■ p.39 **Riemann tensor: 4つの基本的性質**

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, w_c \equiv R_{abc}{}^d\, w_d \tag{1}$$

左辺で w_c は任意の dual vector であり、derivative の交換子は (1,3)型tensor を定義するので右辺のように表すことができる。

明らかに

$$(\natural 1) \qquad R_{bac}{}^d = -R_{abc}{}^d \tag{2}$$

(3.2.17),(3.2.14):

$$\begin{aligned}
\nabla_{[a} \nabla_b\, w_{c]} &= \partial_{[a} (\nabla_b\, w_{c]}) - \Gamma^d{}_{[a\,b}\, (\nabla_{|d|}\, w_{c]}) - \Gamma^d{}_{[a\,b}\, (\nabla_{c]}\, w_d) \\
&= \partial_{[a} (\nabla_b\, w_{c]}) \qquad \because \quad \Gamma^c{}_{a\,b} = \Gamma^c{}_{b\,a} \\
&= \partial_{[a}\, \partial_b\, w_{c]} - \partial_{[a} \big(\Gamma^d{}_{b\,c]}\, w_d \big) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\therefore \quad 0 &= 2\, \nabla_{[a} \nabla_b\, w_{c]} \\
&= \nabla_{[a} \nabla_b\, w_{c]} - \nabla_{[b} \nabla_a\, w_{c]} \\
&= R_{[abc]}{}^d\, w_d \\
(\natural 2) \qquad \therefore \quad R_{[abc]}{}^d &= 0
\end{aligned} \tag{3}$$

(3.2.15):

$$\begin{aligned}
0 &= (\nabla_a \nabla_b - \nabla_b \nabla_a)\, g_{cd} = R_{abc}{}^e\, g_{ed} + R_{abd}{}^e\, g_{ce} \\
&= R_{abcd} + R_{abdc} \\
(\natural 3) \qquad \therefore \quad R_{abdc} &= -R_{abcd}
\end{aligned} \tag{4}$$

(3.2.20): (3.2.14),(3.2.15) より

$$0 = 3\, R_{[abc]\,d} \;=\; R_{abcd} + R_{\cancel{c}abd} + R_{\cancel{b}cad} \tag{5}$$

$$\begin{aligned}
\text{(5) で } a \leftrightarrow d \text{ の置換} \quad 0 &= R_{dbca} + R_{cdba} + R_{bcda} = R_{\cancel{b}dac} - R_{cdab} - R_{\cancel{b}cad} \\
\text{(5) で } b \leftrightarrow d \text{ の置換} \quad 0 &= R_{adcb} + R_{cadb} + R_{dcab} = -R_{\cancel{a}dbc} - R_{\cancel{c}abd} - R_{cdab} \\
\text{(5) で } c \leftrightarrow d \text{ の置換} \times -1 \quad 0 &= -R_{abdc} - R_{dabc} - R_{bdac} = R_{abcd} + R_{\cancel{a}dbc} - R_{\cancel{b}dac}
\end{aligned}$$

$$\begin{aligned}
\text{上の四式の和を取ると} \qquad 0 &= 2\, (R_{abcd} - R_{cdab}) \\
\therefore \quad R_{abdc} &= \therefore \quad R_{cdab}
\end{aligned} \tag{6}$$

(3.2.24) Bianchi identity:

$$(\natural 4) \qquad \nabla_{[a} R_{b\,c]d}{}^e \;=\; 0 \tag{7}$$

Riemann tensorの自由度勘定：

$(\natural 1)$ ～ $(\natural 4)$ の性質は（時）空間の次元に関係が無い。n次元であるとすると、$(\natural 1)$,$(\natural 3)$より自由度は ${}_n\mathrm{C}_2 \times {}_n\mathrm{C}_2$. 更に$(\natural 2)$は ${}_n\mathrm{C}_3 \times n$個の恒等式なので、Riemann tensor の自由度は少なくとも

$$\begin{aligned}
{}_n\mathrm{C}_2 \times {}_n\mathrm{C}_2 - n\, {}_n\mathrm{C}_3 &= n^2\, (n-1)^2/4 - n^2\, (n-1)\, (n-2)/6 \\
&= n^2\, (n^2-1)/12
\end{aligned} \qquad \Rightarrow$$

n	2	3	4
自由度	1	6	20

にまで減る。$n=4$であれば $16 \times 15/12 = \mathbf{20}$となる。加えてBianchi恒等式$(\natural 4)$がある。

■ **Ricci tensor, Ricci scalar**

(3.2.26): (5)より

$$\begin{aligned} 0 &= (R_{abcd}+R_{cabd}+R_{bcad})\,g^{bd} \\ &= R_{ac}+0-R_{ca} \\ \therefore\quad & R_{ac}=R_{ca} \end{aligned}$$

(3.2.28): Weyl tensor

$$\begin{aligned} B_{abcd}\,g^{bd}\equiv g_{a\,[c}\,g_{d]b}\,g^{bd} &= \frac{1}{2}\,(g_{ac}\,g_{db}-g_{ad}\,g_{cb})\,g^{bd} \\ &= \frac{1}{2}\,(n\,g_{ac}-g_{ad}\,g_c{}^d) \\ &= \frac{(n-1)}{2}\,g_{ac} \\ A_{abcd}\,g^{bd}\equiv(g_{a[c}\,R_{d]b}-g_{b[c}\,R_{d]a})\,g^{bd} &= \frac{1}{2}\,(g_{ac}\,R_{db}-g_{ad}\,R_{cb}+g_{bd}\,R_{ca}-g_{bc}\,R_{da})\,g^{bd} \\ &= \frac{1}{2}\,(g_{ac}\,R-R_{ca}+n\,R_{ca}-R_{ac}) \\ &= \frac{n-2}{2}R_{ac}+\frac{1}{2}\,g_{ac}\,R \\ \underbrace{(R_{abcd}-\alpha A_{abcd}-\beta\,B_{abcd}\,R)}_{\equiv C_{abcd}}\,g^{bd} &= \left\{1-\alpha\,\frac{(n-2)}{2}\right\}R_{ac}-\frac{\alpha+\beta(n-1)}{2}\,g_{ac}\,R \\ \alpha=\frac{2}{n-2},\ \ \beta=-\frac{\alpha}{n-1}=-\frac{2}{(n-1)\,(n-2)}\quad\Rightarrow &= 0 \end{aligned}$$

(3.2.32) : Einstein tensor

Bianchi 恒等式(7)は

$$\begin{aligned} \nabla_a\,R_{bcd}{}^e+\nabla_b\,R_{cad}{}^e+\nabla_c\,R_{abd}{}^e &= 0 \\ \text{上式}\times g^{bd}\ \Rightarrow\quad \nabla_a\,R_c{}^e-\nabla_b\,R_{ac}{}^{be}-\nabla_c\,R_a{}^e &= 0 \\ \text{上式}\times g_{be},\ \ b\leftrightarrow c\ \Rightarrow\quad \nabla_a\,R_{bc}-\nabla_b\,R_{ac}+\nabla_d\,R_{abc}{}^d &= 0 \\ \\ \text{上式}\times g^{bc}\ \Rightarrow\quad \nabla_a\,R-2\,\nabla_b\,R_a{}^b &= 0 \\ \\ \therefore\quad G_{ab}\equiv R_{ab}-\frac{1}{2}\,g_{ab}\,R\quad\Rightarrow\quad \nabla^a\,G_{ab} &= 0 \end{aligned}$$