2201-MTL106: ASSIGNMENT-5

Q1. Let $g : [0, \infty) \mapsto (0, \infty)$ be a function such that $g(x) \geq b > 0$ for $x \geq a$. Let X be a non-negative random variable such that $\mathbb{E}[g(X)]$ exists. Show that

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[g(X)]}{b}.$$

Q2. Let X be a Binomial $B(n, p)$ random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then show the followings:

a) For $\lambda > 0$ and $b > 0$, $\mathbb{P}(X - np > nb) \leq \mathbb{E}\big[\exp(\lambda(X - np - nb))\big]$.

b) For any $\epsilon > 0$, $\mathbb{P}(X \geq np + \epsilon\sqrt{np(1-p)}) \leq \frac{1}{1+\epsilon^2}$.

c) For all $\epsilon > 0$, $\mathbb{P}(|X - np| \leq n\epsilon)$ tends to 1.

Q3. Show that $X_n \stackrel{\mathbb{P}}{\to} X$ if and only if $\lim_{n\to\infty} \mathbb{E}\big(1 \wedge |X_n - X|\big) = 0$.

Q4. Let $\{X_n\}$ be a sequence of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given by

$$X_n := \sqrt{n}\mathbf{1}_{(0,\frac{1}{n})}(U), \quad U \sim \mathcal{U}(0, 1).$$

Show that $X_n \stackrel{\mathbb{P}}{\to} 0$ but $X_n \stackrel{2}{\nrightarrow} 0$.

Q5. Prove or disprove: $X_n \stackrel{\mathbb{P}}{\to} 0 \implies \mathbb{E}(X_n) \to 0$ and $\mathrm{Var}(X_n) \to 0$.

Q6. Let $\{X_n\}$ be a sequence of random variables that is monotonically increasing, i.e., $X_n(\omega) \leq X_{n+1}(\omega)$ for all $\omega \in \Omega, n \in \mathbb{N}$. If $X_n \stackrel{\mathbb{P}}{\to} X$, then show that $X_n \stackrel{a.s}{\to} X$.

Q7. Let $X_n \stackrel{d}{\to} X$ with $X = a$ a.e. Then show that $X_n \stackrel{\mathbb{P}}{\to} a$.

Q8. Prove or disprove:

$$X_n \stackrel{d}{\to} X,\ Y_n \stackrel{\mathbb{P}}{\to} c,\ c \in \mathbb{R}, \implies X_n + Y_n \stackrel{d}{\to} X + c,\ X_nY_n \stackrel{d}{\to} cX.$$

Q9. Let $Y, \{X_n\}$ be random variables such that for each fixed $\tau > 0$, $X_n + \tau Y \stackrel{d}{\to} X + \tau Y$. Show that $X_n \stackrel{d}{\to} X$.

Q10. Let $\{X_j\}$ be a sequence of i.i.d. random variables with X_j in L^1. Let $Y_j = e^{X_j}$. Show that $\left(\prod_{i=1}^n Y_i\right)^{\frac{1}{n}}$ converges to a constant $\alpha = e^{\mathbb{E}[X_1]}$.

Q11. Let $\{X_j\}$ be a sequence of i.i.d. non-negative random variables with $\mathbb{E}[X_1] = 1$ and $\mathrm{Var}(X_1) = \sigma^2 \in (0, \infty)$. Show that

$$\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \stackrel{d}{\to} Y, \quad Y \sim \mathcal{N}(0, 1).$$

Q12. Use CLT to show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Q13. Let $\{X_i\}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_i = 1) = \frac{3}{4}$ and $\mathbb{P}(X_i = 0) = \frac{1}{4}$. Let $Y_i = X_i + X_i^2$. Use CLT to evaluate $\mathbb{P}\Big(\sum_{i=1}^{80} Y_i > 100\Big)$.

Q14. Let $\{X_i\}$ be a sequence of **i.i.d** non-negative random variables with mean 4 and variance 16. Calculate:

$$\lim_{n\to\infty} \mathbb{E}\Big[\cos\left(\sqrt{S_n} - 2\sqrt{n}\right)\Big],$$

where $S_n := \sum_{i=1}^n X_i$.

Q15. Let $\{X_n\}$ be a sequence of **i.i.d** random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with uniform distribution on $(-1, 1)$. Let

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2 + X_i^3}.$$

Show that $\sqrt{n}Y_n$ converges in distribution as $n \to \infty$. Let $\phi_n(t)$ be the characteristic function of $\sqrt{n}Y_n$. Calculate $\lim\limits_{n\to\infty} \phi_n(2)$.

Q16. Let $X_n \xrightarrow{\mathbb{P}} X$. Show that the characteristic function ϕ_{X_n} converges pointwise to ϕ_X.