2201-MTL106: ASSIGNMENT-1

Q1. Let $\Omega = \{0, 1, 2, \ldots\}$. Let $\mathcal{F}$ be the collection of subsets of Ω that are either finite or whose complement is finite. Is $\mathcal{F}$ a σ-field? Justify your answer.

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Q2. Let $\mathcal{F}$ be a σ-field of subsets of Ω and suppose $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-field of B. Is $\mathcal{G}$ σ-field of subsets of Ω.

Q3. For $\Omega = \{1, 2, 3, 4\}$ find three different σ-algebras $\mathcal{F}_n$, $n = 1, 2, 3$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$. Moreover, define a set function $\mathbb{P} : \mathcal{F}_3 \to [0, 1]$ such that $\big(\Omega, \mathcal{F}_3, \mathbb{P}\big)$ becomes a probability space.

Q4. Ler $A_r : r \geq 1$ be events such that $\mathbb{P}(A_r) = 1$ for all r. Show that $\mathbb{P}\Big(\cap_{r=1}^{\infty} A_r\Big) = 1$.

Q5. Let $A_1, A_2, \ldots, A_N$ be a system of independent events i.e.,

$$\mathbb{P}\Big(\cap_{i=1}^{r} A_i\Big) = \prod_{i=1}^{r} \mathbb{P}(A_i), \quad r = 2, 3, \ldots, N$$

Assume that $\mathbb{P}(A_n) = \frac{1}{n+1}, \quad n = 1, 2, \ldots, N.$

a) Find the probability that exactly one of the A_i's occur?
b) Find the probability that atmost two A_i's occur?

Q6. An urn contains balls numbered from 1 to N. A ball is randomly drawn.

i) What is the probability that the number on the ball is divisible by 3 or 4?
ii) What happens to the probability in the previous question when $N \to \infty$?

Q7. Show that, for any events A and B,

$$\mathbb{P}\big(A | A \cup B\big) \geq \mathbb{P}(A|B).$$

Q8. Only two factories manufacture ball pens. 20% of pens from factory I and 5% of pens from factory II are defective. Factory I produces twice as many pens as factory II each week.

a) What is the probability that a pen, randomly chosen from a week's production, is satisfactory?
b) If the choosen pen is defective, what is the probability that it came from factory I?

Q9. Consider the flights starting from Delhi to Kolkata. In these flights, 90% leave on time and arrive on time, 6% leave on time and arrive late, 1% leave late and arrive on time and 3% leave late and arrive late. What is the probability that, given a flight late, it will arrive on time?

Q10. A baised coin, with probability $\frac{1}{3}$ of head turning up, is tossed repeatedly. Let p_n be the probability that even number of heads has occurred after n tosses (zero is an even number). Show that

$$p_0 = 1, \quad p_n = \frac{1}{3}(1 - p_{n-1}) + \frac{2}{3} p_{n-1}, \quad n \geq 1.$$

Solve this difference equation.

Q11. To build a working system, one needs to randomly pick 3 components out of 50 available components, some of which are defective. If any of the selected component does not work, then the system also does not work. What is the probability of building a working system if we know that there are 6 faulty components.

Q12. A pile of 8 playing cards has 4 aces, 2 kings and 2 queens. A second pile of 8 playing cards has 1 ace, 4 kings and 3 queens. You conduct an experiment in which you randomly choose a card from the first pile and place it on the second pile. The second pile is then shuffled and you randomly choose a card from the second pile. If the card drawn from the second deck was an ace, what is the probability that the first card was also an ace?

Q13. The coefficients a, b and c of the quadratic equation $ax^2 + bx + c = 0$ are determined by rolling a fair die three times in a row. What is the probability that both the roots of the equation are real?

Q14. Let A_1, A_2 and A_3 are independent events. Prove that

$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = 1 - \prod_{i=1}^{3}(1 - \mathbb{P}(A_i)).$$

Q15. Let A and B are two independent events. Prove or disprove that A and $B^{\complement}$, $A^{\complement}$ and $B^{\complement}$ are independent events.

Q16. We roll a die 5 times. Let A_{ij} be the event that the ith and jth rolls produce the same number. Show that the events $\{A_{ij} : 1 \leq i < j \leq 5\}$ are pairwise independent but not independent.

Q17. Let $\Omega = \{1, 2, 3, \ldots, 7\}$, $\mathcal{F}$ be the power set of Ω, and $\mathbb{P}(A) = \frac{|A|}{7}$ for all $A \in \mathcal{F}$. Show that, if A and B are independent events, then at least one of A and B is either empty set or Ω.

Q18. Two fair dice are rolled. Show that the event that their sum is 7 is independent of the score shown by the first die.