2201-MTL106: ASSIGNMENT-6

Q.1) Let $X(t)$ be a co-variance stochastic process with zero mean, and Θ be a uniformly distributed random variable on $(0, 2\pi)$ such that it is independent of $X(t)$. Define a stochastic process

$$Y(t) = X(t)\cos(\eta t + \Theta)$$

for some positive constant η. Explain whether $Y(t)$ is co-variance stationary or not.

Q.2) Let A and B are given uncorrelated random variables with mean 0 and variance 1. Define a stocastic process

$$X(t) = A\cos(wt) + B\sin(wt), \quad w \geq 0.$$

Is $X(t)$ co-variance stationary process? justify your answer.

Q.3) Let $\{X(t) : t \in [0, T]\}$ be a stochastic process with independent incrments such that $\mathbb{E}[X(t)] = 0$ and $\mathbb{E}[X^2(t)] = 5t^2$ for all $t \in [0, T]$. Find the upper bound of

$$\big|\mathbb{E}[X(t)X(t+h)]\big| \quad \text{for } h > 0,\ t \in [0, T-h].$$

Q.4) Let $\{Y_n : n \geq\}$ be a sequence of independent random variables with

$$\mathbb{P}(Y_n = 1) = p, \quad \mathbb{P}(Y_n = -1) = 1 - p.$$

Let X_n be defined by

$$\begin{cases} X_0 = 0 \\ X_{n+1} = X_n + Y_{n+1}, \quad n \geq 0\,. \end{cases}$$

i) Check whether $\{X_n\}$ is a Markov chain or not.
ii) If it is Markov chain, then find its transition probability matrix.

Q.5) Let $\xi_0, \xi_1, \xi_2, \ldots$ be integer-valued independent random variables. Let $S = \{1, 2, \ldots, N\}$ and $X_0 \in S$ be another random variable independent of $\{\xi_n\}$. Define a new random variables

$$X_{n+1} := f(X_n, \xi_n) \quad n \geq 0,$$

where $f : S \times \mathbb{Z} \to S$ is a certain function. Show that $\{X_n\}$ is a Markov chain.

Q.6) Let X_0 be an integer-valued random variable such that $\mathbb{P}(X_0 = 0) = 1$. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of iid random variables, independent of X_0 such that

$$\mathbb{P}(\xi_n = 1) = p, \quad \mathbb{P}(\xi_n = -1) = q, \quad \mathbb{P}(\xi_n = 0) = 1 - (p + q).$$

Define the new random variables

$$X_n = \max\{0, X_{n-1} + \xi_n\}, \quad n \geq 0.$$

Prove that $\{X_n\}_{n\geq 0}$ form a Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

Q.7) Three person (denoted by $1, 2, 3$) arranged in a circle play a game of throwing a ball to one another. At each stage, the person having the ball is equally likely to throw it into any one of the other two person. Suppose that X_0 denotes the person who had the ball initially and $\{X_n : n \geq 1\}$ denotes the person who had the ball after n throws.

a) Show that $\{X_n : n \geq 1\}$ is a Markov chain.
b) Find the transition probability matrix P, and calculate $\mathbb{P}\big(X_2 = 1 \big| X_0 = 1\big)$.
c) Find the transition probability matrix P if the number of person is m (≥ 4).

Q.8) Let $\{X_n : n \geq 0\}$ be a Markov chain with state space $S = \{1, 2, 3\}$, transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$ and initial distribution $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

a) Compute $\mathbb{P}(X_3 = 2)$.
b) Compute $\mathbb{P}(X_3 = 1, X_2 = 2, X_1 = 3, X_0 = 2)$.

Q.9) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

a) Classify the states of the chain as transient, +ve recurrent or null recurrent.
b) Show that if the process start at state 3, then the expected number of times in state 1, 2 and 3 before being absorbed are 2, 4 and 4 respectively.

Q.10) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/8 & 1/4 & 1/8 & 1/2 \end{pmatrix}$. Check whether all the states are ergodic or not.

Q.11) Let $P = (p_{ij})$ be the transition probability matrix of an irreducible Markov chain with $P^2 = P$. Show that the Markov chain is recurrent and aperiodic.

Q.12) Suppose that the pattern of *sunny* and *rainy* days in a certain place is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8 , and every rainy day is followed by another rainy day with probability 0.6.

a) Calculate the probability that the day after tomorrow will be a rainy day given that today is a sunny day.
b) What is the probability that June 15, 2051 will be a rainy day.

Q.13) Suppose that employees of a company exhibit 4 states of mind: 1 (suicidal); 2 (severe depression); 3 (mild depression); 4 (seeking for professional psychiatric help). Admit changes in state of mind can be modeled as a Markov chain $\{X_n : n \geq 0\}$ with one-step transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

i) Draw the state transition diagram for this DTMC model.
ii) Find the expected number of changes of state of mind until a employee seeks for professional psychiatric help, considering the initial state $X_0 = 2$.
iii) Compute the probability the employee will eventually be suicidal starting from state $X_0 = 3$?

Q.14) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition probability matrix $P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix}$.

a) Classify the states of the chain.
b) Calculate $p_{ij}^{(n)}$ for large n.

Q.15) Let $\{X_n : n \geq 0\}$ be a discrete-time Markov chain (DTMC) with state space $S = \{1, 2, 3\}$ and transition probability matrix $P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix}$.

a) Check whether the chain is ergodic or not, and find $p_{31}^{(2)}$.
b) Examine whether there exists a stationary distribution of the given DTMC or not.
c) Calculate $p_{13}^{(n)}$ for large n.

Q.16) Let $\{X_n : n \geq 0\}$ be a ergodic, irreducible finite Markov chain with transition probability matrix $P = (p_{ij})$ such that $\sum\limits_{i \in S} p_{ij} = 1$ for all $j \in S$. Calculate its limiting probabilities i.e., $p_{ij}^{(n)}$ for large n.