Assignment

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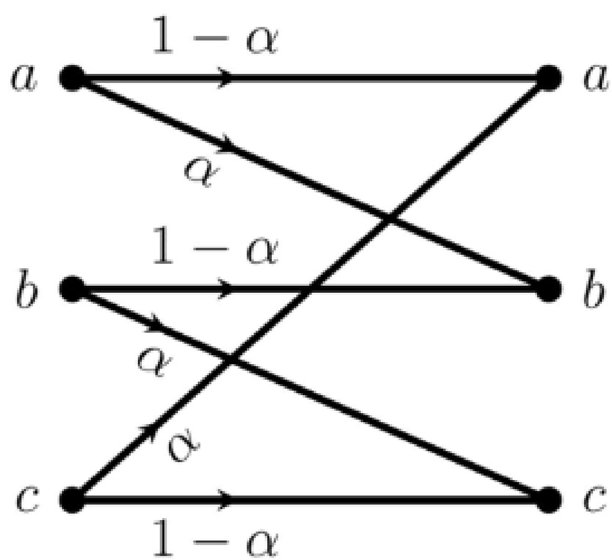
Question:- The transition diagram of a discrete memoryless channel with three input symbols and three output symbols is shown in the figure. The transition probabilities are as marked. The parameter α lies in the interval $[0.25, 1]$. The value of α for which the capacity of this channel is maximized, is
Solution:



Variable	Description	Value
X	Input variable	$X = \{0, 1, 2\}$
Y	Output variable	$Y = \{0, 1, 2\}$
C	Channel Capacity	C
I	Mutual Information	I
H	Entropy	H

$$C = \max_{p(X,Y)} I(X, Y) \tag{1}$$

$$I(X, Y) = \sum_{x,y} p(x, y) \log_2 \frac{p(x, y)}{p(x)\, p(y)} \tag{2}$$

$$= \sum_{x,y} p(x, y) \log_2 \frac{p(y|\, x)}{p(y)} \tag{3}$$

$$= -\sum_{x,y} p(x, y) \log_2 p(y) + \sum_{x,y} p(x, y) \log_2 p(y|\, x) \tag{4}$$

$$= -\sum_{y} p(y) \log_2 p(y) - \left(-\sum_{x,y} p(x, y) \log_2 p(y|\, x)\right) \tag{5}$$

$$= H(Y) - H(Y|\, X) \tag{6}$$

Now,

$$p_Y(0) = p_Y(1) = p_Y(2) = \frac{1}{3} \tag{7}$$

$$H(Y) = -\sum_{y=0}^{2} p_Y(y) \log_2 p_Y(y) \tag{8}$$

$$= -\frac{1}{3} \log_2 \frac{1}{3} \times 3 \tag{9}$$

$$= \log_2 3 \tag{10}$$

$$H(Y|\, X) = -\sum_{x=0}^{2} \sum_{y=0}^{2} p_X(x)\, p_{Y|X}(y|\, x) \log_2 \left(p_{Y|X}(y|\, x)\right) \tag{11}$$

$$= -\sum_{x=0}^{2} p_X(x) \sum_{y=0}^{2} p_Y(y|\, X) \log_2 \left(p_Y(y|\, X)\right) \tag{12}$$

$$= -3 \times \frac{1}{3} \left(\alpha \log_2 \alpha + (1-\alpha) \log_2 (1-\alpha)\right) \tag{13}$$

Using (10) and (13) in (6)

$$I(X, Y) = \log_2 3 + \alpha \log_2 \alpha + (1-\alpha) \log_2 (1-\alpha) \tag{14}$$

$$\implies \frac{d}{d\alpha} I(X, Y) = \left(\log_2 \alpha - \log_2 (1-\alpha)\right) \tag{15}$$

For maxima or minima $\frac{d}{d\alpha} I(X, Y) = 0$

$$\log_2 \alpha - \log_2 (1-\alpha) = 0 \tag{16}$$

$$\alpha = \frac{1}{2} \tag{17}$$

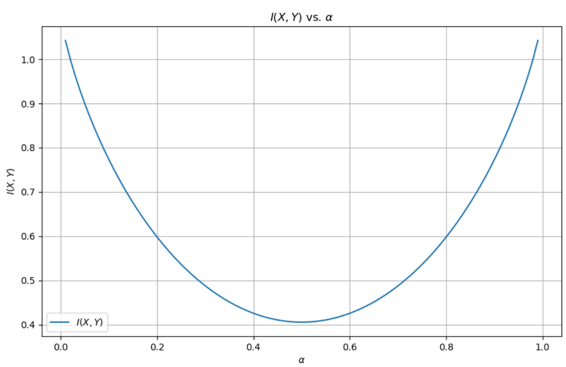
$$\frac{d^2}{d\alpha^2} I(X, Y) = \frac{1}{\log 2} \left(\frac{1}{\alpha} + \frac{1}{1-\alpha}\right) \tag{18}$$

$$\frac{d^2}{d\alpha^2} I(X, Y)\Big|_{\alpha=\frac{1}{2}} = \frac{4}{\log 2} \tag{19}$$

$$\geq 0 \tag{20}$$

At $\alpha = \frac{1}{2}, I(X, Y)$ is minimum
As $I(X, Y)$ is maximum at $\alpha = 0, 1$
But $\alpha \in [0.25, 1]$,

$$\alpha = 1 \tag{21}$$



$I(X, Y)$ vs. α
$I(X, Y)$
α
0.0
0.2
0.4
0.6
0.8
1.0
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$I(X, Y)$