2201-MTL106: ASSIGNMENT-4

Q1. Is the function $F(x, y) = 1 - e^{-xy},\ 0 \le x, y < +\infty$ joint distribution function of some pair of random variables?

Q2. Suppose that the random variables X and Y have joint distribution function

$$F_{(X,Y)}(x, y) = \begin{cases} 0 & \text{if } x < 0 \\ (1 - e^{-x})\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y)\right. \text{if } x \ge 0. \end{cases}$$

Show that X and Y are jointly continuous.

Q3. Let (X, Y) be a two-dimensional discrete type random variables with joint pmf

$$p_{(X,Y)}(x, y) = \begin{cases} cxy, & x, y \in \{1, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathbb{P}(1 \le X \le 2, Y \le 2)$ and $\mathbb{P}(Y = 3)$.

Q4. Let X and Y be independent random variables. The range of X is $\{1, 3, 4\}$ and the range of Y is $\{1, 2\}$. Partial information on the probability mass function is as follows: $p_X(3) = 0.5,\ p_Y(2) = 0.6$, and $p_{(X,Y)}(4, 2) = 0.18$.

i) Determine $p_X(\cdot)$, $p_Y(\cdot)$ and $p_{(X,Y)}(\cdot, \cdot)$ completely.
ii) Determine $\mathbb{P}(|X - Y| \ge 2)$.

Q5. Let X and Y be independent random variable each taking values -1 and 1 with probability $\frac{1}{2}$, and let $Z = XY$.

a) Show that X, Y and Z are pairwise independent.
b) Are they independent ? Justify your answer.

Q6. Let X and Y be independent random variables such that $X \sim \mathcal{G}(\alpha),\ Y \sim \mathcal{G}(\beta)$ with $\alpha \ne \beta$. Show that

$$\mathbb{P}(X + Y = z) = \frac{\alpha\beta}{\alpha - \beta}\Big((1 - \beta)^{z-1} - (1 - \alpha)^{z-1}\Big).$$

Q7. Amit and Supriya work independently on a problem in Tutorial Sheet 4 of MTL106. The time for Amit to complete the problem is exponential distributed with mean 5 minutes. The time for Supriya to complete the problem is exponential distributed with mean 4 minutes. Given that Amit requires more than 1 minutes, what is the probability that he finishes the problem before Supriya?

Q8. The random variable X represents the amplitude of cosine wave; Y represents the amplitude of a sine wave. Both are independent and uniformly distributed over the interval $(0, 1)$. Let R represent the amplitude of their resultant, i.e., $R^2 = X^2 + Y^2$ and θ represent the phase angle of the resultant, i.e., $\theta = \tan^{-1}(\frac{Y}{X})$. Find the joint and marginal pdfs of θ and R.

Q9. Let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal random variables. Show that $Y := \sum_{i=1}^{n} Z_i$ follows a chi-square distribution with n-degrees of freedom i.e., $Y \sim \chi^2_{(n)}$.

Q10. Let X and Y be jointly continuous random variables with joint pdf

$$f_{(X,Y)}(x, y) = \frac{1}{2}(x + y)e^{-(x+y)}, \quad x, y \ge 0.$$

Define $Z := X + Y$. Find the density function and mean of Z.

Q11. Let X and Y be independent random variables with $X \sim \mathcal{U}[0,1]$ and $Y \sim \mathcal{U}[0,1]$. Calculate $\mathrm{Cov}(U,V)$, where the random variables U and V are given by

$$U := \min\{X,Y\}, \quad V := \max\{X,Y\}\,.$$

Q12. Let X and Y be the independent random variables with $X, Y \sim \mathrm{Exp}(1)$.

a) Find the pdf of the random variables

$$X \pm Y,\ XY,\ \frac{X}{Y},\ \min\{X,Y\},\ \max\{X,Y\},\ \frac{X}{X+Y}.$$

ii) Let $U := X + Y$ and $V := X - Y$. Find the conditional pdf of V given $U = u$ for some fixed $u > 0$.

iii) Are U and $W := \frac{X}{X+Y}$ independent? Justify your answer.

Q13. Let X and Y be random variables with zero mean and correlation ρ. Show that

$$\mathbb{E}\big[\mathrm{Var}(Y|X)\big] \le (1-\rho^2)\mathrm{Var}(Y)\,.$$

Q14. Let X and Y be two random variables with joint density function

$$f_{(X,Y)}(x,y) = cx(y-x)e^{-y}, \quad 0 \le x \le y < +\infty.$$

i) Find the conditional probability density functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

ii) Show that $\mathbb{E}[X|Y] = \frac{Y}{2}$ and $\mathbb{E}[Y|X] = X + 2$.

Q15. For **i.i.d** random variables X and Y, show that $U := X + Y$ and $V := X - Y$ are uncorrelated but not necessarily independent. Are U and V independent if $X, Y \sim \mathcal{N}(0,1)$? Give a proper reason.

Q16. Let X and Y have joint density function

$$f_{(X,Y)}(x,y) = 2e^{-x-y}, \quad 0 < x < y < +\infty.$$

Find their marginal density functions and covariance.

Q17. Let X and Y be independent random variables with common density function f.

a) Show that $\tan^{-1}(\frac{Y}{X})$ has the uniform distribution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ if and only if

$$\int_{-\infty}^{\infty} f(x)f(xy)|x|\,dx = \frac{1}{\pi(1+y^2)}, \quad y \in \mathbb{R}\,.$$

b) Find one such f.

Q18. Let X and Y be jointly distributed with density function

$$f_{(X,Y)}(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the $\mathrm{Cov}(X,Y)$ and correlation coefficient.

Q19. Let X and Y be two random variables such that $\rho_{(X,Y)} = \frac{1}{2}$, $\mathrm{Var}(X) = 1$ and $\mathrm{Var}(Y) = 4$. Compute $\mathrm{Var}(X - 3Y)$.