Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Quiz 1

Time: 20 minutes **Date: 30/01/2022**
Max. Marks: 10

Note: The exam is closed-book, and all the questions are compulsory.

1. Consider the following game of chance. You pay 2 dollars and roll a fair die. Then you receive a payment according to the following schedule. If the event $A = \{1, 2, 3\}$ occurs, then you will receive 1 dollar. If the event $B = \{4, 5\}$ occurs, you receive 2 dollars. If the event $C = \{6\}$ occurs, then you will receive 6 dollars. Let X denotes your profit. Construct the sample space $(\Omega, \mathcal{F})$ and show that X is a random variable. Find PMF and CDF of X. What is the average profit you can make if you participate this game?

   (5 marks)

2. Suppose a point is chosen from the interior of a right angle triangle whose vertices are $(0, 0)$, $(2, 0)$, and $(0, 2)$. Let X be defined as the distance from the point chosen to the base. Find the probability density function of X.

   (5 marks)

Department of Mathematics

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Minor Examination

Time: 1 hour 15 minutes **Date: 16/02/2022**
Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

1. (*a*) Let X and Y be independent Poisson random variables with parameters $\lambda_1 > 0$, $\lambda_1 > 0$, respectively. Show that the conditional distribution of X given $X + Y = k$ is a binomial distribution for every positive integer k.

   (*b*) Consider a random variable with PMF given by

$$P\{X = x\} = \begin{cases} \frac{1}{18}, & x = 1, 3 \\ \frac{16}{18}, & x = 2 \end{cases}$$

   Find the value of k for which the general bound given by Chebyshev's inequality cannot be improved; k denotes the constant used in Chebyshev's inequality.

   (4+4 marks)

2. Suppose a box has 3 balls labeled 1,2, and 3. Two balls are selected without replacement from the box. Let X be the number on the first ball and let Y be the number on the second ball. Compute conditional PMF of Y given $X = 1$, $\text{Cov}(X, Y)$ and $\rho(X, Y)$.

   (5 marks)

3. Let (X_1, X_2, X_3) be an RV with joint PMF

$$P\{X_1 = x_1, X_2 = x_2, X_3 = x_3\} = \begin{cases} \frac{1}{4} & \text{if } (x_1, x_2, x_3) \in A \\ 0 & \text{otherwise,} \end{cases}$$

   where $A = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$. Are X_1, X_2, X_3 pairwise independent? Are $X_1 + X_2$ and X_3 independent?

   (4 marks)

4. Let $X_1, X_2, \cdots, X_n$ be independent random variables which follow uniform distribution on $(0, 1)$. Find the (a) PDF of $Y_1 = \max(X_1, X_2, \cdots, X_n)$, (b) PDF of $Y_2 = \min(X_1, X_2, \cdots, X_n)$, c) $E(Y_1)$ and $E(Y_2)$.

   (3+3+2 marks)

5. Let X and Y be independent random variables which follow standard normal distribution. Find the joint PDF of $X + Y$ and $X - Y$. Are $X + Y$ and $X - Y$ independent? What is the conditional PDF of $X + Y$ given $Y = y$?

   (5 marks)

Set A

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Minor Examination

Time: 1 hour **Date: 22/09/2021**

Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

Q.1 The following questions can have multiple correct answers. Write all the correct answers. The marks will be awarded only if you write all correct answers.

(a) Let the random variables X and Y defined on sample space $(\Omega, \mathcal{F})$ have same PMF. Then,

(i) CDF of X and Y are same, (ii) Characteristic function of X and Y are same, (iii) $X(\omega) = Y(\omega),\ \forall\, \omega \in \Omega$

(b) Let the random variables X and Y are such that $E(XY) = E(X)E(Y)$. Then,

(i) Cov(X, Y)= 0, (ii) X and Y are independent , iii) Cov$(X - Y, Y)$ =Var(Y).

(c) Let $X : \Omega \to \mathbb{R}$ be a random variable defined on probability space $(\Omega, \mathcal{F}, P)$. Let Q be a probability distribution of X and $\mathcal{B}$ is Borel σ-field. Then,

(i) $Q : \mathcal{B} \to [0, 1]$, (ii) $Q : \mathbb{R} \to [0, 1]$, (iii) $Q((-\infty, x)) = P\{X \leq x\}$, (iv) Q is continuous.

(d) Let X_1, X_2, X_3, X_4 be pairwise independent random variables and g_1, g_2 are Borel measurable functions. Then,

(i) $g(X_1)$ and $g(X_2)$ are independent, (ii) X_1, X_2, X_3 are independent (iii) X_1 and X_2 are independent, (iv) $g(X_1)$, $g(X_2)$, $g(X_3)$ are independent.

(e) Let the CDF of random variable X is given by

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{x}{2} \text{ if } 0 \leq x < 1 \\ \frac{2}{3} \text{ if } 1 \leq x < 2 \\ \frac{11}{12} \text{ if } 2 \leq x < 3 \\ 1 \text{ if } 3 \leq x. \end{cases}$$

Then, (i) $P\{X = 1\} = \frac{1}{6}$, (ii) $P\{X = 1\}$ cannot be found from given information, (iii) $P\{X \leq 1\} = \frac{1}{2}$, (iv) X is a discrete random variable.

(1+1+1+1+1 marks)

Q.2 Give the final answer to the following questions. The justification of the answers is not required. The step marking is not applicable in these questions.

(a) Let X be a continuous random variable with PDF $\alpha = \frac{3}{5}$; $\beta = \frac{6}{5}$

$$f(x) = \begin{cases} \alpha + \beta x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = \frac{3}{5}$, find the values of α and β.

$\frac{2}{5}$ $\frac{3}{5}$

(2 marks)

(b) Let X be a discrete random variable with MGF $M_X(t) = \alpha + \beta e^{5t}$, $E(X) = 3$. Find i) α, β, ii) PMF of X.

X	0	5
$P_X(x)$	2/5	3/5

(1+1 marks)

(c) A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 43, 30, 20, and 55 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on the bus of randomly selected driver. Compute $E[X]$ and $E[Y]$. $= 37$

$= 41.71$

(2 marks)

(d) A fair coin is tossed three times. Let X = number of heads in three tossings, and Y = difference, in absolute value, between number of heads and number of tails. What is conditional PMF of X given $Y = 3$.

$$P_{X/3}(x) = \begin{cases} 0.5, & x = 0, 3 \\ 0, & \text{other} \end{cases}$$

(2 marks)

(e) Consider an experiment having three possible outcomes, 1, 2, 3, that occur with probabilities 1/2, 1/4, and 1/4, respectively. Suppose two independent repetitions of the experiment are made and let X_i, $i = 1, 2, 3$, denote the number of times the outcome i occurs. What is the PMF of $X_1 + X_2$?

X_1+X_2	0	1	2
$P_{X_1+X_2}(x)$	$\frac{1}{16}$	$\frac{6}{16}$	$\frac{9}{16}$

(2 marks)

Q.3 The following questions are descriptive type. Please provide detailed answers.

1. Let X, Y be iid RVs with common PDF

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

$$g(u,v) = \begin{cases} 2e^{-2u}, & u>0, \ u>v>-u \\ 0, & \text{oth} \end{cases}$$

$$g(v/u=u) = \begin{cases} \frac{1}{2u}, & -u<v<u \\ 0, & \text{oth} \end{cases}$$

$$g(v) = \begin{cases} e^{-2v}, & v>0 \\ e^{2v}, & v<0 \end{cases}$$

Let $U = X + Y$ and $V = X - Y$. Find,
i) Joint PDF of U and V, ii) Marginals of U and V, iii) conditional PDF of V given $U = u$, for some fixed $u > 0$.

$$g(u) = \begin{cases} 4ue^{-2u}, & u>0 \\ 0, & \text{oth} \end{cases}$$

(8 marks)

2. Suppose that two buses, A and B, operate on a route. A person arrives at a certain bus stop on this route at time 0. Let X and Y be the arrival times of buses A and B, respectively, at this bus stop. Suppose that X and Y are independent random variables and density function of X is given by

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } x \in [0, 3] \\ 0, & \text{otherwise,} \end{cases}$$

and PDF of Y is given by

$$f(y) = \begin{cases} \frac{1}{4}, & \text{if } y \in [0, 4] \\ 0, & \text{otherwise.} \end{cases}$$

What is the probability that bus A will arrive before bus B? $= \frac{15}{24}$

(3 marks)

3. Let X be a positive RV of the continuous type with PDF $f(\cdot)$. Find the PDF of the RV $U = \frac{X}{(1+X)}$. If, in particular, X has the PDF

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } 0 \leq x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

what is the PDF of U? $= \begin{cases} \frac{1}{3(1-u)^2}, & 0<u<3/4 \\ 0, & \text{oth} \end{cases}$

(4 marks)

Set B

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Minor Examination

Time: 1 hour **Date: 22/09/2021**

Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

Q.1 The following questions can have multiple correct answers. Write all the correct answers. The marks will be awarded only if you write all correct answers.

(a) Let the random variables X and Y defined on sample space $(\Omega, \mathcal{F})$ have same PMF. Then,

(i) Characteristic function of X and Y are same, (ii) $X(\omega) = Y(\omega)$, $\forall\ \omega \in \Omega$, (iii) CDF of X and Y are same.

(b) Let the random variables X and Y are such that $E(XY) = E(X)E(Y)$. Then,

i) Cov$(X - Y, Y)$ =Var(Y), (ii) X and Y are independent, (iii) Cov(X, Y)= 0.

(c) Let $X : \Omega \to \mathbb{R}$ be a random variable defined on probability space $(\Omega, \mathcal{F}, P)$. Let Q be a probability distribution of X and $\mathcal{B}$ is Borel σ-field. Then,

(i) $Q((-\infty, x)) = P\{X \leq x\}$, (ii) $Q : \mathbb{R} \to [0, 1]$, (iii) $Q : \mathcal{B} \to [0, 1]$, (iv) Q is continuous.

(d) Let X_1, X_2, X_3, X_4 be pairwise independent random variables and g_1, g_2 are Borel measurable functions. Then,

(i) X_1 and X_2 are independent, (ii) X_1, X_2, X_3 are independent, (iii) $g(X_1)$, $g(X_2)$, $g(X_3)$ are independent, (iv) $g(X_1)$ and $g(X_2)$ are independent.

(e) Let the CDF of random variable X is given by

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{x}{2} \text{ if } 0 \leq x < 1 \\ \frac{2}{3} \text{ if } 1 \leq x < 2 \\ \frac{11}{12} \text{ if } 2 \leq x < 3 \\ 1 \text{ if } 3 \leq x. \end{cases}$$

Then, (i) $P\{X \leq 1\} = \frac{1}{2}$, (ii) X is ~~a~~ discrete random variable, (iii) $P\{X = 1\}$ cannot be found from given information, (iv) $P\{X = 1\} = \frac{1}{6}$.

(1+1+1+1+1 marks)

Q.2 Give the final answer to the following questions. The justification of the answers is not required. The step marking is not applicable in these questions.

(a) Let X be a continuous random variable with PDF

$$f(x) = \begin{cases} \alpha + \beta x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$\alpha = 1$; $\beta = 0$

If $E(X) = \frac{1}{2}$, find the values of α and β.

(2 marks)

$\frac{1}{2}$ $\frac{1}{2}$

$\parallel$ $\parallel$

(b) Let X be a discrete random variable with MGF $M_X(t) = \alpha + \beta e^{6t}$, $E(X) = 3$. Find i) α, β, ii) PMF of X.

X	0	6
$P_X(x)$	$\frac{1}{2}$	$\frac{1}{2}$

(1+1 marks)

(c) A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 38, 20, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on the bus of randomly selected driver. Compute $E[X]$ and $E[Y]$. $= 37$

40.16 (2 marks)

(d) A fair coin is tossed three times. Let $X =$ number of heads in three tossings, and $Y =$ difference, in absolute value, between number of heads and number of tails. What is conditional PMF of X given $Y = 1$.

$$p_{X/1}(x) = \begin{cases} 0.5, & x = 1, 2 \\ 0, & oth \end{cases}$$

(2 marks)

(e) Consider an experiment having three possible outcomes, 1, 2, 3, that occur with probabilities 1/5, 2/5, and 2/5, respectively. Suppose two independent repetitions of the experiment are made and let X_i, $i = 1, 2, 3$, denote the number of times the outcome i occurs. What is the PMF of $X_1 + X_2$?

X_1+X_2	0	1	2
$p_{X_1+X_2}(x)$	$\frac{4}{25}$	$\frac{12}{25}$	$\frac{4}{25}$

(2 marks)

Q.3 The following questions are descriptive type. Please provide detailed answers.

1. Let X, Y be iid RVs with common PDF

$$f(x) = \begin{cases} 3e^{-3x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

$$g(u,v) = \begin{cases} \frac{9}{2} e^{-3u}, & u>0, \ u>v>-u \\ 0, & oth \end{cases}$$

$$g(v/u=u) = \begin{cases} \frac{1}{2u}, & -u<v<u \\ 0, & oth \end{cases}$$

$$g(v) = \begin{cases} \frac{3}{2} e^{-3v}, & v>0 \\ \frac{3}{2} e^{3v}, & v<0 \end{cases}$$

Let $U = X + Y$ and $V = X - Y$. Find,
i) Joint PDF of U and V, ii) Marginals of U and V, ii) conditional PDF of V given $U = u$, for some fixed $u > 0$.

$$g(u) = \begin{cases} 9u e^{-3u}, & u>0 \\ 0, & oth \end{cases}$$

(8 marks)

2. Suppose that two buses, A and B, operate on a route. A person arrives at a certain bus stop on this route at time 0. Let X and Y be the arrival times of buses A and B, respectively, at this bus stop. Suppose that X and Y are independent random variables and density function of X is given by

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in [0, 4] \\ 0, & \text{otherwise,} \end{cases}$$

and PDF of Y is given by

$$f(y) = \begin{cases} \frac{1}{5}, & \text{if } y \in [0, 5] \\ 0, & \text{otherwise.} \end{cases}$$

What is the probability that bus A will arrive before bus B? $= \frac{3}{5}$

(3 marks)

3. Let X be a positive RV of the continuous type with PDF $f(\cdot)$. Find the PDF of the RV $U = \frac{X}{(1+X)}$. If, in particular, X has the PDF

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq x \leq 4, \\ 0, & \text{otherwise,} \end{cases}$$

what is the PDF of U? $= \begin{cases} \frac{1}{4(1-u)^2}, & 0<u<\frac{4}{5} \\ 0, & oth \end{cases}$

(4 marks)

Set C

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Minor Examination

Time: 1 hour **Date: 22/09/2021**
Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

Q.1 The following questions can have multiple correct answers. Write all the correct answers. The marks will be awarded only if you write all correct answers.

(a) Let the random variables X and Y defined on sample space $(\Omega, \mathcal{F})$ have same PMF. Then,

(i) CDF of X and Y are same, (ii) Characteristic function of X and Y are same, (iii) $X(\omega) = Y(\omega),\ \forall\, \omega \in \Omega$.

(b) Let the random variables X and Y are such that $E(XY) = E(X)E(Y)$. Then,

i) $\text{Cov}(X-Y, Y) = \text{Var}(Y)$, (ii) $\text{Cov}(X,Y) = 0$, (iii) X and Y are independent.

(c) Let $X : \Omega \to \mathbb{R}$ be a random variable defined on probability space $(\Omega, \mathcal{F}, P)$. Let Q be a probability distribution of X and $\mathcal{B}$ is Borel σ-field. Then,

(i) $Q : \mathcal{B} \to [0,1]$, (ii) Q is continuous, (iii) $Q((-\infty, x)) = P\{X \le x\}$, (iv) $Q : \mathbb{R} \to [0,1]$.

(d) Let X_1, X_2, X_3, X_4 be pairwise independent random variables and g_1, g_2 are Borel measurable functions. Then,
(i) X_1 and X_2 are independent, (ii) $g(X_1)$ and $g(X_2)$ are independent, (iii) $g(X_1)$, $g(X_2)$, $g(X_3)$ are independent, (iv) X_1, X_2, X_3 are independent.

(e) Let the CDF of random variable X is given by

$$F(x) = \begin{cases} 0 \ \text{ if } \ x < 0 \\ \frac{x}{2} \ \text{ if } \ 0 \le x < 1 \\ \frac{2}{3} \ \text{ if } \ 1 \le x < 2 \\ \frac{11}{12} \ \text{ if } \ 2 \le x < 3 \\ 1 \ \text{ if } \ 3 \le x. \end{cases}$$

Then, (i) $P\{X \le 1\} = \frac{1}{2}$, (ii) $P\{X = 1\} = \frac{1}{6}$, (iii) $P\{X = 1\}$ cannot be found from given information, (iv) X is a discrete random variable.

(1+1+1+1+1 marks)

Q.2. Give the final answer to the following questions. The justification of the answers is not required. The step marking is not applicable in these questions.

(a) Let X be a continuous random variable with PDF $\alpha = \frac{9}{7}$; $\beta = -\frac{6}{7}$

$$f(x) = \begin{cases} \alpha + \beta x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = \frac{3}{7}$, find the values of α and β.

$\frac{4}{7}$ $\frac{3}{7}$

(2 marks)

(b) Let X be a discrete random variable with MGF $M_X(t) = \alpha + \beta e^{7t}$, $E(X) = 3$. Find i) α, β, ii) PMF of X.

X	0	7
$P_X(x)$	4/7	3/7

(1+1 marks)

(c) A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 50, 30, 20, and 48 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on the bus of randomly selected driver. Compute $E[X]$ and $E[Y]$. ≈ 37

$= 41.24$

(2 marks)

(d) A fair coin is tossed three times. Let $X =$ number of heads in three tossings, and $Y =$ difference, in absolute value, between number of heads and number of tails. What is conditional PMF of X given $Y = 3$. $P_{X/3}(x) = \begin{cases} 0.5, & x = 0,3 \\ 0, & \text{oth} \end{cases}$

(2 marks)

(e) Consider an experiment having three possible outcomes, 1, 2, 3, that occur with probabilities 1/6, 1/3, and 1/2, respectively. Suppose two independent repetitions of the experiment are made and let X_i, $i = 1, 2, 3$, denote the number of times the outcome i occurs. What is the PMF of $X_1 + X_2$?

X_1+X_2	0	1	2
$P_{X_1+X_2}(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(2 marks)

Q.3 The following questions are descriptive type. Please provide detailed answers.

1. Let X, Y be iid RVs with common PDF

$$f(x) = \begin{cases} 4e^{-4x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

$g(u,v) = \begin{cases} 8e^{-4u}, & u > 0, u > v > -u \\ 0, & \text{oth} \end{cases}$

$g(v/u=u) = \begin{cases} \frac{1}{2u}, & -u < v < u \\ 0, & \text{oth} \end{cases}$

$g(v) = \begin{cases} 2e^{-4v}, & v > 0 \\ 2e^{4v}, & v < 0 \end{cases}$

Let $U = X + Y$ and $V = X - Y$. Find,
i) Joint PDF of U and V, ii) Marginals of U and V, iii) conditional PDF of V given $U = u$, for some fixed $u > 0$. $g(u) = \begin{cases} 16u\, e^{-4u}, & u > 0 \\ 0, & \text{oth} \end{cases}$

(8 marks)

2. Suppose that two buses, A and B, operate on a route. A person arrives at a certain bus stop on this route at time 0. Let X and Y be the arrival times of buses A and B, respectively, at this bus stop. Suppose that X and Y are independent random variables and density function of X is given by

$$f(x) = \begin{cases} \frac{1}{5}, & \text{if } x \in [0, 5] \\ 0, & \text{otherwise,} \end{cases}$$

and PDF of Y is given by

$$f(y) = \begin{cases} \frac{1}{6}, & \text{if } y \in [0, 6] \\ 0, & \text{otherwise.} \end{cases}$$

What is the probability that bus A will arrive before bus B? $= \frac{7}{12}$

(3 marks)

3. Let X be a positive RV of the continuous type with PDF $f(\cdot)$. Find the PDF of the RV $U = \frac{X}{(1+X)}$. If, in particular, X has the PDF

$$f(x) = \begin{cases} \frac{1}{5}, & \text{if } 0 \le x \le 5, \\ 0, & \text{otherwise,} \end{cases}$$

what is the PDF of U? $= \begin{cases} \frac{1}{5(1-u)^2}, & 0 < u < \frac{5}{6} \\ 0, & \text{oth} \end{cases}$

(4 marks)

Set D

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Minor Examination

Time: 1 hour **Date: 22/09/2021**
Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

Q.1 The following questions can have multiple correct answers. Write all the correct answers. The marks will be awarded only if you write all correct answers.

(a) Let the random variables X and Y defined on sample space $(\Omega, \mathcal{F})$ have same PMF. Then,

(i) $X(\omega) = Y(\omega)$, $\forall\, \omega \in \Omega$, (ii) CDF of X and Y are same, (iii) Characteristic function of X and Y are same.

(b) Let the random variables X and Y are such that $E(XY) = E(X)E(Y)$. Then,

(i) X and Y are independent , (ii) Cov(X,Y)= 0, iii) Cov$(X-Y, Y)$ =Var(Y).

(c) Let $X : \Omega \to \mathbb{R}$ be a random variable defined on probability space $(\Omega, \mathcal{F}, P)$. Let Q be a probability distribution of X and $\mathcal{B}$ is Borel σ-field. Then,

(i) $Q : \mathbb{R} \to [0,1]$ (ii) $Q : \mathcal{B} \to [0,1]$ (iii) $Q((-\infty, x)) = P\{X \leq x\}$ (iv) Q is continuous.

(d) Let X_1, X_2, X_3, X_4 be pairwise independent random variables and g_1, g_2 are Borel measurable functions. Then,
(i) X_1, X_2, X_3 are independent (ii) $g(X_1)$ and $g(X_2)$ are independent (iii) $g(X_1), g(X_2), g(X_3)$ are independent (iv) X_1 and X_2 are independent.

(e) Let the CDF of random variable X is given by

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{x}{2} \text{ if } 0 \leq x < 1 \\ \frac{2}{3} \text{ if } 1 \leq x < 2 \\ \frac{11}{12} \text{ if } 2 \leq x < 3 \\ 1 \text{ if } 3 \leq x. \end{cases}$$

Then, (i) X is a discrete random variable, (ii) $P\{X = 1\} = \frac{1}{6}$, (iii) $P\{X = 1\}$ cannot be found from given information, (iv) $P\{X \leq 1\} = \frac{1}{2}$.

(1+1+1+1+1 marks)

Q.2 Give the final answer to the following questions. The justification of the answers is not required. The step marking is not applicable in these questions.

(a) Let X be a continuous random variable with PDF $\alpha = 0,\ \beta = 3$

$$f(x) = \begin{cases} \alpha + \beta x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = \frac{3}{4}$, find the values of α and β. (2 marks)

(b) Let X be a discrete random variable with MGF $M_X(t) = \alpha + \beta e^{4t}$, $E(X) = 3$. Find i) α, β, ii) PMF of X. (1+1 marks)

X	0	4
$P_X(x)$	$\frac{1}{4}$	$\frac{3}{4}$

(c) A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on the bus of randomly selected driver. Compute $E[X]$ and $E[Y]$. $= 37$

$= 39.28$ (2 marks)

(d) A fair coin is tossed three times. Let $X =$ number of heads in three tossings, and $Y =$ difference, in absolute value, between number of heads and number of tails. What is conditional PMF of X given $Y = 1$. $p_{X/Y=1}(x) = \begin{cases} 0.5, & x = 1, 2 \\ 0, & \text{oth} \end{cases}$

(2 marks)

(e) Consider an experiment having three possible outcomes, 1, 2, 3, that occur with probabilities 1/3, 1/3, and 1/3, respectively. Suppose two independent repetitions of the experiment are made and let X_i, $i = 1, 2, 3$, denote the number of times the outcome i occurs. What is the PMF of $X_1 + X_2$?

x_1+x_2	0	1	2
$p_{x_1+x_2}(x)$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$

(2 marks)

Q.3 The following questions are descriptive type. Please provide detailed answers.

1. Let X, Y be iid RVs with common PDF

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

$g(u,v) = \begin{cases} \frac{1}{2} e^{-u}, & u > 0, u > v > -u \\ 0, & \text{oth} \end{cases}$

$g(v/u=u) = \begin{cases} \frac{1}{2u}, & -u < v < u \\ 0, & \text{oth} \end{cases}$

$g(v) = \begin{cases} \frac{1}{2} e^{-v}, & v > 0 \\ \frac{1}{2} e^{v}, & v < 0 \end{cases}$

Let $U = X + Y$ and $V = X - Y$. Find,
i) Joint PDF of U and V, ii) Marginals of U and V, iii) conditional PDF of V given $U = u$, for some fixed $u > 0$.

(8 marks)

2. Suppose that two buses, A and B, operate on a route. A person arrives at a certain bus stop on this route at time 0. Let X and Y be the arrival times of buses A and B, respectively, at this bus stop. Suppose that X and Y are independent random variables and density function of X is given by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, 2] \\ 0, & \text{otherwise,} \end{cases}$$

and PDF of Y is given by

$$f(y) = \begin{cases} \frac{1}{3}, & \text{if } y \in [0, 3] \\ 0, & \text{otherwise.} \end{cases}$$

What is the probability that bus A will arrive before bus B? $= \frac{2}{3}$

(3 marks)

3. Let X be a positive RV of the continuous type with PDF $f(\cdot)$. Find the PDF of the RV $U = \frac{X}{(1+X)}$. If, in particular, X has the PDF

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

what is the PDF of U? $= \begin{cases} \frac{1}{2(1-u)^2}, & 0 < u < \frac{2}{3} \\ 0, & \text{oth} \end{cases}$ (4 marks)

A

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Quiz 1

Time: 20 minutes

Max. Marks: 10

Date: 11/09/21

Note: The exam is closed-book, and all the questions are compulsory.

1. Let the PMF of a random variable X is given by

$$P\{X = k\} = e^{-20}\frac{20^k}{k!}, \quad k = 0, 1, 2, \cdots, \infty.$$

Show that (a) $P\{X \leq 10\} \leq \frac{1}{5}$; (b) $P\{X \geq 40\} \leq \frac{1}{20}$.

(3 + 3 marks)

2. Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{(u, v) \in \mathbb{R}^2 | 0 \leq u \leq 1, 0 \leq v \leq 1\}$, and $\mathcal{F}$ is a Borel σ-field on Ω, and $P(A) = \frac{\text{area of } A}{\text{area of } \Omega}$ for every $A \in \mathcal{F}$. Define a random variable $X : \Omega \to \mathbb{R}$ such that $X(u, v) = \frac{u+v}{4}$ for all $(u, v) \in \Omega$. Find the probability density function of X.

(4 marks)

(1)

(a) $E(X) = 20$; $Var(X) = 20$

$P\{|X-20| \geq 20\} \leq \frac{1}{5}$

$P\{X \leq 10\} \leq \frac{1}{5}$

(b) $P\{|X-20| \geq 20\} \leq \frac{1}{20}$

$P\{X \geq 40\} \leq \frac{1}{20}$

(2)

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ 8x^2, & 0 \leq x < \frac{1}{4} \\ 1 - \frac{1}{2}(2-4x)^2, & \frac{1}{4} \leq x < \frac{1}{2} \\ 1, & x \geq \frac{1}{2} \end{cases}$$

$$f(x) = \begin{cases} 16x, & 0 < x < \frac{1}{4} \\ 4(2-4x), & \frac{1}{4} \leq x < \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

B

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Quiz 1

Time: 20 minutes
Max. Marks: 10

Date: 11/09/21

Note: The exam is closed-book, and all the questions are compulsory.

1. Let the PMF of a random variable X is given by

$$P\{X = k\} = e^{-16}\frac{16^k}{k!}, \quad k = 0, 1, 2, \cdots, \infty.$$

Show that (a) $P\{X \le 8\} \le \frac{1}{4}$; (b) $P\{X \ge 32\} \le \frac{1}{16}$.

(3 + 3 marks)

2. Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{(u, v) \in \mathbb{R}^2 | 0 \le u \le 1, 0 \le v \le 1\}$, and $\mathcal{F}$ is a Borel σ-field on Ω, and $P(A) = \frac{\text{area of } A}{\text{area of } \Omega}$ for every $A \in \mathcal{F}$. Define a random variable $X : \Omega \to \mathbb{R}$ such that $X(u, v) = \frac{u+v}{3}$ for all $(u, v) \in \Omega$. Find the probability density function of X.

(4 marks)

(1) a) $E(x) = 16$; $Var(x) = 16$

$P\{|x - 16| \geq 8\} \leq \frac{1}{4}$

$P\{x \leq 8\} \leq \frac{1}{4}$

(b)

$P\{|x - 16| \geq 16\} \leq \frac{1}{16}$

$P\{x \geq 32\} \leq \frac{1}{16}$.

(2)

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{9}{2}x^2, & 0 \leq x < \frac{1}{3} \\ 1 - \frac{1}{2}(2-3x)^2, & \frac{1}{3} \leq x < \frac{2}{3} \\ 1, & x \geq 2/3 \end{cases}$$

$$f(x) = \begin{cases} 9x, & 0 < x < \frac{1}{3} \\ 3(2-3x), & \frac{1}{3} < x < \frac{2}{3} \\ 0 & - \text{otherwise} \end{cases}$$

C

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Quiz 1

Time: 20 minutes
Max. Marks: 10

Date: 11/09/21

Note: The exam is closed-book, and all the questions are compulsory.

1. Let the PMF of a random variable X is given by

$$P\{X = k\} = e^{-12}\frac{12^k}{k!}, \quad k = 0, 1, 2, \cdots, \infty.$$

Show that (a) $P\{X \le 6\} \le \frac{1}{3}$; (b) $P\{X \ge 24\} \le \frac{1}{12}$.

(3 + 3 marks)

2. Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{(u, v) \in \mathbb{R}^2 | 0 \le u \le 1, 0 \le v \le 1\}$, and $\mathcal{F}$ is a Borel σ-field on Ω, and $P(A) = \frac{\text{area of } A}{\text{area of } \Omega}$ for every $A \in \mathcal{F}$. Define a random variable $X : \Omega \to \mathbb{R}$ such that $X(u, v) = \frac{u+v}{2}$ for all $(u, v) \in \Omega$. Find the probability density function of X.

(4 marks)

(1) (a) $E(x) = 12$; $Var(x) = 12$

$P\{|x - 12| \ge 6\} \le \frac{1}{3}$

$P\{x \le 6\} \le \frac{1}{3}$

(b)

$P\{|x - 12| \ge 12\} \le \frac{1}{12}$

$P\{x \ge 24\} \le \frac{1}{12}$

(2) $F_x(x) = \begin{cases} 0 & , x < 0 \\ 2x^2 & , 0 \le x < \frac{1}{2} \\ 4x - 2x^2 - 1 & , \frac{1}{2} \le x < 1 \\ 1 & , x \ge 1 \end{cases}$

$f(x) = \begin{cases} 4x & , 0 < x < \frac{1}{2} \\ 4 - 4x & , \frac{1}{2} < x < 1 \\ 0 & , \text{otherwise}. \end{cases}$

D

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Quiz 1

Time: 20 minutes
Max. Marks: 10

Date: 11/09/21

Note: The exam is closed-book, and all the questions are compulsory.

1. Let the PMF of a random variable X is given by

$$P\{X=k\} = e^{-8}\frac{8^k}{k!}, \quad k = 0,1,2,\cdots,\infty.$$

Show that (a) $P\{X \le 4\} \le \frac{1}{2}$; (b) $P\{X \ge 16\} \le \frac{1}{8}$.

(3 + 3 marks)

2. Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{(u,v) \in \mathbb{R}^2 | 0 \le u \le 1, 0 \le v \le 1\}$, and $\mathcal{F}$ is a Borel σ-field on Ω, and $P(A) = \frac{\text{area of } A}{\text{area of } \Omega}$ for every $A \in \mathcal{F}$. Define a random variable $X : \Omega \to \mathbb{R}$ such that $X(u,v) = \frac{u+v}{5}$ for all $(u,v) \in \Omega$. Find the probability density function of X.

(4 marks)

(1)

(a) $E(X) = 8$, $Var(X) = 8$

$P\{|X-8| \geq 4\} \leq \frac{1}{2}$

$P\{X \leq 4\} \leq \frac{1}{2}$

(b) $P\{|X-8| \geq 8\} \leq \frac{1}{8}$

$P\{X \geq 16\} \leq \frac{1}{8}$

(2)

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{25}{2}x^2, & 0 \le x < \frac{1}{5} \\ 1 - \frac{1}{2}(2-5x)^2, & \frac{1}{5} \le x < \frac{2}{5} \\ 1, & x \geq 2/5 \end{cases}$$

$$f(x) = \begin{cases} 25x, & 0 < x < \frac{1}{5} \\ 5(2-5x), & \frac{1}{5} \le x < \frac{2}{5} \\ 0, & \text{otherwise} \end{cases}$$

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour   **Minor 1 Examination**   Max. Marks: 25

Name: Entry Number: Signature:

Multiple Selection Questions: **Section 1**   ($1 \times 5 = 5$ **marks**)

Each of the following questions 1 and 2 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **1 mark** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

1. Let $\Omega = \{a, b, c, d\}$. Which of the following sets are NOT σ-fields on Ω.
   (A) $\{\emptyset, \{a\}, \{b\}, \{c, d\}, \Omega\}$ (B) $\{\emptyset, \{a\}, \{b, c, d\}, \Omega\}$
   (C) $\{\emptyset, \{a, b, d\}, \{c, d\}, \Omega\}$ (D) $\{\emptyset, \{b\}, \{a, c, d\}, \Omega\}$ Answer: A, C

2. Let X be uniform distributed random variable on the interval $(0, 1)$. Suppose $Y = g(X)$ is uniform distributed random variable on the interval (a, b) with $-\infty < a < b < \infty$. Then, $g(X)$ is (A) $aX + (b - a)$ (B) $a + (b - a)X$ (C) $bX + (a - b)$ (D) $b + (a - b)X$
   Answer: B, D

3. The probability generating function for $B(n, p)$ is
   (A) $(qt+p)^n$ (B) $(pt+1-p)^n$ (C) $(pt+p)^n$ (D) $(qt+1-p)^n$ Answer: B

4. Which of the following distributions NOT satisfy the memoryless property?
   (A) Poisson   (B) Bernoulli   (C) Geometric   (D) Exponential   Answer: A, B

5. If X is a random variable on a measurable space $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the largest σ-field on Ω, then which of the following real-valued functions will be random variables on this measurable space $(\Omega, \mathcal{F})$. (A) X^- (B) $|X|$ (C) X^2 (D) X^+ Answer: A, B, C, D

——————————Rough Work ——————————

Numeric Type Questions: **Section 2** **($5 \times 2 = 10$ marks)**

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 6 to 10. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or an incorrect answer.

6. Let $\Omega = \{s_1, s_2, s_3, s_4\}$ and $P\{s_1\} = \frac{1}{5}$, $P\{s_2\} = \frac{1}{10}$, $P\{s_3\} = \frac{2}{5}$, $P\{s_4\} = \frac{3}{10}$. Define, $A_n = \begin{cases} \{s_1, s_3\} & \text{if } n \text{ is odd} \\ \{s_2, s_4\} & \text{if } n \text{ is even} \end{cases}$. Find $P(\limsup A_n)$. Answer(F/D): 1

7. Suppose that the number of passengers for a limousine pickup is thought to be either 1, 2, 3, or 4, each with equal probability, and the number of pieces of luggage of each passenger is thought to be 1 or 2, with equal probability, independently for different passengers. What is the probability that there will be five or more pieces of luggage? Answer(F/D): 23/64

8. Suppose X has a geometric distribution with parameter p. Then, the moment generating function for X is given by: Answer(E): $pe^t/(1-qe^t)$

9. Let $X \sim \mathrm{P}(\lambda)$ such that $P(X = 0) = e^{-3}$. Find $Var(X)$. Answer(E): 3

10. Let X be a normal distributed random variable with mean 0 and variance 9. Then, the pdf of $h(X)$ is given by:

$\mu = 0$ $\sigma^2 = 9$

Answer(E):

$y = e^x$

$$f_Y(y) = \begin{cases} \frac{1}{y\sigma\sqrt{2\pi}} e^{-\left(\frac{\log y - \mu}{\sigma\sqrt{2}}\right)^2} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Rough Work

Subjective Type Questions: **Section 3** ($2 \times 5 = 10$ **marks**)

Write the answer in the same page provided for the questions 11 and 12. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps.

11. (a) Write the Kolmogorov axiomatic definition of probability. (2 marks)

(b) Let $\Omega = \Re$ and $\mathcal{F}$ be the Borel σ-field on $\Re$. For each interval $I \subseteq \Re$ with end points c and d ($c \leq d$), let

$$P(I) = \int_c^d \frac{1}{\pi} \frac{1}{1+x^2} dx \ .$$

Does P define a probability on the measurable space $(\Omega, \mathcal{F})$? Justify your answer. (3 marks)

Answer: (a) Let E be a random experiment, Ω be the collection of all possible outcomes of E and F be the σ-field on Ω.

The function P defined on F such that

(i) $P(A) \geq 0 \ \forall A \in F$] - 1

(ii) $P(\Omega) = 1$

(iii) If A_i's are mutually exclusive events in F then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ - 1

Then P is called a probability function.

b) (i) For any interval I

$P(I) = \int_c^d \frac{1}{\pi} \frac{1}{1+x^2} dx = \frac{1}{\pi}(\tan^{-1} d - \tan^{-1} c)$

since $\tan^{-1}(\ )$ is monotonically increasing and $d \geq c$ — 1

$\therefore P(I) \geq 0$

(ii) $P(\Omega) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+x^2} {}^3 dx = \frac{1}{\pi}(\tan^{-1}\infty - \tan^{-1}-\infty) = 1$ — 1

(iii) for disjoint intervals I_1, I_2 - - -

$P(\bigcup_{i=1}^{\infty} I_i) = \int_{\bigcup I_i} \frac{1}{\pi} \frac{1}{1+x^2} dx = \int_{I_1} \frac{1}{\pi (1+x^2)} dx + \text{- - - -} + \text{--} = \sum_{i=1}^{\infty} P(I_i)$ — 1

12. (a) Let X be a continuous type random variable with pdf

$$f(x) = \begin{cases} \beta + \alpha x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = 3/5$, find the value of α and β. (2 marks)

(b) Prove that, for any random variable X, $E(X^2) \geq [E(X)]^2$. When does one have equality? (2 + 1 marks)

Answer: a) $E(X) = 3/5 \Rightarrow \int_0^1 x(\beta + \alpha x^2)\,dx = 3/5$

$\Rightarrow \beta \frac{x^2}{2} + \alpha \frac{x^4}{4} \Big|_0^1 = 3/5$

Also, $\int_0^1 f(x)dx = 1 \Rightarrow \int_0^1 \beta + \alpha x^2 \; dx = 1$

$\Rightarrow \beta x + \frac{\alpha x^3}{3} \Big|_0^1 = 1 \Rightarrow \frac{\beta}{1} + \frac{\alpha}{3} = 1$ - 1

$\Rightarrow \beta = 1 - \frac{\alpha}{3}$ and $\frac{\beta}{2} + \frac{\alpha}{4} = 3/5 \Rightarrow \alpha = 6/5$ - 1
and $\beta = 3/5$

b) 1.) $\text{var}\, X = E[(X-\mu)^2]$
Since $(X-\mu)^2 \geq 0$ for all possible values of X
$\Rightarrow \text{var}\, X \geq 0$ - 1

2.) $\text{var}\, X = E(X^2) - (E(X))^2 \geq 0$
$\Rightarrow E(X^2) \geq (E(X))^2$ - 1

3) For equality $E(X^2) = (E(X))^2$

$\Rightarrow X$ is a degenerate random variable ie.
$P(X = k) = 1$ for some constant $k \in \mathbb{R}$. 1

ϕ, Ω

Department of Mathematics

MAL 250 (Introduction to Probability Theory and Stochastic Processes)

Minor 1 Test (I Semester 2013 - 2014)

Time allowed: 1 hour — Max. Marks: 25

1. (a) Let $\Omega = \{a, b, c, d\}$. Find three different σ-fields $\{F_n\}$ for $n = 0, 1, 2$ such that $F_0 \subset F_1 \subset F_2$. (3 marks)

   (b) The first generation of particles is the collection of off-springs of a given particle. The next generation is formed by the off-springs of these members. If the probability that a particle has k off springs (splits into k parts) is p_k, where $p_o = 0.4$, $p_1 = 0.3$, $p_2 = 0.3$. Assume particles act independently and identically irrespective of the generation. Find the probability that there is only one particle in second generation. (2 marks)

2. Consider the random variable X that represents the number of people who are hospitalized or die in a single head-on collision on the road in front of IIT Delhi main gate in a year. The distribution of such random variables are typically obtained from historical data. Without getting into the statistical aspects involved, let us suppose that the cumulative distribution function of X is as follows:

x	0	1	2	3	4	5	6	7	8	9	10
F(x)	0.250	0.546	0.898	0.932	0.955	0.972	0.981	0.989	0.995	0.998	1.000

Find (a) $P(X = 10)$ 0.002 (b) $P(X \le 5 / X > 2)$ 0.726 (c) $E(X)$. 1.519 (1 + 2 + 2 marks)

3. Assume that, taxis are waiting in a queue for passengers to come. Passengers for these taxis arrive according to a Poisson process with an average of 60 passengers per hour. A taxi departs as soon as two passengers have been collected or 3 minutes have expired since the first passenger has got in the taxi. Suppose you get in the taxi as first passenger. What is the distribution of your waiting time for the departure? Also, find its variance. (2 + 2 marks)

4. State True or False with valid reasons for the following statements. Without valid reasons, marks will NOT be given.

   (a) The probabilities that a student in MAL 250 will expends a high, medium or low amount of effort in studying are 0.50, 0.30 and 0.20, respectively. Given that the student expends a high, medium or low amount of effort, the respective conditional probabilities of getting grade A in this courses are 0.90, 0.40 and 0.05. Then the probability that the student will get grade A in the course is 0.58.

   (b) Consider a parallel system with identical components each with reliability 0.8. If the reliability of the system is to be at least 0.99, then the minimum number of components in this system is 3.

   (c) Define the $(100p)$th percentile of a random variable X is the smallest value of x such that $F(x) = P(X \le x) \ge p$. Then, 50th percentile is called the mode of X.

   (d) Let X be a continuous random variable with pdf $f(x) = \dfrac{1}{\pi(1 + x^2)}$, $-\infty < x < \infty$. Then, Var$(X) = \pi$.

   (1 + 1 + 1 + 1 marks)

5. Pick the **odd one out** with valid reasons for the following statements. Without **valid reasons**, marks will NOT be given. +1

   Discrete

   (a) (1) Bernoulli distribution (2) Binomial distribution (3) Poisson distribution (4) uniform distribution

   (b) (1) Gamma distribution (2) Exponential distribution (3) Poisson distribution (4) Erlang distribution

   (1 + 1 marks)

6. A point X is chosen at random in the interval $[-2, 1]$. Find the pdf of $Y = X^2$. (5 marks)

$\int_{-2}^{1}$ $=$ $x+2$

Department of Mathematics
MTL 106/MAL 250 (Introduction to Probability Theory and Stochastic Processes)
Minor 1 Test (II Semester 2014 - 2015)

Time allowed: 1 hour **Max. Marks: 25**

1. (a) Write the axiomatic definition of probability. (2 marks)

   (b) Consider $\Omega = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Let $\mathcal{F}$ be the largest σ-field over Ω. Define

   $$P(R) = \text{area of } R = (b-a)(d-c)$$

   where R is the rectangular region that is a subset of Ω of the form $R = \{(u,v) : a \leq u < b, c \leq v < d\}$. Let T be the triangular region $T = \{(x,y) : x \geq 0, y \geq 0, x+y < 1\}$. Show that T is an event, and find $P(T)$, using the axioms. (1+2 marks)

2. A random walker starts at 0 on the x-axis and at each time unit moves 1 step to the right or 1 step to the left with probability 0.5. Find the probability that, after 4 moves, the walker is more than 2 steps from the starting position. (3 marks)

3. A student arrives to the bus stop at 6:00 AM sharp, knowing that the bus will arrive in any moment, uniformly distributed between 6:00 AM and 6:20 AM.

   (a) What is the probability that the student must wait more than five minutes? (2 marks)

   (b) If at 6:10 AM the bus has not arrived yet, what is the probability that the student has to wait at least five more minutes? (2 marks)

4. State **True** or **False** with valid reasons for the following statements. Without **valid reasons**, marks will NOT be given.

   (a) A box contains a double-headed coin, a double-tailed coin and an unbiased coin. A coin is picked at random and flipped. It shows a head. The conditional probability that it is the double-headed coin is 0.5 .

   (b) Define the $(100p)$th percentile of a random variable X is the smallest value of x such that $P(X \leq x) \geq p$. Then, 50th percentile is called the *median* of X.

   (c) Consider the following game: you flip an unbiased coin, until the first head appears. If the head appears on the nth flip of the coin, you will receive 2^n rupees. The expected gain for playing the game is 0.5.

   (d) The characteristic function $\phi_X(t)$ of a random variable X satisfies the property $\phi_{-X}(t) = \overline{\phi_X(-t)}$ where bar denotes complex conjugation.

   (1 + 1 + 1 + 1 marks)

5. Let X be a random variable with $N(0, \sigma^2)$. Find the moment generating function for the random variable X. Deduce the moments of order n about zero for the random variable X from the above result.

   (2 + 2 marks)

6. (a) Let X be a uniformly distributed random variable on the interval $[a,b]$ where $-\infty < a < b < \infty$. Find the distribution of the random variable $Y = \frac{X-\mu}{\sigma}$ where $\mu = E(X)$ and $\sigma^2 = Var(X)$. Also, find $P(-2 < Y < 2)$. (3 + 1 marks)

   (b) Suppose Shimla's temperature is modeled as a random variable which follows normal distribution with mean 10 Celcius degrees and standard deviation 3 Celcius degrees. Find the mean if the temperature of Shimla were expressed in Fahreneit degrees. (1 mark)

Department of Mathematics
MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Minor 1 (I Semester 2015 - 2016)

Time allowed: 1 hour **Max. Marks: 25**

1. (a) Write axiomatic definition of probability.
   (b) Show that the conditional probability $P(A/B)$ satisfies the three axioms of probability.

(3 + 3 marks)

2. Let X be a random variable such that $P(X = 2) = \frac{1}{4}$ and its distribution function is given by

$$F_X(x) = \begin{cases} 0, & x < -3 \\ \alpha(x+3), & -3 \leq x < 2 \\ \frac{3}{4}, & 2 \leq x < 4 \\ \beta x^2, & 4 \leq x < 8/\sqrt{3} \\ 1, & x \geq 8/\sqrt{3} \end{cases}.$$

   (a) Find α, β if 2 is the only jump discontinuity of F.
   (b) Compute $P(X < 3/X \geq 2)$.

(1 + 1 + 2 marks)

3. Suppose the length of a telephone conversation between two persons is a random variable X with cumulative distribution function

$$P(X \leq t) = \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-0.04t}, & 0 \leq t < \infty \end{cases},$$

where the time is measured in minutes.

   (a) Given that the conversation has been going on for 20 minutes, compute the probability that it continues for at least another 10 minutes.
   (b) Show that, for any $t > 0$, $E(X/X > t) = t + 25$.

(3 + 2 marks)

4. Consider a random variable X with $E(X) = 1$ and $E(X^2) = 1$.
   (a) Find $E[(X - E(X))^4]$ if it exists.
   (b) Find $P(-1/2 < X \leq 3)$ and $P(X = 0)$.

(3 + 1 + 1 marks)

5. Suppose that X is a continuous random variable with pdf $f_X(x) = e^{-x}$ for $x > 0$. Define $Y = \begin{cases} X, & X < 1 \\ \frac{1}{X}, & X \geq 1 \end{cases}$.
   (a) Discuss whether the distribution of Y is discrete or continuous or mixed type.
   (b) Determine the pmf/pdf as applicable to this case.

(1 + 4 marks)

Department of Mathematics
Minor I Examination
MTL 106: Probability and Stochastic Processes

Venue: LH 121 **Date: 29-08-2017** **Time 2:30 – 3:30 PM** **Full Marks 20**

Q1. **(i)** What are the three axioms for defining probability of an event E?
(Note: These are called **Kolmogorov's Axioms of Probability**)

(ii) Prove or Disprove: Conditional probability of Event A given that Event B has occurred, i.e. P (A | B) satisfies the above axioms.

(iii) Suppose X and Y are independent and identically distributed (***iid***) random variables both following **Bin(2, 0.4).** Draw the graph of the Cumulative Distribution Function (***cdf***) of Z = X + Y.

[3 + 3 + 2 = 8]

Q2. Suppose X is a random variable with pdf $k.e^{\frac{-(x-2)^2}{2}}$ $\forall\, x \in (2, \infty)$

(i) Find the value of *k*.

(ii) Calculate the Expected value of X.

(iii) Obtain the MGF of X

[1.5 + 1.5 + 3 = 6]

Q3. **(i)** Let X be a random variable following **Beta₁ (2, 3)** distribution. Note that the pdf of a random variable **X ~ Beta₁ (m, n)** is:

$$f(x) = \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}, \qquad for\ 0 < x < 1$$

Note that $\beta(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Find Mean and Variance of X.

(ii) Suppose X is a Binomial (10, p) random variable, where p is determined by the trial of a variable P ~ ***BETA₁* (2, 3).**

What is the Expected value of X?

[2.5 + 3.5 = 6]

$(x-2)^2 = 2z$
$(x-2)\, dx = x\,dz$
$dx = \frac{dz}{\sqrt{2z}}$

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MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour **Minor 1 Examination** **Max. Marks: 25**

Name: Jitender singh Entry Number: 2016EE10450 Signature: jitender singh.

Multiple Selection Questions: **Section 1** (2 × 2 = 4 **marks)**

Each of the following questions 1 and 2 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **2 marks** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

1. Based on the probability concepts, which of the following statements are NOT TRUE?
   (A) Ω is the collection of few possible outcomes of a random experiment.
   (B) $(\Omega, \mathcal{F}, P)$ is a probability space. (C) $\mathcal{F}$ is a σ-field on the subset of Ω. 2
   (D) P is a measure. Answer: A (2 marks)

2. Which of the following distributions NOT satisfy the memoryless property? 0
   (A) Exponential (B) Poisson (C) Bernoulli (D) Geometric Answer: ABC (2 marks)

Numeric Type Questions: **Section 2** (6 × 2 = 12 **marks)**

Write the answer upto 4 decimal places or in fraction or the expression for the following questions 3 to 8. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or an incorrect answer.

3. Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{a, d\}, \Omega\}$ and P a function from $\mathcal{F}$ to $[0, 1]$ with $P(\{a\}) = \frac{2}{5}$, $P(\{b, c\}) = \frac{2}{7}$ and $P(\{d\}) = \beta$. The value of β such that P to be a probability on $(\Omega, \mathcal{F})$. Answer: $\frac{11}{35}$ 2 (2 marks)

4. Consider a gambler who on each independent bet either wins 1 with probability $\frac{1}{4}$ or losses 1 with probability $\frac{3}{4}$. The gambler will quit either when he or she is winning a total of 10 or after 50 plays. The probability the gambler plays exactly 13 times?
   Answer: 0 0 (2 marks)

5. The first generation of particles is the collection of offsprings of a given particle. The next generation is formed by the offsprings of these members. If the probability that a particle has k offsprings is p_k, where $p_0 = \frac{1}{2}, p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{6}$. Assume that particles act independently and identically irrespective of the generation. Find the probability that there is no particle in third generation. Answer: 1 0 (2 marks)

6. Let $\Omega = \{1, 2, 3\}$. Let $\mathcal{F}$ be a σ-algebra on Ω, so that $X(w) = w - 1$ is a random variable. Then, $\mathcal{F}$ is given by
   Answer: $\mathcal{F} = \{\phi, \{1\}, \{2,3\}, \{1,2\}, \{3\}, \Omega\}$ 0 (2 marks)

7. Consider a random variable X with $E(X) = 1$ and $E(X^2) = 1$. What is the value of $P(0.1 < X < 1.3)$? Answer: 1 2 (2 marks)

8. The moment generating function (MGF) of a random variable X is given by $M_X(t) = \frac{1}{3} + \frac{1}{6}e^t + \frac{1}{4}e^{2t} + \frac{1}{4}e^{3t}$. If μ is the mean and σ^2 is the variance of X, what is the value of $P(\mu < X < \mu + \sigma)$? Answer: $\frac{1}{4}$ (2 marks)

2

Comprehensive Type Questions: **Section 3** **($3 \times 3 = 9$ marks)**

Each of the following questions 9 to 11 has some subparts. Each subparts has four options out of which one is the correct answer. -0.5 **mark** for incorrect answers of 1 mark and -1 **mark** for incorrect answers of 2 marks question. For no answer, **0 mark**.

9. A club basketball team will play a 50-game season. Twenty four of these games are aginst class A teams and 26 are against class B teams. The outcomes of all the games are independent. The team will win each game against a class A opponent with probability 0.4 and it will win each game against a class B opponent with probability 0.6. Let X_A and X_B denote, respectively, the number of victories against class A and class B teams. Let X denote its total victories in the season.

   (a) What is the distributions of X_A?
   (A) Poisson (B) Bernoulli (C) Geometric (D) Binormail Answer: D (1 mark) 1

   (b) What is the relationship between X_A, X_B and X?
   (A) $X = -X_A - X_B$ (B) $X = X_B - X_A$ (C) $X = X_A - X_B$ (D) $X = X_A + X_B$
   Answer: D (1 mark) 1

   (c) What is the distribution of X?
   (A) Geometric (B) Poisson (C) Binormail (D) Bernoulli Answer: (1 mark)

10. Let X be a continuous type random variable with strictly increasing CDF F_X.

   (a) Which is the one of following distributions of X?
   (A) Standard normal (B) Uniform $(0, 1)$ (C) Gamma (D) Exponential 1
   Answer: A (1 mark)

   (b) What is the type of distribution have the random variable $Y = -\ln(F_X(X))$?.
   (A) Standard normal (B) Gamma (C) Exponential (D) Uniform $(0, 1)$ 2
   Answer: C (2 marks)

11. Let X be a continuous type random variable having the pdf $f(x) = \begin{cases} 0, & -\infty < x \le 0 \\ \frac{1}{2}, & 0 < x \le 1 \\ \frac{1}{kx^2}, & 1 < x < \infty \end{cases}$ 1

   (a) Find k? (A) 4 (B) 3 (C) 5 (D) 2 Answer: D (1 mark)

   (b) Find the pdf of $Y = \frac{1}{X}$?

   (A) $f(y) = \begin{cases} 0, & -\infty < y \le 0 \\ \frac{1}{2}, & 0 < y \le 1 \\ \frac{1}{3y^2}, & 1 < y < \infty \end{cases}$ (B) $f(y) = \begin{cases} 0, & -\infty < y \le 0 \\ \frac{1}{2}, & 0 < y \le 1 \\ \frac{1}{4y^2}, & 1 < y < \infty \end{cases}$ 2

   (C) $f(y) = \begin{cases} 0, & -\infty < y \le 0 \\ \frac{1}{2}, & 0 < y \le 1 \\ \frac{1}{2y^2}, & 1 < y < \infty \end{cases}$ (D) $f(y) = \begin{cases} 0, & -\infty < y \le 0 \\ \frac{1}{4}, & 0 < y \le 1 \\ \frac{3}{4}, & 1 < y < 2 \end{cases}$

   Answer: C (2 marks)

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Minor Examination

Time: 1 hour **Date: 22/09/2021**

Max. Marks: 30

Note: The exam is closed-book, and all the questions are compulsory.

Q.1 The following questions can have multiple correct answers. Write all the correct answers. The marks will be awarded only if you write all correct answers.

(a) Let the random variables X and Y defined on sample space $(\Omega, \mathcal{F})$ have same PMF. Then,

(i) CDF of X and Y are same, (ii) Characteristic function of X and Y are same, (iii) $X(\omega) = Y(\omega), \ \forall\, \omega \in \Omega$

(b) Let the random variables X and Y are such that $E(XY) = E(X)E(Y)$. Then,

(i) Cov(X,Y)= 0, (ii) X and Y are independent , iii) Cov$(X-Y,Y)$ =Var(Y).

(c) Let $X : \Omega \to \mathbb{R}$ be a random variable defined on probability space $(\Omega, \mathcal{F}, P)$. Let Q be a probability distribution of X and $\mathcal{B}$ is Borel σ-field. Then,

(i) $Q : \mathcal{B} \to [0,1]$, (ii) $Q : \mathbb{R} \to [0,1]$, (iii) $Q((-\infty, x)) = P\{X \le x\}$, (iv) Q is continuous.

(d) Let X_1, X_2, X_3, X_4 be pairwise independent random variables and g_1, g_2 are Borel measurable functions. Then,
(i) $g(X_1)$ and $g(X_2)$ are independent, (ii) X_1, X_2, X_3 are independent (iii) X_1 and X_2 are independent, (iv) $g(X_1)$, $g(X_2)$, $g(X_3)$ are independent.

(e) Let the CDF of random variable X is given by

$$F(x) = \begin{cases} 0 \ \text{ if } \ x < 0 \\ \frac{x}{2} \ \text{ if } \ 0 \le x < 1 \\ \frac{2}{3} \ \text{ if } \ 1 \le x < 2 \\ \frac{11}{12} \ \text{ if } \ 2 \le x < 3 \\ 1 \ \text{ if } \ 3 \le x. \end{cases}$$

Then, (i) $P\{X = 1\} = \frac{1}{6}$, (ii) $P\{X = 1\}$ cannot be found from given information, (iii) $P\{X \le 1\} = \frac{1}{2}$, (iv) X is a discrete random variable.

(1+1+1+1+1 marks)

Q.2 Give the final answer to the following questions. The justification of the answers is not required. The step marking is not applicable in these questions.

(a) Let X be a continuous random variable with PDF

$$f(x) = \begin{cases} \alpha + \beta x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = \frac{3}{5}$, find the values of α and β.

(2 marks)

(b) Let X be a discrete random variable with MGF $M_X(t) = \alpha + \beta e^{5t}$, $E(X) = 3$. Find i) α, β, ii) PMF of X.

(1+1 marks)

(c) A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 43, 30, 20, and 55 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on the bus of randomly selected driver. Compute $E[X]$ and $E[Y]$.

(2 marks)

(d) A fair coin is tossed three times. Let X = number of heads in three tossings, and Y = difference, in absolute value, between number of heads and number of tails. What is conditional PMF of X given $Y = 3$.

(2 marks)

(e) Consider an experiment having three possible outcomes, 1, 2, 3, that occur with probabilities 1/2, 1/4, and 1/4, respectively. Suppose two independent repetitions of the experiment are made and let X_i, $i = 1, 2, 3$, denote the number of times the outcome i occurs. What is the PMF of $X_1 + X_2$?

(2 marks)

Q.3 The following questions are descriptive type. Please provide detailed answers.

1. Let X, Y be iid RVs with common PDF

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let $U = X + Y$ and $V = X - Y$. Find,
i) Joint PDF of U and V, ii) Marginals of U and V, iii) conditional PDF of V given $U = u$, for some fixed $u > 0$.

(8 marks)

2. Suppose that two buses, A and B, operate on a route. A person arrives at a certain bus stop on this route at time 0. Let X and Y be the arrival times of buses A and B, respectively, at this bus stop. Suppose that X and Y are independent random variables and density function of X is given by

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } x \in [0, 3] \\ 0, & \text{otherwise,} \end{cases}$$

and PDF of Y is given by

$$f(y) = \begin{cases} \frac{1}{4}, & \text{if } y \in [0, 4] \\ 0, & \text{otherwise.} \end{cases}$$

What is the probability that bus A will arrive before bus B?

(3 marks)

3. Let X be a positive RV of the continuous type with PDF $f(\cdot)$. Find the PDF of the RV $U = \frac{X}{(1+X)}$. If, in particular, X has the PDF

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } 0 \leq x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

what is the PDF of U?

(4 marks)

Department of Mathematics, IIT Delhi

MTL106: Re-minor Exam.

Time: 2 hour **Date:** 18-04-2021 **Total Marks:** 40

Q.1) a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A, B \in \mathcal{F}$ such that $\mathbb{P}(A)\mathbb{P}(B) \neq 0$. Show that if $\mathbb{P}(A \big| B) > \mathbb{P}(A)$, then $\mathbb{P}(B \big| A) > \mathbb{P}(B)$.

b) In a certain region, one in every thousand people is infected by HIV virus that causes AIDS. Tests for presence of virus are fairly accurate but not perfect. It has been observed that, if someone actually has HIV, the probability of testing positive is 0.94. Find the probability that someone chosen at random from the population has HIV and tests positive.

c) Let X be Binomial $B(p, n)$. For what value of j is $\mathbb{P}(X = j)$ the greatest?

2+2+3 marks

Q.2) a) The life of a certain type of bulb is normally distributed with mean 34 days and variance 16 days.

i) What is the probability that such a bulb lasts over 40 days?

ii) What is the probability that it lasts between 30 and 35 days?

iii) Given that it has survived 30 days, what is the conditional probability that the bulb survives another 10 days?

You may use the following: for any $Z \sim \mathcal{N}(0, 1)$, $\mathbb{P}(Z \leq 1.5) = 0.9332$, $\mathbb{P}(Z < 0.25) = 0.6$, $\mathbb{P}(Z \leq 1) = 0.84$, $\mathbb{P}(Z \leq 0.26) = 0.6026$ and $\mathbb{P}(Z \leq -0.4) = 0.3446$.

b) Let X be a continuous random variable with distribution function F. Show that $Y = F(X)$ is uniformly distributed random variable on $[0, 1]$.

(2+2+2)+2 marks

Q.3) a) For any $X \sim \mathcal{N}(0, 1)$, show that

$$\mathbb{P}(X > x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x > 0.$$

b) Let X be a non-negative continuous random variable having a non-increasing probability density function $f_X(\cdot)$. Show that

$$f_X(x) \leq \frac{2\mathbb{E}[X]}{x^2}, \quad x > 0.$$

c) Let X be a random variable with pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and characteristic function $\phi_X(\cdot)$. Show that

$$\phi_X(\lambda) = \frac{1}{2}(1 + i), \quad i^2 = -1.$$

2+3+ 3 marks

Q.4) a) Let X and Y be two dependent random variables with common means 0, variances 1 and correlation coefficient ρ. Show that

$$\mathbb{E}\big[\max\{X^2, Y^2\}\big] \leq 1 + \sqrt{1-\rho^2}.$$

b) For any $X \sim \mathcal{N}(2, 16)$, prove that

$$\mathbb{E}[(X-2)g(X)] = 16\mathbb{E}[g'(X)]$$

when both sides exist for some function g.

c) Show that the random variables $U := X + Y$ and $V := X - Y$ are uncorrected, where X and Y are **i.i.d.** random variables. Explain whether U and V always be independent or not.

3+2+(2+2) marks

Q.5) a) Is the function

$$F(x,y) := \begin{cases} 0, & x < 0, \text{ or } y < 0, \text{ or } x + y < 1 \\ 1, & \text{otherwise} \end{cases}$$

the joint distribution function of some pair of random variables? Explain it.

b) Let X and Y be independent exponential random variables with parameter λ. Define

$$U = X + Y, \quad V = \frac{X}{Y}.$$

i) Find the joint density function of U and V.

ii) Examine whether U and V are independent or not.

2 + (5+1) marks

Best of Luck!!!

Department of Mathematics

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Quiz 2

Time: 20 minutes **Date: 08/03/2022**
Max. Marks: 10

Note: The exam is closed-book, and all the questions are compulsory.

1. (i) The lifetime of a color television picture tube is a normal random variable with mean 8.2 years and standard deviation 1.4 years. What percentage of such tubes lasts
   (a) more than 10 years;
   (b) less than 5 years;
   (c) between 5 and 10 years?

   (1+1+1 marks)

   (ii) Let X have the normal density with mean μ and variance is 0.25. Find a constant c such that

   $$P(|X - \mu| \leq c) = 0.9$$

   .

   (2 marks)

2. A person stands on the street and sells newspapers. Assume that each of the people passing by buys a newspaper independently with probability $\frac{1}{4}$. Let X denote the number of people passing past the seller during the time until he sells his first 150 copies of the newspaper. Using CLT, find $P(X \leq 400)$ approximately.

   (5 marks)

TABLE A1 *Standard Normal Distribution Function:* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Department of Mathematics

MTL 106 (Probability and Stochastic Processes)

Quiz 1

Time: 1 hour 15 minutes **Date: 01/11/20**
Max. Marks: 13

Note: The exam is closed-book, and all the questions are compulsory.

1. Consider the following game of chance. You pay 2 dollars and roll a fair die. Then you receive a payment according to the following schedule. If the event $A = \{1, 2, 3\}$ occurs, then you will receive 1 dollar. If the event $B = \{4, 5\}$ occurs, you receive 2 dollars. If the event $C = \{6\}$ occurs, then you will receive 6 dollars. Let X denotes your profit. Construct the sample space $(\Omega, \mathcal{F})$ and show that X is a random variable. Find PMF and CDF of X. What is the average profit you can make if you participate this game?

(3 marks)

2. The base and altitude of a right angle triangle are obtained by picking points randomly from $[0, a]$ and $[0, b]$, respectively. Show that the probability that the area of the triangle so formed will be less than $\frac{ab}{4}$ is $\frac{(1+\ln 2)}{2}$.

(3 marks)

3. (a) Let X be any random variable and suppose that the MGF $M(t)$ of X exists for every $t > 0$. Then for any $t > 0$,

$$P\{tX > s^2 + \log M(t)\} < e^{-s^2}.$$

(1.5 marks)

(b) For a random variable X with PDF

$$f(x; \lambda) = \begin{cases} \frac{e^{-x} x^{\lambda}}{\lambda!}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda \geq 0$ is an integer, show that

$$P\{0 < X < 2(\lambda + 1)\} \geq \frac{\lambda}{\lambda + 1}.$$

(2.5 marks)

4. Let X be an integer-valued random variable having distribution function F, and let Y be a random variable with PDF

$$h(y) = \begin{cases} 1 & \text{if } y \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Define, the integer-valued random variable Z in terms of Y by

$$Z = m, \quad \text{iff} \quad F(m - 1) < Y \leq F(m),$$

for any integer m. Are probability distributions of Z and X same ? If they are not same, what is the relation between them?

(3 marks)

APAR AHUJA 2019CS10465

A1. $\Omega = \{1, 2, 3, 4, 5, 6\}$

define $\mathcal{F} = \mathcal{P}(\Omega)$ or power set of Ω.

$X : \Omega \to \mathbb{R}$.

Given; $X(1) = -1$
$X(2) = -1$
$X(3) = -1$
$X(4) = 0$
$X(5) = 0$
$X(6) = 4$.

$$X^{-1}((-\infty, x]) = \begin{cases} \phi & x < -1 \\ \{1,2,3\} & -1 \le x < 0 \\ \{1,2,3,4,5\} & 0 \le x < 4 \\ \Omega & 4 \le x \end{cases}$$

As $\phi, \{1,2,3\}, \{1,2,3,4,5\}, \Omega \in \mathcal{F}$. Hence X is a random variable

CDF :
$$F_X(x) = \begin{cases} \mathbb{P}(\phi) = 0 & x < -1 \\ \mathbb{P}\{1,2,3\} = 1/2 & -1 \le x < 0 \\ \mathbb{P}\{1,2,3,4,5\} = 5/6 & 0 \le x < 4 \\ \mathbb{P}(\Omega) = 1 & 4 \le x \end{cases}$$

PMF : $p_X(-1) = \frac{1}{2}$, $p_X(0) = \frac{1}{3}$, $p_X(4) = \frac{1}{6}$.

$\Sigma p_X = p_X(-1) + p_X(0) + p_X(4) = \underline{1}$.

Avg. profit -

$$E[x] = \sum x p_x$$

$$= (-1)\cdot\frac{1}{2} + 0\cdot\frac{1}{3} + 4\cdot\frac{1}{6}$$

$$= -\frac{1}{2} + \frac{2}{3} = \boxed{\frac{1}{6} = E[x]}$$

APAR AHUJA

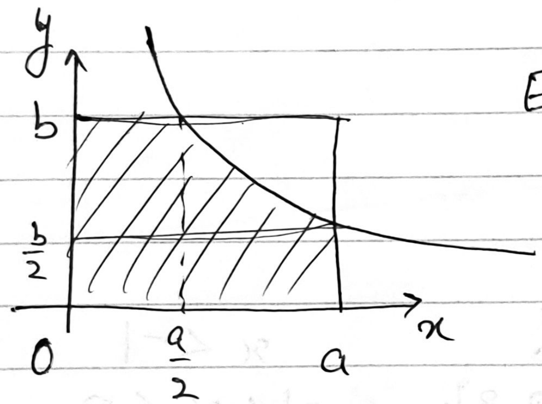
A2. Let Base be x,
Altitude be y.

Define $\Omega = \{ (x,y) \mid 0 \le x \le a, 0 \le y \le b \}$

$\mathcal{F}$ = Borel over $[0,a] \times [0,b]$

E: ar(Δ) $< \frac{ab}{4}$ or $\frac{xy}{2} < \frac{ab}{4}$ | $0 \le x \le a$, $0 \le y \le b$

$\Rightarrow y < \frac{ab}{2x}$.



$E: \left(y < \frac{ab}{2x}\right) \cap (0 \le x \le a) \cap (0 \le y \le b)$

$P(E) = \frac{\text{measure}(E)}{\text{measure}(\Omega)}$

measure$(\Omega) = ab$

$$\text{measure}(E) = b \cdot \frac{a}{2} + \int_{a/2}^{a} \frac{ab}{2x}\, dx = \frac{ab}{2}(1 + \ln 2)$$

$$P(E) = \frac{\text{measure}(E)}{\text{measure}(\Omega)} = \frac{\cancel{ab}/2\,(1+\ln 2)}{\cancel{ab}}$$

$$\boxed{P(E) = \frac{(1+\ln 2)}{2}}$$

Hence proved.

APAR AHUJA

A3. (a)

For $P\{tX > s^2 + \log(E[e^{tX}])\}$;

$$\{tX > s^2 + \log(E[e^{tX}])\} = \{tX > \log(e^{s^2} E[e^{tX}])\}$$
$$= \{e^{tX} > e^{s^2} E[e^{tX}]\}$$

$$P\{e^{tX} > e^{s^2} E[e^{tX}]\} \le \frac{\cancel{E[e^{tX}]}}{e^{s^2}\cancel{E[e^{tX}]}} = e^{-s^2}$$

$\left[\text{using } P(Y \ge a) \le \frac{E[Y]}{a}\right]$

Hence proved.

(b) Given; X | $f(x,\lambda) = \begin{cases} \frac{e^{-x} x^{\lambda}}{\lambda!} & x > 0 \\ 0 & \text{o.w.} \end{cases}$
PDF

$$E[X] = \int_0^{\infty} \frac{x e^{-x} x^{\lambda}}{\lambda!}\, dx = \frac{1}{\lambda!} \int_0^{\infty} e^{-x} x^{\lambda+1}\, dx$$
$$= \frac{(\lambda+1)!}{\lambda!} = \lambda + 1.$$

$$\{0 < X < 2(\lambda+1)\}^c = \{(X \le 0) \cup (X \ge 2(\lambda+1))\}$$
$$= \{\, |X - \lambda - 1| \ge \lambda + 1 \}$$

$$\therefore P\{|X - \lambda - 1| \ge \lambda + 1\} \le \frac{\operatorname{var}(X)}{(\lambda+1)^2}$$

APAR AHUJA.

$$E[x^2] = \int_0^\infty \frac{x^2 e^{-x} x^\lambda}{\lambda!}\,dx = (\lambda+1)(\lambda+2)\frac{\cancel{\lambda!}}{\cancel{\lambda!}}$$

$$\mathrm{var}(x) = (\lambda+1)(\lambda+2) - (\lambda+1)^2 = \lambda+1.$$

$$\therefore P\{|X-\lambda-1| \geq \lambda+1\} \leq \frac{\lambda+1}{(\lambda+1)^2} = \frac{1}{\lambda+1}.$$

$$\Downarrow$$

use $P(A^c) = 1 - P(A)$

$$1 - P\{0 < X < 2(\lambda+1)\} \leq \frac{1}{\lambda+1}$$

$$1 - \frac{1}{\lambda+1} = \frac{\lambda}{\lambda+1} \leq P\{0 < X < 2(\lambda+1)\}$$

Hence proved.

A4. Given; X: integer val. R.V.
d.f is F.

$$Y: RV \Rightarrow \text{pdf. } h(y) = \begin{cases} 1 & y \in (0,1) \\ 0 & \text{o.w.} \end{cases}$$

Z: integer val. R.V.

Now lets see the cd.f. of $Z \Rightarrow$

$$F_Z(m) = \mathbb{P}\{Z \leq m\} = \mathbb{P}\{Y \leq F(m)\}$$

$$= \int_{-\infty}^{F(m)} h(y)\,dy$$

$$= F(m) - 0$$

$$\boxed{F_Z(m) = F(m)}$$

Hence Z and X have the same probability distributions. !

Note: $\{Z \leq m\} = \bigcup \{Z = i\} \; \forall \, i \in \{m, m-1, \ldots -\infty\}$

$$= \bigcup_i \{F(i-1) < Y \leq F(i)\}$$
$$i \in \{m, m-1, \ldots -\infty\}$$

$$= \{0 \leq Y \leq F(m)\}$$

$$= \{Y \leq F(m)\}$$

MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 1 (Basic Probability)

1. Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition noted. This is continued until two consecutive defectives are produced or four items have been checked, which ever occurs first. Describe the sample space for this experiment.

2. Let $\Omega = \{0, 1, 2, \ldots\}$. Let $\mathcal{F}$ be the collection of subsets of Ω that are either finite or whose complement is finite. Is $\mathcal{F}$ a σ-field? Justify your answer.

3. Consider $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Let $\mathcal{F}$ be the largest σ-field over Ω. Define, for any event R, $P(R) =$ area of $R = (b-a)(d-c)$ where R is the rectangular region that is a subset of Ω of the form $R = \{(u, v) : a \leq u < b, c \leq v < d\}$. Let T be the triangular region $T = \{(x, y) : x \geq 0, y \geq 0, x + y < 1\}$. Show that T is an event, and find $P(T)$, using the axiomatic definition of probability.

4. Let $A_1, A_2, \ldots, A_N$ be a system of completely independent events (i.e., $P(\cap_{j=1}^r A_{i_j}) = \prod_{j=1}^r P(A_{i_j}), \quad r = 2, 3, \ldots, N$). Assume that $P(A_n) = \frac{1}{n+1}, \quad n = 1, 2, \ldots, N$.
   (a) Find the probability that exactly one of the A_i's occur?
   (b) Find the probability that atmost two A_i's occur?

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, A_2, \ldots, A_n \in \mathcal{F}$ with $P(\cap_{i=1}^{n-1} A_i > 0)$. Prove that
$$P\left(\cap_{i=1}^n A_i\right) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \ldots P\left(A_n \mid \cap_{i=1}^{n-1} A_i\right)$$

6. If $A_1, A_2, \ldots, A_n$ are n events, then show that $\sum_{i=1}^n P(A_i) - \sum_{i \neq j, i=1}^n P(A_i \cap A_j) \leq P\left(\cup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i)$.

7. Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{a, d\}, \Omega\}$ and P a function from $\mathcal{F}$ to $[0, 1]$ with $P(\{a\}) = \frac{2}{7}$, $P(\{b, c\}) = \frac{3}{5}$ and $P(\{d\}) = \beta$. The value of β such that P to be a probability on $(\Omega, \mathcal{F})$.

8. Prove that, for any two events A and B, $P(A \cap B) \geq 1 - P(A) - P(B)$.

9. Consider a gambler who on each independent bet either wins 1 with probability $\frac{1}{3}$ or losses 1 with probability $\frac{2}{3}$. The gambler will quit either when he or she is winning a total of 10 or after 50 plays. The probability the gambler plays exactly 14 times?

10. Let $\Omega = \{4, 3, 2, 1\}$.
    (a) Find three different σ-algebras $\{\mathcal{F}_n\}$ for $n = 1, 2, 3$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$.
    (b) Further, create a set function $P : \mathcal{F}_3 \to \mathcal{R}$ such that, $(\Omega, \mathcal{F}_3, P)$ is a probability space.

11. Suppose that the number of passengers for a limousine pickup is thought to be either 1, 2, 3, or 4, each with equal probability, and the number of pieces of luggage of each passenger is thought to be 1 or 2, with equal probability, independently for different passengers. What is the probability that there will be five or more pieces of luggage?

12. Let $\Omega = \mathcal{R}$ and $\mathcal{F}$ be the Borel σ-field on $\mathcal{R}$. For each interval $I \subseteq \mathcal{R}$ with end points a and b ($a \leq b$), let $P(I) = \int_a^b \frac{1}{\pi} \frac{1}{1+x^2} dx$. Does P define a probability on the measurable space $(\Omega, \mathcal{F})$? Justify your answer.

13. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{A_n\}$ be a nondecreasing sequence of elements in $\mathcal{F}$. Prove that
$$P\left(\lim_{n \to \infty} A_n\right) = \lim_{n \to \infty} P(A_n).$$

14. Let $\Omega = \{s_1, s_2, s_3, s_4\}$ and $P\{s_1\} = \frac{1}{6}$, $P\{s_2\} = \frac{1}{5}$, $P\{s_3\} = \frac{1}{3}$, $P\{s_4\} = \frac{3}{10}$. Define, $A_n = \begin{cases} \{s_1, s_3\} & \text{if n is odd} \\ \{s_2, s_4\} & \text{if n is even} \end{cases}$.
    Find $P(\liminf A_n)$, $P(\limsup A_n)$.

15. Let w be a complex cube root of unity with $w \neq 1$. A fair die is thrown three times. If x, y and z are the numbers obtained on the die. Find the probability that $w^x + w^y + w^z = 0$.

16. An urn contains balls numbered from 1 to N. A ball is randomly drawn.

    (a) What is the probability that the number on the ball is divisible by 3 or 4?

    (b) What happens to the probability in the previous question when $N \to \infty$?

17. Consider the flights starting from Delhi to Bombay. In these flights, 90% leave on time and arrive on time, 6% leave on time and arrive late, 1% leave late and arrive on time and 3% leave late and arrive late. What is the probability that, given a flight late, it will arrive on time?

18. Let A and B are two independent events. Prove or disprove that A and B^c, A^c and B^c are independent events.

19. Pick a number x at random out of the integers 1 through 30. Let A be the event that x is even, B that x is divisible by 3 and C that x is divisible by 5. Are the events A, B and C pairwise independent? Further, are the events A, B and C mutually independent?

20. Let $C_i, i = 1, 2, \dots, k$ be the partition of sample space Ω. For any events A and B, find $\sum_{i=1}^{k} P(C_i/B)P(A/(B \cap C_i))$.

21. The first generation of particles is the collection of off-springs of a given particle. The next generation is formed by the off-springs of these members. If the probability that a particle has k off springs (splits into k parts) is p_k, where $p_0 = 0.4$, $p_1 = 0.3$, $p_2 = 0.3$, find the probability that there is no particle in second generation. Assume particles act independently and identically irrespective of the generation.

22. A and B throw a pair of unbiased dice alternatively with A starting the game. The game ends when either A or B wins. A wins if he throws 6 before B throws 7. B wins if he throws 7 before A throws 6. What is the probability that A wins the game? Note that "A throws 6" means the sum of values of the two dice is 6. Similarly "B throws 7".

23. In a meeting at the UNO 40 members from under-developed countries and 4 from developed ones sit in a row. What is the probability no two adjacent members are representatives of developed countries.

24. A random walker starts at 0 on the x-axis and at each time unit moves 1 step to the right or 1 step to the left with probability 0.5. Find the probability that, after 4 moves, the walker is more than 2 steps from the starting position.

25. The coefficients a, b and c of the quadratic equation $ax^2 + bx + c = 0$ are determined by rolling a fair die three times in a row. What is the probability that both the roots of the equation are real? What is the probability that both roots of the equation are complex?

26. An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities assumed to be known: $P(A \text{ fails}) = 0.20$, $P(A \text{ and } B \text{ both fail}) = 0.15$, $P(B \text{ fails alone}) = 0.15$. Evaluate the following probabilities (a) $P(A \text{ fails}/B \text{ has failed})$ (b) $P(A \text{ fails alone } /A \text{ or } B \text{ fail})$.

27. An aircraft has four engines in which two engines in each wing. The aircraft can land using atleast two engines. Assume that the reliability of each engine is $R = 0.93$ to complete a mission, and that engine failures are independent.

    (a) Obtain the mission reliability of the aircraft.

    (b) If at least one functioning engine must be on each wing, what is the mission reliability?

28. Four lamps are located in circular. Each lamp can fail with probability q, independently of all the others. The system is operational if no two adjacent lamps fail. Obtain an expression for system reliability?.

29. An urn contains b black balls and r red balls. One of the ball is drawn at random, but when it is put back in the urn c additional balls of the same colour are put in with it. Now suppose that we draw another ball. Find the probability that the first ball drawn was black given that the second ball drawn was red?

30. The base and altitude of a right triangle are obtained by picking points randomly from $[0, a]$ and $[0, b]$ respectively. Find the probability that the area of the triangle so formed will be less than $ab/4$?

31. A batch of N transistors is dispatched from a factory. To control the quality of the batch the following checking procedure is used; a transistor is chosen at random from the batch, tested and placed on one side. This procedure is repeated until either a pre-set number $n(n < N)$ of transistors have passed the test (in which case the batch is accepted) or one transistor fails (in this case the batch is rejected). Suppose that the batch actually contains exactly D faulty transistors. Find the probability that the batch will be accepted.

MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 2 (Random Variable)

1. Consider a probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{0, 1, 2\}$, $\mathcal{F} = \{\emptyset, \{0\}, \{1, 2\}, \Omega\}$, $P(\{0\}) = 0.5 = P(\{1, 2\})$. Give an example of a real-valued function on Ω that is NOT a random variable. Justify your answer.

2. Let $\Omega = \{1, 2, 3\}$. Let $\mathcal{F}$ be a σ-algebra on Ω, so that $X(w) = w + 2$ is a random variable. Find $\mathcal{F}$

3. Do the following functions define distribution functions.
   (a) $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$ (b) $F(x) = (\frac{1}{\pi})\tan^{-1}\text{x}, -\infty < x < \infty$ (c) $F(x) = \begin{cases} 0, & -\infty < x < 0 \\ 1 - e^{-x}, & 0 \le x < \infty \end{cases}$

4. Consider the random variable X that represents the number of people who are hospitalized or die in a single head-on collision on the road in front of a particular spot in a year. The distribution of such random variables are typically obtained from historical data. Without getting into the statistical aspects involved, let us suppose that the cumulative distribution function of X is as follows:

x	0	1	2	3	4	5	6	7	8	9	10
$F(x)$	0.250	0.546	0.898	0.932	0.955	0.972	0.981	0.989	0.995	0.998	1.000

   Find (a) $P(X = 10)$ (b) $P(X \le 5/X > 2)$.

5. Let X be a rv having the cdf: $F(x) = \begin{cases} 0, & x < -1 \\ \frac{1+x}{9}, & -1 \le x < 0 \\ \frac{2+x^2}{9}, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$ Find $P(X \in E)$ where E is $(-1, 0] \cup (1, 2)$.

6. Let X be a random variable with cumulative distribution function given by: $F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{25}, & 1 \le x < 2 \\ \frac{x^2}{10}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$.
   Determine the cumulative discrete distribution functions F_d and one continuous F_c such that: $F_X(x) = \alpha F_d(x) + \beta F_c(x)$.

7. Let X be a rv such that $P(X = 2) = \frac{1}{4}$ and its CDF is given by $F_X(x) = \begin{cases} 0, & x < -3 \\ \alpha(x+3), & -3 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 4 \\ \beta x^2, & 4 \le x < 8/\sqrt{3} \\ 1, & x \ge 8/\sqrt{3} \end{cases}$.
   (a) Find α, β if 2 is the only jump discontinuity of F.
   (b) Compute $P(X < 3/X \ge 2)$.

8. An airline knows that 5 percent of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that passengers come to airport are independent with each other. What is the probability that there will be a seat available for every passenger who shows up?

9. The probability of hitting an aircraft is 0.001 for each shot. Assume that the number of hits when n shots are fired is a random variable having a binomial distribution. How many shots should be fired so that the probability of hitting with two or more shots is above 0.95?

10. A reputed publisher claims that in the handbooks published by them misprints occur at the rate of 0.0024 per page. What is the probability that in a randomly choosen handbook of 300 pages, the third misprint will occur after examining 100 pages?

11. Let $0 < p < 1$ and N be a positive integer. Let $X \sim B\left(N, \frac{p}{N}\right)$. Find $\lim_{N\to\infty}\left(1-\frac{p}{N}\right)^N$, if it exists.

12. In a torture test a light switch is turned on and off until it fails. If the probability that the switch will fail any time it is turned 'on' or 'off' is 0.001, what is probability that the switch will fail after it has been turned on or off 1200 times?.

13. Let X be a Poisson random variable with parameter λ. Show that $P(X = i)$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ.

14. For what values of α, p does the following function represent a probability mass function $p_X(x) = \alpha p^x, x = 0, 1, 2, \ldots$. Prove that the random variable having such a probability mass function satisfies the following memoryless property $P(X > a + s/X > a) = P(X \geq s)$.

15. Consider a random experiment of choosing a point in the annular disc of inner radius r_1 and outer radius r_2 $(r_1 < r_2)$. Let X be the distance of chosen point from the center of annular disc. Find the pdf of X.

16. Let X be an absolutely continuous random variable with density function f. Prove that the random variables X and $-X$ have the same distribution function if and only if $f(x) = f(-x)$ for all $x \in \Re$.

17. The life time (in hours) of a certain piece of equipment is a continuous random variable X, having pdf $f_X(x) = \begin{cases} \frac{xe^{-x/100}}{10^4}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$. If four pieces of this equipment are selected independently of each other from a lot, what is the probability that atleast two of them have life length more than 20 hours?.

18. Suppose that f and g are density function and that $0 < \lambda < 1$ is a constant. (a) Is $\lambda f + (1-\lambda)g$ a probability density function? (b) Is fg (i.e., $fg(x) = f(x)g(x)$) a probability density function? Explain.

19. A system has a very large number (can be assumed to be infinite) of components. The probability that any one of these component will fail in the interval (a, b) is $e^{-a/T} - e^{-b/T}$, independent of others, where $T > 0$ is a constant. Find the mean and variance of the number of failures in the interval $(0, T/4)$.

20. A student arrives to the bus stop at 6:00 AM sharp, knowing that the bus will arrive in any moment, uniformly distributed between 6:00 AM and 6:20 AM.
    (a) What is the probability that the student must wait more than five minutes?
    (b) If at 6:10 AM the bus has not arrived yet, what is the probability that the student has to wait at least five more minutes?

21. The time to failure of certain units is exponentially distributed with parameter λ. At time $t = 0$, n identical units are put in operation. The units operate, so that failure of any unit is not affected by the behavior of the other units. For any $t > 0$, let N_t be the random variable whose value is the number of units still in operation time t. Find the distribution of the random variable N_t.

22. Consider the marks of MTL 106 examination. Suppose that marks are distributed normally with mean 76 and standard deviation 15. 15% of the best students obtained A as grade and 10% of the worst students fail in the course. (a) Find the minimum mark to obtain A as a grade. (b) Find the minimum mark to pass the course.

23. Suppose that the life length of two electronic devices say D_1 and D_2 have normal distributions $\mathcal{N}(40, 36)$ and $\mathcal{N}(45, 9)$ respectively. (a) If a device is to be used for 45 hours, which device would be preferred? (b) If it is to be used for 42 hours which one should be preferred?

24. Let X be a rv with cdf $F(x) = \begin{cases} 0, & -\infty < x < 0 \\ p + (1-p)(1-e^{-\lambda x}), & 0 \leq x < 4 \\ 1, & 4 \leq x < \infty \end{cases}$ with $0 < p < 1$ and $\lambda > 0$. Find the mean of X.

25. Let X be a random variable having a Poisson distribution with parameter λ. Prove that, for $n = 1, 2, \ldots$ $E[X^n] = \lambda E[(X+1)^{n-1}]$.

26. Prove that for any random variable X, $E[X^2] \geq [E[X]]^2$. Discuss the nature of X when one have equality?

27. Let X be the random variable such that $P(a \leq X \leq b) = 1$, where $-\infty < a < b < \infty$. Show that $Var(X) \leq \dfrac{(b-a)^2}{4}$.

28. Consider a random variable X with $E(X) = 1$ and $E(X^2) = 1$.

    (a) Find $E[(X - E(X))^4]$ if it exists.

    (b) Find $P(0.4 < X < 1.7)$ and $P(X = 0)$.

29. Let X be a continuous type random variable with pdf $f(x) = \begin{cases} \alpha + \beta x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$ If $E(X) = 3/5$, find the value of α and β.

30. Let $X \sim \mathrm{P}(\lambda)$ such that $P(X = 0) = e^{-1}$. Find $Var(X)$.

MTL106 Probability Review Notes

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Contents

1 Probability Basics

1.1 Important Terminologies

1. Sample Space (Ω): The set of all possible results of a random experiment.
2. σ-field: A collection $\mathcal{F}$ of subsets of Ω which satisfies
   (i) $\omega \in \mathcal{F}$
   (ii) If $A \in \mathcal{F}$ then $\bar{A} \in \mathcal{F}$
   (iii) Union of countable elements of $\mathcal{F}$ belongs to $\mathcal{F}$, i.e. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

3. Event: Any element of $\mathcal{F}$
4. Samples: Any element of Ω
5. Borel σ-field ($\mathcal{B}$) on $\mathbb{R}$: The collection of Borel sets on $\mathbb{R}$. Borel Sets are those sets which can be formed from countable union, countable intersection and relative complement of open intervals.

1.2 Axiomatic Definition

A real valued set function P defined on a sigma field $\mathcal{F}$ over σ which satisfies:

1. $P(A) \geq 0 \; \forall \; A \in \mathcal{F}$
2. $P(\Omega) = 1$
3. If $A_1, A_2, ...$ are mutually disjoint events in $\mathcal{F}$ then

$$P\left(\bigcup A_i\right) = \sum P(A_i)$$

Note: The triplet $(\Omega, \mathcal{F}, P)$ is called a Probability Space.

1.3 Properties of Probability

1. $P(\phi) = 0$
2. $P(A^c) = 1 - P(A)$
3. For any $A, B \in \mathcal{F}$, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

4. If $A, B \in \mathcal{F}$ and $A \subseteq B$ then $P(A) \leq P(B)$ and $P(B \backslash A) = P(B) - P(A)$
5. Let $\{A_n\}$ is an increasing (decreasing) sequence of events in $\mathcal{F}$, i.e. $A_n \subseteq A_{n+1}$ ($A_{n+1} \subseteq A_n$). Then,

$$P\left(\lim_{n \longrightarrow \infty} A_n\right) = \lim_{n \longrightarrow \infty} P(A_n)$$

1.4 Classical Definition

A probability space $(\Omega, \mathcal{F}, P)$ with finite Ω, $\mathcal{F} = 2^{\Omega}$ and $P(\{\omega\}) = \frac{1}{|\Omega|} \; \forall \; \omega \in \Omega$ is called a Laplace probability space. Probability of any event is given by,

$$P(A) = \frac{|A|}{|\Omega|}$$

1.5 More Concepts

1. Conditional Probability: The probability of the event B under the condition A is defined as

$$P(A/B) = \frac{P(A \cup B)}{P(B)}$$

2. Independent Events: Two events are independent iff

$$P(A \cap B) = P(A)P(B)$$

(i) Pairwise Independent: Sequence of events $\{A_i\}$ is pairwise independent if $P(A_i)P(A_j) = P(A_i \cap A_j)\ \forall\ i \neq j$.

(ii) Mutually Independent: Sequence of events $\{A_i\}$ is pairwise independent if $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2)...P(A_n)$.

3. Total Probability Theorem: For mutually disjoint and mutually exhaustive events $A_1, A_2, ...A_n$ we have for any event B,

$$P(B) = \sum_n P(B \backslash A_i)P(A_i)$$

4. Baye's Theorem: For any event $B \in \mathcal{F}$ with $P(B) > 0$, for mutually disjoint and mutually exhaustive events $A_1, A_2, ...A_n$ we have,

$$P(A_i \backslash B) = \frac{P(A_i)P(B \backslash A_i)}{\sum_n P(B \backslash A_i)P(A_i)}$$

2 Random Variable

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real valued function $X : \Omega \longrightarrow \mathbb{R}$ is said to be a random variable iff

$$X^{-1}\{(-\infty, x]\} \in \mathcal{F}\ \forall\ x \in \mathbb{R}$$

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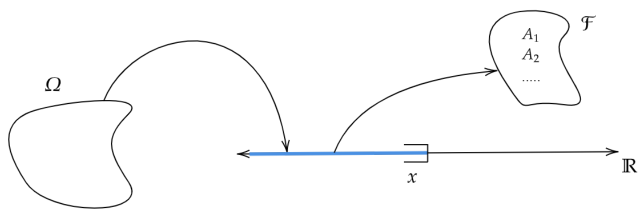

Figure 1: Collect the set of outcomes $\omega \in \Omega$ which under the mapping X gives values from $(-\infty, x]$. If this lies in $\mathcal{F}$ then X is a random variable.

2.1 Distribution Function

A real valued function F satisfying:

1. $0 \leq F(x) \leq 1\ \forall\ x \in \mathbb{R}$
2. F is monotonically increasing function.
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
4. F(x) is right continuous.

2.1.1 Cumulative Distribution Function

We define the distribution function F_X for a probability space $(\Omega, \mathcal{F}, P)$ and random variable X as,

$$F_X(x) = P(X \leq x),\ x \in \mathbb{R}$$

Alternately,

$$F_X(x) = P(\omega \in \Omega \mid X(\omega) \leq x)$$

2.2 Types of Random Variables

2.2.1 Discrete RV

If the CDF of the Random Variable (RV) has countable no. of (left) discontinuities then it is a discrete RV. For a discrete RV, the CDF is given by:

$$F_X(x) = \sum_{x_i \leq x} P(X = x_i)$$

The probability mass function (PMF) of a discrete RV is defined as, $\mathcal{P}(x) = P(X = x_i)$ when $x = i$ and 0 at other points. Properties:

1. $\mathcal{P}(x) \geq 0 \; \forall \; x \in \mathbb{R}$
2. $\sum_x \mathcal{P}(x) = 1$

2.2.2 Continuous Type RV

If the CDF of the RV is continuous in x then it is of continuous type. We represent the CDF as,

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

The function f is called the **probability density function** of the continuous type RV. Note that $f(x) = F'(x)$. It satisfies:

1. $f(t) > 0 \; \forall \; t \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f(t)dt = 1$

2.2.3 Mixed Type RV

If a RV is continuous in some intervals and has countable no. of discontinuities as well.

2.3 Probability Distribution Relations

1. $P(a < X \leq b) = F(b) - F(a)$
2. $P(a \leq X \leq b) = F(b) - F(a) + P(X = a)$
3. $P(a < X < b) = F(b) - F(a) - P(x = b)$
4. For a continuous type RV,
$$P(a \leq X \leq b) = \int_a^b f(t)dt$$
5. For a small interval $(x, x + \Delta x)$,
$$P(x \leq X \leq x + \Delta x) \approx f(x)\Delta x$$

2.4 Function of Random Variable

A function Y defined as, $Y = g(X)$ is a random variable if g is a Borel measurable function. (Note that every continuous and piecewise continuous function is Borel measurable.)

2.4.1 Distribution of Y

We can find the distribution of the random variable Y as follows:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$

From here, we may determine the distribution of Y. Note that if g is strictly monotonic and differentiable then the following holds: $f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right|$, where $y = g(X)$, and 0 otherwise.

3 Mean and Variance

3.1 Mean/Expectation

Let $(\Omega, \mathcal{F}, P)$ be a probability space. If X is discrete type, and $\sum_i |X_i| P(X = x_i)$ is finite, then X has an expected value given by,

$$E(X) = \sum_i X_i P(X = x_i)$$

If X is continuous type with PDF f and $\int_{-\infty}^{\infty} |x| f(x) dx$ is finite, then the expected value is given by:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Properties:

1. $E(c) = c$, where c is a constant.
2. $E(aX + b) = aE(X) + b$
3. If $P(X \geq 0) = 1$, $E(X) \geq 0$ if it exists.
4. If X is continuous type with the CDF $F(X)$ then $E(X)$ is given by,

$$E(X) = \int_0^{\infty} (1 - F(x))\, dx - \int_{-\infty}^{0} (F(x))\, dx$$

5. If X is discrete type, then,

$$E(X) = \sum_{k=0}^{n} (1 - F_X(k)))$$

3.2 Variance

Let X be a random variable with $E(X) = \mu$ exists. The second order moment about mean is called variance, defined as,

$$Var(X) = E\left((x - \mu)^2\right)$$

On simplifying we also get,

$$Var(X) = E(X^2) - (E(X))^2$$

Properties:

1. $Var(c) = 0$ where c is a constant.
2. If $Var(X)$ exists then $Var(X) \geq 0$.
3. If $P(X = \alpha) = 1$ then $E(X) = \alpha$ and $Var(X) = 0$.
4. $Var(aX + b) = a^2 Var(X)$

3.3 Higher Order Moments

3.3.1 n-th Order Moment about Mean

For a RV X for which n-th order moments about mean exist, we denote by

$$\mu_n = E\left((x-\mu)^n\right), n = 1, 2, 3...$$

3.3.2 n-th Order Moment about Origin

For a RV X for which n-th order moments about origin exist, we denote by

$$\mu_n' = E\left(x^n\right), n = 1, 2, 3...$$

Note that if μ_n' exists then μ_r' also exists for all $r < n$.

3.3.3 Relation between Moments about Mean and Origin

If X is an RV for which n-th order moments exist then,

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (\mu_k')(-\mu_1')^{n-k}$$

3.4 Moment Inequalities

3.4.1 Markov's Inequality

Let X be a <u>non-negative</u> RV and $E(X)$ exists. For all $t > 0$ we have,

$$P(X \geq t) \leq \frac{E(X)}{t}$$

3.4.2 Chebyshev's Inequality

Let X be a RV with $E(X) = \mu$ and $Var(X) = \sigma^2$ exists. Then for $\epsilon > 0$ we have,

$$P\left(|X - c| \geq \epsilon\right) \leq \frac{E\left((x-c)^2\right)}{\epsilon^2}$$

In particular, when we have $c = \mu$ then,

$$\boxed{P\left(|X - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2}}$$

3.4.3 Jensen's Inequality

Let X be a RV with $E(X)$ exists. Let g be any convex function. Then we must have,

$$E(g(X)) \geq g(E(X))$$

4 Generating Functions

4.1 Probability Generating Function

Let X is a non-negative integer valued RV with $P_k = P(X = k)$ then we can define the PGF as,

$$G_X(s) = \sum_{k=0}^{\infty} P_k s^k$$

Remarks:

1. Converges in $s \in (-1, 1)$
2. $G_x(s) = E(s^X)$
3. $G_X^{(r)}(s)$ represents the r-th derivative of G_X. Evaluating it at zero gives factorial moments of r-th order, i.e.
$$G_X^{(r)}(1) = E\left(x(x-1)...(x-r+1)\right)$$
4. We can find the probabilities from the k-th derivatives of the PDF, i.e.
$$P(X = k) = \frac{1}{k!} G_X^{(k)}(0)$$
5. If X and Y have the same PGF for all s then X and Y have the same distribution.

4.2 Moment Generating Function

Let X is a RV such that $E(e^{tx})$ exists for t in some interval including zero. Then,

$$M_X(t) = E(e^{tx})$$

If X is a discrete RV then we have,

$$M_X(t) = \sum_k P(X = k) e^{kt}$$

Remarks:

1. One can get the n-th order moments about the moment from the n-th derivative of the MGF at origin, i.e.
$$E(X^n) = M_X^{(n)}(0)$$
2. If X and Y have the same MGF for all t then X and Y have the same distribution.

4.3 Characteristic Function

Let X is a random variable. Characteristic function is defined as

$$\psi_X(t) = E(e^{itx})$$

Characteristic function is complex valued, always exists. Remarks:

1. $\psi_X(0) = 1$
2. $|\psi_X(t)| \leq 1 \ \forall t$
3. If $E(X^n)$ exists then
$$E(X^n) = \frac{1}{i^n} \psi_X^{(n)}(0)$$
4. If X and Y have the same characteristic function t then X and Y have the same distribution.

5 Random Vectors

A collection of n random variables $(X_1, X_2, ...X_n)$ over a probability space $(\Omega, \mathcal{F}, P)$ is called a random vector.

5.1 Joint Distribution

For a 2-D random vector, the joint CDF is given by:

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)\ ,\ x_1, x_2 \in \mathbb{R}$$

It satisfies:

1. $F(x_1, x_2)$ is non-decreasing, continuous from the right w.r.t each of the coordinates (x_1, x_2).
2. When $x_1 \to \infty, x_2 \to \infty$ then $F(x_1, x_2) \to 1$.
3. When $x_1 \to -\infty$ or $x_2 \to -\infty$ then $F(x_1, x_2) \to 0$.
4. For every $(a, c), (b, d)$ s.t. $a < b$ and $c < d$ we have,

$$F(b, d) - F(b, c) - F(a, d) + F(a, c) > 0$$

The distribution of one of the random variables constituting a random vector is called a **marginal distribution**. From a joint CDF $F(x_1, x_2)$, we can obtain the marginal distribution of X_1 as,

$$F_{X_1}(x_1) = \lim_{x_2 \to \infty} F(x_1, x_2)$$

For discrete type random variables, the joint PMF is similarly defined and satisfies,

1. $P(x_1, x_2) \geq 0$, $P(x_1, x_2) \leq 1$
2. $\sum_{X_1} \sum_{X_2} P(x_1, x_2) = 1$

By summing up the PMF over one of the coordinates, we can get the marginal distribution of the other random variable, i.e.

$$P_{X_1}(x_1) = \sum_{X_2} P(x_1, x_2)$$

5.2 Joint PDF

Let X, Y be continuous type RVs with joint distribution F. Then the joint PDF is defined as,

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

Or, in another form -

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(t, s) dt ds$$

We can find the marginal PDF from the joint PDF by integrating over $\mathbb{R}$ over one of the coordinates.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, s) ds$$

Properties of joint PDF:

1. $f(x, y) \geq 0\ \forall\ x, y$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, s) dt ds = 1$

6 Independent Random Variables

We say that X and Y are independent random variables if and only if

$$F(x,y) = F_X(x)F_Y(y) \ \forall \ (x,y) \in \mathbb{R}^2$$

For discrete type random variables, a necessary and sufficient condition is,

$$P(X = x_i, Y = y_i) = P(X = x_i)P(Y = y_i)$$

For continuous type random variables, another condition involving PDFs is:

$$f(x,y) = f_X(x)f_Y(y)$$

Note that for independent random variables, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. Note that the converse is **not** necessarily true.

6.1 Functions of Independent RVs

Let X, Y be independent random variables. Let g, h be Borel-measurable functions (continuous functions are Borel-measurable). Then, $g(X)$ and $h(Y)$ are also independent random variables.

6.2 i.i.d Random Variables

We say $\{X_n\}$ is a sequence of independently, identically distributed RVs with common distribution if each (X_i, X_j) , $i \neq j$ are independent and the distributions of $X_1, X_2, ...$ are the same.

7 Conditional Distributions

7.1 Discrete RVs

For two random variables X, Y of discrete type, condiitonal PMF of X given $Y = y$ is given by:

$$P_{X/Y}(x/y) = P(X = x/Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

As long as $P(Y = y) > 0$

7.2 Continuous RVs

The conditional PDF of X given $Y = y$ is given by,

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)}$$

For all y with $f_Y(y) > 0$.

8 Functions of Random Variables

For any collection of random variables $X_1, X_2, ...X_n$ and a Borel-measurable function g, we have $Y = g(X_1, X_2, ...X_n)$ is a random variable.

8.1 Distribution of Function of RV

Consider any 2D continuous type RV with joint PDF $f(x, y)$. Define $Z = H_1(X, Y)$ and $W = H_2(X, Y)$. Assuming that H_1, H_2 are Borel-measurable, we can solve for the distribution of Z, W under the assumptions:

- It is possible to solve $z = H_1(x, y)$ and $w = H_2(x, y)$ uniquely for x, y in terms of z, w. Let the solution be $x = g_1(z, w)$ and $y = g_2(z, w)$.
- The partial derivatives of x, y wrt z, w exist and are continuous.

Then the joint PDF of Z, W can be written as,

$$f_{Z,W}(z, w) = f_{X,Y}\left(g_1(z, w), g_2(z, w)\right) |J(z, w)|$$

Where, $J(z, w)$ is a normalizing constant given by,

$$J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

8.2 Important Results

For some common types of functions of RVs, the distributions can be found out as follows:

1. If $Z = X + Y$ then
$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z - x)dx$$
2. If $U = X - Y$ then
$$f_U(u) = \int_{-\infty}^{\infty} f(u + y, y)dy$$
3. If $V = XY$ then
$$f_V(v) = \int_{-\infty}^{\infty} f\left(x, \frac{v}{x}\right) . \frac{1}{|x|}dx$$
4. If $W = \frac{X}{Y}$ then
$$f_W(w) = \int_{-\infty}^{\infty} f(yw, y).|y|dy$$

9 Covariance and Correlation

The covariance of two random variables is defined as

$$cov(X, Y) = \mathbb{E}\left((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Properties of covariance:

1. $cov(aX, Y) = acov(X, Y)$
2. $cov(X + Y, Z) = cov(X, Z) + cov(Y, Z)$
3. $cov(\sum X_i, Y) = \sum cov(X_i, Y)$
4. If X, Y are independent, $cov(X, Y) = 0$.

9.1 Variance Formula

The variance of a sum of random variables can be expressed as:

$$var(\sum X_i) = \sum var(X_i) + \sum \sum cov(X_i, X_j)$$

9.2 Correlation Coefficient

The correlation coefficient $\rho(X, Y)$ is defined as,

$$\rho(X, Y) = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

Here, $\sigma_X = \sqrt{var(X)}$. Note that $|\rho(X, Y)| \leq 1$.

10 Limiting Distributions

10.1 Convergence in Distribution

Let $X_1, X_2, ...X_n$ be a sequence of random variables with CDF $F_1, F_2, ...$ respectively. We say that $\{X_n\}$ converges in distribution to X i.e. $X_n \xrightarrow{d} X$ if:

$$\lim_{n \longrightarrow \infty} F_n(x) = F(x)$$

Where F is the CDF of the random variable X.

10.2 Convergence in Probability

Let $X_1, X_2, ...X_n$ be a sequence of RVs. We say that $X_n \xrightarrow{p}$ if for any $\epsilon > 0$ we have:

$$\lim_{n \to \infty} P\left(|X_n - X| > \epsilon\right) = 0$$

Note that here we need the RV or at least the distribution of X beforehand.

10.3 Convergence in rth Moment

Let $r > 0$ and $\{X_n\}$ be a sequence of random variables defined on a probability space such that $\mathbb{E}(|X_n|^r)$ exists and finite $\forall$ n. We say that $\{X_n\}$ converges to X in rth moment if

$$\lim_{n \to \infty} \mathbb{E}\left(|X_n - X|^r\right) = 0$$

10.4 Convergence Almost Surely

Let $\{X_n\}$ be a sequence of random variables. We say that $X_n \xrightarrow{a.s.} X$ if

$$P\left(\lim_{n \to \infty} X_n = X\right) = 1$$

In other words,

$$P(A) = P\left(\{\omega \in \Omega | X_n(w) \to X\}\right) = 1$$

11 Laws of Large Numbers

11.1 Weak Law of Large Numbers

Let $X_1, X_2, ...X_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Then for any $\epsilon > 0$ we have,

$$P\left(\left|\frac{X_1 + X_2 + ...X_n}{n} - \mu\right| > \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}$$

The proof follows by Chebyshev's inequality.

11.2 Strong Law of Large Numbers

Let $X_1, X_2, ...X_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Then the random variable $\bar{X}$ defined as

$$\bar{X} = \frac{X_1 + X_2 + ... + X_n}{n}$$

satisfies $\bar{X} \xrightarrow{a.s.} \mu$, i.e. converges almost surely to the mean.

12 Central Limit Theorem

Let $X_1, X_2, ...$ be a sequence of random variables with mean μ and variance σ^2 defined on a probability space. Define

$$Z_n = \frac{\sum X_i - \mathbb{E}(\sum X_i)}{\sqrt{var(\sum X_i)}}$$

Then for larger n, we have that Z_n is approximately standard normal distributed, i.e.

$$P(Z_n \leq z) \approx \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{-t^2}{2}}\, dt$$

13 Common Discrete Distributions

13.1 Bernoulli Distribution

Bernoulli distribution is observed while tossing a coin or other simple experiments. PMF:

$$P(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \end{cases}$$

13.2 Binomial Distribution

Binomial distribution PMF:

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, 2...$$

Properties:

- Denoted by $X \sim B(n, p)$. n, p are the parameters of the distribution.
- Binomial distribution can be decomposed as a sum of n Bernoulli distributed random variables.
- Mean: $\mathbb{E}(X) = np$
- Variance: $var(X) = np(1-p)$
- MGF: $M(t) = (pe^t + 1 - p)^n$

13.3 Geometric Distribution

PMF:

$$P(k) = (1-p)^{k-1}p, k = 1, 2, ...$$

Properties:

- In a sequence of Bernoulli trials, the 1st success is a geometric distribution.
- Denoted by $X \sim Geo(p)$.
- Mean: $\mathbb{E}(X) = \frac{1}{p}$
- Variance: $var(X) = \frac{1-p}{p^2}$
- MGF: $M(t) = \frac{pe^t}{1-(1-p)e^t}$

13.4 Negative Binomial Distribution

PMF:

$$P(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, k = r, r+1, ...$$

Properties:

- In a sequence of Bernoulli trials, the probability of getting r-th success in k-th trial is a NB distribution.
- Denoted by $X \sim NB(r, p)$.
- Mean: $\mathbb{E}(X) = \frac{r}{p}$
- Variance: $var(X) = \frac{r(1-p)}{p^2}$
- MGF: $M(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r$

13.5 Poisson Distribution

PMF:

$$P(k) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2...$$

Properties:

- Denoted by $X \sim \text{Poisson}(\lambda)$.
- Mean: $\mathbb{E}(X) = \lambda$
- Variance: $var(X) = \lambda$
- MGF: $M(t) = e^{\lambda(e^t-1)}$

14 Common Continuous Distributions

14.1 Uniform Distribution

PDF:

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Properties:

- Denoted by $X \sim U(a, b)$.
- Mean: $\mathbb{E}(X) = \frac{a+b}{2}$
- Variance: $var(X) = \frac{(b-a)^2}{12}$
- MGF: $M(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$

14.2 Exponential Distribution

PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Properties:

- Denoted by $X \sim exp(\lambda)$.
- Mean: $\mathbb{E}(X) = \frac{1}{\lambda}$
- Variance: $var(X) = \frac{1}{\lambda^2}$
- MGF: $M(t) = \frac{1}{1-\frac{t}{\lambda}}, t < \lambda$
- Exponential distribution satisfies Markov property or memoryless property, i.e.

  $$P(X > t + s / x > s) = P(X > t)$$

  Geometric distribution also has this property.

14.3 Gamma Distribution

PDF:

$$f(x) = \begin{cases} \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\tau(r)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Properties:

- Denoted by $X \sim \text{Gamma}(r, \lambda)$.
- It represents sum of r exponentially distributed RVs.
- Mean: $\mathbb{E}(X) = \frac{r}{\lambda}$
- Variance: $var(X) = \frac{r}{\lambda^2}$
- MGF: $M(t) = \left(\frac{1}{1-\frac{t}{\lambda}}\right)^r, t < \lambda$

14.4 Normal Distribution

PDF:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$$

Properties:

- Denoted by $X \sim N(\mu, \sigma^2)$.
- If we consider $Z = \frac{X-\mu}{\sigma}$ then we have Z is standard normal distributed, i.e. $Z \sim N(0, 1)$ with

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$$

- Mean: μ
- Variance: σ^2
- MGF: $M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$
