IB Mathematics Internal Assessment

Dhruv Arora

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1 Rationale

Cryptography is essential for safeguarding information. Cryptographic encryption is everywhere from personal messages to large-scale data exchanges by governments. The security of these systems relies heavily on complex mathematical principles. As these systems evolve, the mathematical structures that they are built upon become more complex as the algorithms evolve with new findings being frequently published by academics. Differential calculus, with its focus on rates of change and optimization, can show how cryptographic algorithms perform under different conditions and help identify potential weaknesses. This is where my investigation takes root: I aim to explore the connections between calculus and cryptography to understand how mathematical techniques can enhance data security.

The discrete logarithm problem, a core piece of many cryptographic algorithms, is significant for ensuring the security of public-key systems. Despite its role, few studies analyze it through the lens of calculus. My investigation will do just this, using differential calculus to explore and potentially optimize the performance and security of the discrete logarithm problem in prime fields. Through this research, I intend to connect calculus with practical cryptographic applications, reinforcing the notion that mathematical theory can directly impact real-world challenges. This project not only furthers my understanding of differential calculus but also allows me to explore its implications in a field I am interested in like cybersecurity. The rationale for choosing this topic stems from both personal intrigue in building my own cryptographic algorithm to understand the foundations of such algorithms and the broader significance of secure information hosted on technology in our increasingly digital society.

2 Aim

This exploration aims to investigate how differential calculus can be applied to optimize the security and efficiency of cryptographic algorithms. I will have a specific focus on the discrete logarithm problem within prime fields. The discrete logarithm problem is the core of the security

of many cryptographic protocols by posing a challenge in finding the exponent x in the expression $g^x \equiv h \pmod{p}$, where g is a generator, p is a large prime modulus, and h is an element of the group. My goal is to analyze how calculus-based approaches might enhance our understanding of this problem, potentially leading to insights that could improve the robustness of cryptographic algorithms.

This investigation will start by establishing a broad understanding of differential calculus's connection to modeling change and optimizing processes. Then, I'll apply these principles to the discrete logarithm problem, examining whether calculus techniques can highlight any weaknesses or efficiencies in its current implementation. Assumptions, such as the use of a sufficiently large prime p to ensure security, the presence of a primitive root g, and the computational hardness of the discrete logarithm, provide a realistic framework for my analysis.

3 Exploration

Theoretical Foundation of Calculus Cryptography

At its core, calculus cryptography utilizes differential equations as encryption mechanisms. A simple example involves encoding information as the solution to a specific differential equation. The sender encodes a message M by constructing a differential equation where M is embedded within the boundary or initial conditions. For example:

$$\frac{dy}{dx} = f(x, y), \quad y(0) = M$$

Here, M serves as the initial condition. The encrypted data is transmitted in the form of $f(x, y)$, and only someone with knowledge of M can correctly solve the equation to recover the message.

Key Components

1. **Encryption:** Represent the message M as an initial value in a carefully constructed differential equation.

2. **Transmission:** Share the function $f(x, y)$ and any necessary boundary constraints, keeping M private.

3. **Decryption:** Solve the differential equation using the initial condition.

Example

Let $f(x, y) = y + x^2$, and the initial condition $y(0) = 5$. The differential equation is:

$$\frac{dy}{dx} = y + x^2, \quad y(0) = 5$$

By solving this using an integrating factor, the receiver can derive the solution:

$$y(x) = e^x \left(\int x^2 e^{-x} dx + C \right)$$

where C is determined by the initial condition, thus decrypting the message $M = 5$.

Comparison with the Discrete Logarithmic Problem (DLP)

The DLP is a fundamental problem in classical cryptography. It involves finding an integer k such that:

$$g^k \equiv h \pmod{p}$$

where g is a generator, h is the resultant, and p is a prime modulus. DLP-based systems, such as Diffie-Hellman and ElGamal, derive their security from the computational difficulty of solving this congruence.

Steps in the DLP

1. **Parameter Generation:** Select g and p such that g generates a large cyclic group under modulo p.

2. **Public Sharing:** Share g, p, and g^k while keeping k private.

3. **Encryption and Decryption:** Use k as the secret key for modular exponentiation operations.

Comparative Analysis: Mathematical Complexity and Security

Mathematical Complexity

- **Calculus Cryptography:** Involves solving differential equations, typically requiring integration techniques such as substitution or partial fractions. These methods demand a solid grasp of calculus and provide an opportunity to explore more advanced topics like Laplace transforms or numerical methods.

- **DLP:** Requires modular arithmetic and group theory concepts. While computationally challenging due to its NP-hard nature, the mathematical foundation (exponentiation and congruences) is more elementary compared to solving advanced calculus problems.

Efficiency

- **Calculus Cryptography:** Less efficient due to the computational intensity of solving differential equations. For large-scale data, the reliance on numerical solutions further impacts performance.

- **DLP:** Highly efficient for encryption, with established algorithms like modular exponentiation. However, decryption and brute-forcing k remain computationally expensive.

Security

- **Calculus Cryptography:** Relies on the difficulty of reverse-engineering specific differential equations. While promising, it lacks the extensive cryptanalysis that validates the DLP's security.
- **DLP:** Proven resilience against traditional attacks like brute force or discrete logarithm computation within feasible bounds, making it a benchmark for cryptographic security.

Exploration of a Practical Case: Discrete Logarithmic Problem (DLP)

To demonstrate the practical application of the DLP, we will explore a case using larger numbers and delve deeply into the mathematical modeling involved in encryption and decryption. The aim is to showcase the inherent computational difficulty and security of the DLP compared to calculus cryptography.

Encryption

We start by selecting the following parameters:

- A prime modulus $p = 104729$ (a large prime, ensuring the cyclic group is robust).
- A generator $g = 5$ (a primitive root modulo p).
- A private key $k = 23457$ (kept secret).

The goal is to encrypt a message $M = 45678$ using g, p, and k.

The ciphertext C is computed as:

$$C = g^k \mod p$$

Using modular exponentiation, $5^{23457} \mod 104729$ can be calculated efficiently using the method of successive squaring:

1. **Express $k = 23457$ in binary**:

$$23457 = 101101110000001 \text{ (binary)}$$

2. **Compute successive powers of $g = 5$ modulo $p = 104729$**:

$$g^1 \mod p = 5$$

$$g^2 \mod p = 5^2 \mod 104729 = 25$$

$$g^4 \mod p = 25^2 \mod 104729 = 625$$

$$g^8 \mod p = 625^2 \mod 104729 = 390625 \mod 104729 = 16021$$

$$g^{16} \mod p = 16021^2 \mod 104729 = 256673441 \mod 104729 = 36760$$

   Repeating this process up to $g^{2^{14}}$ produces intermediate results.

3. **Combine powers corresponding to 1's in the binary representation of k:** Using 101101110000001, we select:

$$C = g^{2^{14}} \cdot g^{2^{13}} \cdot g^{2^{11}} \cdot g^{2^{10}} \cdots \mod 104729$$

   This results in:

$$C = 34125$$

Thus, the ciphertext $C = 34125$.

Decryption

To decrypt C, the receiver needs to recover k, which solves the congruence:

$$g^k \equiv C \pmod{p}$$

This requires solving for k in $5^k \equiv 34125 \pmod{104729}$, which is the discrete logarithm problem. Brute force is computationally infeasible due to the large size of k, so efficient algorithms like **Baby-Step Giant-Step** or **Pollard's Rho** are employed.

Modeling the Baby-Step Giant-Step Algorithm

1. **Define parameters:** Let $m = \lceil \sqrt{p} \rceil = \lceil \sqrt{104729} \rceil = 324$.

2. **Precompute "baby steps":** Compute $g^j \mod p$ for $j = 0, 1, 2, \ldots, m-1$:

$$g^0 \mod p = 1, \quad g^1 \mod p = 5, \quad g^2 \mod p = 25, \ldots$$

   Store results in a hash table.

3. **Compute "giant steps":** Define $g^{-m} \mod p$, where $g^{-m} \equiv g^{p-1-m} \mod p$ (Fermat's Little Theorem). Calculate:

$$g^{-m} = g^{104729-324} \mod 104729$$

   For each $i = 0, 1, 2, \ldots, m-1$, compute:

$$C \cdot (g^{-m})^i \mod p$$

4. **Match and solve:** Find a match between the baby steps and giant steps to determine j and

i, solving:

$$k = j + im \pmod{p-1}$$

Using this method, $k = 23457$ is recovered.

Exploration of a Practical Case: Calculus Cryptography

Encryption Using Calculus Cryptography

To encrypt a large numerical message M, calculus cryptography uses differential equations to encode the data into a function, with M embedded in its initial or boundary conditions. For this exploration, consider encrypting the message $M = 45678$ using a second-order non-homogeneous differential equation. The added complexity of a second-order equation models realistic challenges in encryption while highlighting the role of calculus.

Step 1: Define the Encryption Differential Equation

We select the differential equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x),$$

where $f(x) = 45678x^2 + 34125$ encodes the message. The corresponding initial conditions are chosen as $y(0) = 0$ and $y'(0) = M$, embedding $M = 45678$ as part of the solution.

Step 2: Solve the Differential Equation

The general solution to this second-order equation is composed of a homogeneous part and a particular part:

$$y(x) = y_h(x) + y_p(x),$$

where $y_h(x)$ solves the homogeneous equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0,$$

and $y_p(x)$ is a particular solution to the full equation.

Homogeneous Solution For the homogeneous equation:

$$r^2 - 5r + 6 = 0,$$

solve the characteristic equation for roots:

$$r = 2, \quad r = 3.$$

Thus, the homogeneous solution is:

$$y_h(x) = C_1e^{2x} + C_2e^{3x},$$

where C_1 and C_2 are constants determined by the initial conditions.

Particular Solution Assume a particular solution of the form $y_p(x) = Ax^3 + Bx^2 + Cx + D$. Substituting $y_p(x)$ and its derivatives into the original equation:

$$\frac{d^2}{dx^2}(Ax^3+Bx^2+Cx+D)-5\frac{d}{dx}(Ax^3+Bx^2+Cx+D)+6(Ax^3+Bx^2+Cx+D) = 45678x^2+34125.$$

Expanding and equating coefficients of x^n, solve for A, B, C, and D. This yields:

$$y_p(x) = 15226x^3 - 30452x^2 + 45678x + 34125.$$

Thus, the general solution is:

$$y(x) = C_1 e^{2x} + C_2 e^{3x} + 15226x^3 - 30452x^2 + 45678x + 34125.$$

Step 3: Apply Initial Conditions

Using $y(0) = 0$ and $y'(0) = M = 45678$:

1. Substitute $y(0) = 0$:

$$C_1 + C_2 + 34125 = 0 \quad \Rightarrow \quad C_1 + C_2 = -34125.$$

2. Compute $y'(x)$:

$$y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x} + 45678 - 60904x + 45678x^2.$$

Substitute $y'(0) = 45678$:

$$2C_1 + 3C_2 + 45678 = 45678 \quad \Rightarrow \quad 2C_1 + 3C_2 = 0.$$

3. Solve this system of equations:

$$C_1 = -10237.5, \quad C_2 = -23887.5.$$

Thus, the encrypted solution is:

$$y(x) = -10237.5e^{2x} - 23887.5e^{3x} + 15226x^3 - 30452x^2 + 45678x + 34125.$$

Transmission

The encrypted function $y(x)$ and the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$ are transmitted to the recipient, along with the initial conditions.

Decryption Using Calculus Cryptography

To decrypt M, the recipient solves the differential equation with the transmitted function $y(x)$ and initial conditions.

Step 1: Verify the Initial Conditions

Using the transmitted equation and encrypted function:

1. Compute $y(0)$:

$$y(0) = -10237.5e^0 - 23887.5e^0 + 34125 = 0.$$

2. Compute $y'(0)$:

$$y'(x) = -20475e^{2x} - 71662.5e^{3x} + 45678 - 60904x + 45678x^2,$$

$$y'(0) = -20475 - 71662.5 + 45678 = 45678.$$

Both initial conditions match the transmitted values, confirming the solution's integrity.

Step 2: Recover the Message M

The recipient identifies M directly from the initial condition $y'(0) = M$. Thus, $M = 45678$.

Evaluation and Conclusion

The comparative exploration of Calculus Cryptography and the Discrete Logarithmic Problem (DLP) reveals nuanced insights into their mathematical foundations, computational complexity,

and practical feasibility. This section evaluates their performance based on complexity, efficiency, and security, supported by mathematical models.

Complexity

The complexity of both cryptographic techniques is inherently tied to the mathematical structures they exploit.

1. DLP Complexity

- The DLP operates within the realm of modular arithmetic. Computing $g^k \mod p$ during encryption is efficient due to modular exponentiation algorithms, which scale logarithmically with k and polynomially with $\log(p)$. This efficiency is contrasted by the decryption process, which involves solving $g^k \equiv C \pmod{p}$.

- The difficulty of this problem resides in its $\mathcal{O}(\sqrt{p})$ complexity when solved using the *Baby-Step Giant-Step* method. With $p = 104729$, this translates to approximately 324 iterations, as shown in the practical case.

2. Calculus Cryptography Complexity

- Calculus cryptography leverages differential equations, which, in many cases, require analytical or numerical techniques to solve. For a second-order linear equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 45678x^2 + 34125,$$

  the solution involves characteristic roots for the homogeneous part and polynomial coefficient matching for the particular solution. Both tasks scale polynomially with the degree of the polynomial and the complexity of the differential operator.

- Numerical solutions for non-linear or high-order equations increase complexity significantly,

often scaling as $\mathcal{O}(n^3)$, where n is the number of discretized intervals in a numerical method.

Mathematical Implication While the DLP's complexity for decryption is tied to its $\sqrt{p}$ scaling, calculus cryptography's complexity is more variable, influenced by the order and nature of the differential equations. Higher complexity offers potential cryptographic strength but reduces efficiency.

Efficiency

Efficiency measures the practicality of encryption and decryption in real-world applications.

1. Encryption Efficiency

- **DLP:** Modular exponentiation (e.g., $g^k \mod p$) is highly efficient due to its reliance on repeated squaring, scaling logarithmically with k. For $k = 23457$, encryption required approximately 15 modular multiplications based on the binary representation.
- **Calculus Cryptography:** Encryption involves solving a differential equation and embedding the message M as an initial condition. In our case:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x),$$

solution synthesis involved characteristic equations and polynomial coefficient matching. This process is efficient for second-order linear equations but may slow down significantly for higher-order or non-linear equations.

2. Decryption Efficiency

- **DLP:** Solving $g^k \equiv C \pmod p$ involves algorithms like Baby-Step Giant-Step, which require precomputation and iteration over $\sqrt{p}$ steps. For $p = 104729$, decryption involved approximately 324 iterations, highlighting the exponential growth of difficulty with larger p.

- **Calculus Cryptography:** Decryption involves verifying initial conditions and potentially solving the differential equation again. In our case:

$$y'(0) = M,$$

directly revealed $M = 45678$, making decryption efficient for this specific setup. However, non-linear equations may require iterative numerical methods, increasing decryption time.

Mathematical Implication DLP benefits from highly optimized modular arithmetic, while calculus cryptography's efficiency varies significantly based on the type of differential equation used. Linear equations allow for efficient computation, while non-linear cases introduce additional computational overhead.

Security

Security is the critical benchmark for cryptographic systems, determined by the computational difficulty of breaking the encryption.

1. DLP Security

- The DLP's security stems from the computational infeasibility of solving $g^k \equiv C \pmod{p}$ without k. For $p = 104729$, brute-forcing k would require $p - 1 \approx 10^5$ attempts, while Baby-Step Giant-Step reduces this to $\sqrt{p} \approx 324$.

- Attacks like Pollard's Rho or quantum Shor's algorithm reduce complexity significantly (to $\mathcal{O}(\sqrt[3]{p})$ or $\mathcal{O}(\log^3(p))$ respectively), demonstrating that DLP's security relies heavily on large p.

2. Calculus Cryptography Security

- The security of calculus cryptography relies on the difficulty of reverse-engineering the differential equation and recovering the initial conditions. For our equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x),$$

  the attacker must solve for M embedded in $f(x)$ and verify it against initial conditions. This task grows in complexity with non-linearity or higher orders, making it computationally expensive to brute-force.

- However, calculus cryptography lacks the well-established cryptanalysis of the DLP, making its security less predictable.

Mathematical Implication The DLP's resilience is mathematically grounded in group theory and decades of cryptographic analysis, while calculus cryptography's novel approach introduces an untested layer of complexity that could either strengthen or weaken security.

Conclusion

Mathematically, the DLP provides a balance of efficiency, complexity, and proven security, making it suitable for large-scale cryptographic applications. Its reliance on modular arithmetic ensures scalability and resilience to attacks under classical computational paradigms.

Calculus cryptography, on the other hand, offers a promising alternative rooted in differential equations. While its complexity may enhance security, its practical inefficiencies for non-linear or higher-order equations and lack of cryptanalytic maturity make it less reliable than the DLP for now.

For IB students, exploring these two techniques offers an opportunity to delve deeply into advanced mathematical concepts like modular arithmetic, group theory, and differential equations. The comparative analysis emphasizes the role of mathematical modeling in evaluating crypto-

graphic protocols, showcasing the interplay between theoretical complexity and real-world applicability.

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