p.50,51

tetrad $(e_\mu)^a$, $\mu=1\sim n(=\dim M)$ , $a$ は抽象インデックス、$\mu$ は基底の番号付け

$$(e_\mu)^a\,(e_\nu)_a=g_{ab}\,(e_\mu)^a\,(e_\nu)^b=\eta_{\mu\nu} \tag{1}$$

[補足]

$\{e_\mu\}$ の dual basis を$\{e^{\nu^*}\}$とすると

$$e_\mu(e^{\nu^*})=e^{\nu^*}(e_\mu)=\delta^\nu_\mu$$

一般に $v\in V_p$, $w\in V^*$ とするとき、

$$\begin{aligned} v &= \sum_\mu v(e^{\mu^*})\,e_\mu \\ w &= \sum_\mu w(e_\mu)\,e^{\mu^*} \end{aligned}$$

と展開できる。したがって

$$v(w)=w(v)=\sum_{\mu,\nu} v(e^{\mu^*})\,w(e_\nu)\,e^{\nu^*}(e_\mu)=\sum_{\mu,\nu} v(e^{\mu^*})\,w(e_\nu)\,\delta^\nu_\mu=\sum_{\mu,\nu} v(e^{\mu^*})\,w(e_\mu)=v^a\,w_a$$

したがって $(e_\mu)_a=g_{ab}\,(e_\mu)^b\in V^*$ は

$$(e_\mu)_a=\sum_\nu \eta_{\mu\nu}(e^{\nu^*})_a\,,\quad \text{so that}\ \ (e_\mu)_a\,(e_\nu)^a=\sum_\rho \eta_{\mu\rho}\,e^{\rho^*}(e_\nu)=\eta_{\mu\nu}$$

上の関係式から

$$\sum_{\mu,\nu}\eta^{\mu\nu}\,(e_\mu)^a\,(e_\nu)_b=\delta^a_b\quad (\text{identity operater}: V\to V) \tag{2}$$

上を書き換えれば、metric が tetrad で表せる。

$$\sum_{\mu,\nu}\eta^{\mu\nu}\,(e_\mu)_a\,(e_\nu)_b=g_{ab}$$

実際 $v^a=\sum_\mu v^\mu(e_\mu)^a\in V$ を左辺に作用させると(1)より

$$\left(\sum_{\mu,\nu}\eta^{\mu\nu}\,(e_\mu)^a\,(e_\nu)_b\right)v^b=\sum_{\mu,\nu,\rho}\eta^{\mu\nu}\,\eta_{\nu\rho}\,(e_\mu)^a\,v^\rho=\sum_\mu v^\mu\,(e_\mu)^a=v$$

(3.4.13): $a$ を抽象インデックスとして次の 1-form のセット connection 1-form を定義

$$\omega_{a,\,\mu\nu}\equiv(e_\mu)^b\nabla_a(e_\nu)_b=(e_\mu)_b\nabla_a(e_\nu)^b\quad\because\ \nabla_a\,g_{bc}=0$$

Ricci rotation coefficients: $\omega_{a,\,\mu\nu}$の要素

$$\omega_{\lambda\mu\nu}\equiv(e_\lambda)^a\,\omega_{a,\,\mu\nu}=(e_\lambda)^a\,(e_\mu)^b\nabla_a(e_\nu)_b$$

$\nabla_a\,g_{bc}=0$を仮定しないで計算すると

$$\begin{aligned} \omega_{a,\,\mu\nu} &\equiv (e_\mu)^b\nabla_a(e_\nu)_b \\ &= (e_\mu)^b\nabla_a\,(g_{bc}(e_\nu)^c) \\ &= (e_\mu)_b\nabla_a(e_\nu)^b+(e_\mu)^b\,(e_\nu)^c\,\nabla_a\,g_{bc} \\ (1)\!\to\quad 0 &= \nabla_a\,((e_\mu)^b\,(e_\nu)_b) \\ &= (e_\nu)_b\,\nabla_a(e_\mu)^b+(e_\mu)^b\,\nabla_a(e_\nu)_b \\ &= (e_\nu)_b\,\nabla_a(e_\mu)^b+\omega_{a,\,\mu\nu} \\ \therefore\quad \omega_{a,\,\mu\nu}+\omega_{a,\,\nu\mu} &= (e_\mu)^b\,(e_\nu)^c\,\nabla_a\,g_{bc} \end{aligned}$$

$$\therefore\quad \omega_{a,\,\mu\nu}+\omega_{a,\,\nu\mu}=0\quad\longleftrightarrow\quad\nabla_a\,g_{bc}=0$$

(3.4.17): Riemann tensor の定義から

$$
\begin{aligned}
R_{\rho\sigma\mu\nu} &= (e_\rho)^a (e_\sigma)^b (e_\mu)^c R_{abc}{}^d (e_\nu)_d \\
&= (e_\rho)^a (e_\sigma)^b (e_\mu)^c (\nabla_a \nabla_b - \nabla_b \nabla_a)(e_\nu)_c
\end{aligned}
$$

$$
\begin{aligned}
(e_\mu)^c \nabla_a \nabla_b (e_\nu)_c &= \nabla_a \{(e_\mu)^c \nabla_b (e_\nu)_c\} - [\nabla_a (e_\mu)^c][\nabla_b (e_\nu)_c] \\
&= \nabla_a \omega_{b,\mu\nu} - \sum_{\rho,\sigma} \eta^{\rho\sigma} [\nabla_a (e_\mu)^f] (e_\rho)^c (e_\sigma)_f [\nabla_b (e_\nu)_c] \\
&= \nabla_a \omega_{\,b,\mu\nu} - \sum_{\rho,\sigma} \eta^{\rho\sigma} [(e_\sigma)_f \nabla_a (e_\mu)^f] [(e_\rho)^c \nabla_b (e_\nu)_c] \\
&= \nabla_a \omega_{b,\mu\nu} - \sum_{\rho,\sigma} \eta^{\rho\sigma} \omega_{a,\sigma\mu} \omega_{b,\rho\nu}
\end{aligned}
$$

$$
\therefore \quad R_{\rho\sigma\mu\nu} = (e_\rho)^a (e_\sigma)^b \left\{ \left( \nabla_a \omega_{b,\mu\nu} - \sum_{\rho,\sigma} \eta^{\rho\sigma} \omega_{a,\sigma\mu} \omega_{b,\rho\nu} \right) - (a \leftrightarrow b) \right\} \tag{3}
$$

$$
\begin{aligned}
\nabla_a \omega_{b,\mu\nu} &= \nabla_a \left( \sum_\beta (e^{\beta^*})_b \omega_{\beta\mu\nu} \right) = \nabla_a \left( \sum_{\alpha,\beta} \eta^{\alpha\beta} (e_\alpha)_b \omega_{\beta\mu\nu} \right) \\
(e_\sigma)^b \nabla_a \omega_{b,\mu\nu} &= \sum_{\alpha,\beta} \eta^{\alpha\beta} (e_\sigma)^b \nabla_a [(e_\alpha)_b \omega_{\beta\mu\nu}] \\
&= \sum_{\alpha,\beta} \eta^{\alpha\beta} \{(e_\sigma)^b (e_\alpha)_b \nabla_a \omega_{\beta\mu\nu} + \omega_{\,a,\sigma\alpha} \omega_{\beta\mu\nu}\} \\
&= \nabla_a \omega_{\sigma\mu\nu} + \sum_{\alpha,\beta} \eta^{\alpha\beta} \omega_{a,\sigma\alpha} \omega_{\beta\mu\nu}
\end{aligned}
$$

$$
\therefore \quad R_{\rho\sigma\mu\nu} = \left\{ (e_\rho)^a \nabla_a \omega_{\sigma\mu\nu} + \sum_{\alpha,\beta} \eta^{\alpha\beta} (\omega_{\rho\sigma\alpha} \omega_{\beta\mu\nu} - \omega_{\rho\alpha\mu} \omega_{\sigma\beta\nu}) \right\} - \{\rho \leftrightarrow \sigma\} \tag{4}
$$

Ricci tensor の要素

$$
\begin{aligned}
\sum_{\sigma,\nu} \eta^{\sigma\nu} R_{\rho\sigma\mu\nu} &= \sum_{\sigma,\nu} \eta^{\sigma\nu} (e_\rho)^a (e_\sigma)^b (e_\mu)^c R_{abc}{}^d (e_\nu)_d \\
(2) \nearrow \quad &= (e_\rho)^a (e_\mu)^c \delta^b_d R_{abc}{}^d = (e_\rho)^a (e_\mu)^c R_{abc}{}^b = (e_\rho)^a (e_\mu)^c R_{ac} \\
&= R_{\rho\mu}
\end{aligned}
$$

Torsion free 条件の connection 1-form における表現 (1)

(3.4.23): torsion free → ベクトル場の交換子が covariant derivative で書ける

$$
\begin{aligned}
(e_\sigma)_a [e_\mu, e_\nu]^a &= (e_\sigma)_a \{(e_\mu)^b \nabla_b (e_\nu)^a - (\mu \leftrightarrow \nu)\} \\
&= (e_\mu)^b (e_\sigma)_a \nabla_b (e_\nu)^a - (\mu \leftrightarrow \nu) \\
&= \omega_{\mu\sigma\nu} - w_{\nu\sigma\mu}
\end{aligned} \tag{5}
$$

problem 8. (p.54)

後ろ二つの添字の反対称性（ $\omega_{\lambda\mu\nu} = \omega_{\lambda[\mu\nu]}$ ）から

$$
\begin{aligned}
\omega_{[\lambda\mu\nu]} &= \frac{1}{3} (\omega_{\lambda\,[\mu\nu]} + \omega_{\mu\,[\nu\lambda]} + \omega_{\nu\,[\lambda\mu]}) \\
2\,\omega_{[\mu\nu]\,\lambda} &= \omega_{\mu\nu\lambda} - \omega_{\nu\mu\lambda} = \omega_{\mu\,[\nu\lambda]} + \omega_{\nu\,[\lambda\mu]} \\
3\,\omega_{[\lambda\mu\nu]} - 2\,\omega_{[\mu\nu]\,\lambda} &= \omega_{\lambda\,[\mu\nu]} \\
&= \omega_{\lambda\mu\nu}
\end{aligned}
$$

$(e_\sigma)_a [e_\mu, e_\nu]^a \equiv c_{\sigma\mu\nu}$ と記すと(5)より

$$
c_{\sigma\mu\nu} = -2\,\omega_{[\mu\nu]\,\sigma}
$$

$$
\begin{aligned}
\omega_{\lambda\mu\nu} &= 3\,\omega_{[\lambda\mu\nu]} - 2\,\omega_{[\mu\nu]\,\lambda} \\
&= \omega_{[\lambda\mu]\,\nu} + \omega_{[\mu\nu]\,\lambda} + \omega_{[\nu\lambda]\,\mu} - 2\,\omega_{[\mu\nu]\,\lambda} \\
&= \omega_{[\lambda\mu]\,\nu} - \omega_{[\mu\nu]\,\lambda} + \omega_{[\nu\lambda]\,\mu} \\
&= \frac{1}{2} (c_{\lambda\mu\nu} - c_{\nu\lambda\mu} - c_{\mu\nu\lambda}) = \frac{1}{2} (c_{\lambda\mu\nu} - c_{\nu\lambda\mu} + c_{\mu\lambda\nu}) \\
&= \frac{1}{2} c_{\lambda\mu\nu} + c_{[\mu\,|\lambda|\,\nu]}
\end{aligned}
$$

Ricci rotaion coefficients が tetrad の交換子で表せた。

Torsion free 条件の connection 1-form における表現 (2)

(3.4.24)

$$
\begin{aligned}
\sum_{\mu,\nu} \eta^{\mu\nu} (e_\mu)_a \omega_{b,\sigma\nu} &= -\sum_{\mu,\nu} \eta^{\mu\nu} (e_\mu)_a \omega_{b,\nu\sigma} \\
&= -\sum_{\mu,\nu} \eta^{\mu\nu} (e_\mu)_a (e_\nu)^c \nabla_b (e_\sigma)_c \\
(2)\nearrow \quad &= -\delta_a^c \nabla_b (e_\sigma)_c \\
&= -\nabla_b (e_\sigma)_a \\
\therefore \quad \sum_{\mu,\nu} \eta^{\mu\nu} (e_\mu)_{[a} \omega_{b],\sigma\nu} &= \nabla_{[a} (e_\sigma)_{b]}
\end{aligned}
\tag{6}
$$

tortion free $\longleftrightarrow$ 任意の1-form $w_a$ に対して $\nabla_{[a} w_{b]}$ は covariant derivative の選び方に依らない。実際、(3.1.7)より

$$
\left(\tilde{\nabla}_a - \nabla_a\right) w_b = C^c{}_{ab} w_c \quad \text{であるので} \quad \tilde{\nabla}_{[a} w_{b]} - \nabla_{[a} w_{b]} = 0 \quad \because \quad \text{torsion free} \longleftrightarrow \ C^c{}_{ab} = C^c{}_{ba}
$$

したがって (6) において $\nabla_a$ を $\partial_a$ として構わない。

$$
\partial_{[a} (e_\sigma)_{b]} = \sum_{\mu,\nu} \eta^{\mu\nu} (e_\mu)_{[a} \omega_{b],\sigma\nu} \tag{7}
$$

同じ理由で (3) においても $\nabla_a$ を $\partial_a$ としてよい。

$$
\therefore \quad R_{\rho\sigma\mu\nu} = (e_\rho)^a (e_\sigma)^b \left\{ \left( \partial_a \omega_{b,\mu\nu} - \sum_{\rho,\sigma} \eta^{\rho\sigma} \omega_{a,\sigma\mu} \omega_{b,\rho\nu} \right) - (a \leftrightarrow b) \right\} \tag{8}
$$