Assignment

EE22BTECH11019 - Dhruv Parashar

Question: Find the sum of n terms of an AP where common difference = d using Contour Integration.
Solution:

Symbol	Value	Description
$x(n)$	$(x(0) + nd)u(n)$	n^{th} term of an A.P
$x(0)$	$x(0)$	1^{st} term of the A.P
d	d	Common difference
$u(n)$	unit step function	$u(n) = 0$ $(n < 0)$ $u(n) = 1$ $(n \geq 0)$
$s(n)$	$\sum_{k=0}^{n} x(k)$	Sum of n terms of an AP

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{1}$$

$$= \sum_{n=-\infty}^{\infty} (x(0) + nd)u(n)z^{-n} \tag{2}$$

$$U(z) = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{3}$$

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{4}$$

$$S(z) = \sum_{n=-\infty}^{\infty} s(n)z^{-n} \tag{5}$$

$$s(n) = x(n) * u(n) \tag{6}$$

$$S(z) = X(z)\, U(z) \tag{7}$$

$$= \left(\frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}\right)\left(\frac{1}{1 - z^{-1}}\right), |z| > 1 \tag{8}$$

By performing inverse Z transform on S(z) using contour integration

$$s(n) = \frac{1}{2\pi j} \oint_C S(z)\, z^{n-1}\, dz \tag{9}$$

$$s(n) = \frac{1}{2\pi j} \oint_C \left(\frac{x(0)\, z^{n-1}}{(1 - z^{-1})^2} + \frac{dz^{n-2}}{(1 - z^{-1})^3}\right) dz \tag{10}$$

For R_1 we can observe that the pole has been repeated twice.

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z - a)^m f(z)\right) \tag{11}$$

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z - 1)^2 \frac{x(0)\, z^{n+1}}{(z - 1)^2}\right) \tag{12}$$

$$= x(0)(n + 1) \lim_{z \to 1} (z^n) \tag{13}$$

$$= x(0)(n + 1) \tag{14}$$

For R_2 we can observe that the pole has been repeated thrice.

$$R_2 = \frac{1}{(2)!} \lim_{z\to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{dz^{n+1}}{(z-1)^3} \right) \tag{15}$$

$$= \frac{d\,(n+1)}{2} \lim_{z\to 1} \frac{d}{dz} \left(z^n\right) \tag{16}$$

$$= \frac{d\,(n+1)\,(n)}{2} \lim_{z\to 1} \left(z^{n-1}\right) \tag{17}$$

$$= \frac{d\,(n)\,(n+1)}{2} \tag{18}$$

$$\implies R = R_1 + R_2 \tag{19}$$

$$= x\,(0)\,(n+1) + \frac{d\,(n)\,(n+1)}{2} \tag{20}$$

Finally,

$$s\,(n) = x\,(0)\,(n+1)\,u\,(n) + d\left(\frac{n\,(n+1)}{2}\right)u\,(n) \tag{21}$$

$$= \frac{n+1}{2}\,(2x\,(0) + nd)\,u\,(n) \tag{22}$$