■ 拡張Eddington-Finkelstein 座標変換

Schwarzschild計量に対するEddington-Finkelstein座標に相当する座標変換として

$$\begin{cases} dt &= dv - \dfrac{r^2+a^2}{\Delta}\,dr \\ d\phi &= d\psi - \dfrac{a}{\Delta}\,dr \end{cases} \tag{1}$$

を与えるような座標変換

$$\begin{aligned} (t,r,\theta,\phi) &\longrightarrow (v,r,\theta,\psi)=(v(t,r),r,\theta,\psi(\phi,r)) \\ (v,r,\theta,\psi) &\longrightarrow (t,r,\theta,\phi)=(t(v,r),r,\theta,\phi(\psi,r)) \end{aligned} \tag{2}$$

を考える。(1)は簡単に積分できて

$$\begin{aligned} \frac{r^2+a^2}{\Delta}=\frac{r^2+a^2}{r^2+a^2-2r} &= 1+\frac{2r-2}{r^2+a^2-2r}+\frac{2}{r^2+a^2-2r} \end{aligned}$$

$$\begin{aligned} \frac{2}{r^2+a^2-2r} &= \frac{2}{(r-1)^2-(1-a^2)} \\ &= \frac{1}{\sqrt{1-a^2}}\left(\frac{1}{r-(1+\sqrt{1-a^2})}-\frac{1}{r-(1-\sqrt{1-a^2})}\right), \quad a<1 \\ &= \frac{1}{\sqrt{1-a^2}}\left(\frac{1}{r-r_+}-\frac{1}{r-r_-}\right) \\ &= \frac{1}{\sqrt{1-a^2}}\frac{d}{dr}\log\left|\frac{r-r_+}{r-r_-}\right| \end{aligned}$$

$$\downarrow$$

$$= 1+\frac{d}{dr}\log|r^2+a^2-2r|+\frac{1}{\sqrt{1-a^2}}\frac{d}{dr}\log\left|\frac{r-r_+}{r-r_-}\right|$$

$$\begin{aligned} dt &= dv-\frac{r^2+a^2}{\Delta}\,dr \\ (\Rightarrow)\quad t &= v-r-\log|r^2+a^2-2r|-\frac{1}{\sqrt{1-a^2}}\log\left|\frac{r-r_+}{r-r_-}\right|+\text{const.} \end{aligned}$$

同じく

$$\frac{a}{\Delta}=\frac{a}{r^2+a^2-2r} = \frac{a}{2\sqrt{1-a^2}}\frac{d}{dr}\log\left|\frac{r-r_+}{r-r_-}\right|$$

なので

$$\begin{aligned} d\phi &= d\psi-\frac{a}{\Delta}\,dr \\ (\Rightarrow)\quad \phi &= \psi-\frac{a}{2\sqrt{1-a^2}}\log\left|\frac{r-r_+}{r-r_-}\right|+\text{const.} \end{aligned}$$