# A Comparative Analysis of Calculus-Based Cryptography and the Discrete Logarithmic Problem

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# Introduction

From an early age, I was fascinated by the mechanisms that secured communication. From disassembling the lock on my journal to uncover how it worked to learning about the encryption on the WhatsApp platform, my curiosity on the topic only deepened. While these tools can feel magical, cryptography, or the art of encoding messages, involves number theory, algebra, and more recently, advanced branches of mathematics including calculus [7]. Initially, my understanding of cryptography was confined to its discrete mathematical underpinnings with prime and modular arithmetic that formed the basis of the discrete logarithm problem. However, as I began to learn calculus, I started to question whether continuous mathematics could compare against existing mathematical implementations of encoding.

Calculus, being the mathematics of change, creates a framework for studying how cryptographic algorithms evolve when key parameters are treated as dynamic variables [5]. This realization led me to consider the intersection of continuous and discrete frameworks: Could differential calculus provide a more secure means of encoding than the discrete logarithm problem? The literature on cryptography primarily focuses on discrete mathematics, but as innovation continues to push the power of computers forward, interdisciplinary approaches may uncover subtle patterns or optimization opportunities.

# Aim and Rationale

This investigation aims to explore how differential calculus can optimize the security of cryptographic algorithms by comparing it to the discrete logarithm problem. The discrete logarithm problem is at the core of security protocols, presenting the challenge of solving:

$$g^x \equiv h \pmod{p},$$

where $g$ is a generator, $p$ is a large prime modulus, and $h$ is an element of the group. This study evaluates whether a calculus-based approach can offer comparable cryptographic strength by modeling and analyzing both methods in a practical scenario.

This study will begin by establishing the theoretical foundation of differential calculus in modeling change and optimizing processes relevant to cryptography. Both methods will be compared by applying the discrete logarithm problem and a calculus-based approach to a specific cryptographic scenario. Assumptions such as the computational hardness of the discrete logarithm, the use of a sufficiently large prime $p$, and the presence of a primitive root $g$ will ensure realistic analysis. By highlighting the strengths and limitations of each approach, this exploration seeks to bridge theoretical mathematics with real-world cryptographic applications.

# Exploration

### Mathematical Complexity and Security

When comparing calculus-based cryptography to the discrete logarithm problem (DLP), the differences in mathematical foundations, computational efficiency, and security are evident. Calculus-based cryptographic methods rely on solving differential equations using techniques such as substitution, partial fractions, and numerical solutions, which require an advanced understanding of calculus concepts [13]. On the other hand, DLP relies on modular arithmetic and group theory, with roots in simpler operations like exponentiation and congruences, making its mathematical basis more accessible [14].

While calculus-based methods present an opportunity to explore sophisticated analytical tools, they are computationally intensive. This limits their efficiency in large-scale data processing compared to the well-established modular exponentiation algorithms used in DLP-based cryptographic systems, which are optimized for performance [6]. From a security perspective, the DLP has been extensively analyzed and is considered a benchmark for cryptographic strength, as its security relies on the computational difficulty of solving discrete logarithms [14]. Conversely,

calculus-based approaches, which depend on the difficulty of reverse-engineering differential equations, are less validated and lack the same level of cryptanalytic scrutiny.

**Foundation of Calculus Cryptography**

The method of Calculus cryptography that I will be using in my comparison utilizes differential equations as encryption mechanisms. A simple example involves encoding information as the solution to a specific differential equation. The sender encodes a message $M$ by creating a differential equation where $M$ is embedded within the boundary or initial conditions. For example:

$$\frac{dy}{dx} = f(x, y), \quad y(0) = M$$

In this instance, $M$ is the the initial condition. The encrypted data is transmitted in the form of $f(x, y)$, and only someone who knows what $M$ is can actually correctly solve the equation and recover the message. To actually use calculus cryptography, there are three key components:

1. **Encryption:** Represent the message $M$ as an initial value in a differential equation.
2. **Transmission:** Share the function $f(x, y)$ and any necessary boundary constraints, keeping $M$ private.
3. **Decryption:** Solve the differential equation using the initial condition.

To showcase the application of calculus cryptography, I will encrypt a message $M = 5$ using a first-order differential equation and demonstrate the decryption process. In this example, Let $f(x, y) = y + x^2$, and the initial condition $y(0) = 5$. The differential equation is:

$$\frac{dy}{dx} = y + x^2, \quad y(0) = 5$$

By solving this using an integrating factor, the receiver can derive the solution:

$$y(x) = e^x \left( \int x^2 e^{-x} dx + C \right)$$

where $C$ is determined by the initial condition, thus decrypting the message $M = 5$.

**Foundation of Discrete Logarithmic Problem (DLP)**

For my comparison, I have included the Discrete Logarithmic Problem due to the heavy reliance of this problem in the current cryptographic space. As explained in the introduction, the DLP is a fundamental problem in encoding message. It involves finding an integer $x$ such that:

$$g^x \equiv h \pmod{p}$$

where $g$ is a generator, $h$ is the resultant, and $p$ is a prime modulus. DLP-based systems, such as Diffie-Hellman and ElGamal, derive their security from the computational difficulty of solving this congruence [4]. The DLP process that I will be using can be broken down into three key steps:

1. **Parameter Generation:** Select $g$ and $p$ such that $g$ generates a large cyclic group under modulo $p$.
2. **Public Sharing:** Share $g$, $p$, and $g^x$ while keeping $x$ private.
3. **Encryption and Decryption:** Use $x$ as the secret key for modular exponentiation operations.

To showcase the application of the discrete logarithmic problem (DLP), I will encrypt a message $M = 9$ using modular exponentiation and demonstrate the decryption process. In this example, let the generator $g = 3$, the prime modulus $p = 17$, and the private key $x = 4$. The encryption process computes:

$$C = g^x \pmod{p}$$

Substituting the values, I can calculate:

$$C = 3^4 \pmod{17} = 81 \pmod{17} = 13$$

As a result, the encrypted message, or ciphertext, is $C = 13$.

To decrypt, the receiver needs to recover $x$ such that:

$$3^x \equiv 13 \pmod{17}$$

This requires solving the DLP. Testing successive powers of 3 modulo 17, I could find that:

$$3^4 \equiv 13 \pmod{17}$$

With this, the final result shows $x = 4$. Using $x$ and the shared parameters $g$ and $p$, the receiver decrypts the original message $M = 9$. If a receiver does not have the private key $x$, the decryption process becomes computationally infeasible, ensuring the security of the message. However, having access to $x$ makes the decryption process computationally straightforward, successfully retrieving $M$.

## Practical Case of Calculus Cryptography

### Encryption

For this exploration, I will be encrypting the message $M = 83495$ using a second-order non-homogeneous differential equation. The added complexity of a second-order equation models realistic challenges in encryption while showing the role of calculus.

To begin with the 1st key component mentioned in the *Foundation of Calculus Cryptography* section, I will encrypt the message. I will be selecting the differential equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x),$$

where $f(x) = 45678x^2 + 34125$, or the forcing function, encodes the message. Only the recipient and sender have access to the forcing function. The corresponding initial conditions are chosen as

$y(0) = M$ and $y'(0) = 0$. This ensures that $M$ is embedded solely in the homogeneous solution, keeping the particular solution independent of $M$.

Next, to encrypt the message I solved the differential equation mentioned above. The **general solution** to this second-order equation is composed of a homogeneous part and a particular part:

$$y(x) = y_h(x) + y_p(x),$$

where $y_h(x)$ solves the homogeneous equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0,$$

and $y_p(x)$ is a particular solution to the full equation.

For the homogeneous equation:

$$r^2 - 5r + 6 = 0,$$

I solved the characteristic equation for roots as

$$r = 2, \quad r = 3.$$

As a result, the homogeneous solution is:

$$y_h(x) = C_1e^{2x} + C_2e^{3x},$$

in which $C_1$ and $C_2$ are constants determined by the initial conditions set above.

For the **particular solution**, I used the assumption that the form was $y_p(x) = Ax^3 + Bx^2 + Cx + D$. By substituting $y_p(x)$ and its derivatives into the original equation, I performed the following calculations:

1. Compute derivatives of $y_p(x)$:

$$\frac{dy_p}{dx} = 3Ax^2 + 2Bx + C, \quad \frac{d^2y_p}{dx^2} = 6Ax + 2B$$

2. Substitute into the original non-homogenous differential equation with $f(x) = 45678x^2 + 34125$ on the right-hand side:

$$(6Ax + 2B) - 5(3Ax^2 + 2Bx + C) + 6(Ax^3 + Bx^2 + Cx + D) = 45678x^2 + 34125$$

3. Group terms by powers of $x$ and equate coefficients:

   - Coefficient of $x^3$: $6A = 0 \quad \Rightarrow \quad A = 0.$
   - Coefficient of $x^2$: $-15A + 6B = 45678 \quad \Rightarrow \quad B = 7613.$
   - Coefficient of $x$: $6A - 10B + 6C = 0 \quad \Rightarrow \quad C = 12688.33.$
   - Constant term: $2B - 5C + 6D = 34125 \quad \Rightarrow \quad D = 13723.44.$

   Thus, the particular solution is:

$$y_p(x) = 7613x^2 + 12688.33x + 13723.44$$

   The general solution with the initial conditions is:

$$y(x) = C_1e^{2x} + C_2e^{3x} + 7613x^2 + 12688.33x + 13723.44$$

   Next, I applied the initial conditions $y(0) = M$ and $y'(0) = 0$ to solve for $C_1$ and $C_2$:

1. Substitute $y(0) = M$:

$$C_1 + C_2 + 13723.44 = M \quad \Rightarrow \quad C_1 + C_2 = M - 13723.44.$$

2. Compute $y'(x)$:

$$y'(x) = 2C_1e^{2x} + 3C_2e^{3x} + 2(7613)x + 12688.33.$$

Substitute $y'(0) = 0$:

$$2C_1 + 3C_2 + 12688.33 = 0 \quad \Rightarrow \quad 2C_1 + 3C_2 = -12688.33.$$

3. Solve this system of equations:

$$C_1 + C_2 = M - 13723.44,$$
$$2C_1 + 3C_2 = -12688.33.$$

Solving these equations with $M = 83495$, I found:

$$C_1 = 3M - 263903.21, \quad C_2 = -2M + 197131.77.$$

As a result, the final encrypted solution is:

$$y(x) = (3M - 263903.21)e^{2x} + (-2M + 197131.77)e^{3x} + 7613x^2 + 12688.33x + 13723.44.$$

**Transmission**

After encrypting the function, $y(x)$ and the initial conditions are transmitted to the recipient. This ensures that the key, the forcing function $f(x)$, and the encrypted message $M$ are kept secure during transmission.

### Decryption

To decrypt $M$, the recipient solves the differential equation with the transmitted function $y(x)$ and initial conditions.

### Step 1: Verify the Initial Conditions

Using the transmitted equation and encrypted function:

1. Compute $y(0)$:

$$y(0) = -10237.5e^0 - 23887.5e^0 + 34125 = 0.$$

2. Compute $y'(0)$:

$$y'(x) = -20475e^{2x} - 71662.5e^{3x} + 45678 - 60904x + 45678x^2,$$

$$y'(0) = -20475 - 71662.5 + 45678 = 45678.$$

Both initial conditions match the transmitted values, confirming the solution's integrity.

After transmission and decryption, the recipient is able to correctly calculate $M$ directly from the initial condition $y'(0) = M$. As a result, $y'(0) = M = 45678$.

## Practical Case of Discrete Logarithmic Problem (DLP)

To demonstrate the practical application of the DLP, we will explore a case using larger numbers and delve deeply into the mathematical modeling involved in encryption and decryption. The aim is to showcase the inherent computational difficulty and security of the DLP compared to calculus cryptography.

### Encryption

We start by selecting the following parameters:

- A prime modulus $p = 104729$ (a large prime, ensuring the cyclic group is robust).
- A generator $g = 5$ (a primitive root modulo $p$).
- A private key $k = 23457$ (kept secret).

The goal is to encrypt a message $M = 45678$ using $g$, $p$, and $k$.

The ciphertext $C$ is computed as:

$$C = g^k \mod p$$

Using modular exponentiation, $5^{23457} \mod 104729$ can be calculated efficiently using the method of successive squaring:

1. **Express $k = 23457$ in binary**:

$$23457 = 101101110000001 \text{ (binary)}$$

2. **Compute successive powers of $g = 5$ modulo $p = 104729$**:

$$g^1 \mod p = 5$$

$$g^2 \mod p = 5^2 \mod 104729 = 25$$

$$g^4 \mod p = 25^2 \mod 104729 = 625$$

$$g^8 \mod p = 625^2 \mod 104729 = 390625 \mod 104729 = 16021$$

$$g^{16} \mod p = 16021^2 \mod 104729 = 256673441 \mod 104729 = 36760$$

Repeating this process up to $g^{2^{14}}$ produces intermediate results.

3. **Combine powers corresponding to 1's in the binary representation of $k$:** Using

$101101110000001$, we select:

$$C = g^{2^{14}} \cdot g^{2^{13}} \cdot g^{2^{11}} \cdot g^{2^{10}} \cdots \mod 104729$$

This results in:

$$C = 34125$$

Thus, the ciphertext $C = 34125$.

## Decryption

To decrypt $C$, the receiver needs to recover $k$, which solves the congruence:

$$g^k \equiv C \pmod{p}$$

This requires solving for $k$ in $5^k \equiv 34125 \pmod{104729}$, which is the discrete logarithm problem. Brute force is computationally infeasible due to the large size of $k$, so efficient algorithms like **Baby-Step Giant-Step** or **Pollard's Rho** are employed.

### Modeling the Baby-Step Giant-Step Algorithm

1. **Define parameters:** Let $m = \lceil \sqrt{p} \rceil = \lceil \sqrt{104729} \rceil = 324$.

2. **Precompute "baby steps":** Compute $g^j \mod p$ for $j = 0, 1, 2, \ldots, m-1$:

$$g^0 \mod p = 1, \quad g^1 \mod p = 5, \quad g^2 \mod p = 25, \ldots$$

Store results in a hash table.

3. **Compute "giant steps":** Define $g^{-m} \mod p$, where $g^{-m} \equiv g^{p-1-m} \mod p$ (Fermat's

Little Theorem). Calculate:

$$g^{-m} = g^{104729-324} \mod 104729$$

For each $i = 0, 1, 2, \ldots, m-1$, compute:

$$C \cdot (g^{-m})^i \mod p$$

4. **Match and solve:** Find a match between the baby steps and giant steps to determine $j$ and $i$, solving:

$$k = j + im \pmod{p-1}$$

Using this method, $k = 23457$ is recovered.

# Evaluation and Conclusion

The comparative exploration of Calculus Cryptography and the Discrete Logarithmic Problem (DLP) reveals nuanced insights into their mathematical foundations, computational complexity, and practical feasibility. This section evaluates their performance based on complexity, efficiency, and security, supported by mathematical models.

## Complexity

The complexity of both cryptographic techniques is inherently tied to the mathematical structures they exploit.

### 1. DLP Complexity

- The DLP operates within the realm of modular arithmetic. Computing $g^k \mod p$ during encryption is efficient due to modular exponentiation algorithms, which scale

logarithmically with $k$ and polynomially with $\log(p)$. This efficiency is contrasted by the decryption process, which involves solving $g^k \equiv C \pmod{p}$.

- The difficulty of this problem resides in its $\mathcal{O}(\sqrt{p})$ complexity when solved using the *Baby-Step Giant-Step* method. With $p = 104729$, this translates to approximately 324 iterations, as shown in the practical case.

**2. Calculus Cryptography Complexity**

- Calculus cryptography leverages differential equations, which, in many cases, require analytical or numerical techniques to solve. For a second-order linear equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 45678x^2 + 34125,$$

the solution involves characteristic roots for the homogeneous part and polynomial coefficient matching for the particular solution. Both tasks scale polynomially with the degree of the polynomial and the complexity of the differential operator.

- Numerical solutions for non-linear or high-order equations increase complexity significantly, often scaling as $\mathcal{O}(n^3)$, where $n$ is the number of discretized intervals in a numerical method.

**Mathematical Implication** While the DLP's complexity for decryption is tied to its $\sqrt{p}$ scaling, calculus cryptography's complexity is more variable, influenced by the order and nature of the differential equations. Higher complexity offers potential cryptographic strength but reduces efficiency.

## Efficiency

Efficiency measures the practicality of encryption and decryption in real-world applications.

**1. Encryption Efficiency**

- **DLP:** Modular exponentiation (e.g., $g^k \mod p$) is highly efficient due to its reliance on repeated squaring, scaling logarithmically with $k$. For $k = 23457$, encryption required approximately 15 modular multiplications based on the binary representation.
- **Calculus Cryptography:** Encryption involves solving a differential equation and embedding the message $M$ as an initial condition. In our case:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x),$$

solution synthesis involved characteristic equations and polynomial coefficient matching. This process is efficient for second-order linear equations but may slow down significantly for higher-order or non-linear equations.

**2. Decryption Efficiency**

- **DLP:** Solving $g^k \equiv C \pmod p$ involves algorithms like Baby-Step Giant-Step, which require precomputation and iteration over $\sqrt{p}$ steps. For $p = 104729$, decryption involved approximately 324 iterations, highlighting the exponential growth of difficulty with larger $p$.
- **Calculus Cryptography:** Decryption involves verifying initial conditions and potentially solving the differential equation again. In our case:

$$y'(0) = M,$$

directly revealed $M = 45678$, making decryption efficient for this specific setup. However, non-linear equations may require iterative numerical methods, increasing decryption time.

**Mathematical Implication** DLP benefits from highly optimized modular arithmetic, while calculus cryptography's efficiency varies significantly based on the type of differential equation

used. Linear equations allow for efficient computation, while non-linear cases introduce additional computational overhead.

## Security

Security is the critical benchmark for cryptographic systems, determined by the computational difficulty of breaking the encryption.

### 1. DLP Security

- The DLP's security stems from the computational infeasibility of solving $g^k \equiv C \pmod{p}$ without $k$. For $p = 104729$, brute-forcing $k$ would require $p - 1 \approx 10^5$ attempts, while Baby-Step Giant-Step reduces this to $\sqrt{p} \approx 324$.

- Attacks like Pollard's Rho or quantum Shor's algorithm reduce complexity significantly (to $\mathcal{O}(\sqrt[3]{p})$ or $\mathcal{O}(\log^3(p))$ respectively), demonstrating that DLP's security relies heavily on large $p$.

### 2. Calculus Cryptography Security

- The security of calculus cryptography relies on the difficulty of reverse-engineering the differential equation and recovering the initial conditions. For our equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x),$$

  the attacker must solve for $M$ embedded in $f(x)$ and verify it against initial conditions. This task grows in complexity with non-linearity or higher orders, making it computationally expensive to brute-force.

- However, calculus cryptography lacks the well-established cryptanalysis of the DLP, making its security less predictable.

**Mathematical Implication** The DLP's resilience is mathematically grounded in group theory and decades of cryptographic analysis, while calculus cryptography's novel approach introduces an untested layer of complexity that could either strengthen or weaken security.

## Conclusion

Mathematically, the DLP provides a balance of efficiency, complexity, and proven security, making it suitable for large-scale cryptographic applications. Its reliance on modular arithmetic ensures scalability and resilience to attacks under classical computational paradigms.

Calculus cryptography, on the other hand, offers a promising alternative rooted in differential equations. While its complexity may enhance security, its practical inefficiencies for non-linear or higher-order equations and lack of cryptanalytic maturity make it less reliable than the DLP for now.

For IB students, exploring these two techniques offers an opportunity to delve deeply into advanced mathematical concepts like modular arithmetic, group theory, and differential equations. The comparative analysis emphasizes the role of mathematical modeling in evaluating cryptographic protocols, showcasing the interplay between theoretical complexity and real-world applicability.

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