2201-MTL106: ASSIGNMENT-2

Q1. A die is tossed two times. Let X be the sum of face values on the two tosses and Y be the absolute value of the difference in face values. What is Ω? Examine whether X and Y are random variables or not.

Q2. Show that set of all random variables on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, denoted by $\mathbb{X}$, is a vector space over $\mathbb{R}$. For finite Ω, write down the basis of the vector space $\mathbb{X}$.

Q3. Do the following functions define distribution functions.

a) $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$ b) $F(x) = \frac{1}{\pi}\tan^{-1}(x), \quad -\infty < x < \infty$ c) $F(x) = \begin{cases} 0, & -\infty < x < 0 \\ 1 - e^{-x}, & 0 \le x < \infty \end{cases}$

Q4. Let F and G be distribution functions. Show that convex combination of F and G is also a distribution function. Is the product FG a distribution function? Is $F(x) + (1 - F(x))\log(1 - F(x))$ a distribution function? Justify your answer.

Q5. Let f and g be density functions. Show that convex combination of f and g is also a density function. Is the product fg a density function?

Q6. Let X be a continuous type random variable with pdf

$$f(x) = \begin{cases} \alpha + \beta x^2, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

If $\mathbb{E}[X] = 4$, find the values of α and β.

Q7. Let X be a non-negative random variable with distribution function $F_X(\cdot)$ given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2\pi}, & 0 \le x < 2\pi \\ 1, & x \ge 2\pi \end{cases}$$

Find the distribution function of $Y := X^2$. Are the random variables X and Y continuous? Justify your answer.

Q8. Let X be a random variable with distribution function $F(x)$ given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \le x < 1 \\ \frac{2}{3}, & 1 \le x < 2 \\ \frac{11}{12}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Find $\mathbb{P}(X > \frac{1}{2})$, $\mathbb{P}(2 < X \le 4)$, $\mathbb{P}(X < 3)$, and $\mathbb{P}(X = 1)$.

Q9. Let X be a random variable with $\mathbb{P}(0 \le X \le c) = 1$. Show that $\text{Var}(X) \le \dfrac{c^2}{4}$.

Q10. Let X be a positive random variable with pdf $f(x)$. Find the pdf of the random variable $U := \frac{X}{1+X}$.

Q11. Let X be a random variable with pdf $f(x)$ given by

$$f(x) = \begin{cases} \frac{\beta 5^\beta}{x^{\beta+1}}, & x \geq 5 \\ 0, & x < 5 \end{cases}$$

for positive constant β. Show that $\mathbb{E}[X^n]$ exists if and only if $n < \beta$. Find the mean and variance of the random variable for $\beta > 2$.

Q12. Let X be an integer-valued random variable with probability generating function (pgf) $G_X(\cdot)$. Find the pgf of $Y := \alpha X + \beta$ for some non-negative constants α, β.

Q13. Let X be a discrete random variable with pgf $G_X(s) = \frac{s}{3}(2+s^2)$. Find the distribution of X.

Q14. Let X be a random variable whose pdf is given by

$$f_X(x) = \begin{cases} e^{-2x} + \frac{1}{2}e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Write down the moment generating function (mgf) for X. Use this mgf to compute the first and second moments of X.

Q15. For the Cauchy pdf $f(x) = \dfrac{1}{\pi}\dfrac{1}{1+x^2}$, $-\infty < x < \infty$, does the mgf exist?

Q16. Let X be a discrete type random variable with mgf $m_X(t) = a + be^{2t} + ce^{4t}$ and $\mathbb{E}[X] = 3$, $\mathrm{Var}(X) = 2$. Find $\mathbb{E}[2^X]$.

Q17. Let $\phi_X(\cdot)$ be the characteristic function of a random variable X. Show that

$$1 - |\phi_X(2u)|^2 \leq 4\Big(1 - |\phi_X(u)|^2\Big).$$

Q18. Let X be a continuous random variable with cdf $F_X(\cdot)$. Find the distribution and characteristic function of the random variable Y defined as $Y := F_X(X)$.