p130. 半径$r=a$の球面上の線素

$$ds^2 = a^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

$\theta=0$ 近傍での局所座標として次の座標変換 $(\theta,\phi)\Rightarrow(x,y)$

$$\begin{cases} x = a\theta\cos\phi \\ y = a\theta\sin\phi \end{cases} \quad\Leftrightarrow\quad \begin{cases} \theta = (x^2+y^2)^{1/2}/a, & (a\theta)^2 = x^2+y^2 \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$a\,d\theta = \frac{1}{a\,\theta}(x\,dx + y\,dy)$$

$$\frac{1}{\cos^2\phi}d\phi = (1+\tan^2\theta)d\phi = \frac{x^2+y^2}{x^2}d\phi = \frac{1}{x^2}(x\,dy - y\,dx) \;\Rightarrow\; a\,d\phi = \frac{1}{a\,\theta^2}(x\,dy/x - y\,dx)$$

$$\sin^2\theta \simeq \theta^2(1-\theta^2/6)^2 \simeq \theta^2(1-\theta^2/3)$$

従って式 (1)は

$$\begin{aligned} ds^2 &\simeq \frac{1}{(a\,\theta)^2}\{(x\,dx+y\,dy)^2 + (1-\theta^2/3)(x\,dy - y\,dx)^2\} \\ &= \frac{1}{(a\,\theta)^2}\{(x^2+y^2)\,(dx^2+dy^2) - \theta^2(y^2\,dx^2 + x^2\,dy^2 - 2\,x\,y\,dx\,dy)/3\} \\ &= (1-\frac{1}{3a^2}y^2)\,dx^2 + (1-\frac{1}{3a^2}x^2)\,dy^2 - \frac{2}{3a^2}x\,y\,dx\,dy \end{aligned}$$

$$\therefore \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{3a^2}y^2, & -\frac{1}{3a^2}x\,y \\ -\frac{1}{3a^2}x\,y & 1-\frac{1}{3a^2}x^2 \end{pmatrix}, \quad (g_{ab})\Big|_{(x,y)=(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{\partial g_{ab}}{\partial x_c}\Big|_{(x,y)=(0,0)} = 0$$

p127. 平坦時空のペンローズ図

動径距離 $r>0$

$$\begin{cases} u = t-r \\ v = t+r \end{cases} \Leftrightarrow \begin{cases} t = (u+v)/2 \\ r = (v-u)/2 \end{cases} \Longrightarrow \boxed{\begin{matrix} u' = \tan^{-1}u \\ v' = \tan^{-1}v \end{matrix}} \Longrightarrow \begin{cases} t' = u'+v' \\ r' = v'-u' \end{cases} \Leftrightarrow \begin{cases} u' = (t'-r')/2 \\ v' = (t'+r')/2 \end{cases}$$

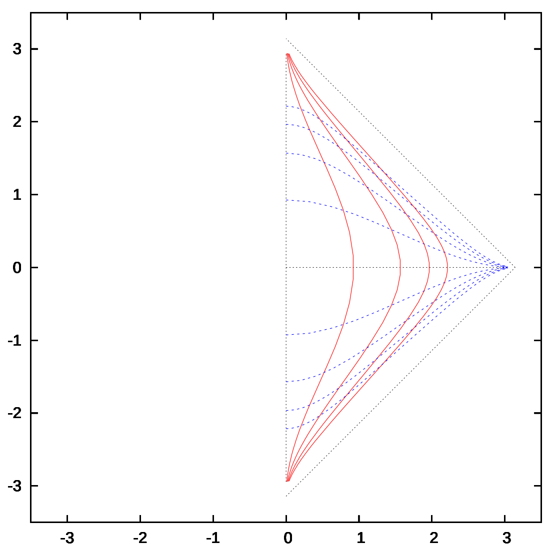
$$r>0 \quad\Rightarrow\quad v>u \quad\Rightarrow\quad v'>u' \quad\Rightarrow\quad r'>0$$

$$-\infty<u<v<\infty \quad\Rightarrow\quad -\pi/2<u'<v'<\pi/2 \quad\Rightarrow\quad -\pi<t'-r'<t'+r'<\pi$$

```
GNUplot] cd "/home/dhoshu/study/phys/notes/gravity_Hartle"
         ~load("p127_pr_diagram.plot")
```

赤： $t=0.5,\ 1.0,\ 1.5,\ 2.0$

青： $r=0.5,\ 1.0,\ 1.5,\ 2.0$



p.130 $\Rightarrow$ Exercise 7.9