NIUS Midterm Report : T16 Study of Helicity Amplitude Formalism

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Abstract

Spinor helicity formalism provides a method for calculating scattering cross sections in high energy QCD calculations. In the first part of this project we have learnt the conventional method of calculating scattering amplitude and cross section and their application in QED. The basic concepts learnt will be used in the second part of the project in which we will learn the helicity amplitude formalism and its application to QCD.

Contents

1 Introduction

Scattering amplitude is the most important quantity in particle physics experiments. The standard way of calculating the scattering amplitude for a process is by drawing Feynman diagrams, using Feynman rules to find the invariant amplitude, summing over the external spin states, converting the spin sum into a trace and using trace theorems to evaluate the trace. If a process has N possible Feynman diagrams then the square of the amplitude will contain N^2 terms. This standard method becomes cumbersome for complex processes. Such extensive calculations can be managed by using the Helicity amplitude technique. In Helicity Amplitude method, the scattering process is decomposed into scattering of helicity eigenstates for which it is simpler to calculate the helicity amplitudes. The individual helicity amplitudes are summed and squared to get the amplitude for the process. In certain situations this greatly simplifies the scattering cross section calculation.

A relativistic theory of quantum mechanics is required to describe the high energy collision experiments carried out in particle accelerators . The Klein-Gordon equation and the Dirac equation are two such attempts at unifying special relativity and quantum mechanics. However these single particle equations cannot describe creation and annihilation of particles. Quantum field theory successfully combines relativity and quantum mechanics and can describe creation and annihilation of particles. QFT has enormous applications in fields like high energy physics and condensed matter physics.

The Klein Gordon equation is discussed in section 2. The Dirac equation, properties of gamma matrices, transformation of spinors, bilinear covariants and free solution to the Dirac equation are discussed in section 3. In section 4 we discuss about scattering cross section, decay rates and Feynman rules for calculating them. We calculate the scattering cross section for $e^-\mu^-$ scattering. In section 5 we quantize the Klein Gordon field and see that it describes non interacting spin 0 bosons. We also quantize a complex scalar field and analyse the resulting theory to illustrate the procedure of canonical quantization.

2 Klein Gordon equation

Consider the Schrodinger equation for a free particle,

$$i\frac{\partial\psi}{\partial t} = \frac{-1}{2m}\bigtriangledown^2\psi = H\psi \tag{1}$$

Clearly the form of this equation changes under spatial rotations as the spatial derivatives change and the time derivative is unaffected. Thus the Schrodinger equation is not Lorentz covariant. Any relativistic theory must be Lorentz covariant according to the principle of relativity. We thus seek a relativistic single particle wave equation which is Lorentz covariant. We may start with the relativistic energy momentum relation $E^2 = \boldsymbol{p}^2 + m^2$ and use Bohr's correspondance principle to replace $E \to i\frac{\partial}{\partial t}$ and $\boldsymbol{p} \to -i\bigtriangledown$. Doing this we get

$$-\frac{\partial^2\psi}{\partial t^2} = (-\bigtriangledown^2 + m^2)\psi \tag{2}$$

$$\left(\frac{\partial^2\psi}{\partial t^2} - \bigtriangledown^2 + m^2\right)\psi = 0 \tag{3}$$

This can be written as,

$$\left(\partial_\mu\partial^\mu + m^2\right)\psi = 0 \tag{4}$$

Equation 4 is the free Klein-Gordon equation. This equation is Lorentz covariant because the D'Alembertian operator $\partial_\mu\partial^\mu$ is Lorentz invariant. Here ψ is a scalar and thus cannot describe spin degrees of freedom. The conjugate of equation 4 is

$$\left(\partial_\mu\partial^\mu + m^2\right)\psi^\star = 0 \tag{5}$$

Multiplying equation 4 by $\psi^\star$, equation 5 by ψ and subtracting them we get

$$\partial_\mu(i(\psi^\star\partial^\mu\psi - (\partial^\mu\psi^\star)\psi)) = 0 \tag{6}$$

where we have multiplied by i to get the probability current density similar to the non-relativistic case. We get a continuity equation of the form $\partial_\mu j^\mu = 0$ with $j^\mu = (j^0, \boldsymbol{j}) = i(\psi^\star \partial^\mu \psi - (\partial^\mu \psi^\star)\psi)$.The density and current density are given by

$$j^0 = \rho = i\left(\psi^\star \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^\star}{\partial t}\right) \qquad \boldsymbol{j} = -i(\psi^\star \bigtriangledown \psi - \psi \bigtriangledown \psi^\star) \tag{7}$$

The eigenfunctions of the Klein Gordon equation are of the form :

$$\psi(\mathbf{x}, t) = Ne^{-i(Et - \mathbf{p.x})} = Ne^{-ip.x} \tag{8}$$

where N is a normalisation constant. Plugging eq.8 into eq. 4 we find that there are two solutions with energies $E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$ meaning that negative energy states also exist.
Using this solution in eq. 7, we get the probability current as

$$j^\mu = 2|N|^2(E, \boldsymbol{p}) \tag{9}$$

This means that the density $\rho = j^0$ can be negative and thus cannot be directly interpreted as probability density. We need an interpretation of states with negative energy and of negative densities.

2.1 Feynman-Stuckelberg interpretation of negative energy states

Consider the phase of the wavefunction $Et - \mathbf{p.x}$. We can replace $t \to -t$ and $-E \to E$ and leave the phase unchanged. Since we reversed time we must replace $\boldsymbol{p} \to -\boldsymbol{p}$. By reversing the charge and momentum we convert an incoming particle into an outgoing antiparticle. This interpretation of the negative energy states of a particle as positive energy states of it's anti particle moving backward in time was given independently by Feynman and Stuckelberg.
We define the electromagnetic current density for a Klein-Gordon particle of charge q as

$$J_\mu^{em} = qj^\mu = 2q|N|^2(E, \boldsymbol{p}) \tag{10}$$

This interpretation of the density as EM current density and not probability density was given by Pauli and Weisskopf. For a positively charged incoming particle with positive energy the EM current density is

$$J_\mu^{em} = 2(+q)|N|^2 p^\mu = 2(+q)|N|^2(E, \boldsymbol{p}) \tag{11}$$

and for a positively charged particle with negative energy

$$J_\mu^{em} = 2(+q)|N|^2(-E, \boldsymbol{p}) = 2(-q)|N|^2(E, -\boldsymbol{p}) \tag{12}$$

From the above equation it is clear that the negative energy states of an incoming particle can be interpreted as positive energy states of outgoing antiparticle by replacing $-E \to E$, $q \to -q$ and $\boldsymbol{p} \to -\boldsymbol{p}$. Also equation 10 takes care of negative densities as the EM charge density can be negative. This is how we deal with the negative densities and negative energies.

3 Dirac equation

The Klein-Gordon equation had the problem of negative probability density. Dirac realized that the problem was the second order time derivative and that a first order equation would give positive density. Following Dirac we look for a single particle relativistic wave equation which is first order in time. To keep space and time on equal footing we assume that the equation is first order in spatial derivatives as well. In Schrodinger representation the time-dependent wave function will be $\psi(x_0, x_1, x_2, x_3)$ and the energy and momentum operators are

$$E = p_0 = i\frac{\partial}{dx_0} = i\frac{\partial}{dt} \qquad p^j = -p_j = -i\frac{\partial}{dx^j} = -i\partial_j \tag{13}$$

We seek an equation of the form

$$(p_0 + \alpha^k p_k - mc\beta)\psi = 0 \tag{14}$$

or equivalently

$$i\frac{\partial\psi}{dt} = (\boldsymbol{\alpha}.\boldsymbol{p} + \beta m)\psi = H\psi \tag{15}$$

with α^k and β independent of $\boldsymbol{p}$ and thus commute with $\boldsymbol{p}$. All space-time points must be equivalent and thus α^k and β should be independent of $\boldsymbol{x}$ and thus commute with $\boldsymbol{x}$. Since α^k and β commute with both $\boldsymbol{x}$ and $\boldsymbol{p}$, they must correspond to some new degree of freedom. Moreover $\alpha^k$and β cannot be constants because if they are constant then the form of equation 15 changes under rotations.

3.1 Properties of α's and β

For the Hamiltonian to be Hermitian we need $\alpha^k$and β to be Hermitian that is

$$(\alpha^k)^\dagger = \alpha^k \qquad and \qquad \beta^\dagger = \beta \tag{16}$$

By applying H twice to ψ we get

$$-\frac{\partial^2\psi}{\partial t^2} = -\left(\Sigma\frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\partial_i\partial_j\right) - im\Sigma(a_i\beta + \beta\alpha_i)\partial_i\psi + \beta^2 m^2\psi$$

We expect this to reduce to the Klein-Gordon equation or the corresponding classical equation $E^2 = \boldsymbol{p}^2 + m^2$. This equation reduces to equation 2 if

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} I \tag{17}$$

$$\alpha_i\beta + \beta\alpha_i = 0 \tag{18}$$

$$\alpha^{i\,2} = \beta^2 = I \tag{19}$$

Thus all the four $\alpha^k$and β anticommute and the square of each is 1. Note that the above argument is not applicable if the mass of particle is 0. Since $\alpha^{i\,2} = \beta^2 = I$ the eigen values of α^k and β can be ± 1. Using equations 17 to equation 19 we get

$$Tr(\alpha^i) = Tr(\alpha^i\beta^2) = -Tr(\beta\alpha^i\beta) = -Tr(\alpha^i\beta^2) = -Tr(\alpha^i) = 0 \tag{20}$$

Similarly

$$Tr(\beta) = 0 \tag{21}$$

Since sum of eigenvalues of a matrix is equal to the trace of that matrix, if $\alpha^k$and β are represented by $N \times N$ matrices then N must be even. Consider N=2. Any 2×2 matrix can be written as a combination of $\{I, \sigma^1, \sigma^2, \sigma^3\}$ where σ^i are the Pauli matrices. Also the three Pauli matrices anticommute with each other. Thus we cannot find four anticommuting 2×2 matrices and the minimum possible N is 4. For N=4 many representations exist.

In the **standard or Dirac-Pauli representation**

$$\alpha^k = \begin{bmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{bmatrix} \qquad and \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{22}$$

In the **chiral or Weyl representation**

$$\alpha^k = \begin{bmatrix} -\sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix} \qquad and \qquad \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{23}$$

where σ^k are the 2×2 Pauli matrices.

In these representations since $\alpha^k$and β are 4×4 matrices ψ must be a column vector with four components, each a function of only the four x's. ψ is called a Dirac spinor or four spinor. We will study the transformation of ψ later. In non relativistic quantum mechanics spin 1/2 particles are described by the Pauli-Schrodinger equation and the solutions of this equation are two component spinors. The solutions of Dirac equation 15 are four component objects and half the solutions correspond to negative energy states. We will use a particular representation only while obtaining explicit solutions of the Dirac equation.

3.2 Dirac equation in covariant form

Multiplying equation 15 from the left by β we get

$$i\beta\frac{\partial\psi}{dt} = (\beta\alpha^k p_k + m)\psi \Rightarrow (i(\beta\partial_0 + i\beta\alpha^k\partial_k) - m)\psi = 0$$

$$\text{Let} \qquad \gamma^0 = \beta \qquad and \qquad \gamma^k = \beta\alpha^k \tag{24}$$

For notational convenience we assume that the index of gamma can be lowered or raised like a normal four vector even though they do not form a four vector. The above equation becomes

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{25}$$

Using the Feynman slash notation $\not{v} = \gamma^0 v^0 - \boldsymbol{\gamma.v}$ where v is a four vector, the above equation becomes

$$(i\not{\partial} - m)\psi = 0 \tag{26}$$

This is the Dirac equation in covariant form.

3.3 Properties of Gamma matrices

From the relations 16 to 19 and the definition 24 of gamma matrices we can get the following relations

$$(\gamma^0)^\dagger = \gamma^0 \qquad\qquad (\gamma^0)^2 = I \tag{27}$$

$$(\gamma^k)^\dagger = -\gamma^k \qquad\qquad (\gamma^k)^2 = -I \tag{28}$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I \tag{29}$$

Using equations 22, 23 and the definition 24 of gamma matrices we get, in the standard representation

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad and \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \tag{30}$$

In chiral representation

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad and \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \tag{31}$$

3.4 Transformation of Dirac spinors

Consider an active Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Under this transformation ∂_μ transforms like a covariant vector as $\partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu$. Let the wavefunction transform as $\psi'(x) = S\psi(\Lambda^{-1}x)$. Then after the transformation the Dirac equation 25 becomes

$$(i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - m)S\psi = 0$$

Multipling on the left byS^{-1}

$$(i\Lambda_\mu{}^\nu S^{-1}\gamma^\mu S\partial_\nu - m)\psi = 0 \tag{32}$$

This equation has the same form as equation 25 if

$$\Lambda_\mu{}^\nu S^{-1}\gamma^\mu S = \gamma^\nu \tag{33}$$

$$\Rightarrow S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu \tag{34}$$

A wavefunction which transforms under a Lorentz transformation as $\psi'(x) = S\psi$ with S obeying equation 34 is called a Dirac spinor or four component Lorentz spinor. Eq.34 ensures that the Dirac equation is Lorentz covariant. The adjoint spinor is defined as $\bar{\psi} = \psi^\dagger\gamma^0$. For parity operation,

$$\Lambda^\nu_\mu = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \tag{35}$$

$$\Rightarrow S_P^{-1}\gamma^0 S_P = \gamma^0 \qquad \text{and} \qquad S_P^{-1}\gamma^k S_P = -\gamma^k \qquad \text{for k=1,2,3.} \tag{36}$$

This is satisfied by $S_P = \gamma^0$. Now consider an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \Delta\omega^\mu{}_\nu$. The condition for Lorentz transformation $\Lambda g \Lambda^T = g$ is satisfied only if

$$\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu} \Rightarrow \Delta\omega^\mu{}_\nu = -\Delta\omega_\nu{}^\mu \tag{37}$$

This means that $\Delta\omega^{\mu\nu}$ has only six independent elements. This is expected as we have 3 boosts and 3 rotations. $\Delta\omega^{\mu\nu}$ corresponding to a boosts v^j along x^j and rotations θ^j along x^j is

$$\Delta\omega^\mu{}_\nu = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & \theta^3 & -\theta^2 \\ -v^2 & -\theta^3 & 0 & \theta^1 \\ -v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \tag{38}$$

We expect that S must be close to the identity transformation and could be expanded in a power series as

$$S = 1 + \tau \qquad and \qquad S^{-1} = 1 - \tau \tag{39}$$

where τ is an infinitesimal. Using equation 39 in 34 we get

$$(1-\tau)\gamma^\mu(1+\tau) = (g^\mu{}_\nu + \Delta\omega^\mu{}_\nu)\gamma^\nu \tag{40}$$

$$\Rightarrow [\gamma^\mu, \tau] = \omega^\mu{}_\nu\gamma^\nu \tag{41}$$

We show below that $\boldsymbol{\tau = \frac{-i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}$ with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ satisfies the above condition.

We can prove the relation by using eq. 29 and eq.37.

$$\begin{aligned}
[\gamma^\rho, \tau] = \frac{\Delta\omega_{\mu\nu}}{8}\{\gamma^\rho[\gamma_\mu,\gamma_\nu] - [\gamma_\mu,\gamma_\nu]\gamma^\rho\} &= \frac{\Delta\omega_{\mu\nu}}{8}\{(\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\rho) + (\gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\nu\gamma^\mu)\} \\
\gamma^\mu\gamma^\nu\gamma^\rho &= -\gamma^\mu\gamma^\rho\gamma^\nu + 2g^{\nu\rho}\gamma^\mu \\
&= \gamma^\rho\gamma^\mu\gamma^\nu + 2(g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu) \\
\Rightarrow (\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\rho) &= 2(g^{\mu\rho}\gamma^\nu - g^{\nu\rho}\gamma^\mu) \\
\text{Similarly,} \qquad (\gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\nu\gamma^\mu) &= 2(g^{\rho\mu}\gamma^\nu - g^{\rho\nu}\gamma^\mu) \\
\Rightarrow [\gamma^\rho, \tau] &= \frac{\Delta\omega_{\mu\nu}}{4}\{(g^{\mu\rho}\gamma^\nu - g^{\nu\rho}\gamma^\mu) + (g^{\rho\mu}\gamma^\nu - g^{\rho\nu}\gamma^\mu)\} \\
&= \frac{1}{4}\{\Delta\omega^\rho{}_\nu\gamma^\nu - \Delta\omega_\mu{}^\rho\gamma^\mu + \Delta\omega^\rho{}_\nu\gamma^\nu - \Delta\omega_\mu{}^\rho\gamma^\mu\} \\
&= \frac{1}{4}\{\Delta\omega^\rho{}_\nu\gamma^\nu - \Delta\omega_\mu{}^\rho\gamma^\mu + \Delta\omega^\rho{}_\nu\gamma^\nu - \Delta\omega_\mu{}^\rho\gamma^\mu\} \\
&= \frac{1}{4}\{\Delta\omega^\rho{}_\nu\gamma^\nu + \Delta\omega^\rho{}_\mu\gamma^\mu + \Delta\omega^\rho{}_\nu\gamma^\nu + \Delta\omega^\rho{}_\mu\gamma^\mu\} \\
&= \Delta\omega^\rho{}_\nu\gamma^\nu
\end{aligned}$$

Thus for a general infinitesimal Lorentz transformation

$$S = I - \frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu} \tag{42}$$

Since any finite transformation can be built up from infinitesimal once we know the form of S for a general Lorentz transformation.

3.5 Bilinear covariants

Define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{43}$$

Using the anticommmutation relation 29 it can be proved that

$$(\gamma^5)^\dagger = \gamma^5 \qquad (\gamma^5)^2 = I \qquad \{\gamma^5, \gamma^\mu\} = 0 \tag{44}$$

In standard representation $\gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and in chiral representation $\gamma^5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$. γ^5 can also be written as

$$\gamma^5 = \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{45}$$

We state a useful result without proof

$$S^\dagger\gamma^0 = b\gamma^0 S^{-1} \tag{46}$$

where $b = 1$ for $\Lambda^{00} > 1$ and $b = -1$ for $\Lambda^{00} < -1$.
To construct the most general form of covariant current we need bilinear quantities of the form $\bar{\psi}(4\times4)\psi$ where (4×4) is a product of gamma matrices. We now consider the transformation properties of bilinear quantities under orthocronous ($\Lambda^{00} > 1$) Lorentz transformations.

1. $\bar{\psi}\psi$ is a scalar.

$\bar{\psi}'\psi' = \psi'^\dagger\gamma^0\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi = \psi^\dagger\gamma^0 S^{-1}S\psi = \bar{\psi}\psi$.

Since $S_P = \gamma^0$ for parity we get, $\bar{\psi}'\psi' = \psi'^\dagger\gamma^0\psi' = \psi^\dagger S_P^\dagger\gamma^0 S_P\psi = \bar{\psi}\psi$.

2. $\bar{\psi}\gamma^\mu\psi$ transforms as a vector.

$\bar{\psi}'\gamma^\mu\psi' = \psi^\dagger S^\dagger\gamma^0\gamma^\mu S\psi = \psi^\dagger\gamma^0(S^{-1}\gamma^\mu S)\psi = \psi^\dagger\gamma^0\Lambda^\mu{}_\nu\gamma^\nu\psi = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$.

From eq.35, we have under parity $\bar{\psi}'\gamma^0\bar{\psi} = \gamma^\mu\psi$ and $\bar{\psi}'\gamma^k\psi' = -\bar{\psi}\gamma^k\psi$.

3. $\bar{\psi}\sigma^{\mu\nu}\psi$ transforms as a tensor.

$$\begin{aligned}
\bar{\psi}'\sigma^{\mu\nu}\psi' &= \frac{i}{2}\bar{\psi}'(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi' = \frac{i}{2}\psi^\dagger S^\dagger\gamma^0(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)S\psi \\
&= \frac{i}{2}\psi^\dagger\gamma^0 S^{-1}(\gamma^\mu(SS^{-1})\gamma^\nu - \gamma^\nu(SS^{-1})\gamma^\mu)S\psi \\
&= \frac{i}{2}\bar{\psi}\left((S^{-1}\gamma^\mu S)(S^{-1}\gamma^\nu S) - (S^{-1}\gamma^\nu S)(S^{-1}\gamma^\mu S)\right)\psi \\
&= \frac{i}{2}\bar{\psi}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho\left((\gamma^\sigma\gamma^\rho) - (\gamma^\rho\gamma^\sigma)\right)\psi \\
&= \Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho\bar{\psi}\sigma^{\sigma\rho}\psi
\end{aligned}$$

4. $S^{-1}\gamma^5 S = (det\Lambda)\gamma^5$.

$$\begin{aligned}
S^{-1}\gamma^5 S &= \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}S^{-1}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)S \\
&= \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}S^{-1}(\gamma^\mu(SS^{-1})\gamma^\nu(SS^{-1})\gamma^\rho(SS^{-1})\gamma^\sigma S \\
&= \left((\frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\delta\Lambda^\nu{}_\theta\Lambda^\rho{}_\phi\Lambda^\sigma{}_\pi)(\gamma^\delta\gamma^\theta\gamma^\phi\gamma^\pi)\right) \\
&= \left(\frac{-i}{4!}\epsilon_{\delta\theta\phi\pi}(det\Lambda)(\gamma^\delta\gamma^\theta\gamma^\phi\gamma^\pi)\right) \\
&= (det\Lambda)\gamma^5
\end{aligned}$$

5. $\bar{\psi}\gamma^5\psi$ transforms as a pseudoscalar.

$\bar{\psi}'\gamma^5\psi' = \psi^\dagger S^\dagger\gamma^0\gamma^5 S\psi = \psi^\dagger\gamma^0(S^{-1}\gamma^5 S)\psi = det(\Lambda)\bar{\psi}\gamma^5\psi$

For parity transformation $(det\Lambda) = -1$ and thus $\bar{\psi}\gamma^5\psi$ changes sign under parity and it remains constant under proper Lorentz transformation ($det\Lambda = 1$).

6. $\bar{\psi}\gamma^5\gamma^\mu\psi$ transforms as an axial vector (pseudovector).

$\bar{\psi}'\gamma^5\gamma^\mu\psi' = \psi^\dagger S^\dagger\gamma^0\gamma^5\gamma^\mu S\psi = \psi^\dagger\gamma^0 S^{-1}\gamma^5 SS^{-1}\gamma^\mu S\psi = (det\Lambda)\Lambda^\mu{}_\nu\bar{\psi}\gamma^5\gamma^\mu\psi$

These quantities are used in constructing currents. For example the weak interaction between an electron and neutrino is represented by a current of the form $j^\mu = \bar{\psi}_e\gamma^\mu\frac{1}{2}(1-\gamma^5)\psi_\nu$. This is know as the V-A (vector-axial vector)form of the weak current.

3.6 Continuity equation

The Dirac equation 25 can be written as

$$i\gamma^0\partial_0\psi + i\gamma^k\partial_k\psi - m\psi = 0 \tag{47}$$

Taking the hermitian conjugate equation and multiplying on the right by γ^0

$$-i\partial_0(\psi^\dagger\gamma^0)\gamma^0 + i\partial_k\psi^\dagger\gamma^k\gamma^0 - m\psi^\dagger\gamma^0 = 0 \tag{48}$$

$$-i\partial_0\bar{\psi}\gamma^0 - i\partial_k\bar{\psi}\gamma^k - m\bar{\psi} = 0 \tag{49}$$

$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \tag{50}$$

This is the adjoint equation of the Dirac equation. Now multiply equation 25 on the left by $\bar{\psi}$ and equation 50 on the right by ψ and adding we get

$$\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\gamma^\mu\psi = \partial_\mu\bar{\psi}\gamma^\mu\psi = 0$$

Thus we get a continuity equation $\partial_\mu j^\mu = 0$ from the Dirac equation with current

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{51}$$

We have already shown that $\bar{\psi}\gamma^\mu\psi$ transforms as a four vector. The timelike component

$$\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \Sigma\psi^\star_\alpha\psi_\alpha \geq 0$$

is positive definite and can be interpreted as probability density. However we use the Pauli-Weissenkopf interpretation and interpret

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi \tag{52}$$

as the electromagnetic current density of electron $j^\mu = (\rho, \boldsymbol{j})$.

3.7 Free particle spinors and helicity

Let us look for four momentum eigensolutions to eq.15 of the form

$$\psi = u(\boldsymbol{p})e^{-ip.x} \tag{53}$$

Substituting ψ in eq.25 we get

$$(\gamma^\mu p_\mu - m)u(\boldsymbol{p}) = 0 \tag{54}$$

$$(\not{p} - m)u(\boldsymbol{p}) = 0 \tag{55}$$

Substituting ψ in eq.15 we get

$$Hu = (\boldsymbol{\alpha}.\boldsymbol{p} + \beta m)u = Eu \tag{56}$$

Note that here $\boldsymbol{p}$ is a 3-vector and not an operator. When $\boldsymbol{p} = 0$ ie., the particle is at rest, this becomes

$$Hu = \beta m u = \begin{bmatrix} mI & 0 \\ 0 & -mI \end{bmatrix}$$

with eigenvalues m, m, -m, -m and corresponding eigenvectors

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{57}$$

The first two solutions represent $E > 0$ fermions and the last two represent $E < 0$ fermions. When momentum is finite and non-zero

$$Hu = (\boldsymbol{\alpha}.\boldsymbol{p} + \beta m)u = \begin{bmatrix} mI & \boldsymbol{\sigma}.\boldsymbol{p} \\ \boldsymbol{\sigma}.\boldsymbol{p} & -mI \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = E \begin{bmatrix} u_a \\ u_b \end{bmatrix} \tag{58}$$

where we have split the four spinor u into two two-component spinors u_a and u_b. This gives us two equations

$$\boldsymbol{\sigma}.\boldsymbol{p}\, u_b = (E - m)u_a \tag{59}$$
$$\boldsymbol{\sigma}.\boldsymbol{p}\, u_a = (E + m)u_b \tag{60}$$

We expect four independent solutions two with $E > 0$ and two with $E < 0$. For the $E > 0$ solutions we take $u_a^{(s)} = \chi^{(s)}$ where $\chi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\chi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and s=1,2. This gives us

$$u_b = \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{(E+m)}\chi^{(s)} \quad \Rightarrow \quad u^{(s)}(\boldsymbol{p}) = N \begin{bmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{(E+m)}\chi^{(s)} \end{bmatrix} \tag{61}$$

where N is normalisation constant.
For the $E < 0$ solutions we take $u_b^{(s)} = \chi^{(s)}$ and from eq.59 we get

$$u_a^{(s)} = \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{(E-m)}\chi^{(s)} = \frac{-\boldsymbol{\sigma}.\boldsymbol{p}}{(|E|+m)}\chi^{(s)} \quad \Rightarrow \quad u^{(s+2)}(\boldsymbol{p}) = N \begin{bmatrix} \frac{-\boldsymbol{\sigma}.\boldsymbol{p}}{(|E|+m)}\chi^{(s)} \\ \chi^{(s)} \end{bmatrix} \tag{62}$$

Our choice of u_a for $E > 0$ solutions and u_b for $E < 0$ solutions are such that the solutions 61 and 62 reduce to solutions 57 when $\boldsymbol{p} = 0$. It can be verified that the solutions are orthogonal

$$u^{(r)\dagger}u^{(s)} = 0 \qquad\qquad r \neq s \tag{63}$$

The solutions $u^{(1,2)}$ describe an electron with definite momentum p and energy E. The solutions $u^{(3,4)}$ describe an electron with definite momentum p and energy $-|E|$. Using the Feynman-Stuckelberg interpretation we can interpret these spinors as positive energy positron spinors

$$u^{(3,4)}(-\boldsymbol{p})e^{(-i(-p).x)} \equiv v^{(2,1)}(\boldsymbol{p})e^{(ip.x)} \tag{64}$$

where $v^{(2,1)}(\boldsymbol{p}) = u^{(3,4)}(-\boldsymbol{p})$ are the positron spinors.
Replacing p by $-p$ in 55 we get

$$(-\not{p} - m)u(-\boldsymbol{p}) = 0 \tag{65}$$
$$\Rightarrow (\not{p} + m)v(\boldsymbol{p}) = 0 \tag{66}$$

Notice that we identify spinor labels 1,2 with negative energy states 4,3. This reverse in order occurs because absence of spin along a particular direction is equivalent to presence of spin in opposite direction.

Helicity

From the solutions of Dirac equation it is clear that the energy is two fold degenerate . This means that there must an observable which commutes with H and $\boldsymbol{P}$ whose eigenvalues can be used to distinguish between two states with same energy E. Consider $\Sigma.\hat{p} = \begin{bmatrix} \boldsymbol{\sigma}.\hat{p} & 0 \\ 0 & \boldsymbol{\sigma}.\hat{p} \end{bmatrix}$. This operator commutes with both H and $\boldsymbol{P}$. We define the helicity operator as $\frac{1}{2}\boldsymbol{\Sigma}.\hat{\boldsymbol{p}}$. It has eigen values $\lambda = \pm\frac{1}{2}$. Because of our choice of $\chi^{(s)}$ we choose $\boldsymbol{p} = (0, 0, p)$. Then

$$\frac{1}{2}\boldsymbol{\sigma}.\hat{\boldsymbol{p}}\chi^{(s)} = \frac{1}{2}\sigma_3\chi^{(s)} = \lambda\chi^{(s)} \tag{67}$$

where $\lambda = +1/2$ or $-1/2$ corresponding to $s = 1$ or 2. Thus we can use helicity to label the solutions and distinguish between solutions with same eigenvalue E. Since both spin and momentum are reversed in going from negative energy particles to positive energy antiparticles the **helicity remains constant**. Moreover since the eigenvalues of $\boldsymbol{\sigma}.\hat{\boldsymbol{p}}$ are ± 1 we have

$$(\boldsymbol{\sigma}.\hat{\boldsymbol{p}})^2 = I \quad \Rightarrow \quad (\boldsymbol{\sigma}.\boldsymbol{p})^2 = |\boldsymbol{p}|^2 \tag{68}$$

Completeness relation and normalisation

We choose box normalization so that there are 2E particles per unit volume ,

$$\int_{\text{unit vol}} \rho dV = \int \psi^\dagger \psi dV = u^\dagger u = 2E \tag{69}$$

From this we get the orthogonality relations

$$u^{(r)\dagger} u^{(s)} = 2E\delta_{rs} \qquad v^{(r)\dagger} v^{(s)} = 2E\delta_{rs} \tag{70}$$

Using equation 61 we get

$$u^{(s)\dagger} u^{(s)} = |N|^2 \left(1 + \left(\frac{\boldsymbol{\sigma}.\boldsymbol{p}}{(E+m)}\right)^2\right) = \frac{2|N|^2 E}{E+m} \tag{71}$$

and similar equation for v. So the normalisation constant is

$$N = \sqrt{E+m} \tag{72}$$

We prove some useful relations below.
1. $\not{p}\not{p} = p^2 = m^2$.

$$\not{p}\not{p} = \gamma^\mu p_\mu \gamma^\nu p_\nu = p_\mu p_\nu \gamma^\mu \gamma^\nu = p_\mu p_\nu \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} = p_\mu p_\nu g^{\mu\nu} = p^2$$

2. $\Lambda_+ = \frac{\not{p}+m}{2m}$ and $\Lambda_- = \frac{-\not{p}+m}{2m}$ are projection operators.

$$\begin{aligned}
\Lambda_+^2 &= \frac{1}{4m^2}(\not{p}^2 + 2m\not{p} + m^2) = \frac{1}{4m^2}(2m^2 + 2m\not{p}) = \frac{\not{p}+m}{2m} = \Lambda_+ \\
\Lambda_-^2 &= \frac{1}{4m^2}(\not{p}^2 - 2m\not{p} + m^2) = \frac{1}{4m^2}(2m^2 - 2m\not{p}) = \frac{-\not{p}+m}{2m} = \Lambda_- \\
\Lambda_+\Lambda_- &= \frac{1}{4m^2}(\not{p}+m)(-\not{p}+m) = \frac{1}{4m^2}(-\not{p}^2 + m) = 0 = \Lambda_-\Lambda_+ \\
\Lambda_+ + \Lambda_- &= \frac{\not{p}+m}{2m} + \frac{-\not{p}+m}{2m} = 1
\end{aligned}$$

From eqs.55 and 66 we can see that Λ_+ and Λ_- project onto spinors of positive and negative energies respectively. Using 55, 66 and 70 the following orthonormality relations can be proved

$$\textbf{3. } \bar{u}^{(r)} u^{(s)} = 2m\delta_{rs}, \qquad \textbf{4. } \bar{v}^{(r)} v^{(s)} = -2m\delta_{rs}, \qquad \textbf{5. } \bar{u}^{(r)} v^{(s)} = \bar{v}^{(r)} u^{(s)} = 0 \tag{73}$$

6. Completeness relations:

$$\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = \not{p} + m \qquad \sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = \not{p} - m \tag{74}$$

Acting on $u^{(s')}$ and $v^{(s')}$ we get

$$\begin{aligned}
\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} (u^{(s')}) &= \sum_{s=1,2} u^{(s)} (\bar{u}^{(s)} u^{(s')}) = 2m \sum_{s=1,2} u^{(s)} \delta_{ss'} = 2m u^{(s')} \\
\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} (v^{(s')}) &= \sum_{s=1,2} u^{(s)} (\bar{u}^{(s)} v^{(s')}) = 0
\end{aligned}$$

The action of $\sum_{s=1,2} u^{(s)} \bar{u}^{(s)}$ on $u^{(s)}$ and $v^{(s)}$ is the same as $\not{p} + m$. Since any general spinor can be written as a superposition of $u^{(s)}$ and $v^{(s)}$ we have $\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = \not{p} + m$. Similarly we can show that $\sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = \not{p} - m$.
These relations will be used in QED calculations.

4 Electrodynamics of spin 1/2 particles

We saw that the free particle solutions $\psi = u(\boldsymbol{p})e^{-ip.x}$ satisfying the Dirac equation 54 describe a free electron of four momentum p^μ. To describe an electron in electromagnetic field A^μ the Dirac equation has to be modified. This can be done in analogy with classical mechanics by replacing $p^\mu \to p^\mu + eA^\mu$ in 54,

$$(\gamma_\mu(p^\mu + eA^\mu) - m)\psi = 0$$
$$\Rightarrow \quad (\gamma_\mu p^\mu - m)\psi = -e\gamma_\mu A^\mu$$

$$\text{Let} \qquad -e\gamma_\mu A^\mu = \gamma^0 V \tag{75}$$
$$\Rightarrow (\gamma_\mu p^\mu - m)\psi = \gamma^0 V\psi \tag{76}$$

Multiplying 76 by γ^0 on the left we get

$$p^0\psi + (-\boldsymbol{\alpha}.\boldsymbol{p} + \beta m)\psi = V\psi$$
$$\Rightarrow \quad i\frac{\partial\psi}{\partial t} = ((\boldsymbol{\alpha}.\boldsymbol{p} - \beta m) + V)\psi = (H + V)\psi$$

This is similar to what we get in non relativistic quantum mechanics when we introduce a perturbation V to the unperturbed Hamiltonian H. This justifies our choice in equation 75.
Now using perturbation theory the transition amplitude for an electron going from an initial state ψ_i to a final state ψ_f can be written in the covariant form as

$$T_{fi} = -i\int d^4x\psi_f^\dagger(x)V\psi_i(x) \tag{77}$$

Using 75 we get

$$T_{fi} = -i\int d^4x\bar{\psi}_f\gamma^0 V\psi_i \tag{78}$$
$$= -i\int d^4x\bar{\psi}_f(-e\gamma_\mu A^\mu)\psi_i \tag{79}$$
$$\Rightarrow T_{fi} = -i\int j_\mu^{fi}A^\mu d^4x \tag{80}$$

where

$$j_\mu^{fi} = -e\bar{\psi}_f\gamma_\mu\psi_i \tag{81}$$
$$= -e\bar{u}_f\gamma_\mu u_i e^{i(p_f - p_i)} \tag{82}$$

is the electromagnetic transition current between initial state i and final state f. Notice that this expression for j_μ^{fi} is similar to equation 52. Here $\bar{u}_f^{(r)}$ is the row spinor corresponding to an outgoing electron in spin state (r) with momentum p_f and $u_i^{(s)}$ is the column spinor corresponding to an incoming electron in spin state (s) with momentum p_i.

4.1 Feynman rules for external fermionic lines and photon propagator

We illustrate the Feynman rules by considering Møller scattering which is electron-electron scattering. This is represented by the Feynman diagram of Fig.1.

We consider one of the electrons (say 1) to be moving in the electromagnetic field created by the other electron (say 2). Using eqs. 81 and 82, the currents associated with the electrons are

$$j_{(1)}^\mu = -e\bar{u}_c\gamma^\mu u_a e^{i(p_c - p_a)} \tag{83}$$
$$j_{(2)}^\mu = -e\bar{u}_d\gamma^\mu u_b e^{i(p_d - p_b)} \tag{84}$$

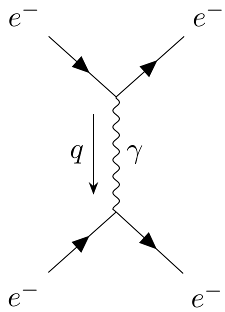

Figure 1: Feynman diagram for $e^-e^- \to e^-e^-$

Using the Maxwells equation in Lorentz gauge, we have

$$\Box^2 A^\mu = j^\mu_{(2)} \tag{85}$$

Since $\Box^2 e^{iq.x} = -q^2 e^{-iq.x}$ we get the field created by electron 2 as

$$A^\mu = \left(\frac{-1}{q^2}\right) j^\mu_{(2)} \tag{86}$$

with $q = p_d - p_b$. Using this in 80 we get

$$\begin{aligned}
T_{fi} &= -i \int j^{(1)}_\mu \left(\frac{-1}{q^2}\right) j^\mu_{(2)} d^4x \\
&= -i(-e\bar{u}_c \gamma_\mu u_a) \left(\frac{-1}{q^2}\right) (-e\bar{u}_d \gamma^\mu u_b) \int e^{i(p_c+p_d-p_a-p_b)} d^4x \\
&= -i(-e\bar{u}_c \gamma_\mu u_a) \left(\frac{-1}{q^2}\right) (-e\bar{u}_d \gamma^\mu u_b)(2\pi)^4 \delta^{(4)}(p_c + p_d - p_a - p_b)
\end{aligned}$$

We define the invariant amplitude $\mathscr{M}$ as

$$T_{fi} = -i(2\pi)^4 \delta^{(4)}(p_c + p_d - p_a - p_b)\mathscr{M} \tag{87}$$

Thus the amplitude assosciated with the Feynman diagram in Fig.1 is

$$-i\mathscr{M} = (ie\bar{u}_c \gamma_\mu u_a) \left(\frac{-ig_{\mu\nu}}{q^2}\right) (ie\bar{u}_d \gamma^\nu u_b) \tag{88}$$

We attach a factor of $\bar{u}$ with outgoing external fermionic lines, u with incoming external fermionic lines, $ie\gamma^\mu$ with vertex and $\left(\frac{-ig_{\mu\nu}}{q^2}\right)$ with internal photon lines of the Feynman diagram. This is shown in Fig.2. $(ie\gamma^\mu)$ is called the **vertex factor** and $\left(\frac{-ig_{\mu\nu}}{q^2}\right)$ is known as the **photon propagator**. By multiplying all the factors associated with a Feynman diagram, we get the corresponding amplitude $-i\mathscr{M}$. Because we are dealing with identical particles, a second diagram is also possible as shown in Fig.3. The amplitudes coresponding to the two diagrams must have a relative minus sign because of the exchange of identical particles. Thus, the invariant amplitude for møller scattering is

$$\mathscr{M} = -e^2(\bar{u}_c \gamma^\mu u_a) \left(\frac{1}{q^2}\right) (\bar{u}_d \gamma_\mu u_b) + e^2(\bar{u}_d \gamma^\mu u_a) \left(\frac{1}{q^2}\right) (\bar{u}_c \gamma_\mu u_b) \tag{89}$$

4.2 Cross section and decay rate

We define the cross section for $a + b \to c + d$ scattering as

$$\text{Cross section} = \frac{W_{fi}}{\text{(initial flux)}} \text{(number of final states)} \tag{90}$$

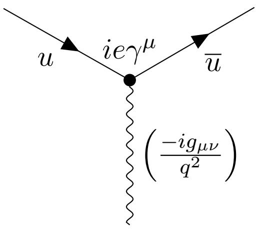

Figure 2: Feynman rules for $-i\mathscr{M}$

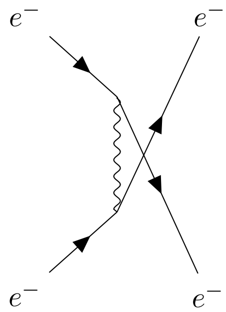

Figure 3: Second diagram for $e^-e^- \to e^-e^-$

Here, W_{fi} is the transition rate per unit volume defined as

$$W_{fi} = \lim_{T\to\infty} \frac{|T_{fi}|^2}{TV} \tag{91}$$

Using 87 and writing one of the delta functions as an integral, we get

$$\begin{aligned} W_{fi} &= \lim_{T\to\infty} \left(\frac{1}{TV}\right)(-i)(2\pi)^4\delta^{(4)}(p_c+p_d-p_a-p_b)|\mathscr{M}|^2 \int_{-T/2}^{T/2}\int_V e^{i(p_c+p_d-p_a-p_b)}d^4x & (92)\\ &= \lim_{T\to\infty} \left(\frac{1}{TV}\right)(-i)(2\pi)^4\delta^{(4)}(p_c+p_d-p_a-p_b)|\mathscr{M}|^2(TV) & (93)\\ &= -i(2\pi)^4\delta^{(4)}(p_c+p_d-p_a-p_b)|\mathscr{M}|^2 & (94)\end{aligned}$$

In the lab frame (b at rest), we define the initial flux as the product of number of beam particles moving per unit area per unit time and the number of target particles per unit volume. That is

$$\text{Initial flux} = \left(\frac{|2\boldsymbol{v_a}|E_a}{V}\right)\left(\frac{2E_b}{V}\right) = 2(|\boldsymbol{v_a}|E_a).(2E_b) \tag{95}$$

We know that if we use box normalisation in single particle quantum mechanics, the three components of momentum are quantized and can be increased only in steps of $(\frac{2\pi}{L})$, where L is the length of the box. Since there are $2E$ particles per unit volume this means that

$$\text{number of states per particle in the range } \boldsymbol{p} \text{ to } \boldsymbol{p}+d^3p = \frac{d^3p}{(2\pi/L)^3 2E} \equiv \frac{Vd^3p}{(2\pi)^3 2E} \tag{96}$$

Thus for particles c and d to get scattered into momentum elements d^3p_C and d^3p_D

$$\text{Number of final states } = \frac{Vd^3p_c}{(2\pi)^3 2E_c}.\frac{Vd^3p_d}{(2\pi)^3 2E_d} = \frac{d^3p_c}{(2\pi)^3 2E_c}.\frac{d^3p_d}{(2\pi)^3 2E_d} \tag{97}$$

Thus the cross section is

$$d\sigma = \frac{|\mathscr{M}|^2}{\mathscr{F}}dQ \tag{98}$$

where

$$dQ = (2\pi)^4\delta^{(4)}(p_c+p_d-p_a-p_b)\frac{d^3p_c}{(2\pi)^3 2E_c}.\frac{d^3p_d}{(2\pi)^3 2E_d} \tag{99}$$

$$\text{and} \qquad \mathscr{F} = (2|\boldsymbol{v_a}|E_a).(2E_b) \tag{100}$$

are the Lorentz invariant phase space factor and incident flux in lab frame respectively.

Cross section can be intuitively understood as follows. In a scattering process let the number of target particles be n_t and the flux of beam be $n_b v_b$ which is the number of beam particles traversing a unit area perpendicular to the beam velocity. Then we expect that the number of particles scattered per unit time n_s must be proportional to n_t and $n_b v_b$. This can be written as

$$n_s = \sigma(n_t)(n_b v_b) \tag{101}$$

where the constant of proportionality σ is related to the intrinsic probability of scattering. This proportionality constant is the cross section for this process. Notice that σ has the dimensions of area. So cross section can be interpreted as the effective area of the beam seen by the target.

For a general collinear collision, the flux factor can be written as

$$\begin{aligned} \mathscr{F} &= (2|\boldsymbol{v_a} - \boldsymbol{v_b}|E_a).(2E_b) && (102)\\ &= 4(|\boldsymbol{v_a}| + |\boldsymbol{v_b}|)E_a E_b && (103)\\ &= 4(|\boldsymbol{p}_a|E_b + |\boldsymbol{p}_b|E_a) && (104) \end{aligned}$$

where we have used the relativistic expression $\boldsymbol{v} = \boldsymbol{p}/E$.

We show below that for a general collinear collision, $\mathscr{F} = 4((p_a.p_b)^2 - m_a^2 m_b^2)^{\frac{1}{2}}$.

$$\begin{aligned} \mathscr{F} &= 4((p_a.p_b)^2 - m_a^2 m_b^2)^{\frac{1}{2}} \\ &= 4\{(E_a E_b - \boldsymbol{p_a}.\boldsymbol{p_b})^2 - (E_a^2 - |p_a|^2)(E_b^2 - |p_b|^2)\}^{\frac{1}{2}} \\ &= 4\{(E_a E_b + |\boldsymbol{p}_a||\boldsymbol{p}_b|)^2 - (E_a^2 - |\boldsymbol{p}_a|^2)(E_b^2 - |\boldsymbol{p}_b|^2)\}^{\frac{1}{2}} \\ &= 4(2E_a E_b|\boldsymbol{p}_a||\boldsymbol{p}_b| + E_a^2|\boldsymbol{p}_b|^2 + E_b^2)|\boldsymbol{p}_a|^2)^{\frac{1}{2}} \\ &= 4(|\boldsymbol{p}_a|E_b + |\boldsymbol{p}_b|E_a) \end{aligned}$$

From this relation it is clear that $\mathscr{F}$ is invariant. It can be shown that in CM frame for the process $a + b \rightarrow c + d$

$$\begin{aligned} dQ &= \frac{1}{4\pi^2}\frac{p_f}{4\sqrt{s}}d\Omega && (105)\\ \mathscr{F} &= 4\pi\sqrt{s} && (106)\\ \left.\frac{d\sigma}{d\Omega}\right|_{CM} &= \frac{1}{64\pi^2 s}\frac{p_f}{p_i}|\mathscr{M}|^2 && (107) \end{aligned}$$

where $d\Omega$ is solid angle about $|\boldsymbol{p}_c|$, $|\boldsymbol{p_a}| = |\boldsymbol{p_b}| = p_i$, $|\boldsymbol{p_c}| = |\boldsymbol{p_d}| = p_f$ and $s = (E_a + E_b)^2$.

Unpolarized cross section

If we are measuring unpolarized cross section, that is we are not measuring the spins of incoming and outgoing particles, then we must do the following replacement

$$|\mathscr{M}|^2 \rightarrow \overline{|\mathscr{M}|^2} = \frac{1}{(2s_a + 1)(2s_b + 1)} \sum_{\text{spin states}} |\mathscr{M}|^2 \tag{108}$$

where we have summed over all possible final spin states and averaged over the initial spin states.

Decay rate and lifetime

Consider the decay of a particle $A \rightarrow 1 + 2 + ... + n$ into n particles. In the CM frame the particle A is at rest so that the flux factor is $2E_A$. Thus the differential decay rate is

$$d\Gamma = \frac{1}{2E_A}|\mathscr{M}|^2 \frac{d^3 p_1}{(2\pi)^3 2E_1}.\frac{d^3 p_2}{(2\pi)^3 2E_2}...\frac{d^3 p_n}{(2\pi)^3 2E_n}(2\pi)^4 \delta^{(4)}(p_c + p_d - p_a - p_b) \tag{109}$$

Differential decay rate is the cross section for a particle decay in CM frame. The total decay rate Γ is the sum of decay rates for all channels. The total decay rate is also equal to the fraction of particles A decayed in unit time. This leads to the exponential decay law for the number of particles A

$$\Gamma = \frac{-dN_A}{dt}/N_A \quad \Rightarrow \quad N_A(t) = N_A(0)e^{\Gamma t} \tag{110}$$

From this it is clear that Γ^{-1} is the lifetime of particle A.

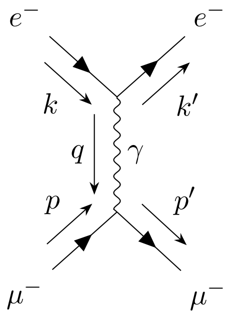

Figure 4: Feynman diagram for $e^-\mu^- \to e^-\mu^-$ showing the four momenta

4.3 $e^-\mu^-$ scattering amplitude

The spin sum in the expression for unpolarized cross section is non-trivial. We explore how this sum can be done using Casimir's trick in which the spin sum is rewritten as a trace by considering $e^-\mu^-$ scattering. Consider the scattering of an electron by a muon represented by the Feynman diagram in Fig. 4. Using the Feynman rules, the invariant amplitude associated with the diagram is

$$\mathscr{M} = -e^2(\bar{u}(k')\gamma^\mu u(k))\left(\frac{1}{q^2}\right)(\bar{u}(p')\gamma_\mu u(p)) \tag{111}$$

Since both electron and muon are spin 1/2 particles we have

$$\begin{aligned} |\mathscr{M}|^2 &= \frac{1}{(2s_a+1)(2s_b+1)}\sum_{\text{spin states}}|\mathscr{M}|^2 && (112)\\ &= \frac{1}{4}\sum_{\text{spin states}}|\mathscr{M}|^2 && (113)\\ &= \frac{e^4}{q^4}L_e^{\mu\nu}L^{muon}_{\mu\nu} && (114)\end{aligned}$$

where

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{e^-\text{ spins}}(\bar{u}(k')\gamma^\mu u(k))(\bar{u}(k')\gamma^\nu u(k))^* \tag{115}$$

and similar expression for $L^{muon}_{\mu\nu}$. We have broken down the spin sum into product of two tensors which involve sum over the spins of just one particle. Note that the second term in the electron tensor of equation 115 is a 1×1 matrix so the hermitian conjugate is the same as complex conjugate. Now

$$(\bar{u}(k')\gamma^\nu u(k))^* = (\bar{u}(k')\gamma^\nu u(k))^\dagger = (u(k')^\dagger\gamma^0\gamma^\nu u(k))^\dagger = u(k)^\dagger\gamma^{\nu\dagger}\gamma^{0\dagger}(u(k')^\dagger)^\dagger = u(k)^\dagger\gamma^0\gamma^\nu u(k')$$

$$\Rightarrow (\bar{u}(k')\gamma^\nu u(k))^* = \bar{u}(k)\gamma^\nu u(k') \tag{116}$$

Let s and s' be the initial and final spin state of electron. Using the above result and writing the matrix product in equation 115 explicitly in terms of matrix elements, we get

$$\begin{aligned} L_e^{\mu\nu} &= \frac{1}{2}\sum_{s,s'}\{\bar{u}^{(s')}(k')_\alpha(\gamma^\mu)_{\alpha\beta}\,u^{(s)}(k)_\beta\}\{\bar{u}^{(s)}(k)_\gamma(\gamma^\nu)_{\gamma\delta}\,u^{(s')}(k')_\delta\}\\ &= \frac{1}{2}\sum_{s'}\{u^{(s')}(k')_\delta\bar{u}^{(s')}(k')_\alpha\}(\gamma^\mu)_{\alpha\beta}\sum_{s}\{u^{(s)}(k)_\beta\bar{u}^{(s)}(k)\}_\gamma(\gamma^\nu)_{\gamma\delta}\\ &= \frac{1}{2}[(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma^\nu]_{\delta\delta}\end{aligned}$$

where we have used the completeness relations 74. Thus the electron tensor can be rewritten as a trace

$$L_e^{\mu\nu} = \frac{1}{2}Tr[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu] \tag{117}$$

Similarly, we get the muon tensor as

$$L_{\mu\nu}^{muon} = \frac{1}{2}Tr[(\not{p}' + M)\gamma^\mu(\not{p} + M)\gamma^\nu] \tag{118}$$

These traces can be evaluated without actually multiplying any matrices by using trace theorems. Using the trace theorems A.1,A.2 and A.3 of Appendix.A we further simplify equation 115

$$\begin{aligned}
L_e^{\mu\nu} &= \frac{1}{2}\{Tr[\not{k}'\gamma^\mu\not{k}\gamma^\nu] + \frac{m^2}{2}\gamma^\mu\gamma^\nu\} \\
&= \frac{1}{2}k'_\sigma k_\rho\, Tr[\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu] + 2m^2 g^{\mu\nu} \\
&= 2\{[k'_\sigma k_\rho(g^{\sigma\nu}g^{\mu\rho} - g^{\mu\nu}g^{\sigma\rho} + g^{\rho\mu}g^{\sigma\nu})] + m^2 g^{\mu\nu}\} \\
&= 2(k'^\mu k^\nu + k'^\nu k^\mu - (k'.k - m^2)g^{\mu\nu})
\end{aligned}$$

Similarly, we get

$$L_{\mu\nu}^{muon} = 2('p^\mu p^\nu + p'^\nu p^\mu - (p'.p - M^2)g^{\mu\nu}) \tag{119}$$

Substuiting this in equation 114 and simplifying we get the spin averaged amplitude for electron muon scattering as

$$\overline{|\mathscr{M}|^2} = \frac{8e^4}{q^4}[(k'.p')(k.p) + (k.p')(k'.p) - m^2(p'.p) - M^2(k'.k) + 2m^2M^2] \tag{120}$$

In the extreme relativistic limit we can ignore the masses and the amplitude becomes

$$\overline{|\mathscr{M}|^2} \approx \frac{8e^4}{q^4}[(k'.p')(k.p) + (k.p')(k'.p)] \tag{121}$$

The Mandelstam variables in this limit become

$$s = (k+p)^2 \approx 2k.p = 2k'.p' \tag{122}$$

$$t = (k-k')^2 \approx -2k.k' = -2p.p' \tag{123}$$

$$u = (k-p')^2 \approx -2k.p' = -2p.k' \tag{124}$$

Thus for high energy unpolarized electron muon scattering

$$\overline{|\mathscr{M}|^2} = 2e^4\left(\frac{s^2+u^2}{t^2}\right) \tag{125}$$

4.4 $e^-e^+ \to \mu^+\mu^-$ cross section

Fig.5 gives the Feynman diagram for the process $e^-e^+ \to \mu^+\mu^-$. Notice that this process is related to the process $e^-\mu^- \to e^-\mu^-$ by crossing symmetry. We should interchange $k' \longleftrightarrow -p$ or equivalently $s \longleftrightarrow t$. Doing this interchange in equation 125 we get in the high energy limit

$$\overline{|\mathscr{M}|^2} = 2e^4\left(\frac{t^2+u^2}{s^2}\right) \tag{126}$$

Now in the CM frame $|\boldsymbol{p}_A| = |\boldsymbol{p}_B| = |\boldsymbol{p}_C| = |\boldsymbol{p}_D| = p$ since the masses can be neglected. Also the Mandelstam variables become

$$s \approx 4p^2, \qquad t \approx -2p^2(1 - cos\theta), \qquad u \approx -2p^2(1 + cos\theta) \tag{127}$$

$$\Rightarrow \left(\frac{t^2+u^2}{s^2}\right) = \frac{4p^4(1-cos\theta)^2 + 4p^4(1+cos\theta)^2}{16p^4} \tag{128}$$

$$= \frac{1}{2}(1 + cos^2\theta) \tag{129}$$

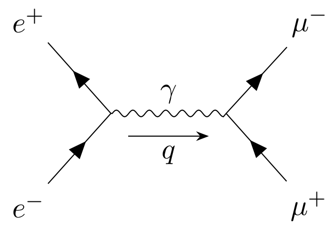

Figure 5: Feynman diagram for $e^-e^+ \to \mu^+\mu^-$

Using equation 107

$$\begin{aligned}\left.\frac{d\sigma}{d\Omega}\right|_{CM} &= \frac{1}{64\pi^2 s}\frac{p_f}{p_i}\overline{|\mathscr{M}|^2} \\ &= \frac{1}{64\pi^2 s}\left[\frac{2e^4}{2}(1+cos^2\theta)\right]\end{aligned}$$

Integrating this over $d\Omega$ we get the total cross section for the process $e^-e^+ \to \mu^+\mu^-$ as

$$\sigma(e^-e^+ \to \mu^+\mu^-) = \left(\frac{4\pi\alpha^2}{3s}\right) \tag{130}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. This result agrees with experiments to about 10 percent. The next term in the perturbation series takes care of almost all the discrepancy.

5 Canonical Quantization

Canonical quantization is a procedure for obtaining a quantum field theory from a classical field theory. In non-relativistic quantum mechanics we quantize a classical system by replacing momentum and position in the classical Hamiltonian by their corresponding operators. Similarly in canonical quantization we start with a classical field and promote the field to field operator. Commutation relations are imposed on the field operator. This procedure works only for free fields (non interacting).

5.1 Klein Gordon field

Consider the Lagrangian for Klein Gordon field,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{131}$$

The corresponding equation of motion is the Klein Gordon equation. The conjugate momentum and the Hamiltonian density are

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi \tag{132}$$

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2) \tag{133}$$

We now replace the field and its momentum by the field operators $\phi \to \hat{\phi}$ and $\pi \to \hat{\pi}$. Motivated by the canonical commutator $[x, p] = i$ we impose the commutation relation

$$[\hat{\phi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \tag{134}$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\pi}(x), \hat{\pi}(y)] = 0 \tag{135}$$

In analogy with the ladder operator formalism of simple harmonic oscillator we write the field and it's conjugate momentum as a sum of plane waves as

$$\hat{\phi}(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} (a_{\boldsymbol{p}} e^{i\boldsymbol{p}.x} + a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}.x}) \tag{136}$$

$$\hat{\pi}(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} (a_{\boldsymbol{p}} e^{i\boldsymbol{p}.x} - a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}.x}) \tag{137}$$

The commutation relations in 134 is equivalent to the following commutation relation

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \tag{138}$$

$$[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = [a^\dagger_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{q}}] = 0 \tag{139}$$

The Hamiltonian is

$$H = \int d^3x \, \mathcal{H} = \frac{1}{2} \int d^3x \, (\pi^2 + (\nabla\phi)^2 + m^2\phi^2) \tag{140}$$

Using equation 136, 137, doing the integral over d^3x to get a delta function and using it to do one momentum integral we get

$$H = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_{\boldsymbol{p}}} \left[(-\omega^2_{\boldsymbol{p}} + \boldsymbol{p}^2 + m^2)(a_{\boldsymbol{p}} a_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}} a^\dagger_{-\boldsymbol{p}}) + (\omega^2_{\boldsymbol{p}} + \boldsymbol{p}^2 + m^2)(a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} + a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}}) \right] \tag{141}$$

Since $\omega^2_{\boldsymbol{p}} = \boldsymbol{p}^2 + m^2$ the first term vanishes and we get

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\boldsymbol{p}} (a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}) \tag{142}$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\boldsymbol{p}} \left[(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}) + \frac{1}{2} (2\pi)^3 \delta^{(3)}(0) \right] \tag{143}$$

The integral of the second term is infinite. This infinity can be removed by normal ordering of operators. A normal ordered string of operators is one in which the annihilation operators are placed to the right. Normal ordering the operators in equation 142 we get

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\boldsymbol{p}} (a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}) = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \hat{n}_{\boldsymbol{p}} \tag{144}$$

where $\hat{n}_{\boldsymbol{p}} = (a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}})$ is the number operator which gives the number of particles in a momentum eigenstate. From the expression 144 we get the commutation relations

$$[H, a^\dagger_{\boldsymbol{p}}] = \omega_{\boldsymbol{p}} a_{\boldsymbol{p}} \dagger \tag{145}$$

$$[H, a^\dagger_{\boldsymbol{p}}] = -\omega_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{146}$$

The quantized Klein Gordon field is similar to a system of independent harmonic oscillators. Let us find the spectrum of the Hamiltonian 144. In analogy with Harmonic oscillators, the state $|0\rangle$ such that $a_{\boldsymbol{p}}|0\rangle = 0$ for all momentum $\boldsymbol{p}$ is called the vacuum state and has zero energy. Other eigenstates can be built from the vacuum state by the application of creation operators. The general state $a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} ... a^\dagger_{\boldsymbol{r}} |0\rangle = |\boldsymbol{pq}...\boldsymbol{r}\rangle$ is an eigenstate of the Hamiltonian with energy $\omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}} + ... \omega_{\boldsymbol{r}}$. This forms the spectrum of H in 144.
Doing similar calculations we get the total momentum operator as

$$\boldsymbol{P} = -\int d^3x \, \pi(\boldsymbol{x}) \bigtriangledown \phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \boldsymbol{p} \, a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{147}$$

Thus the operator $a^\dagger_{\boldsymbol{p}}$ creates a momentum $\boldsymbol{p}$ and energy $\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$. Since the excitations $a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} ... a^\dagger_{\boldsymbol{r}} |0\rangle$ are discrete entities satisfying the relativistic energy momentum relation we can call them

particles. $a^\dagger_{\boldsymbol{p}}$ creates particles with momentum $\boldsymbol{p}$ and energy $E_{\boldsymbol{p}} = \omega_{\boldsymbol{p}}$. Since the operators $a^\dagger_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{q}}$ commute the two particle states $a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}}|0\rangle$ and $a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}}|0\rangle$ with the two particles interchanged are equal. So the particles obey Bose-Einstein statistics.So the Hamiltonian in eqn 144 is same as the Hamiltonian for a system of non interacting identical bose particles.
Let us try to interpret the field operator $\phi(\boldsymbol{x})$. We normalize the vacuum state as $\langle 0|0\rangle = 1$. We choose the normalization factor for single particle states as $|\boldsymbol{p}\rangle = \sqrt{2E_{\boldsymbol{p}}} a^\dagger_{\boldsymbol{p}}|0\rangle$ so that $\langle \boldsymbol{q}|\boldsymbol{p}\rangle = 2E_{\boldsymbol{p}}$ is Lorentz invariant. Now using eqn 136

$$\phi(\boldsymbol{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-i\boldsymbol{p}.\boldsymbol{x}} |\boldsymbol{p}\rangle \tag{148}$$

Except for a factor of $\frac{1}{2E_{\boldsymbol{p}}}$ this expression is the same as the non relativistic expression of the eigenstate $|\boldsymbol{x}\rangle$. Thus $\phi(\boldsymbol{x})$ acting on the vacuum state produces a particle at position $\boldsymbol{x}$. Note that

$$\langle 0|\phi(\boldsymbol{x})|\boldsymbol{p}\rangle = e^{i\boldsymbol{p}.\boldsymbol{x}} \tag{149}$$

is the position space representation of the single particle wavefunction of $|\boldsymbol{p}\rangle$ as expected from non relativistic quantum mechanics.

5.2 Complex scalar field

Consider the Lagrangian for two non interacting real scalar fields ,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2 \tag{150}$$

Introducing a complex field $\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ we can write the Lagrangian as

$$\mathcal{L} = (\partial^\mu \psi^\dagger \partial_\mu \psi) - m^2\psi^\dagger\psi \tag{151}$$

We can treat the real and complex parts of the field as two degrees of freedom or equivalently consider the field and its complex conjugate as two degrees of freedom. We take the latter approach. The momenta conjugate to ψ and $\psi^\dagger$ are

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \partial^0\psi^\dagger, \qquad \pi_{\psi^\dagger} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^\dagger)} = \partial^0\psi \tag{152}$$

The Hamiltonian density is

$$\mathcal{H} = \sum_{\sigma=\psi,\psi^\dagger} \pi_\sigma \partial^0\sigma - \mathcal{L} \tag{153}$$

$$= \partial^0\psi^\dagger\partial_0\psi + \bigtriangledown\psi^\dagger. \bigtriangledown\psi + m^2\psi^\dagger\psi \tag{154}$$

We promote the fields to field operators and impose the commutation relation

$$[\hat{\psi}(t,\boldsymbol{x}), \hat{\pi}_\psi(t,\boldsymbol{y})] = [\hat{\psi}^\dagger(t,\boldsymbol{x}), \hat{\pi}_{\psi^\dagger}(t,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{155}$$

All other commutators vanish. We expand the complex field operator in a sum of plane waves as

$$\hat{\psi}(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} (b_{\boldsymbol{p}} e^{i\boldsymbol{p}.x} + c^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}.x}) \tag{156}$$

$$\hat{\psi}^\dagger(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} (b^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}.x} + c_{\boldsymbol{p}} e^{i\boldsymbol{p}.x}) \tag{157}$$

$$\hat{\pi}(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} (i) \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} (b^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}.x} - c_{\boldsymbol{p}} e^{i\boldsymbol{p}.x}) \tag{158}$$

$$\hat{\pi}(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} (b_{\boldsymbol{p}} e^{i\boldsymbol{p}.x} - c^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}.x}) \tag{159}$$

Note that in the expansion 136 we used only one type of operator ($a_{\boldsymbol{p}}$), because we wanted $\hat{\phi}$ to be Hermitian as the field ϕ is a scalar. Since ψ is a complex field we use two different types of operator ($b_{\boldsymbol{p}}$ and $c_{\boldsymbol{p}}$) in the expansion 156. It can be shown that equation 155 is equivalent to

$$[b_{\boldsymbol{p}}, b_{\boldsymbol{p}}^\dagger] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \tag{160}$$

$$[c_{\boldsymbol{p}}, c_{\boldsymbol{p}}^\dagger] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \tag{161}$$

and all other commutators vanish. Proceeding similar to Klein Gordon field we get the normal ordered Hamiltonian as

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}}(b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}} + c_{\boldsymbol{p}}^\dagger c_{\boldsymbol{p}}) = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}}(\hat{n}_{\boldsymbol{p}}^{(b)} + \hat{n}_{\boldsymbol{p}}^{(c)}) \tag{162}$$

We can see that the operators $b_{\boldsymbol{p}}^\dagger$ and $c_{\boldsymbol{p}}^\dagger$ create two different types of particles with the same mass and spin zero. They are interpreted as particle and antiparticle.

Noether's theorem and conserved charge

In classical mechanics the Noether's theorem states that for every symmetry there is a conserved current. Let a continuous symmetry transformation $\phi \to \phi + D\phi$ change the Lagrangian only by a four divergence($D\mathcal{L} = \partial_\mu W^\mu$). Then there exists a conserved current $J_N^\mu = \Pi^\mu D\phi - W^\mu$ ie., $\partial^\mu J_\mu = 0$. This gives raise to a conserved charge $Q_N = \int d^3x\, J_N^0$.
Notice that the Lagrangian in 151 does not change under the transformation

$$\psi = e^{i\alpha}\psi, \qquad \psi^\dagger = e^{-i\alpha}\psi^\dagger \tag{163}$$

ie., $D\mathcal{L} = 0 \Rightarrow W^\mu = 0$. For an infinitesimal phase change $\delta\alpha$,

$$\psi \to \psi + i\psi\delta\alpha, \qquad \psi^\dagger \to \psi^\dagger - i\psi^\dagger\delta\alpha \tag{164}$$

The conserved Noether current is

$$\begin{aligned} J_N^\mu &= \sum_{\sigma=\psi,\psi^\dagger} \Pi_\sigma^\mu D\sigma \\ &= i\left[(\partial^\mu\psi^\dagger)\psi - (\partial^\mu\psi)\psi^\dagger\right] \end{aligned}$$

By promoting the fields to field operators we get the Noether current operator

$$\hat{J}_N^\mu = i\left[(\partial^\mu\hat{\psi}^\dagger)\hat{\psi} - (\partial^\mu\hat{\psi})\hat{\psi}^\dagger\right] \tag{165}$$

The conserved charge operator is

$$\begin{aligned} \hat{Q}_N &= \int d^3x\, \hat{J}_N^0 \\ &= i\int d^3x\, \left[(\partial^0\hat{\psi}^\dagger)\hat{\psi} - (\partial^0\hat{\psi})\hat{\psi}^\dagger\right] \\ &= \int d^3x\, (\hat{\pi}_\psi\hat{\psi} - \hat{\psi}^\dagger\hat{\pi}_{\psi^\dagger}) \end{aligned}$$

Inserting the mode expansions 156 to 159 and simplifying we get

$$\hat{Q}_N = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2}\right)(-b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}} + c_{\boldsymbol{p}} c_{\boldsymbol{p}}^\dagger - b_{\boldsymbol{p}} b_{\boldsymbol{p}}^\dagger + c_{\boldsymbol{p}}^\dagger c_{\boldsymbol{p}}) \tag{166}$$

Normal ordering the operator we get

$$\hat{Q}_N = \int \frac{d^3p}{(2\pi)^3}(-b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}} + c_{\boldsymbol{p}}^\dagger c_{\boldsymbol{p}}) = N_c - N_b \tag{167}$$

Thus the number of antiparticles minus the number of particles is a conserved quantity in the complex scalar field theory.

6 Summary

In the first part of this project I have gained basic knowledge of relativistic quantum mechanics and quantum field theory. We studied the Dirac equation in detail and looked at bilinear covariants which are used in writing Lagrangians. Using the Dirac equation and perturbation theory we arrived at the Feynman rules for QED. Using the Feynman rules we calculated the scattering amplitude for Møller scattering and $e^-\mu^-$ scattering. Then using crossing symmetry we calculated the scattering cross section for $e^-e^+ \rightarrow \mu^-\mu^+$. These calculations illustrate the standard way of calculating scattering cross sections. At last we studied canonical quantization of fields as an introduction to quantum field theory. We did canonical quantization of two free fields- the Klein-Gordon field and the complex scalar field.

Appendix A

Trace theorems

Trace of various combinations of gamma matrices occur frequently in calculation of the spin averaged amplitudes. The following trace theorem will be useful in QED calculations. They can be proved by using the anticommutation relation of gamma matrices.

$$\text{Trace of product of odd number of } \gamma^{\mu} \text{ 's vanish} \tag{A.1}$$

$$Tr(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu} \tag{A.2}$$
$$Tr(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4[g^{\mu\sigma}g^{\nu\rho} - g^{\nu\sigma}g^{\mu\rho} + g^{\rho\sigma}g^{\mu\nu}] \tag{A.3}$$
$$Tr(\not{a}\not{b}\not{c}\not{d}) = 4[(a.b)(c.d) - (a.c)(b.d) + (a.d)(b.c)] \tag{A.4}$$
$$Tr(\gamma^5) = 0 \tag{A.5}$$
$$Tr(\gamma^5\not{a}\not{b}) = 0 \tag{A.6}$$
$$Tr(\gamma^5\not{a}\not{b}\not{c}\not{d}) = 4i\epsilon_{\mu\nu\rho\sigma}a^{\mu}b^{\nu}c^{\rho}d^{\sigma} \tag{A.7}$$
$$Tr(\gamma^{\mu}\gamma^{\nu}\gamma_{\mu}\gamma_{\nu}) = -32 \tag{A.8}$$
$$Tr(\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}\gamma_{\mu}\gamma_{\nu}) = Tr(\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}\gamma_{\mu}\gamma_{\nu}) = 16g^{\rho\sigma} \tag{A.9}$$
$$Tr(\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}\gamma^{\lambda}\gamma_{\mu}\gamma^{\tau}\gamma_{\nu}) = -32g^{\rho\lambda}g^{\sigma\tau} \tag{A.10}$$

The following relations will be useful in simplifying traces

$$\gamma_{\mu}\gamma^{\mu} = 4 \tag{A.11}$$
$$\gamma_{\mu}\not{a}\gamma^{\mu} = -2\not{a} \tag{A.12}$$
$$\gamma_{\mu}\not{a}\not{b}\gamma^{\mu} = 4a.b \tag{A.13}$$
$$\gamma_{\mu}\not{a}\not{b}\not{c}\gamma^{\mu} = -2\not{c}\not{b}\not{a} \tag{A.14}$$

Appendix B

Mandelstam variables

In a two body scattering event $A + B \to C + D$ it is convenient to use Lorentz invariant combinations of four momenta defined below.

$$s = \left(\frac{P_A + P_B}{c}\right)^2, \qquad t = \left(\frac{P_A - P_C}{c}\right)^2, \qquad u = \left(\frac{P_A - P_D}{c}\right)^2 \tag{B.1}$$

These are known as the Mandelstam variables. From this definition the following equations can be proved

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 \tag{B.2}$$

$$E_{total}^{CM} = E_A^{CM} + E_B^{CM} = E_C^{CM} + E_D^{CM} = \sqrt{s}c^2 \tag{B.3}$$

$$E_A^{CM} = \frac{(s + m_A^2 - m_B^2)c^2}{2\sqrt{s}} \tag{B.4}$$

$$E_A^{lab} = \frac{(s - m_A^2 - m_B^2)c^2}{2m_B} \tag{B.5}$$

$$P_A.P_B = \frac{1}{2}(s - m_A^2 - m_B^2)c^2 \tag{B.6}$$

$$P_A.P_C = \frac{1}{2}(-t + m_A^2 + m_C^2)c^2 \tag{B.7}$$

$$P_A.P_D = \frac{1}{2}(-u + m_A^2 + m_D^2)c^2 \tag{B.8}$$

The last three equations will be useful in calculating Feynman amplitudes.

Proof:

1) Expanding the terms in the bracket in the definition of s, t and u we get

$$\begin{aligned}
s + t + u &= \frac{1}{c^2}(3P_A^2 + P_B^2 + P_C^2 + P_D^2 + 2P_A(P_B - P_C - P_D)) \\
&= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 - 2\frac{P_A^2}{c^2} \\
&= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 - 2m_A^2 \\
&= m_A^2 + m_B^2 + m_C^2 + m_D^2
\end{aligned}$$

2) Since $\boldsymbol{p_A} + \boldsymbol{p_B} = 0$ in CM frame,

$$\begin{aligned}
sc^2 &= (P_A^{CM} + P_B^{CM})^2 \\
&= \frac{1}{c^2}(E_A^{CM} + E_B^{CM})^2 - (\boldsymbol{p_A} + \boldsymbol{p_B})^2 \\
&= \frac{1}{c^2}(E_A^{CM} + E_B^{CM})^2 \\
E_{total}^{CM} &= E_A^{CM} + E_B^{CM} = E_C^{CM} + E_D^{CM} = \sqrt{s}c^2
\end{aligned}$$

3) Also in CM frame since $\boldsymbol{p_A} + \boldsymbol{p_B} = 0$,

$$
\begin{aligned}
\boldsymbol{p_A}.\boldsymbol{p_A} &= \boldsymbol{p_B}.\boldsymbol{p_B} \\
\Rightarrow \frac{E_A^2}{c^2} - m_A^2 c^2 &= \frac{E_B^2}{c^2} - m_B^2 c^2 \\
\Rightarrow (E_A^{CM} - E_B^{CM}) &= \frac{(m_A^2 - m_B^2)c^2}{\sqrt{s}}
\end{aligned}
$$

Using equation B.3 and solving for E_A^{CM} we get

$$
E_A^{CM} = \frac{(s + m_A^2 - m_B^2)c^2}{2\sqrt{s}}
$$

4) Let B be at rest in the lab frame ie., $\boldsymbol{p_B} = 0$. In this frame

$$
\begin{aligned}
sc^2 &= (P_A^{lab} + P_B^{lab})^2 \\
\Rightarrow (s - m_A^2 - m_B^2)c^2 &= \frac{2}{c^2}(E_A^{lab}.E_B^{lab}) - 2\boldsymbol{p_A}.\boldsymbol{p_B} \\
\Rightarrow (s - m_A^2 - m_B^2)c^2 &= 2(E_A^{lab}.m_B) \\
\Rightarrow E_A^{lab} &= \frac{(s - m_A^2 - m_B^2)c^2}{2m_B}
\end{aligned}
$$

5) Expanding the bracket in the definition of s,

$$
\begin{aligned}
sc^2 &= P_A^2 + P_B^2 + 2P_A.P_B \\
\Rightarrow +2P_A.P_B &= sc^2 - (m_A^2 + m_B^2)c^2 \\
\Rightarrow P_A.P_B &= \frac{1}{2}(s - m_A^2 - m_B^2)c^2
\end{aligned}
$$

The remaining two equations can be proved similarly.

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