \sin B$
$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$$\lim_{h\to 0} \frac{\cosh\cdot\cos\frac{\pi}{6} - \sin h\sin\frac{\pi}{6} - \sqrt{3}\sin h\cos\frac{\pi}{6} + \cosh\sin\frac{\pi}{6}}{\pi - 6\left(\frac{6h+\pi}{6}\right)}$$

$$\lim_{h\to 0} \frac{\cosh\cdot\frac{\sqrt{3}}{2}h - \sin h\,\frac{1}{2} - \sqrt{3}\left(\sin h\,\frac{\sqrt{3}}{2} + \cos h\cdot\frac{1}{2}\right)}{\cancel{\pi} - 6h + \cancel{\pi}}$$

$$\lim_{h\to 0} \frac{\cos\frac{\sqrt{3}}{2}h - \sin\frac{h}{2} - \sin\frac{3h}{2} - \cos\frac{\sqrt{3}}{2}h}{-6h}$$

$$\lim_{h\to 0} \frac{\cancel{-}\sin\frac{4h}{2}}{\cancel{-}6h}$$

$$\lim_{h\to 0} \frac{\sin 4h}{3+2h}$$

$$\frac{1}{3}\lim_{h\to 0}\frac{\sin h}{h} = \frac{1}{3}\times 1 = \frac{1}{3}$$

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4. $$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}}\right]$$

By rationalizing both numerator & Denominator

$$\lim_{x\to\infty}\left[\frac{(\cancel{x^2}+5-\cancel{x^2}+3)(\sqrt{x^2+3}+\sqrt{x^2+1})}{(x^2+3-x^2-1)(\sqrt{x^2+5}+\sqrt{x^2-3})}\right]$$

$$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}} \times \frac{\sqrt{x^2+5}+\sqrt{x^2-3}}{\sqrt{x^2+5}+\sqrt{x^2-3}} \times \frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2-3}}\right]$$

$$\lim_{x\to\infty}\left[\frac{(x^2+5-x^2+3)(\sqrt{x^2+3}+\sqrt{x^2+1})}{(x^2+3-x^2-1)(\sqrt{x^2+5}+(\sqrt{x^2-3})}\right]$$

$$\lim_{x\to\infty}\frac{\overset{4}{\cancel{8}}(\sqrt{x^2+3}+\sqrt{x^2+1})}{\underset{1}{\cancel{2}}(\sqrt{x^2+5}+\sqrt{x^2-3})}$$

$$4\lim_{x\to\infty}\frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)}+\sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After applying limit we get,
$= 4$

Q2 ::

$$f(x) = \begin{cases} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}, & \text{for } 0 < x \le \dfrac{\pi}{2} \\ \dfrac{\cos x}{\pi - 2x}, & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases} \text{ at } x = \frac{\pi}{2}$$

$$f(\pi/2) = \frac{\sin 2\left(\frac{\pi}{2}\right)}{\sqrt{1-\cos 2(\pi/2)}} \quad \therefore f(\pi/2) = 0$$

f at $x = \frac{\pi}{2}$ define.

(ii) $\lim\limits_{x \to \frac{\pi}{2}^+} f(x) = \lim\limits_{x \to \pi/2^+} \dfrac{\cos x}{\pi - 2x}$

By substituting Method

$x - \frac{\pi}{2} = h$

$x = h + \frac{\pi}{2}$ where $h \to 0$

$$\lim_{h \to 0} \frac{\cos(h + \pi/2)}{\pi - 2\left(h + \frac{\pi}{2}\right)}$$

$$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - 2\left(\frac{2h+\pi}{2}\right)}$$

$$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{2}\right)}{-2h}$$

$$\lim_{h \to 0} \frac{\cos h \cdot \cos\frac{\pi}{2} - \sin h \cdot \sin\frac{\pi}{2}}{-2h}$$

$$\lim_{h \to 0} \frac{\cos h \cdot 0 - \sin h}{-2h}$$

$$\lim_{h \to 0} \frac{\cancel{-}\sin h}{\cancel{-}2h} = \frac{1}{2} \lim_{x \to 0} \frac{\sin h}{h}$$

$$= \frac{1}{2}$$

$$\lim_{x \to \pi/2} \cancel{f(x) = \lim_{x \to \pi/2^+} \frac{\cos x}{\pi - 2x}}$$

RHL $f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$

using $\sin 2x = 2\sin x \cdot \cos x$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{2 \sin x \cdot \cos x}{\sqrt{2 \sin^2 x}}$$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{2 \cancel{\sin x} \cdot \cos x}{\sqrt{2}\, \cancel{\sin x}}$$

$$\frac{2}{\sqrt{2}} \lim_{x \to \frac{\pi}{2}^-} \cos x$$

$\therefore$ LHL $\neq$ RHL

$\therefore$ f is not continuous at $x = \frac{\pi}{2}$

5 (ii)

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & 0 < x < 3 \\ x+3 & 3 \le x \le 6 \\ \frac{x^2-9}{x+3} & 6 \le x < 9 \end{cases}$$ at $x=3$ & $x=6$

at $x = 3$

(i) $\cancel{f(3) = \frac{x^2-9}{x-3} = 0}$

$\cancel{f}$ at $x = 3$ define.

ii $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x+3$

$f(3) = x+3 = 3+3 = 6.$

f is define at $x = 3$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} \frac{x^2-9}{x-3}$$

$\therefore$ LHL = RHL

f is continuous at $x = 3$

for $x = 6$

$$f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{\cancel{27}\,3}{\cancel{9}\,1} = 3$$

$$\lim_{x \to 6^+} \frac{x^2-9}{x+3}$$

$$\lim_{x \to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$$

$$\lim_{x \to 6^+} x-3 = 6-3 = 3$$

$$\lim_{x \to 6^+} x+3 = 3+6 = 9$$

$\therefore$ LHL $\neq$ RHL

f is not continuous.

6. (i) $f(x) = \left.\begin{matrix} \dfrac{1-\cos 4x}{x^2} & x < 0 \\ = k & x = 0 \end{matrix}\right\}$ at $x=0$

sol^n: f is continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0} \frac{1-\cos 4x}{x^2} = k.$$

$$\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = k$$

$$2\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = k$$

$$2\lim_{x\to 0} \left(\frac{\sin 2x}{x^2}\right) = k$$

$$2(2)^2 = k$$

$$\underline{\underline{k = 8}}$$

(ii) $f(x) = \left.\begin{matrix} (\sec^2 x)^{\cot^2 x} & , x \neq 0 \\ = k & x = 0 \end{matrix}\right\}$ at $x = 0$

sol^n:

$$f(x) = (\sec^2 x)^{\cot^2 x}$$

using: $\sec^2 - \tan^2 x = \sec^2 x = 1$

$\therefore \sec^2 x = 1 + \tan^2 x$

R

$$\cot^2 x = \frac{1}{\tan^2 x}$$

$$\therefore \lim_{x\to 0} (\sec^2 x)^{\cot^2 x}$$

$$\lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$$

we know that

$$\lim_{x\to 0} (1+px)^{\frac{1}{px}} = e$$

$$\therefore k = e //$$

(iii) $f(x) = \left.\begin{matrix} \dfrac{\sqrt{3}-\tan x}{\pi - 3x} & , x \neq \frac{\pi}{3} \\ = k & x = \frac{\pi}{3} \end{matrix}\right\}$ at $x = \frac{\pi}{3}$

$$x = h + \frac{\pi}{3}$$

where $h \to 0$

$$f\left(\frac{\pi}{3}+h\right) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3\left(\frac{\pi}{3}+h\right)}$$

12.
$$\lim_{h\to 0} \frac{\sqrt{3} - \tan\left(\pi/3 + h\right)}{\pi - 3\left(\frac{\pi}{3} + h\right)}$$

using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$$\lim_{h\to 0} \frac{\sqrt{3} - \dfrac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3}\cdot \tan h}}{\pi - \pi - 3h}$$

$$\lim \frac{\dfrac{\sqrt{3}\left(1 - \tan\frac{\pi}{3}\cdot \tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3} \times \tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3\cdot \tan h) - (\sqrt{3} + \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3\tan h) - \sqrt{3} - \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{-4\tan h}{-3h(1 - \sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{4\tan h}{3h(1 - \sqrt{3}\tan h)}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h} \lim_{h\to 0} \frac{1}{(1 - \sqrt{3}\tan h)}$$

$$= \frac{4}{3}\left(\frac{1}{(1 - \sqrt{3}(0))}\right)$$

$$= \frac{4}{3}\left(\frac{1}{1}\right) = \frac{4}{3}$$

7. (i) $f(x) = \begin{cases} \dfrac{1 - \cos 3x}{x \cdot \tan x} & x \neq 0 \\ 9 & x = 0 \end{cases}$ at $x = 0$

$$f(x) = \frac{1 - \cos 3x}{x \tan x}$$

$$\lim_{x\to 0} = \frac{\dfrac{2\sin^2\frac{3x}{2}}{x^2} \times x^2}{\dfrac{x \cdot \tan x}{x^2} \times x^2}$$

$$= \frac{2\lim_{x\to 0}\left(\frac{3}{2}\right)^2}{1} = \frac{2\times 9}{4} = \frac{9}{2}$$

$\lim_{x \to 0} f(x) = \frac{9}{2} \quad g = f(0)$

$\therefore$ f is not continuous at $x=0$

Redefine function.

$$f(x) = \begin{cases} \dfrac{1-\cos 3x}{x \tan x} & x \neq 0 \\ \dfrac{9}{2} & x = 0 \end{cases}$$

Now $\lim_{x \to 0} f(x) = f(0)$.

f has removable discontinuity at $x=0$

7. (ii) $$f(x) = \begin{cases} \dfrac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2} & x \neq 0 \\ \dfrac{\pi}{6} & x = 0 \end{cases}$$ at $x=0$

$$\lim_{x \to 0} \frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$$

$$\lim_{x \to 0} \frac{e^{3x}-1}{x} \cdot \lim_{x \to 0} \sin\left(\frac{\pi x}{180}\right)$$

$$\lim_{x \to 0} 3 \cdot \frac{e^{3x}-1}{3x} \quad \lim_{x \to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$3 \lim_{x \to 0} \frac{e^{3x}-1}{3x} \quad \lim_{x \to 0} \sin\left(\frac{\pi x}{180}\right)$$

f is continuous at $x=0$.

8. $f(x) = \dfrac{e^{x^2} - \cos x}{x^2} \quad x=0$

is continuous at $x=0$

$$\lim_{x \to 0} f(x) = f(0)$$

$$= \lim_{x \to 0} \frac{e^{x^2} - \cos x + 1 - 1}{x^2}$$

$$\lim_{x \to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x \to 0} \frac{1-\cos x}{x^2}$$

Multiply with 2 a Num & Deno.

$$= 1 + 2 \times \frac{1}{4} = \frac{3}{2} = f(0)$$

9. $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$, $x \neq \frac{\pi}{2}$

$f(x)$ is continuous at $x = \frac{\pi}{2}$.

$$\lim_{x \to \pi/2} \frac{2 - 1 + \sin x}{\cos^2 x \left(\sqrt{2} + \sqrt{1+\sin x}\right)}$$

$$= \lim_{x \to \pi/2} \frac{1 + \sin x}{(1 - \sin x)\left(\sqrt{2} + \sqrt{1 + \sin x}\right)}$$

$$= \lim_{x \to \pi/2} \frac{1}{(1 - \sin x)(\sqrt{2}x)\sqrt{1 + \sin x}}$$

$$= \frac{1}{2(\sqrt{2} + \sqrt{2})}$$

$$= \frac{1}{2(2\sqrt{2})}$$

$$= \frac{1}{4\sqrt{2}} = f\left(\frac{\pi}{2}\right) = \frac{1}{4\sqrt{2}}$$

4/12/19

Practical No. 2

Topic: Derivation:

Q1. Show that the following function defined from R to R differentiable.

(i) $\cot x$.

$f(x) = \cot x$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan a}{x - a} = \lim_{x \to a} \frac{\tan a - \tan x}{(x-a)\tan x \cdot \tan a}$$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$f'(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(a+h-a)\tan(a+h)\tan a}$$

formula $\tan(A - B) = \dfrac{\tan A - \tan B}{h \times \tan(a+h)\tan a}$

$$= \lim_{h \to 0} \frac{\tan(a - x - h) \cdot (1 + \tan a + \tan(a+h))}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h\to 0} \frac{-\tan h}{h} \times \frac{1}{1+\frac{\tan a \tan(a+h)}{\tan(a+h)\tan a}}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a} = \frac{-\sec^2 a}{\tan^2 a}$$

$$= \frac{-1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a} = -\operatorname{cosec}^2 a$$

$\therefore f'(a) = -\operatorname{cosec}^2 a$

$\therefore f$ is differentiable $\forall a \in R$

(ii) $\operatorname{cosec} x$

$f(x) = \operatorname{cosec} x$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x\to a} \frac{\sin a - \sin x}{(x-a)\sin a \sin x}$$

put $x - a = h$

$x = a+h$ as $x \to a$, $h \to 0$

$$f'(a) = \lim_{h\to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \sin(a+h)}$$

formula: $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right) \times \sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \cdot \sin(a+h)}$$

$$= -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$$

$$= -\frac{\cos a}{\sin^2 a} = -\cot a \cdot \operatorname{cosec} a$$

(iii) $\sec x$

$f(x) = \sec x$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a} \frac{\cos a - \cos x}{(x-a)\cos a \cos x}$$

Put $x - a = h$

$x = a+h$

as $x \to a$

$$f'(a) = \lim_{h\to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cdot \cos(a+h)}$$

formula:

$$-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\lim_{h\to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \cos a \cdot \cos(a+h)}$$

$$= -\frac{1}{2} \times \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cdot \cos(a+0)}$$

$$= -\frac{1}{2} \times \frac{-2\sin a}{\cos a \times \cos a}$$

$$= \tan a \sec a$$

Q2. If $f(x) = 4x+1$, $x \le 2$

$= x^2+5$, $x > 0$ at $x = 2$, then find function is differentiable or not.

Soln:

LHD

$$= \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x+1-(4\times 2+1)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4(x-2)}{x-2} = 4$$

RHD

$$\lim_{x\to 2^+} \frac{x^2+5-9}{x-2}$$

$$\lim_{x\to 2^+} \frac{x^2-4}{\cancel{x}-2} = \lim_{x\to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= 2+2 = 4$$

$\therefore$ RHD = LHD

f is differentiable at $x = 2$.

Q3. If $f(x) = 4x+7$, $x < 3$

$= x^2+3x+1$, $x \ge 3$

find f is differentiable or not

Soln:

RHD:

$$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3^+} \frac{x^2+3x+1-(3^2+3\cdot 3+1)}{x-3}$$

$$\lim_{x\to 3^+} \frac{x^2+3x+1-19}{x-3} = \lim_{x\to 3^+} \frac{x^2+3x-18}{x-3}$$

$$\lim_{x\to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$\lim_{x\to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{x-3}} = 3+6 = 9$$

LHD:

$$\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3^-} \frac{4x+7-19}{x-3}$$

$$\lim_{x\to 3^-} \frac{4x-12}{x-3} = \lim_{x\to 3^-} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}} = 4$$

RHD $\ne$ LHD

$\therefore$ f is not differentiable at $x = 3$

Qu. If $f(x) = 8x - 5 \quad , x \leq 2$

$= 3x^2 - 4x + 7 \quad , x > 2$ at $x = 2$

Then find f is differentiable or not.

soln:

RHD:

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{(3x + 2)(x - 2)}{(x - 2)}$$

$$= 3 \times 2 + 2 = 8$$

$$= 8$$

LHD:

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$\lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$\lim_{x \to 2^-} \frac{8x - 16}{x - 2} = \lim_{x \to 2^-} \frac{8\cancel{(x - 2)}}{\cancel{x - 2}} = 8$$

LHD = RHD

$\therefore$ f is differentiable at $x = 3$.

11/12/19

Practical No. 3

Topic: Application of Derivative.

1. Find the interval in which function is increasing or decreasing.

(i) $f(x) = x^3 - 5x - 11$ (ii) $f(x) = x^2 - 4x$

(iii) $f(x) = 2x^3 + x^2 - 20x + 4$

(iv) $f(x) = x^3 - 27x + 5$

(v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

2. Find the intervals in which function is concave upwards & concave downwards.

(i) $y = 3x^2 - 2x^3$

(ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

(iii) $y = x^3 - 27x + 5$

(iv) $y = 69 - 24x - 9x^2 + 2x^3$

(v) $y = 2x^3 + x^2 - 20x + 4$.

1. $f(x) = x^3 - 5x - 11$

f is increasing iff

$f'(x) > 0$

$\because f(x) = x^3 - 5x - 11$

$f'(x) = 3x^2 - 5$

$3x^2 - 5 > 0$

$x = \pm\sqrt{\frac{5}{3}}$

$\therefore x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

Now ~~if~~ f is decreasing iff

$f'(x) < 0$

$3x^2 - 5 < 0$

$x = \pm\sqrt{\frac{5}{3}}$

$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

b. $f(x) = x^2 - 4x$

f is increasing iff

$f'(x) > 0$

$f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$2x - 4 > 0$

$2(x-2) > 0$

$x - 2 > 0$

$x = 2.$

$\therefore \quad x \in (2, \infty)$

f is decreasing iff

$f'(x) < 0$

$2x - 4 < 0$

$2(x-2) < 0$

$x - 2 < 0$

$x = 2$

$\therefore \quad x \in (-\infty, 2)$

c. $f(x) = 2x^3 + x^2 - 20x + 4$

f is increasing iff

$f'(x) > 0$

$f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$6x^2 + 2x - 20 > 0$

$6x^2 + 12x - 10x - 20 > 0$

$6x(x+2) - 10(x+2) > 0$

$(x+2)(6x-10) > 0$

$x = -2, \frac{5}{3}$

f is decreasing iff

$f'(x) < 0$

$6x^2 + 2x - 20 < 0$

$(x+2)(6x-10) < 0$

$x = -2, \frac{5}{3}$

$x \in (-2, \frac{5}{3})$

d. $f(x) = x^3 - 27x + 5$

f is increasing iff

$f'(x) > 0$

$f'(x) = x^3 - 27x + 5$

$= 3x^2 - 27$

$3(x^2 - 9) > 0$

$x^2 - 9 > 0$

$x = 3, -3$

f is decreasing iff

$f'(x) < 0$

$3x^2 - 27 < 0$

$3(x^2 - 9) < 0$

$x^2 - 9 < 0$

$x = 3, -3$

$\therefore x \in (-3, 3)$

e. $f(x) = 69 - 24x - 9x^2 + 2x^3$

f is increasing iff

$f'(x) > 0$

$f(x) = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = -24 - 18x + 6x^2 > 0$

$6(-4 - 3x + x^2) > 0$

$x^2 - 3x - 4 > 0$

$(x-4)(x+1) > 0$

$x = 4, -1$

$x \in (-\infty, -1) \cup (4, \infty)$

f is decreasing iff

$f'(x) < 0$

$-24 - 18x + 6x^2 < 0$

$6(-4 - 3x + x^2) < 0$

$(x-4)(x+1) < 0$

$x = 4, -1$

$x \in (-1, 4)$

Q2

a. $y = 3x^2 - 2x^3$

$y = f(x)$

$f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$f''(x) = 6 - 12x$

f is concave upward iff

$f''(x) > 0$

$6 - 12x > 0$

$6(1 - 2x) > 0$

$(1 - 2x) > 0$

$-(2x - 1) > 0$

$x \in (-\infty, \frac{1}{2})$

f is concave downward iff

$f''(x) < 0$

$6(1 - 2x) < 0$

$-(2x - 1) < 0$

$x \in (\frac{1}{2}, \infty)$

b. $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$y = f(x)$
$f(x) = x^4 - 6x^3 + 12x^2 + 5x + 7$
$f'(x) = 4x^3 - 18x^2 + 24x + 5$
$f''(x) = 12x^2 - 36x + 24$

f is concave upward iff
$f''(x) > 0$
$12x^2 - 36x + 24 > 0$
$12(x^2 - 3x + 2) > 0$
$x^2 - 3x + 2 > 0$
$(x-2)(x-1) > 0$
$x = 2, 1$

$x \in (-\infty, 1) \cup (2, \infty)$

f is concave downward iff
$f''(x) < 0$
$12x^2 - 36x + 24 < 0$
$12(x^2 - 3x + 2) < 0$
$x^2 - 3x + 2 < 0$
$(x-2)(x-1) < 0$
$x = 2, 1$
$x \in (1, 2)$

c. $y = x^3 - 27x + 5$
$y = f(x)$
$f(x) = x^3 - 27x + 5$
$f'(x) = 3x^2 - 27$
$f''(x) = 6x$

f is concave upward iff
$f''(x) > 0$
$6x > 0$
$x > 0$

$x = 0$
$x \in (0, \infty)$

f is concave downward iff
$f''(x) < 0$
$6x < 0$
$x < 0$
$x = 0$

$x \in (-\infty, 0)$

d. $f(x) = 69 - 24x - 9x^2 + 2x^3$
f is concave upward iff
$f(x) = 69 - 24x - 9x^2 + 2x^3$
$f'(x) = \cancel{8} - 24 - 18x + 6x^2$
$f''(x) = -18 + 12x$
~~f is concave downward iff~~

$f''(x) > 0$
$-18 + 12x > 0$
$6(2x - 3) > 0$
$x = 3/2$ $x \in (\frac{3}{2}, \infty)$

f is concave downwards iff

$f''(x) < 0$

$-18 + 12x < 0$

$6(2x-3) < 0$

$2x - 3 < 0$

$x = \frac{3}{2}$

$x \in (-\infty, \frac{3}{2})$

e. $y = 2x^3 + x^2 - 20x + 4$

$y = f(x)$

$f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff

$f''(x) > 0$

$2(6x+1) > 0$

$6x + 1 > 0$

$x = -1/16$

$\therefore x \in (-1/16, \infty)$

f is concave downward iff

$f''(x) < 0$

$12x + 2 < 0$

$2(6x+1) < 0$

$x = -1/6$

$\therefore x \in (-\infty, -1/6)$

18/12/19

Practical No. 4

Topic: Application of Derivative & Newton's method.

Q1. Find minimum and maximum value of the following function.

(i) $f(x) = x^2 + \frac{16}{x^2}$

(ii) $f(x) = 3 - 5x^3 + 3x^5$

(iii) $f(x) = 2x^3 - 3x^2 + 1$

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

Q2. Find the root of following equation by Newton's method.

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

(ii) $f(x) = x^3 - 4x - 9$ in $[2,3]$

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1,2]$.

Q1

1. $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

For maxima / minima

$f'(x) = 0$

$2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$\because \cancel{f''(2) =} f''(-2) = 2 + \frac{96}{(\pm 2)^4} = 2 + \frac{96}{16} = 8 > 0$

$\therefore$ $f(x)$ is minimum at $x = \pm 2$

$f(2) = 8$ is the minimum value.

(ii) soln:

$f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = 15x^4 - 15x^2$

For maxima / Minima

$f'(x) = 15x^4 - 15x^2 = 0$

$\therefore x^4 - x^2 = 0.$

$\therefore x^2(x^2 - 1) = 0$

$x = 0, -1, 1$

$f''(x) = 60x^3 - 30x$

$f''(0) = 0$

$f''(-1) = -60 + 30 = -30 < 0$

$f''(1) = 60 - 30 = 30 > 0$

$\therefore f(x)$ is maximum at -1 and minimum at 1

$f(-1) = 3 + 5 - 3 = 5$

$f(1) = 3 - 5 + 3 = 1$

(iii) soln:

$f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

For maxima / minima

$f'(x) = 0.$

$\therefore 3x^2 - 6x = 0$

$\therefore x^2 - 2x = 0$

$x(x-2) = 0$

$\therefore x = 0, 2$

$f''(x) = 6x - 6$

$f''(0) = -6 > 0$

$f''(2) = 6 > 0$

$\therefore$ $f(x)$ is maximum at $x=0$ and minimum at $x=2$

$\therefore$ $f(0) = 1$, $f(2) = -3$

(iv) solⁿ:

$f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

for maxima/minima

$\therefore$ $f'(x) = 0$

$\therefore$ $6x^2 - 6x - 12 = 0$

$\therefore$ $x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x+1)(x-2) = 0$

$x = -1, 2$

$f''(x) = 12x - 6$

$f''(-1) = -12 - 6 = -18 < 0$

$f''(2) = 24 - 6 = 18 > 0$

$\therefore$ $f(x)$ is maximum at $x=1$ and minimum at $x=2$

$\therefore$ $f(-1) = 8$

$f(x) = -19$

Q2

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

$f'(x) = 3x^2 - 6x - 55$

$x_0 = 0$

$f(x_0) = 9.5$

$f'(x) = -55$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{9.5}{-55} = 0.1727$$

$f(x_1) = -0.0828$

$f'(x_1) = -55.9467$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{-0.0828}{-55.9467}$$

$x_2 = 0.1712$

$f(x_2) = 0.0011$

$f'(x_2) = -55.9392$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.1712 - \frac{0.0011}{-55.9393}$$

$\therefore$ $x_3 = 0.1712$

$\therefore$ $x = 0.1712$ is the root of the equation.

(ii) solⁿ:

$f(x) = x^3 - 4x - 9$

$f'(x) = 3x^2 - 4$

$f(2) = -9$

$f(3) = 6$

$\therefore$ 6 is ~~maximum~~ close to 0 and the number be,

$x_0 = 3$

$f(x_0) = 6$

$f'(x_0) = 23$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_1)} = 2.7391$$

$f(x_2) = 17.9835$

$\therefore x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 2.707 - \dfrac{0.6085}{17.9835}$

$x_3 = 2.7065$

$f(x_3) = -0.0005$

$f'(x_3) = 17.9757$

$\therefore x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$

$= 2.7065 - \dfrac{0.0005}{17.9757}$

$x_4 = 2.7065$

$\therefore$ 2.7065 is the root of the equation.

(iii) Sol^n:

$f(x) = x^3 - 1.8x^2 - 10x + 17 \quad [1,2]$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = 1 - 1.8 - 10 + 17 = 6.2$

$f(2) = -2.2$

$\therefore$ -2.2 is close to 0 a the number be

$\therefore x_0 = 2$

$f(x_0) = -2.2$

$f'(x_0) = -5.2$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 2 - \dfrac{-2.2}{-5.2}$

$x_1 = 1.5769$

$f(x_1) = (1.5769)^3 - 1.8(1.5769)^2 - 10(1.5769) + 17$

$= 0.6762$

$f'(x) = -8.217$

$\therefore x_2 = x_1 - \dfrac{f(x)}{f'(x)}$

$= 1.5769 - \dfrac{0.6762}{-8.217}$

$x_2 = 1.6592$

$\therefore f(x_2) = 0.0204$

$f'(x_2) = -7.7143$

$\therefore x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 1.6592 - \dfrac{0.0204}{-7.7143}$

$x_3 = 1.6618$

$\therefore f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$\therefore f(x_3) = 0$

$\therefore$ 1.6618 is the root of the function.

Practical No. 5

Q1. Solve the following Integration.

(i) $I = \int \frac{dx}{\sqrt{x^2+2x-3}}$ (ii) $\int (4e^{3x}+1)\,dx$

(iii) $\int (2x^2 - 3\sin x + 3\sqrt{x})\,dx$

(iv) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$ (v) $\int t^7 \sin(2t^4)\,dt$

(vi) $\int \sqrt{x}\,(x^2-1)\,dx$ (vii) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

(viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx$ (ix) $\int e^{\cos^2 x} \cdot \sin 2x\,dx$

(x) $\int \left(\frac{x^2-2x}{x^3-3x^2+1}\right) dx$

(i) sol^n:

$$I = \int \frac{1}{\sqrt{x^2+2x-3}}\,dx$$

$$= \int \frac{1}{\sqrt{x^2+2x+1-4}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2-2^2}}\,dx$$

$$I = \ln\left|x+1 + \sqrt{x^2+2x+3}\right| + c$$

(ii) sol^n:

$$I = \int (4e^{3x}+1)\,dx$$

$$= 4\int e^{3x}\,dx + \int 1\,dx$$

$$I = \frac{4e^{3x}}{3} + x + c$$

(iii) $I = \int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

$$= 2\int x^2\,dx - 3\int \sin x\,dx + 5\int x^{1/2}\,dx$$

$$= \frac{2x^3}{3} + 3\cos x + \frac{5\times 2}{3} x^{3/2} + c$$

$$I = \frac{2x^3}{3} + 3\cos x + \frac{10}{3} x\sqrt{x} + c$$

(iv) $I = \int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

Put $\sqrt{x} = t$.

$\therefore \frac{1}{2\sqrt{x}} = \frac{dt}{dx}$ $\therefore \frac{dx}{\sqrt{x}} = 2dt$

$$I = \int \frac{(x)^6 + 3(\sqrt{x})^2 + 4}{\sqrt{x}}\,dx$$

$$= 2\int t^6 + 3t^2 + 4\,dt$$

$$= 2\left[\frac{t^7}{7} + \frac{8t^3}{3} + 4t\right] + c$$

$$= 2\left[\frac{x^{7/2}}{7} + x^{3/2} + 4x^{1/2}\right] + c$$

(v) $I = \int t^7 \sin(2t^4)\, dt$

$= \int t^4 \sin(2t^4) \times t^3\, dt$

Put $t^4 = x$

$4t^3 = \frac{dx}{dt}$

$t^3\, dt = \frac{1}{4}\frac{dx}{dt}$

$I = \frac{1}{4}\int x \cdot \sin(2x)\, dx$

$\therefore I = \frac{1}{4}\left[x\int \sin 2x\, dx - \int\left(\left(\frac{d}{dx}x\right)\int \sin 2x\, dx\right) dx\right]$

$= \frac{1}{4}\left[\cancel{-x\cdot\frac{\cos 2x}{2}} + \frac{1}{2}\int \cos 2x\, dx\right]$

$= \frac{1}{16}\sin 2x - x\cdot\frac{\cos 2x}{8} + c$

$I = \frac{1}{16}\sin 2t^4 - t^4 \cdot \frac{\cos 2t^4}{8} + c$

(vi) $I = \int \sqrt{x}\,(x^2 - 1)\, dx$

$\therefore I = \int x^2\sqrt{x}\, dx - \int \sqrt{x}\, dx$

$= \int x^{5/2}\, dx - \int x^{1/2}\, dx$

$= \frac{x^{7/2}}{7/2} - \frac{x^{3/2}}{3/2} + c$

$= \frac{2}{7}x^3\sqrt{x} - \frac{2}{3}x\sqrt{x} + c$

(vii) $I = \int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right) dx$

$\frac{1}{x^2} = t$

$\therefore \frac{-2}{x^3} = \frac{dt}{dx}$

$\therefore \frac{dx}{x^3} = \frac{-1}{2}\, dt$

$I = \frac{-1}{2}\int \sin t\, dt$

$= \frac{-1}{2}\cdot \cos t + c$

$= \frac{\cos t}{2} + c$

$I = \frac{\cos(1/x^2)}{2} + c$

(viii) $I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}} \, dx$

Put $\sin x = t$

$\therefore \cos x \, dx = dt$

$\therefore I = \int \frac{1}{t^{2/3}} \, dt$

$= \int t^{-2/3} \, dt$

$= \frac{t^{-2/3+1}}{-2/3+1} + c$

$= 3t^{1/3} + c$

$I = 3\sqrt[3]{\sin x} + c$

(ix) $I = \int e^{\cos^2 x} \cdot \sin 2x \cdot dx$

Put $\cos^2 x = t$

$\therefore -2\cos x \sin x = \frac{dt}{dx}$

$\therefore \sin 2x \, dx = -dt$

$\therefore I = -\int e^t \, dt$

$= -e^t + c$

$\therefore I = -e^{\cos^2 x} + c$

(x) $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

$x^3 - 3x^2 + 1 = t$

$\therefore 3x^2 - 6x = \frac{dt}{dx}$

$\therefore (x^2 - 2x) \, dx = \frac{dt}{3}$

$\therefore I = \frac{1}{3} \int \frac{1}{t} \, dt$

$= \frac{1}{3} \log |t| + c$

$I = \frac{1}{3} \log |x^3 - 3x^2 + 1| + c$

Practical NO. 6.

Application of integration & Numerical Integration.

Q1 Find the length of the following curve:

1. $x = t\sin t$, $y = 1 - \cos t$, $t \in [0, 2\pi]$
2. $y = \sqrt{4-x^2}$, $x \in [-2, 2]$
3. $y = x^{3/2}$ in $[0, 4]$
4. $x = 3\sin t$, $y = 3\cos t$ $t \in [0, 2\pi]$
5. $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1, 2]$

Q2. Using simpson's Rule solve the following.

1. $\int_0^2 e^{x^2}\, dx$ with $n = 4$.
2. $\int_0^4 x^2\, dx$ with $n = 4$.
3. $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ with $n = 6$.

2. $y = \sqrt{4-x^2}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}}(-2x)$

$= \frac{-x}{\sqrt{4-x^2}}$

$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_{-2}^{2} \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$

$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$

$= 2\int_{-2}^{2} \frac{1}{\sqrt{2-x^2}}\, dx$

$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$

$= 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$

$= 2\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right.$

$\therefore \boxed{L = 2\pi}$

3. $y = x^{3/2} \quad x \in [0, 4]$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

$\therefore L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

$= \int_0^4 \sqrt{1 + \frac{9}{4} x} \, dx$

$= \frac{1}{2} \int_0^4 \sqrt{4 + 9x} \, dx$

$= \frac{1}{2} \left[\frac{(4+9x)^{3/2}}{3/2} \times \frac{1}{9} \right]_0^4$

$= \frac{1}{27} \left[(4 + 9x)^{3/2} \right]_0^4$

$= \frac{-1}{27} \left[(4+0)^{3/2} - (4+36)^{3/2} \right]$

$L = \frac{1}{27} \left(40^{3/2} - 8 \right)$

4. $x = 3\sin t \quad , \quad y = 3\cos t$

$\frac{dx}{dt} = 3\cos t \quad , \quad \frac{dy}{dt} = -3\sin t$

$L = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \, dt$

$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2} \, dt$

$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t} \, dt$

$= \int_0^{2\pi} 3\sqrt{1} \, dt$

$= 3 \int_0^{2\pi} 1 . \, dt$

$= 3 \, (2\pi)$

$L = 6\pi$

5. $x = \frac{1}{6} y^3 + \frac{1}{2y}$

$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$

$L = \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$

$= \int_0^2 \sqrt{1 + \frac{(y^3 - 1)}{4y^2}} \, dy$

$$= \int_1^2 \sqrt{\frac{(y^4-1)+2y^3}{4y^4}}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\, dy$$

$$= \int_1^2 \frac{(y^4+1)^2}{(2y^2)^2}\, dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\, dy.$$

$$= \frac{1}{2}\int_1^2 y^2\, dy + \frac{1}{2}\int_1^2 y^{-2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} + \frac{y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{17}{6}\right] = \frac{17}{12}$$

Q2.

1. $\int_0^2 e^{x^2}\, dx$, $n = 4$

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} = 0.5$$

x	0	0.5	1	1.5	2
y	1	1.284	2.7183	9.4877	54.5982

$$\int_0^2 e^{x^2}\, dx = \frac{0.5}{3}\Big[(1+54.5982) + 4(1.284+9.4877) + 2(2.7183)\Big]$$

$$= \frac{0.5}{3}\left[55.5982 + 43.0868 + 54.36\right]$$

$$= 17.3535$$

2. $\int_0^4 x^2\, dx$ $n = 4$

$$h = \frac{4-0}{4} = 1$$

x	0	1	2	3	4
y	0	1	4	9	16

$$\int_0^4 x^2\, dx = \frac{1}{3}\left[0 + 16 + 4(1+9) + 2(4)\right]$$

$$= \frac{1}{3}\left[0 + 16 + 4(10) + 8\right]$$

$$= \frac{64}{3} = 21.3333$$

3. $\int_0^{\pi/3} \sqrt{\sin x}\, dx \qquad n = 6$

$h = \dfrac{\pi/3 - 0}{6} = \dfrac{\pi}{18}$

x	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$
y		0.4167	0.5843	0.7071	0.8017	0.8752	0.9306

$$\int_0^{\pi/3} \sqrt{\sin x}\, dx = \frac{\pi/18}{3}\left[0.4167 + 0.9306 + 4(0.4167 + 0.7071 + 0.8752) + 2(0.5843 + 0.8017)\right]$$

$$= \frac{\pi/18}{3}\left[1.3473 + 4(1.999) + 2(1.3865)\right]$$

$$= \frac{\pi}{54}\left[1.3473 + 7.996 + 2.773\right]$$

$$= \frac{\pi}{54}(12.1163)$$

$$= 0.7049$$

22/01/2022

Practical No. 7

Differential Equation

1. Solve the following differential equation.

1. $x\dfrac{dy}{dx} + y = e^x$

2. $e^x\dfrac{dy}{dx} + 2e^x y = 1$

3. $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y$

4. $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$

5. $e^2x\dfrac{dy}{dx} + 2e^{2x}y = 2x$

6. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dx = 0$

7. $\dfrac{dy}{dx} = \sin^2(x-y+1)$

8. $\dfrac{dy}{dx} = \dfrac{2x+3y-1}{6x+9y+6}$

Q1. $x \cdot \frac{dy}{dx} + y = e^x$

$= \frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

$P(x) = 1/x \qquad q(x) = \frac{e^x}{x}$

$I.F = e^{\int 1/x \, dx}$

$= x$

$y(IF) = \int q(x) \, (IF) \, dx + c$

$xy = \int \frac{e^x}{x} \, x \, dx + c$

$= \int e^x dx + c$

$xy = e^x + c$

2. $e^x \frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + 2y = e^{-x}$

$p(x) = 2 \qquad q(x) = e^{-x}$

$IF = e^{\int p(x) dx}$

$= e^{\int 2 dx}$

$= e^{2x}$

$y(IF) = \int Q(x) \, (IF) \, dx + c$

$y \cdot e^{2x} = \int e^{-x} \, e^{2x/x} + c$

$= \int e^{-x+2x} dx + c$

$= \int e^x dx + c$

$y \cdot e^{2x} = e^x + c$

3. $\frac{dy}{dx} + \frac{2}{x} y = \frac{\cos x}{x^2}$

$p(x) = 2/x \qquad q(x) = \cos x / x^2$

$If = e^{\int p(x) dx}$

$= e^{\int 2/x \, dx}$

$= e^{2/x}$

$= e^{\ln x^2}$

$If = x^2$

$y(IF) = \int q(x) \, (IF) \, dx + c$

$= \int \frac{\cos^2 x}{x^2} \, x^2 \, dx + c$

$= \int \cos x + c$

$x^2 y = -\sin x + c$

4. $x \cdot \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \frac{3}{x} y = \frac{\sin x}{x^2}$

$p(x) = 3/x \qquad q(x) = \sin x / x^2$

$IF = e^{\int P(x)\,dx}$

$= e^{\int 3 \cdot \frac{1}{x}\,dx}$

$= e^{3\ln x}$

$IF = x^3$

$y(IF) = \int Q(x)(IF)\,dx + c$

$x^3 y = \int \frac{\sin x}{x^3} x^3\,dx + c$

$= \int \sin x\,dx + c$

$x^3 y = -\cos x + c$

5. $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$P(x) = 2 \quad Q(x) = 2x e^{-2x}$

$IF = e^{\int P(x)\,dx}$

$= e^{\int 2\,dx}$

$= e^{2x}$

$y(IF) = \int Q(x)(IF)\,dx + c$

$= \int 2x\, e^{2x} \cdot e^{-2x}\,dx + c$

$ye^{2x} = \int 2x\,dx + c$

$\therefore ye^{2x} = x^2 + c$

6. $\sec^2 x \cdot \tan y\,dx + \sec^2 y \tan x\,dy = 0.$

$\sec^2 x \cdot \tan y\,dx = \sec^2 y \cdot \tan x\,dy.$

$\frac{\sec^2 x\,dx}{\tan x} = \frac{-\sec^2 y\,dy}{\tan y}$

$\therefore \int \frac{\sec^2 x\,dx}{\tan} = \int \frac{\sec^2 y\,dy}{\tan y}$

$\therefore \log|\tan x| = -\log|\tan y| + 1$

$\therefore \log|\tan x \cdot \tan y| = c$

$\tan x \cdot \tan y = e$

7. $\frac{dy}{dx} = \sin^2(x - y - 2) - x$

Differentiating both side.

$x - y + 1 = v.$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$1 - \frac{dv}{dx} = \frac{dy}{dx}$

$1 - \frac{dv}{dx} = \sin^2 v$

$\frac{dy}{dx} = \cos^2 x$

$\int \sec^2 v\,dv = \int dx$

$\tan v = x + c$

$\tan(x+y-1) = x + c$

8. $\dfrac{dy}{dx} = \dfrac{2x+3y-1}{6x+9y+6}$

put $2x + 3y = v$

$\therefore 2 + 3\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{dv}{dx} - 2\right)$

$\dfrac{1}{3}\left(\dfrac{dv}{dx} - 2\right) = \dfrac{1}{3}\left(\dfrac{v-1}{v+2}\right)$

$\dfrac{dv}{dx} \quad \dfrac{v-1+2v+4}{v+2} = \dfrac{3v+3}{v+2} = 3\left(\dfrac{v+1}{v+2}\right)$

$\int \dfrac{v+1}{v+1}\, dx + \int \dfrac{1}{v+1}\, dv = 3x + c$

$v + \log|v+1| = 3x + c$

$\therefore 2x + 3y + \log|2x+3y+1| = 3x + c$

$3y = x - \log|2x+3y+1| + c$

05/02/22

Practical No. 8

Euler's Method

* Using Euler's method find the following: 77

1. $\dfrac{dy}{dx} = y + e^x - 2$; $y(0) = 2$, $h = 0.5$, find $y(2)$
2. $\dfrac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$, find $y(1)$
3. $\dfrac{dy}{dx} = \sqrt{\dfrac{x}{y}}$, $y(0) = 1$, $h = 0.2$, find $y(1)$
4. $\dfrac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$, find $y(2)$ for $h = 0.5$ & $h = 0.25$
5. $\dfrac{dy}{dx} = \sqrt{xy} + 2$, $y(1) = 1$, find $y(1.2)$ with $h = 0.2$

Q1. $f(x, y) = y + e^x - 2$

$y(0) = 2 \quad x = 0.5$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.2231
2	1	3.2231	3.9414	5.1938
3	1.5	5.1938	7.6755	9.0315

$y(2) = 9.0315$

2.

$\frac{dy}{dx} = 1+y^2 = f(x,y)$

$y(0) = 0 \quad h = 0.2$

x	x_n	y_n	$f(x_n,y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.405
2	0.4	0.408	1.1665	0.6413
3	0.6	0.6413	1.4113	0.9236
4	0.8	0.9236	1.8529	1.2942

$y(1) = 1.2942$

3.

$f(x,y) = \sqrt{\frac{x}{y}} \quad ; \quad y(0) = 1 \quad ; \quad h = 0.2$

x	x_n	y_n	$f(x_n,y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2106
3	0.6	1.2106	0.704	1.3514
4	0.8	1.3514	0.7694	1.5053

$y(1) = 1.5053$

4.

$f(x,y) = 3x^2+1 \quad ; \quad y(1) = 2 \quad ; \quad h = 0.5$

x	x_n	y_n	$f(x_n,y_n)$	y_{n+1}
1	1	2	4	4
2	1.5	4	7.75	7.875

$y(2) = 7.875$

$h = 0.25$

x	x_n	y_n	$f(x_n,y_n)$	y_{n+1}
1	1	2	4	4
2	1.25	3	5.6875	4.4219
3	1.5	4.4219	7.75	6.3594
4	1.75	6.3594	10.1875	8.9063

$y(1) = 8.9063$

5.

$f(x,y) = \sqrt{xy} + 2 \quad ; \quad y(1) = 1 \quad ; \quad h = 0.2$

x	x_n	y_n	$f(x_n,y_n)$	y_{n+1}
1	1	1	3	1.6

$y(1.2) = 1.6$

Practical NO. 9

Topic : Limits & partial-order derivative.

1. Evaluate the following limits.

(i) $\lim\limits_{(x,y)\to(-4,-1)} \dfrac{x^3 - 3y + y^2 - 1}{xy + 5}$

(ii) $\lim\limits_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2 + y^2 - 4x)}{x + 3y}$

(iii) $\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x^2 - y^2 z^2}{x^3 - x^2 y^2}$

2. Find f_x, f_y for each of the following

(i) $f(x,y) = xy\, e^{x^2+y^2}$

(ii) $f(x,y) = e^x \cos y$

(iii) $f(x,y) = x^3 y^2 - 3x^2 y + y^3 + 1$

3. Using definition find values of f_x, f_y at $(0,0)$ for : $f(x,y) = \dfrac{2x}{1+y^2}$

4. Find all second order partial derivatives of function. Also verify whether $f_{xy} = f_{yx}$

(i) $f(x,y) = \dfrac{y^2 - xy}{x^2}$ (ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

(iii) $f(x,y) = \sin(xy) + e^{x+y}$

6. Find the linearization of $f(x,y)$ at given point:

(i) $f(x,y) = \sqrt{x^2 + y^2}$ at $(1,1)$

(ii) $f(x,y) = 1 - x + y\sin x$ at $\left(\frac{\pi}{2}, 0\right)$

(iii) $f(x,y) = \log x + \log y$ at $(1,1)$

Q1.

(i) $\lim\limits_{(x,y)\to(-4,-1)} \dfrac{x^3 - 3y + y^2 - 1}{xy + 5}$

$\dfrac{(-4)^3 - 3(-1) + (-1)^2 - 1}{(-4)(-1) + 5}$

$\dfrac{64 + 3 + \cancel{1} - \cancel{1}}{4 + 5}$

$\dfrac{67}{9}$

(ii) $\lim\limits_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2 + y^2 - 4x)}{x + 3y}$

$\dfrac{(0+1)((2)^2 + (0)^2 - 4(2))}{2 + 3(0)}$

$$\frac{1(4+0-8)}{2} = \frac{-4}{2} = -2$$

(iii) $\lim\limits_{(x,y,z)\to(1,1,1)} \frac{x^2 - y^2 z^2}{x^3 - x^2 y^2}$

$$\frac{(1)^2 - (1)^2(1)^2}{(1)^3 - (1)^2(1)(1)}$$

$$\frac{1-1}{1-1} = \frac{0}{0}$$

... (solⁿ not defined)

Q2.

(i) $f(x,y) = xye^{x^2+y^2}$

$= x \cdot y \cdot e^{x^2} \cdot e^{y^2}$

$f\{x\} = \frac{\partial}{\partial x} x \cdot y \cdot e^{x^2} \cdot e^{y^2}$

$= y \cdot e^{y^2} \cdot \frac{d}{dx} x \cdot e^{x^2}$

$= ye^{y^2}\left[x \frac{\partial}{\partial x} e^{x^2} + e^{x^2} \frac{\partial}{\partial x} y\right]$

$= y \cdot e^{y^2}\left[2x^2 e^{x^2} + e^{x^2}\right]$

$= (2x^2+1) \cdot ye^{x^2+y^2}$

$fy = xe^{x^2} \frac{\partial}{\partial y} y \cdot e^{y^2}$

$= x \cdot e^{x^2}\left[y \frac{\partial}{\partial y} e^{y^2} + e^{y^2} \frac{\partial}{\partial y} y\right]$

$= (2y^2+1)\; x \cdot e^{x^2+y^2}$

(ii) $f(x,y) = e^x - \cos y$

$fx = \frac{\partial}{\partial x} e^x - \cos y$

$= e^x - \cos y$

$fy = \frac{\partial}{\partial y} e^x - \cos y$

$= -e^x \cdot \sin y$

(iii) $f(x,y) = x^3 y^2 - 3x^2 y + y^3 + 1$

$f\{x\} = \frac{\partial}{\partial x}(x^3 y^2 - 3x^2 y + y^3 + 1)$

$= 3x^2 y^2 - 6xy$

$fy = \frac{\partial}{\partial y}(x^3 y^2 - 3x^2 y + y^3 + 1)$

$= 2x^3 y - 3x^2 + 3y^2$

Q3. (i) $f(x,y) = \frac{2x}{1+y^2}$

$f\{x\} = \frac{1}{1+y^2} \frac{\partial}{\partial x}(2x)$

$= \frac{2}{1+y^2}$

$fx(0,0) = \frac{2}{1+(0)^2} = 2$

$\therefore f_y = 2x \frac{\partial}{\partial y}\left(\frac{1}{1+y^2}\right)$

$= 2x \cdot \frac{-1}{(1+y^2)^2} \cdot 2y$

$= \frac{2-4xy}{(1+y^2)^2}$

$f_y(0,0) = \frac{-4\times 0\times 0}{(1+0^2)^2} = 0$

Q4. $f(x,y) = \frac{y^2-xy}{x^2}$

$f(x) = \frac{\partial}{\partial x}\left(\frac{y^2-xy}{x^2}\right)$

$= -2y^2x^{-3} + \frac{y}{x^2}$

$= \frac{y}{x^2} - \frac{2y^2}{x^3}$

$f_y = \frac{2y-x}{x^2}$

$\therefore f_{xx} = \frac{\partial}{\partial x} f_x$

$= \frac{\partial}{\partial x}\left(\frac{y}{x^2} - \frac{2y^2}{x^3}\right)$

$= \frac{6y^2}{x^4} - \frac{2y}{x^3}$

$\therefore f_{yy} = \frac{\partial}{\partial y} f_y$

$= \frac{2}{x^2}$

$\therefore f_{xy} = \frac{\partial}{\partial y} f(x) = \frac{\partial}{\partial y}\left(\frac{y}{x^2} - \frac{2y^2}{x^3}\right)$

$= \frac{1}{x^2} - \frac{4y}{x^3}$

$\therefore f_{xy} = f_{yx}$

(ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$f_x = \frac{\partial}{\partial x}\left(x^3 + 3x^2y^2 - \log(x^2+1)\right)$

$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$

$f_y = \frac{\partial}{\partial y}\left(x^3 + 3x^2y^2 - \log(x^2+1)\right)$

$= 0 + 6yx^2 - 0$

$= 6x^2y$

$\therefore f_{xx} = \frac{\partial}{\partial x}(f_x)$

$= \frac{\partial}{\partial x}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 6x + 6y^2 - \frac{4x - 2x^2 + 2}{(x^2+1)^2}$

$\therefore f_{yy} = \frac{\partial}{\partial y}(f_y)$

$= \frac{\partial}{\partial y}(6x^2y)$

$= 6x^2$

$\therefore f_{xy} = \frac{\partial}{\partial y} f_x$

$= \frac{\partial}{\partial y}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 12xy$

$\therefore f_{yx} = \frac{\partial}{\partial x} f_y$

$= \frac{\partial}{\partial x}(6x^2y)$

$= 12xy$

$\therefore f_{xy} = f_{yx}$

(iii).

$f(x,y) = \sin(xy) + e^{x+y}$

$= \sin(xy) + e^x \cdot e^y$

$f_x = \frac{\partial}{\partial x}(\sin(xy) + e^x \cdot e^y)$

$= y\cos(xy) + e^x \cdot e^y$

$f_y = \frac{\partial}{\partial y}(\sin(xy) + e^x \cdot e^y)$

$= x\cos(xy) + e^x \cdot e^y$

$\therefore f_{xx} = \frac{\partial}{\partial x} f_x$

$= \frac{\partial}{\partial x}\, y\cos(xy) + e^x \cdot e^y$

$= -y^2 \sin(xy) + e^x \cdot e^y$

$\therefore f_{yy} = \frac{\partial}{\partial y} f_y$

$= \frac{\partial}{\partial y}(x\cos(xy) + e^x \cdot e^y)$

$= -x^2 \cdot \sin^2(xy) + e^x \cdot e^y$

$\therefore f_{xy} = \frac{\partial}{\partial y} f_x$

$= \frac{\partial}{\partial y}(y\cos(xy) + e^x \cdot e^y)$

$= -xy\sin(xy) + \cos(xy) + e^x \cdot e^y$

$\therefore f_{yx} = \frac{\partial}{\partial x} f_y$

$= \frac{\partial}{\partial x}(x\cos(xy) + e^x \cdot e^y)$

$= -xy\sin(xy) + \cos(xy) + e^x \cdot e^y$

$\therefore f_{xy} = f_{yx}$

Q5.

(i) $f(x,y) = \sqrt{x^2+y^2}$ $\quad (a,b) = (1,1)$

$f_x = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$ $\qquad f_y = \frac{2y}{2\sqrt{y^2+x^2}}$

$= \frac{x}{\sqrt{x^2+y^2}}$ $\qquad = \frac{y}{\sqrt{x^2+y^2}}$

$f_x(1,1) = \frac{1}{\sqrt{2}}$ $\qquad = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= \sqrt{2} + \frac{x-1}{\sqrt{2}} + \frac{y-1}{\sqrt{2}}$

$= \sqrt{2} + \frac{x+y-2}{\sqrt{2}}$

(ii) $f(x,y) = 1-x+y\sin x$ $\quad (a,b) = (\pi/2, 0)$

$f(\pi/2, 0) = 1 - \frac{\pi}{2} + 0 \cdot \sin\frac{\pi}{2}$

$= \frac{2-\pi}{2}$

$f_x = -1 + y\cos x$ $\qquad f_y = \sin x$

$f_y(\pi/2, 0) = \sin \pi/2$

$= 1$

$f_x(\pi/2, 0) = -1 + 0 \times \cos \pi/2$

$= -1$

$\therefore L(x,y) = f(\pi/2, 0) + f_x(\pi/2,0)(x-\pi/2) + f_y(\pi/2,0)(y-0)$

$= \frac{2-\pi}{2} + (-1)\left(x - \frac{\pi}{2}\right) + 1(y)$

$= 1 - \frac{\pi}{2} - x + \frac{\pi}{2} + y$

$= 1 - x + y$

(iii) $f(x,y) = \log x + \log y$ $\quad (a,b) = 1,1$

$f(1,1) = \log(1) + \log(1)$

$= 0$

$f_x = \frac{1}{x}$ $\qquad f_y = \frac{1}{y}$

$f_x(1,1) = 1$ $\qquad f_y(1,1) = 1$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 0 + 1(x-1) + 1(y-1)$

$= x+y-2$

AV
22/01/2020

Practical NO. 10

Q1. Find the directional derivative of the given vector at the given points.

(i) $f(x,y) = x + 2y - 3$ at $\bar{u} = 3\hat{i} - \hat{j}$, $a(1,-1)$

(ii) $f(x,y) = y^2 - 4x + 1$ at $\bar{u} = \hat{i} + 5\hat{j}$, $a(3,4)$

(iii) $f(x,y) = 2x + 3y$ at $\bar{u} = 3\hat{i} + 4\hat{j}$, $a(7,2)$

Q2. Find gradient vector for the following functions at the given point.

(i) $f(x,y) = x^y + y^x$, $a = (1,1)$

(ii) $f(x,y) = (\tan^{-1}x) - y^2$, $a = (1,-1)$

(iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$.

Q3. Find the equation of tangent and normal of each of the following curves.

(i) $x^2\log y + e^{xy} = 2$ at $(1,0)$

(ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2,-2)$

Q4. Find the equation of tangent and normal line to each of the following curves. surfaces.

(i) $x^2 - 2yz + 3y + xz = 7$ at $(2,1,0)$

(iii) $3xyz - x - y + z = -4$ at $(1,-1,2)$

Q5. Find the local maximum & minimum for the following function.

(i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$.

(ii) $f(x,y) = 2x^4 + 3x^2y - y^2$

(i) Here, $u = 3\hat{i} - \hat{j}$ is not a unit vector.

$|\bar{u}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$.

unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$
$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3 = 1 - 2 - 3 = -4$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(1 + \frac{3}{\sqrt{10}}\right), \left(-1 - \frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h\to 0} \frac{\cancel{-4} + \cancel{h}/\sqrt{10} + \cancel{4}}{\cancel{h}} = \frac{1}{\sqrt{10}}$

(ii) $f(x,y) = y^2 - 4x + 1$

$u = \hat{i} + 5\hat{j}$

$\because \bar{u} = \frac{\bar{u}}{|u|}, \frac{3\hat{i} + 5\hat{j}}{\sqrt{1^2 + 5^2}}$

$\because \bar{u} = \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(0) = (u)^2 - 4(3) + 1$

$= 16 - 12 + 1$

$= 5$

$f(a+hu) = f\left((3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\right)$

$= f\left(\left(3 + \frac{h}{\sqrt{26}}\right), \left(4 + \frac{5h}{\sqrt{26}}\right)\right)$

$= \left(4 + \frac{\sqrt{5h}}{26}\right)^2 - 4\left(5 + \frac{h}{\sqrt{21}}\right)$

$= 16 + \frac{40h}{\sqrt{26}} + \frac{25h^2}{26} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} - \frac{36h}{\sqrt{26}} + 5$

$D_u f(x) = \lim \frac{f(a+hu) - f(0)}{h}$

$= \lim_{h \to 0} \frac{\frac{25h^2}{26} - \frac{36h}{\sqrt{26}} + 5 - 5}{h}$

$= \lim_{h \to 0} \frac{25h^2}{26} - \frac{36}{\sqrt{26}}$

$= \frac{36}{\sqrt{26}}$

(iii) $f(x,y) = 2x + 3y$ a $(1,2)$

$u = 3\hat{i} + 4\hat{j}$

$u = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{3^2 + 4^2}} (3\hat{i} + 4\hat{j})$

$= \frac{1}{\sqrt{25}} (3\hat{i} + 4\hat{j})$

$\bar{u} = (3/25, 4/5)$

$f(0) = 2(1) + 3(2)$

$= 2 + 6$

$f(a) = 8$

$f(a+h+i) = f((1,2) + h(3/5, 4/5))$

$= f\left(\left(1 + \frac{3h}{5}\right), (2 + 4h/5)\right)$

$= 2(1 + 2h/5) + 3(2 + 4h/5)$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$

$= \frac{18h}{5} + 8$

$D_u f(a) = \lim_{x \to 12} \frac{f(a+hu) - f(a)}{2}$

$D_u f(a) = \lim_{x \to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$

$= \lim_{x \to 10} \cdot \frac{18x/5}{x}$

$= \frac{18}{5}$

Q2.

(i) $f(x,y) = x^y + y^x$

$f_x = yx^{y-1} + y^x \cdot \log y$.

$f_y = \frac{d}{dy} x^y + y^x$

$f_y \cdot xy^{x-1} + x^y . \log x$

$\nabla f(x,y) = f(f_y, f_x)$

$\nabla f(x,y) = (yx^{y-1} + y^x \log y, \; xy^{x+1}, \; x^y \log y)$

$\nabla f(1,1) = (1(1)^{2-1} + 1 \log 1, \; 1(1)^{2-1} + 1^2 \log 1$

$\nabla f(1,1) = (1,1)$

(ii) $f(x,y) = (\tan^{-1} x) y^2$

$f_x = \frac{d}{dx} (\tan^{-1} x) y^2$

$f_x = \frac{y^2}{1+x^2}$

$f_y = \frac{d}{dy} (\tan^{-1} x) y^2$

$= 2y \tan^{-1} x$

$\nabla f(x,y) = f(f_x, f_y)$

$= \epsilon \left(\frac{y^2}{1+x^2}, \; 2y + \tan^{-1} x \right)$

$\nabla f(-1,1) = \left(\frac{1^2}{1+(-1)^2}, \; 2(-1) \tan^{-1} x \right)$

$= \left(\frac{1}{1+1}, \; -2 \times \frac{\pi}{4} \right)$

$\nabla f(1-1) = \left(\frac{1}{2}, -\frac{\pi}{2} \right)$

(iii) $f(x,y,z) = xy^2 - e^{x+y+z}$

$f_x = y^2 - e^{x+y+z}$

$f_y = x^2 - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= (y^2 - e^{x+y+z}, \; x^2 - e^{x+y+z}, \; xy - e^{x+y+z})$

$\nabla f(1,-1,0) = (-1, -1, -2)$

Q3.

(i) $x^2 \cos y + e^{xy} = 2 = 0$

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$f_x = 2x\cos y + ye^{xy}$
$f_y = -x^2\sin y + xe^{xy}$

$= f_x(x - x_0) + f_y(y - y_0) = 0$

$= (2x\cos y + ye^{xy}) (x-1) + (-x^2\sin y + xe^{xy})(y-0) = 0.$

$= 2x^2\cos y + xye^{xy} + 2x\cos y - y^{e^{y}} - (x^2\sin x - xe^{xy}) = 0.$

$= (1, 0, 40) = (1, 0)$

(ii) $f(x,y) = x^2 + y^2 - 2x + 3y + 2 = 0$

$f_x = 2x - 2 \qquad f_x(x, -2) = 2$

$f_y = 2y + 3 \qquad f_y(2, -2) = 1$

Tangent:

$f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)$

$2(x-2) + 1(y+2) = 0$

$2x - 4 + y + 2 = 0$

$2x + y - 2 = 0$

Normal:

$x - 2y + d = 0$

$2 - 2(-2) + d = 0$

$\therefore d = 2$

$\therefore x - 2y + 2 = 0$

Q4. (i) $f(x,y,z) = x^2 - 2y^2 + 3y + 2x - 7$

$f_x = 2x + 2 \qquad f_x(x_0, y_0, z_0) = 2(2) + 0 = 4$

$f_y = -2x + 3 \qquad f_y(x_0, y_0, z_0) = 2(0) + 2 = 2$

$f_z = -2y + x \qquad f_z(x_0, y_0, z_0) = -2(1) + 2 = 4$

Tangent:

$= f_x(x - x_0) + f_y(x_0 y_0 z_0)(y - y_0) + f_z(x_0 y_0 z_0)(z - z_0)$

$= 4(x - 2) + 3(y - 1) + 0(z - 0) = 0$

$= 4x - 8 + 3y - 3 = 0$

$= 4x + 3y - 11 = 0$

Normal:

$$\frac{x - x_0}{f_x(x_0 y_0 z_0)} = \frac{y - y_0}{f_y(x_0 y_0 z_0)}$$

$$\frac{x-2}{4} = \frac{y-1}{1} = \frac{z-0}{0}$$

(ii) $f(x,y,z) = 3xyz - x - y + z + 4 = 0$

$f_x = 3yz - 1 \qquad f_x(x_0 y_0 z_0) = 3(-1)(2) - 1 = -7$

$f_y = 3xz - 1 \qquad f_y(x_0 y_0 z_0) = 3(1)(2)(-1) = 5$

$f_z = 3xy + 1 \qquad f_z(x_0 y_0 z_0) = 3(1)(-1) + 1 = -2$

Tangent:

$f_x(x_0 y_0 z_0)(x-x_0) + f_y(x_0 y_0 z_0)(y-y_0) + f_z(x_0 y_0 z_0) = 0$

$-7(x-1) + 5(y+1) - 2(x-2) = 0$

$-7x + 7 + 5y + 5 - \underset{2x}{\cancel{}} + 4 = 0$

$-7x + 5y - 2x + 16$

$7x - 5y + 2x - 16 = 0$

Normal:

$$\frac{x-x_0}{f_x(x_0 y_0 z_0)} = \frac{y-y_0}{f_y(x_0 z_0 y_0)} = \frac{z-z_0}{f_z(x_0 y_0 z_0)}$$

$$= \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

(iii) $f(x, y, z) = 3xyz - x - y + z + 4 = 0$

$f_x = 3x - 1$ $\quad f_x(x_0\ y_0\ z_0) = 3(-1)(2) - 1 = -7$

$f_y = 3x^2 - 1$ $\quad f_y(x_0\ y_0\ z_0) = 3(1)(2) - 1 = 5$

$f_z = 3yx + 1$ $\quad f_z(x_0\ y_0\ z_0) = 3(1)(-1) + 1 = -2$

Tangent:

$f_x(x_0 y_0 z_0)(x-x_0) + f_y(x_0 y_0 z_0)(y-y_0) + f_z(x_0 y_0 z_0)$

$-7(x-1) + 5(y+1) - 2(x-2) = 0$

$-7x + 7 + 5y + 5 - 2x + 4 = 0$

$-7x + 5y - 2z + 16$

Q5. $f(x, y) = 3x^2 + y^2 - 3xy + 6x - 4y.$

$f_x = 6x - 3y + 6.$

$f_y = 2y - 3x - 4.$

$f_x = 0$ $\quad f_y = 0$

$6x - 3y = -6$, $\quad 3x - 2y = -4$, $\quad 6x + 4y = -8$

$\therefore y = 2$

$\therefore x = 0$

$(x, y) = (0, 2)$

$r = f_{xx} = 6.$

$s = f_{xy} = -3$

$t = f_{yy} = 2$

$rt - s^2 = 6(2) - (-3)^2 - 12 - 4 = 3 > 0$

$r > 0$

f is maximum at $(0, 2)$

$f(0, 2) = 3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$

$= 4 + 0 - 0 + 0 - 8$

$f(0, 2) = -4$

(ii) $f(x, y, z) = 2x^2 + 3x^2 y - y^2$

$f_y = 8x^3 + 6xy$

$f_y = -2y + 3x^2$

$f_y = 0$

$8x^3 + 6xy = 0$.

$x(8x^2 + 6y) = 0$

$(x-y) = (0,0)$ is a root

$f_y = 0$

$-2y + 3(0) = 0$

$\therefore y = 0$

$8x^2 + 6y = 0$

$x^2 = \frac{-2}{3} y$

$f_y = -2y - \frac{2}{3} \times 3y$

$= -4y = 0$

$y = 0$

$x^6 = 0$.

$x = 0$

$(x, y) = (0,0)$ is the only root

$f_{x_1} = 24x^2 + 6y = 0$

$f_{x_1} = 6x = 0$

$f_{x_1} = -2 + 0 = -2$

$rk - f^2 - (0)(0) - (2)^2$

$= rt - f^2 = -4 < 0$

$(0,0)$ is the saddle point.

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