Mathematics and the Craft of Chess

Introduction

From the age of seven, I have been an avid chess player. Chess is considered a mathematically driven game, but there is one component of chess that is not often associated with mathematics: the elegant craftsmanship of the physical pieces themselves.

Figure 1: Chess pieces

In my early algebra classes, while I was calculating the volume of various conical and spherical objects, I often wondered how the volume of more complex symmetrical figures would be computed. I was provided the answer to this question in my IB Mathematics class, in the form of solids of revolution. While finding the volume of a symmetrical vase, I recognized the shape to be somewhat similar to a chess pawn. I realized that certain chess pieces, such as the bishop, pawn, and queen, could be formed by revolving a complex curve around the axis of a graph. This realization led me to my aim: **What is the volume of a standard pawn in the game of chess?**

Rationale

To determine the volume of a chess pawn, I will need to define it by a function. As such, I will need to plot it on the Cartesian plane. I will use the computer graphing software LoggerPro to translate the image of a pawn into the Cartesian plane, and by applying nonlinear regression, derive an equation for the pawn.

I will then use that curve to calculate the volume of a solid of revolution when the curve is rotated around the X axis, which will represent the total volume of the piece.

To verify the validity of my results, I will calculate the true volume of the pawn using a method known as *water displacement*[1], which is a method that will produce a higher level of accuracy.

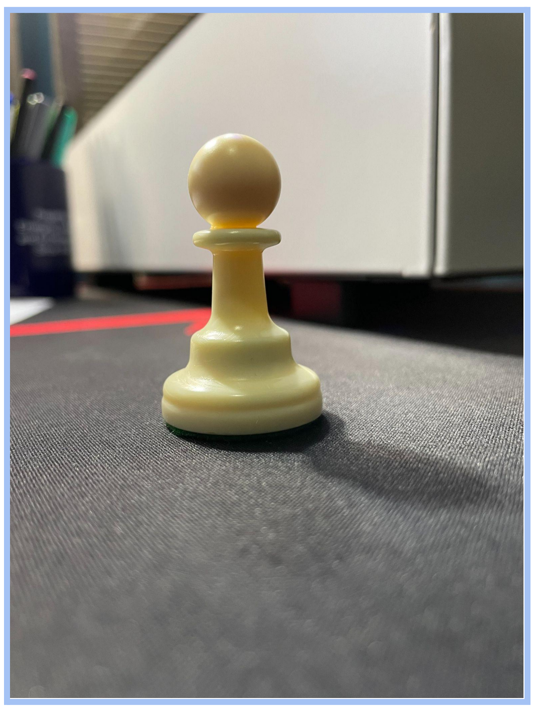

Figure 2: Photograph of a real pawn (taken by me)

[1] https://en.wikipedia.org/wiki/Displacement_(fluid)

Investigation

To begin plotting the image in the Cartesian plane, I took a picture of a real chess pawn against a solid background. I then used the image processing tool remove.bg to convert the image into a transparent PNG, which allowed me to directly insert it into a graphing software such as LoggerPro.

Figure 3: Transparent image obtained from remove.bg

With the transparent PNG file, I inserted a Photo with Image Analysis object into LoggerPro. After enlarging the image to match the scale of my graph, I added a Point Series corresponding to my image. I then manually traced the image by clicking my cursor along its lines, which translated the points to the Cartesian plane.

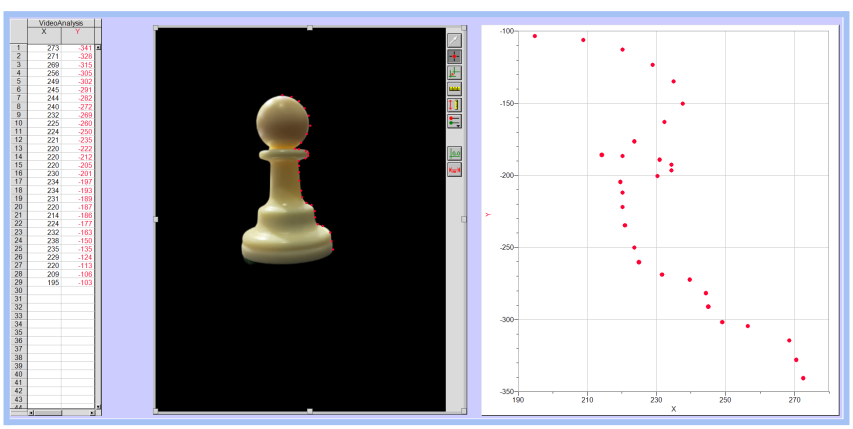

Figure 4: Initial pawn trace and corresponding visualization on Cartesian plane

Tracing the general shape gave me the X and Y values for my scaled image.

Table 1: Initial Coordinate Positions	
Initial X-pos (units)	Initial Y-pos (units)
273.3	-345.3
271.3	-328.3
269.3	-315.3
251.3	-302.3
243.3	-279.3
230.3	-264.3
222.3	-237.3
219.3	-213.3
228.3	-202.3
239.3	-190.3
216.3	-184.3
228.3	-174.3
239.3	-155.3
233.3	-133.3
225.3	-117.3
209.3	-105.3
193.3	-102.3

Next, I attempted to use LoggerPro's nonlinear regression approach to model a function for my image. However, because my Cartesian point values did not represent a function (because the X values had multiple Y values), I was not able to obtain a reasonably accurate equation model (the highest correlation being 0.4421). Therefore, I chose to swap the X and Y coordinates in order to obtain the inverse of each point on the plane, which would allow me to produce a more accurate equation model.

Table 2: Inverse Coordinate Positions	
Inverse X-pos (units)	Inverse Y-pos (units)
-345.3	273.3
-328.3	271.3
-315.3	269.3
-302.3	251.3
-279.3	243.3
-264.3	230.3
-237.3	222.3
-213.3	219.3
-202.3	228.3
-190.3	239.3
-184.3	216.3
-174.3	228.3
-155.3	239.3
-133.3	233.3
-117.3	225.3
-105.3	209.3
-102.3	193.3

I found this method to be somewhat inaccurate, because the three dimensional curves of the object were not properly encapsulated in the photo I took. I elected to modify my approach, instead holding the physical piece up to my screen and tracing the object as before. I found this method to have somewhat lower accuracy due to human error, but it was a better representation of the three dimensional piece I was analyzing.

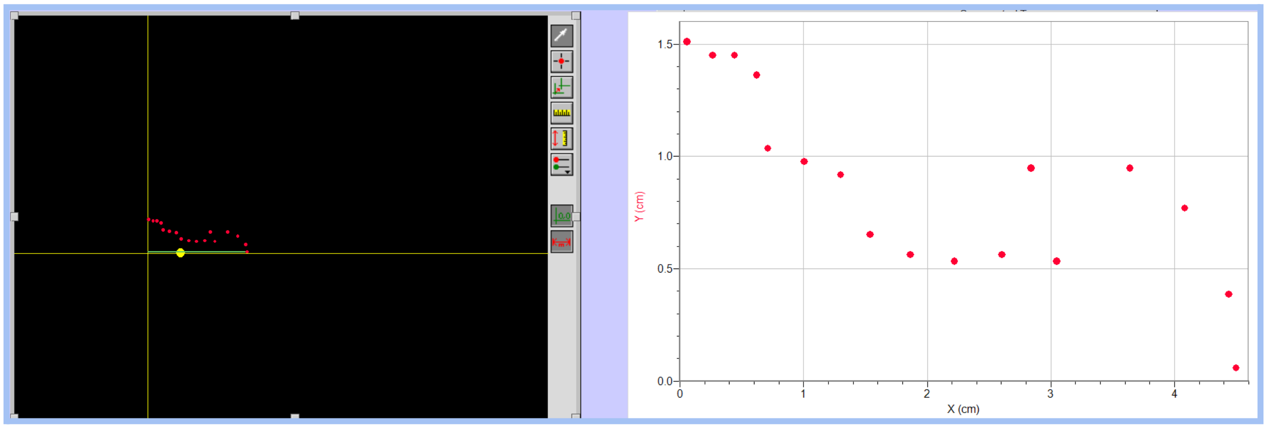

Figure 5: Pawn trace using real piece and corresponding visualization on Cartesian plane

At this point, I ran into an obstacle. I needed to compute the volume of the piece in cubic centimeters, and the image I was using had an arbitrary scale. Fortunately, I was able to use LoggerPro's scale tool to define a measurement for the picture. I used a ruler to measure the real piece[2] from one end to the other, and came up with 4.50 cm.

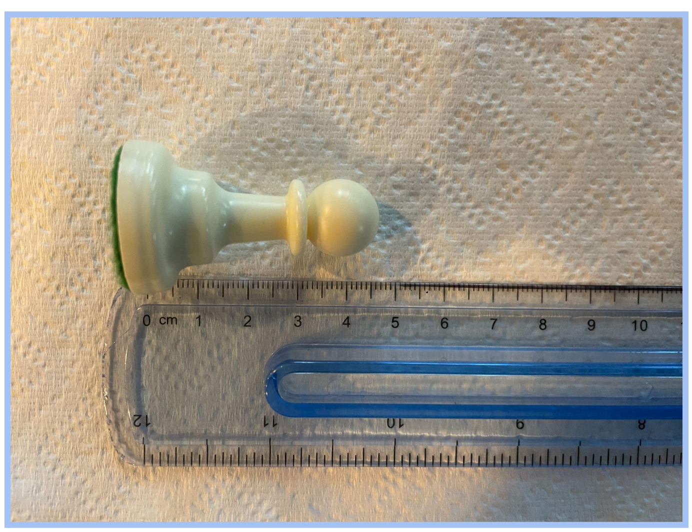

Figure 6: Measuring the pawn

[2] Numbers here are not recorded to 4 significant digits because of measurement uncertainty. (+/- 0.05 cm)

I traced this distance in LoggerPro and equated it to 4.50 cm, which resulted in a scaled[3] adjustment of all the X and Y values.

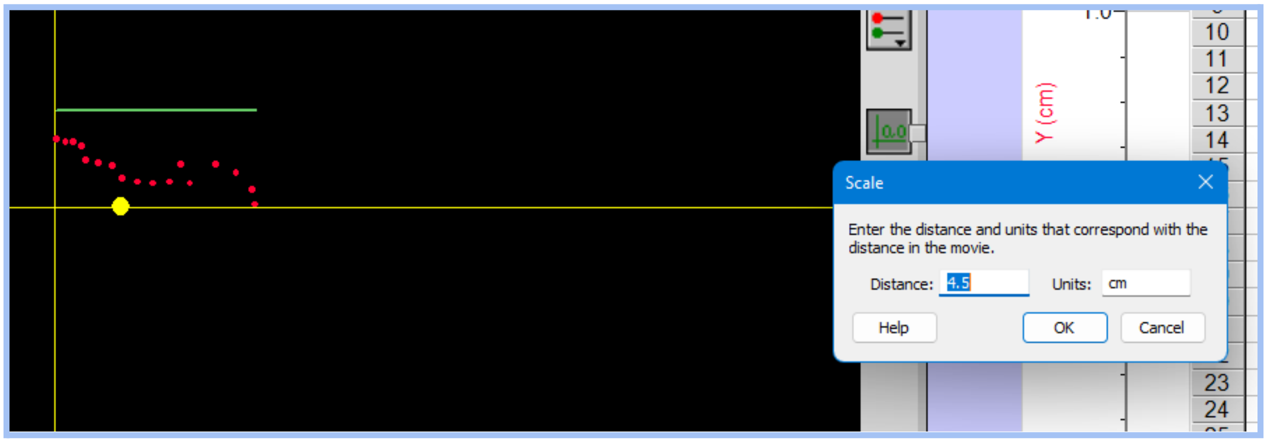

Figure 7: Scaling the graph

To further simplify the calculation of an equation, I set the origin of my graph to be directly underneath and to the left of the bottommost point of my image trace. This would ensure the entire equation would be constrained to the first quadrant of the Cartesian plane, and simplify the coefficients of a polynomial equation model. This would also allow me to easily rotate the equation around the X axis to produce a shape similar to that of a pawn.

[3] The green bar depicted in the image is equivalent to 4.50 cm. LoggerPro automatically scales the values.

Table 3: Final Adjusted Coordinate Positions	
Adjusted X-pos (cm)	Adjusted Y-pos (cm)
0.05921	1.509
0.2664	1.450
0.4440	1.450
0.6217	1.361
0.7105	1.036
1.006	0.9769
1.302	0.9177
1.539	0.6513
1.865	0.5625
2.220	0.5328
2.605	0.5625
2.842	0.9473
3.049	0.5328
3.641	0.9473
4.085	0.7697
4.440	0.3848
4.500	0.05921

Upon adjusting the X and Y values, I was able to calculate various equation models for my image. Due to the curvature of the image, a linear fit would not apply, so I opted to use nonlinear regression to model the image. As I tested various polynomial models, I noticed that as the degree of the polynomial increased, the higher the correlation with the actual coordinate values was.

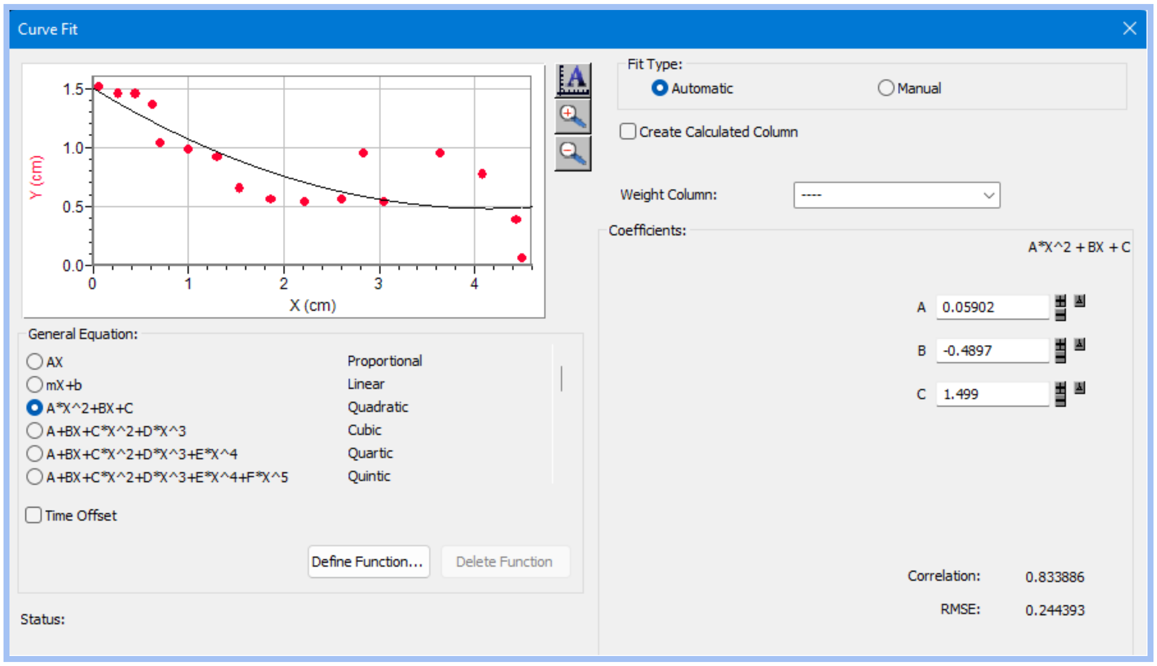

Figure 8: Quadratic polynomial model for secondary pawn trace

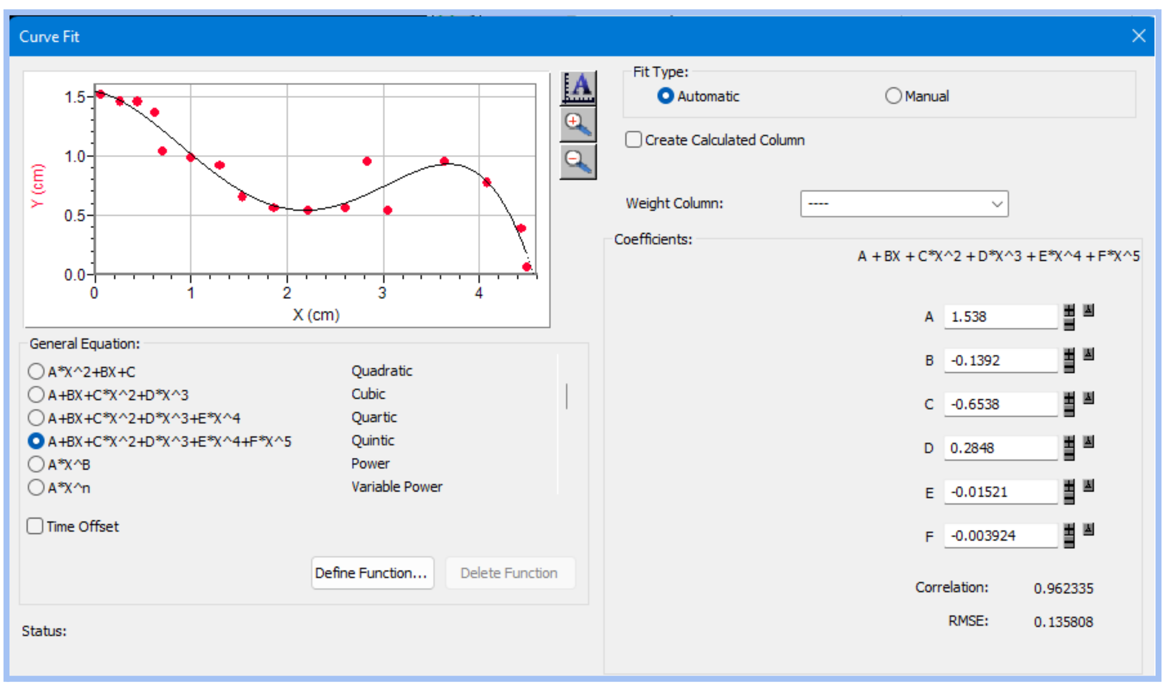

Figure 9: Quintic polynomial model for secondary pawn trace

The highest preset model in LoggerPro was a quintic polynomial, so I manually inputted a preset for a decic (tenth degree) polynomial to test my theory.

$$p(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + Hx^7 + Ix^8 + Jx^9 + Kx^{10}$$

Eq. 1: Decic polynomial model applied by nonlinear regression

My findings proved correct, as the correlation factor for the decic polynomial model was 0.9702 while it was merely 0.8339 for a quadratic, and 0.9623 for a quintic.

Even with a correlation of 0.9702 however, the equation model was not perfect; many of the coordinate points were outliers, and the contour did not match the shape of the pawn. Additionally, the resulting equation was very complicated, and would have been too difficult to integrate. Therefore, I decided to use the quintic polynomial model. Using LoggerPro's nonlinear regression approach, I derived the following equation:

$$f(x) = -0.003924x^5 - 0.01521x^4 + 0.2848x^3 - 0.6538x^2 - 0.1392x + 1.538$$

Eq 2: Quintic polynomial equation for secondary pawn trace calculated through nonlinear regression (LoggerPro)

This equation, when revolved around the X axis, produced the following solid:

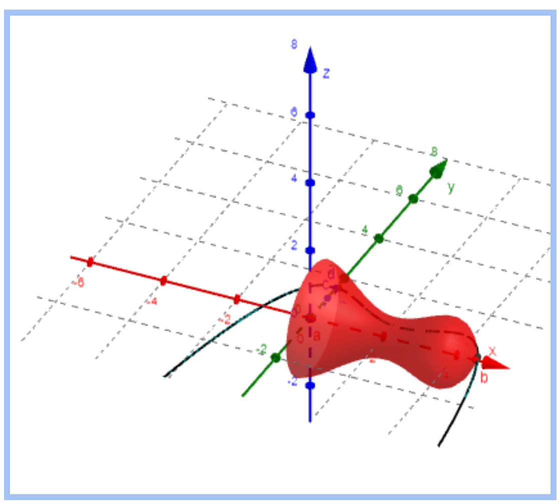

Figure 10: Solid of Revolution produced by revolving Eq 2 around the X axis[4]

[4] https://www.geogebra.org/m/zBRtUVfR

I applied the following formula to determine the volume of this solid:

$$V = \pi \int_{a}^{b} (f(x))^2 dx$$

Eq. 3: Volume of a solid of revolution about the X axis

Where:

π is a constant

a is the lower bound (lowest x-value in Table 3)

b is the upper bound (highest x-value in Table 3)

f(x) is the derived equation of the component (*Eq. 2)*

V is the volume of the solid

Integrating Eq. 2

Substituting *Eq. 2* for $f(x)^2$ in *Eq. 3* we get:

$$V = \pi \int_{0.05921}^{4.500} (-\, 0.003924x^5 - 0.01521x^4 + 0.2848x^3 - 0.6538x^2 - 0.1392x + 1.538)^2 dx$$

After expanding *f(x)*, we arrive at:

$$f(x)^2 = 0.00001549x^{10} + 0.0001194x^9 - 0.002004x^8 - 0.003533x^7 + 0.1021x^6$$
$$- 0.3802x^5 + 0.3014x^4 + 1.055x^3 - 1.992x^2 - 0.4282x + 2.365$$

Let us simplify this equation's coefficients[5] using *Eq. 1*.

Next, we apply the Sum Rule, shown by *Eq. 4*.

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Eq. 4: Sum Rule of Integration

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[5] See page 10.

$$V = \pi\left(K\int_{0.05921}^{4.500} x^{10}dx + J\int_{0.05921}^{4.500} x^{9}dx + I\int_{0.05921}^{4.500} x^{8}dx + H\int_{0.05921}^{4.500} x^{7}dx + G\int_{0.05921}^{4.500} x^{6}dx + \right.$$

$$\left. F\int_{0.05921}^{4.500} x^{5}dx + E\int_{0.05921}^{4.500} x^{4}dx + D\int_{0.05921}^{4.500} x^{3}dx + C\int_{0.05921}^{4.500} x^{2}dx + B\int_{0.05921}^{4.500} x\,dx + \int_{0.05921}^{4.500} A\,dx\right)$$

This produces:

$$V = \pi\left[\frac{Kx^{11}}{11} + \frac{Jx^{10}}{10} + \frac{Ix^{9}}{9} + \frac{Hx^{8}}{8} + \frac{Gx^{7}}{7} + \frac{Fx^{6}}{6} + \frac{Ex^{5}}{5} + \frac{Dx^{4}}{4} + \frac{Cx^{3}}{3} + \frac{Bx^{2}}{2} + Ax\right]_{0.05921}^{4.500}$$

Plugging back in our values and applying the fundamental theorem of calculus[6] gives us:

$$V = 3.485\pi$$

Solving this equation, we get our final answer using this method to be:

$V = 10.95\ cm^3$

As seen in Figure 10, the solid produced by this model is an imperfect representation of the real pawn. As such, I opted to split the piece into *components* and find the individual volume of each before adding them together to determine the total volume. The next step was to trace each segment and find an equation to model each one. Using the same PhotoAnalysis object as before, I added a new Point Series for each segment of the piece. To determine where a segment would end, I used the following methodology:

- First, I traced the segment.
- Second, I used a regression curve based on my prior knowledge of parent functions to derive an equation model for the segment.

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[6] https://mathworld.wolfram.com/FundamentalTheoremsofCalculus.html

- I analyzed the correlation. If it was higher than 0.9900, I started a new segment from the endpoint of that segment. If not, I removed a single point from the segment and returned to the first step.

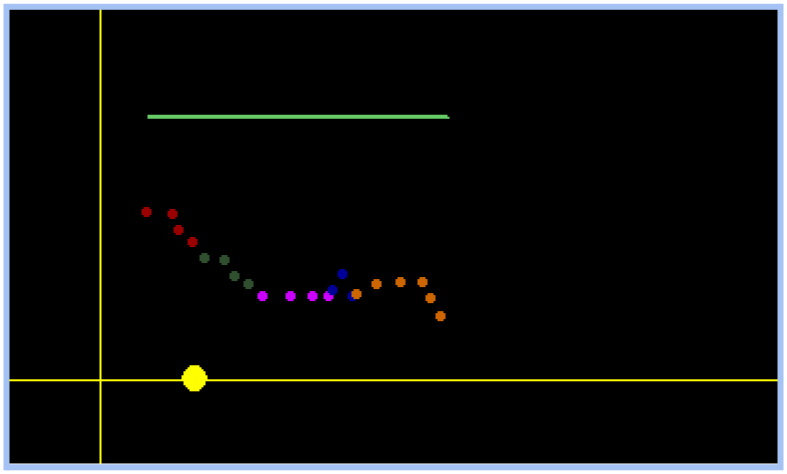

Figure 11: Segmented pawn trace

I ended up with five segments, each representing a portion of the contour of the pawn.

Table 4: X and Y coordinates for each section of the segmented pawn trace (all units in cm)											
X	Y	X1	Y1	X2	Y2	X3	Y3	X4	Y4	X5	Y5
0.05921	1.509	0.05921	1.510								
0.2664	1.450	0.2664	1.450								
0.4441	1.450	0.4441	1.450								
0.6217	1.361	0.6217	1.361								
0.7105	1.036			0.7105	1.036						
1.006	0.9770			1.006	0.9770						
1.302	0.9178			1.302	0.9178						
1.539	0.6513					1.539	0.6513				
1.865	0.5625					1.865	0.5625				
2.220	0.5329					2.220	0.5329				

Table 4 (ctd): X and Y coordinates for each section of the segmented pawn trace (all units in cm)											
2.605	0.5625					2.605	0.5625	2.605	0.5625		
2.842	0.9474							2.842	0.9474		
3.049	0.5329							3.049	0.5329	3.049	0.5329
3.641	0.9474									3.641	0.9474
4.085	0.7697									4.085	0.7697
4.440	0.3849									4.440	0.3849
4.500	0.0592									4.500	0.0592

When rotated about the X axis, each resulting shape would resemble a segment of the pawn, and when spliced together, would resemble the full pawn itself. As shown in Figure 12, the segmented pawn trace was a far more realistic representation of the real pawn.

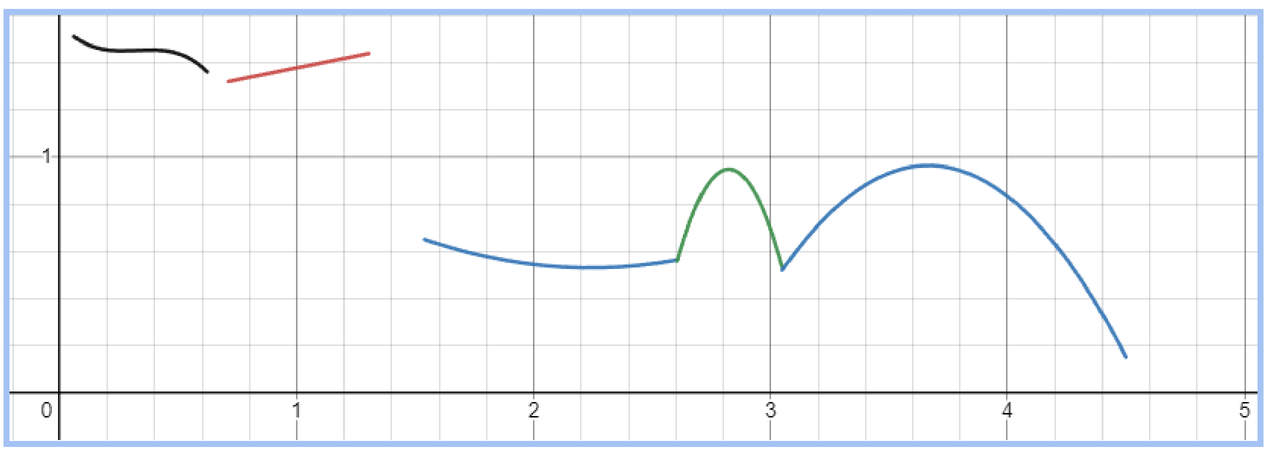

Figure 12: Segmented pawn trace visualized in Desmos[7]

To find the volume of the pawn, I needed to integrate each component separately, using the starting and ending X coordinates as the bounds[8]. The volume of each component was given by *Eq. 3.*[9]

[7] https://www.desmos.com/calculator/
[8] See Table 5, overleaf.
[9] See page 11.

$$g\,(x) = -\,3.825x\text{^}3 + 3.688x\text{^}2 - 1.142x + 1.565$$

Eq. 5: Derived equation for segment 1

$$h\,(x) = 0.2x + 1.178$$

Eq. 6: Derived equation for segment 2

$$i\,(x) = 0.2443x\text{^}2 - 1.094x + 1.756$$

Eq. 7: Derived equation for segment 3

$$j(x) = -\,8.163x\text{^}2 + 46.09x - 64.11$$

Eq. 8: Derived equation for segment 4

$$k\,(x) = -\,1.165x\text{^}2 + 8.541x - 14.69$$

Eq. 9: Derived equation for segment 5

Table 5: Component Equation Bounds

Component	Lower Bound	Upper Bound
1	0.05921	0.6217
2	0.7105	1.303
3	1.539	2.605
4	2.605	3.049
5	3.049	4.500

Integrating Component 1[10]

$$V_1 = \pi \int_{0.05921}^{0.6217} (-3.825x^3 + 3.688x^2 - 1.142x + 1.565)^2 dx$$

Using the same process demonstrated on page 12 and applying the Sum Rule, we arrive at:

$$V_1 = 1.178\pi$$

Solving this equation, we get the volume of Component 1 to be:

$$V_1 = 3.701\ cm^3$$

Integrating Component 2

$$V_2 = \pi \int_{0.7105}^{1.303} (0.2x + 1.178)^2 dx$$

Using the same process demonstrated on page 12 and applying the Sum Rule, we arrive at:

$$V_2 = 1.128\pi$$

Solving this equation, we get the volume of Component 2 to be:

$$V_2 = 3.544\ cm^3$$

Integrating Component 3

$$V_3 = \pi \int_{1.539}^{2.605} (0.2443x^2 - 1.049x + 1.756)^2 dx$$

Using the same process demonstrated on page 12 and applying the Sum Rule, we arrive at:

$$V_3 = 0.3369\pi$$

Solving this equation, we get the volume of Component 3 to be:

$$V_3 = 1.058\ cm^3$$

[10] In the interest of succinctness, and due to the similarity of these equations and the steps of their integration to the steps shown on page 12, the full process of integration was not included here.

Integrating Component 4

$$V_4 = \pi \int_{2.605}^{3.049} (-8.163x^2 + 46.09x - 64.11)^2 dx$$

Using the same process demonstrated on page 12 and applying the Sum Rule, we arrive at:

$$V_4 = 0.3008\pi$$

Solving this equation, we get the volume of Component 3 to be:

$$V_4 = 0.9449\ cm^3$$

Integrating Component 5

$$V_5 = \pi \int_{3.049}^{4.500} (-1.165x^2 + 8.541x - 14.69)^2 dx$$

Using the same process demonstrated on page 12 and applying the Sum Rule, we arrive at:

$$V_5 = 0.8724\pi$$

Solving this equation, we get the volume of Component 3 to be:

$$V_5 = 2.741\ cm^3$$

With the values of these five integrals, we substitute into *Eq. 10:*

$$V_{total} = V_1 + V_2 + V_3 + V_4 + V_5$$

Eq. 10: Total volume of the solid

$$V_{total} = 3.701 + 3.544 + 1.058 + 0.9449 + 2.741$$

$$V_{total} = \mathbf{11.99\ cm^3}$$

Water Displacement Method

Upon calculating this value, I elected to corroborate my findings by using water displacement to calculate the real-world volume of the pawn. The water displacement method allows for all the water molecules to perfectly occupy all the spaces around the object[11], and due to the properties of water, the exact amount of water molecules are then displaced. Because one milliliter of water is equal to one cubic centimeter,[12] this method will provide a high degree of accuracy.

I devised the following methodology[13]:

1. Fill a cup with water until it overflows.
2. Wait until the water in the cup stabilizes (is exactly full)
3. Place a container underneath the cup.
4. Holding the object by the tip, slowly lower it into the cup until fully submerged.
5. Once all the water that exits the cup has fallen into the container, measure[14] the volume of water in the container in milliliters (mL).

Using this methodology, I ran five trials. Table 6 depicts the results.

Figures 13 and 14: Implementation of methodology and materials used in procedure

[11] https://homework.study.com
[12] https://en.wikipedia.org/wiki/Displacement_(fluid)
[13] See Figure 13.
[14] See Figure 14.

Table 6: Water Displacement Method	
Trial Number	Water Displacement
1	11.50
2	11.25
3	11.75
4	12.00
5	12.25
Average	**11.75**

Calculating Percent Error[15]

Eq. 11 shows the formula for percent error.

$$\delta = \frac{|\upsilon_a - \upsilon_e|}{\upsilon_e} \times 100\%$$

Eq. 11: Percent Error

Where:

δ = percent error

υ_a = actual observed value

υ_e = expected value

Our observed values are the values of V obtained using the first two methods, while the expected value is the value of V obtained using water displacement. Plugging these values into *Eq. 11,* we get:

$$\delta = \frac{|10.95 - 11.75|}{11.75} \times 100\%$$

$$\delta = 6.809\%$$

[15] https://chem.libretexts.org/

Percent Error (δ) for method 1

$$\delta = \frac{|11.99 - 11.75|}{11.75} \times 100\%$$

$$\delta = 2.043\%$$

Percent Error (δ) for method 2

Five percent is generally accepted to be a reasonable percent error[16], and as such, I can determine that method 2 was a more accurate method to calculate the volume of the pawn.

Evaluation and Conclusion

Before analyzing my results, certain considerations must be evaluated. My investigation made various assumptions that may affect the validity of the results. First, I assumed that the layer of material which makes up the piece is extremely thin. The actual piece has added weights and thicker materials. While the volume will remain the same, some of that volume by definition is occupied in a real piece. I also assumed the pawn was a standard chess piece. Manufacturing variances around the world would likely produce pieces with slightly differing volumes. Furthermore, I assumed that the piece was perfectly circular and symmetrical. Imperfections during the manufacturing process would likely render this untrue in a real-world piece, although only by a miniscule margin. Additionally, the chess pieces used in world championship games are handmade, meaning there is some inconsistency due to human imperfection.

Secondly, my methodology had some limitations. My initial process for tracing the contour of the pawn was flawed, because it did not represent the three dimensional curvature. My second

[16] Ibid.

process was a better representation, but since I did not have an electronic visual to follow, there was likely some inaccuracy due to human error. Furthermore, using nonlinear regression to calculate an equation for the entire pawn as a whole produced an inaccurate representation, and even the segmented approach was not a perfect depiction. As such, there is a reasonable level of uncertainty in my calculations. Throughout my investigation, I reported measurements to four significant digits. This sacrificed some accuracy for the purposes of formatting, but not enough to alter the final result by a significant amount.

Ultimately, I was able to achieve the standards outlined in my aim. Using integral calculus, I was able to compute the volume of the pawn by simply tracing it. I wanted to measure my validity, so I used the water displacement method to determine the real volume of the pawn. Based on the results of both methods,[17] I conclude that my calculated answer is an accurate measure of the true volume of a standard chess pawn.

[17] My initial result was 10.95, my revised result was 11.99, and the value I determined using water displacement (real-world value) was 11.75. My percent error for the initial result was approximately 7%, and my percent error for the revised result was approximately 2%.

References

Desmos Studio PBC. (n.d.). *Let's learn together.* Desmos. Retrieved November 11, 2022, from https://www.desmos.com/

Fundamental theorems of calculus. from Wolfram MathWorld. (n.d.). Retrieved November 12, 2022, from https://mathworld.wolfram.com/FundamentalTheoremsofCalculus.html

Libretexts. (2022, August 3). 3.13: Percent error. Chemistry LibreTexts. Retrieved November 12, 2022, from https://chem.libretexts.org/Bookshelves/Introductory_Chemistry/Introductory_Chemistry_(CK-12)/03%3A_Measurements/3.13%3A_Percent_Error

Lindner, A. (n.d.). *Solid of Revolution*. GeoGebra. Retrieved November 11, 2022, from https://www.geogebra.org/m/zBRtUVfR

Symbolab . (n.d.). Retrieved November 11, 2022, from https://www.symbolab.com/

Vernier. (n.d.). *Logger pro® 3*. LoggerPro. Retrieved November 11, 2022, from https://www.vernier.com/product/logger-pro-3/

Water Displacement. Study.com. (n.d.). Retrieved November 11, 2022, from https://homework.study.com/

Wikimedia Foundation. (2022, October 14). *Displacement (fluid)*. Wikipedia. Retrieved November 11, 2022, from https://en.wikipedia.org/wiki/Displacement_(fluid)