Assignment

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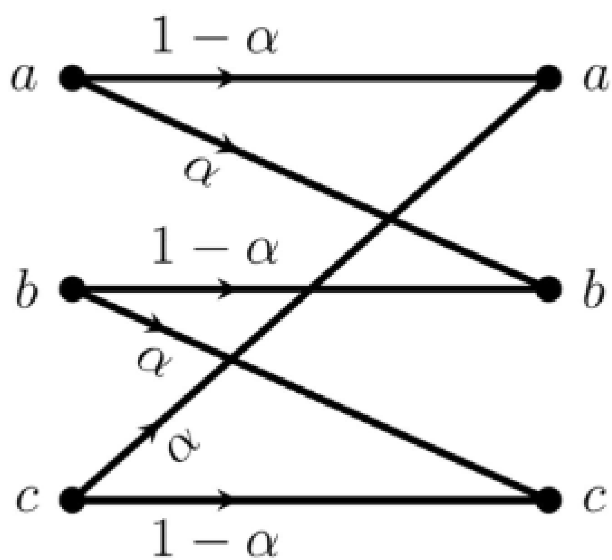
Question:- The transition diagram of a discrete memoryless channel with three input symbols and three output symbols is shown in the figure. The transition probabilities are as marked. The parameter α lies in the interval [0.25, 1]. The value of α for which the capacity of this channel is maximized, is
Solution:



Variable	Description	Value
X	Input variable	$X = \{0, 1, 2\}$
Y	Output variable	$Y = \{0, 1, 2\}$
C	Channel Capacity	C
I	Mutual Information	I
H	Entropy	H

$$C = \max_{p(X,Y)} I(X, Y) \tag{1}$$

$$I(X, Y) = H(Y) - H(Y|X) \tag{2}$$

Now,

$$\Pr(Y = 0) = \Pr(Y = 1) = \Pr(Y = 2) = \frac{1}{3} \tag{3}$$

$$H(Y) = -\sum_{k=0}^{2} \Pr(Y = k) \log_2 \Pr(Y = k) \tag{4}$$

$$= -\frac{1}{3} \log_2 \frac{1}{3} \times 3 \tag{5}$$

$$= \log_2 3 \tag{6}$$

$$H(Y|X) = -\sum_{k_1=0}^{2} \sum_{k_2=0}^{2} \Pr(X = k_1) \Pr(Y = k_2|X) \log_2 (\Pr(Y = k_2|X)) \tag{7}$$

$$= -\sum_{k_1=0}^{2} \Pr(X = k_1) \sum_{k_2=0}^{2} \Pr(Y = k_2|X) \log_2 (\Pr(Y = k_2|X)) \tag{8}$$

$$= -3 \times \frac{1}{3} (\alpha \log_2 \alpha + (1 - \alpha) \log_2 (1 - \alpha)) \tag{9}$$

Using (6) and (9) in (2)

$$I(X, Y) = \log_2 3 + \alpha \log_2 \alpha + (1 - \alpha) \log_2 (1 - \alpha) \tag{10}$$

$$\implies \frac{d}{d\alpha} I(X, Y) = (\log_2 \alpha - \log_2 (1 - \alpha)) \tag{11}$$

For maxima or minima $\frac{d}{d\alpha} I(X, Y) = 0$

$$\log_2 \alpha - \log_2 (1 - \alpha) = 0 \tag{12}$$

$$\alpha = \frac{1}{2} \tag{13}$$

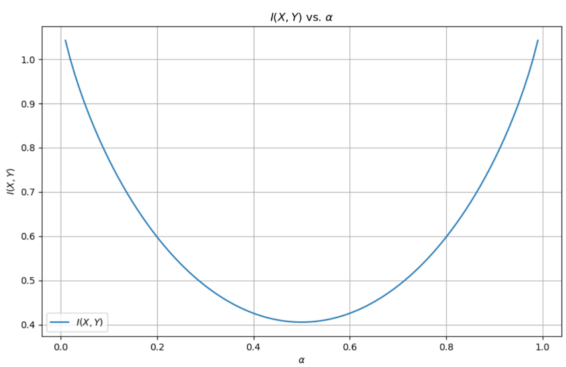
$$\frac{d^2}{d\alpha^2} I(X, Y) = \frac{1}{\log 2} \left(\frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) \tag{14}$$

$$\frac{d^2}{d\alpha^2} I(X, Y)\Big|_{\alpha=\frac{1}{2}} = \frac{4}{\log 2} \tag{15}$$

$$\geq 0 \tag{16}$$

At $\alpha = \frac{1}{2}, I(X, Y)$ is minimum
As $I(X, Y)$ is maximum at $\alpha = 0, 1$
But $\alpha \in [0.25, 1]$,

$$\alpha = 1 \tag{17}$$


$I(X, Y)$ vs. α
1.0
0.9
0.8
0.7
0.6
0.5
0.4
$I(X, Y)$
0.0
0.2
0.4
0.6
0.8
1.0
α
$I(X, Y)$