Answer

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Question: In a locality 'A', the probability of a convective storm event is 0.7 with a density function,

$$f_{X_1}(x_1) = e^{-x_1}, \quad x_1 > 0 \tag{1}$$

The probability of tropical cyclone-induced storm in the same location is given by the density function,

$$f_{X_2}(x_2) = 2e^{-2x_2}, \quad x_2 > 0 \tag{2}$$

The probability of occuring more than 1 unit of storm event is

Solution:

Laplace Transform

Let X be a random variable such that

$$X = X_1 + X_2 \tag{3}$$

Given,

$$f_{X_1}(x) = e^{-x}u(x) \tag{4}$$
$$f_{X_2}(x) = 2e^{-2x}u(x) \tag{5}$$

Where, $u(x)$ is unit step function.

CDF of X_1:

$$F_{X_1}(x) = \int_{-\infty}^{x} f_{X_1}(x)\,dx \tag{6}$$
$$= \int_{0}^{x} e^{-x}dx \tag{7}$$
$$= 1 - e^{-x} \tag{8}$$

CDF of X_2:

$$F_{X_2}(x) = \int_{-\infty}^{x} f_{X_2}(x)\,dx \tag{9}$$
$$= \int_{0}^{x} 2e^{-2x}dx \tag{10}$$
$$= 1 - e^{-2x} \tag{11}$$

Now,

$$M_X(s) = M_{X_1}(s) \cdot M_{X_2}(s) \tag{12}$$
$$M_{X_1}(s) = \int_{-\infty}^{\infty} f_{X_1}(x) \cdot e^{-sx}dx \tag{13}$$
$$= \int_{0}^{\infty} e^{-x} \cdot e^{-sx}u(x)\,dx \tag{14}$$
$$= \frac{1}{s+1} \tag{15}$$

Region of Convergence(ROC) of $M_{X_1}(s)$:

$$Re(s) > -1 \tag{16}$$

Now,

$$M_{X_2}(s) = \int_{-\infty}^{\infty} f_{X_2}(x) \cdot e^{-sx}dx \tag{17}$$
$$= \int_{0}^{\infty} 2e^{-2x} \cdot e^{-sx}u(x)\,dx \tag{18}$$
$$= \frac{2}{s+2} \tag{19}$$

ROC of $M_{X_2}(s)$:

$$Re(s) > -2 \tag{20}$$

Using (15) and (19) in (12)

$$M_X(s) = \frac{1}{s+1} \times \frac{2}{s+2} \tag{21}$$
$$= \frac{2}{(s+1)(s+2)} \tag{22}$$

$$p_X(x) = L^{-1}[M_X(s)] \tag{23}$$
$$= L^{-1}\left[\frac{2}{(s+1)(s+2)}\right] \tag{24}$$
$$= 2L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \tag{25}$$

ROC of laplace transform:

$$Re(s) > -1 \cap Re(s) > -2 \tag{26}$$
$$\implies Re(s) > -1 \tag{27}$$

So, now

$$p_X(x) = 2\left(e^{-x} - e^{-2x}\right)u(x) \tag{28}$$

CDF of X:

$$F_X(x) = \int_{-\infty}^{x} p_X(x)\,dx \tag{29}$$
$$= 2\int_{-\infty}^{x} \left(e^{-x} - e^{-2x}\right)u(x)\,dx \tag{30}$$

But $p_X(x)$ is integrable for $x > 0$

$$F_X(x) = 2\int_0^x \left(e^{-x} - e^{-2x}\right) dx \tag{31}$$
$$= 2\left(-e^{-x} + \frac{1}{2}e^{-2x}\right)\Bigg|_0^x \tag{32}$$
$$= 2\left(-e^{-x} + \frac{1}{2}e^{-2x} + \frac{1}{2}\right) \tag{33}$$
$$= -2e^{-x} + e^{-2x} + 1 \tag{34}$$

Now,

$$\Pr(X > 1) = F_X(1) \tag{35}$$
$$= -2e^{-1} + e^{-2} + 1 \tag{36}$$
$$= 0.39 \tag{37}$$

Steps for simulation the given distribution

1) CDF of X_1 is given as:

$$F_{X_1}(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases} \tag{38}$$

2) Declare a function inverse cdf ($I(u)$) such that its input is any random number and output is random variable whose cdf equals that of the given distribution
For x $\leq$ 0

$$u = 0 \tag{39}$$
$$\because x \leq 0 \tag{40}$$
$$u \leq 0 \tag{41}$$

For x $>$ 0

$$u = 1 - e^{-x} \tag{42}$$
$$e^{-x} = 1 - u \tag{43}$$
$$x = -\ln(1-u) \tag{44}$$
$$\because x > 0 \tag{45}$$
$$u > 0 \tag{46}$$

$$I(u) = \begin{cases} 0 & u \leq 0 \\ -\ln(1-u) & u > 0 \end{cases} \tag{47}$$

3) Define three arrays random_vars , cdf_values , theoretical_cdf_values to store random variables, simulated cdf values and theoretical cdf values
4) Generate random numbers using rand() and calling inverse cdf funtion to generate our random variable
5) Calling cdf function to calculate the cdf of the generated random variable
6) Storing the random variable,theoretical cdf and generated cdf into their respective arrays
7) Storing the data of these three array into a .dat file
8) Plotting these .dat file in python
9) Repeat all these steps for X_2 as well as X

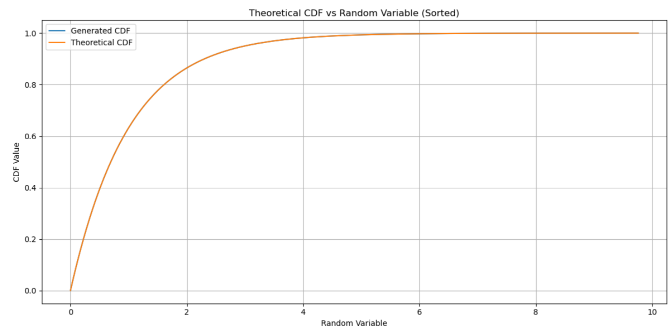

Fig. 9. Theoretical vs Simulation Analysis wrt X_1

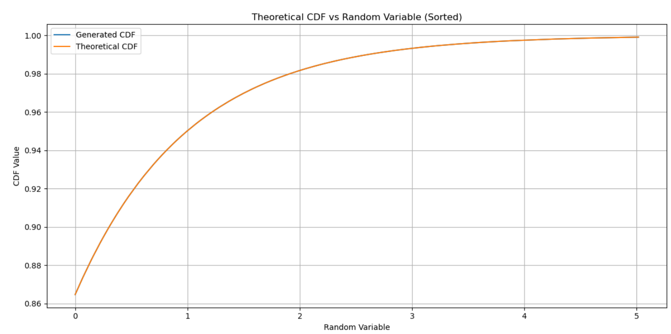

Fig. 9. Theoretical vs Simulation Analysis wrt X_2

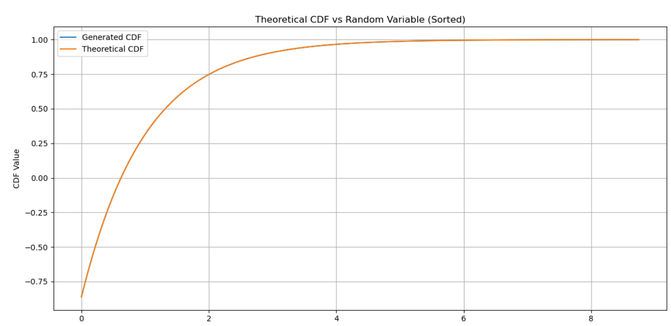

Fig. 9. Theoretical vs Simulation Analysis wrt X