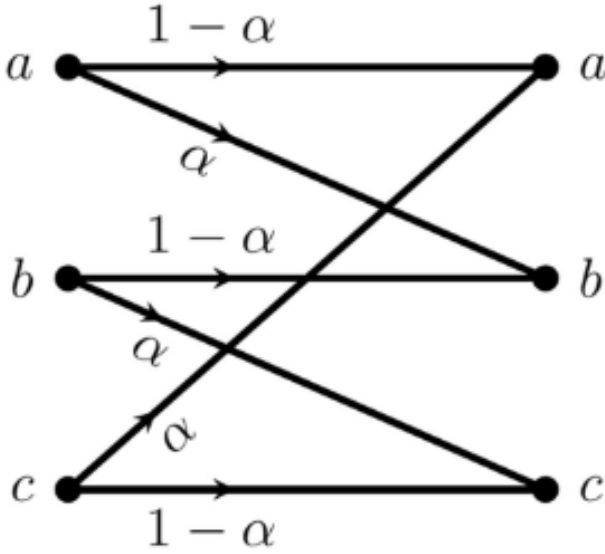


Assignment

Dhruv Parashar-EE22BTECH11019

Question:- The transition diagram of a discrete memoryless channel with three input symbols and three output symbols is shown in the figure. The transition probabilities are as marked. The parameter α lies in the interval $[0.25, 1]$. The value of α for which the capacity of this channel is maximized, is

Solution:



Variable	Description	Value
X	Input variable	$X = \{0, 1, 2\}$
Y	Output variable	$Y = \{0, 1, 2\}$
C	Channel Capacity	C
I	Mutual Information	I
H	Entropy	H

Now,

$$\Pr(Y = 0) = \Pr(Y = 1) = \Pr(Y = 2) = \frac{1}{3} \quad (3)$$

$$H(Y) = - \sum_{k=0}^2 \Pr(Y = k) \log_2 \Pr(Y = k) \quad (4)$$

$$= -\frac{1}{3} \log_2 \frac{1}{3} \times 3 \quad (5)$$

$$= \log_2 3 \quad (6)$$

$$H(Y|X) = - \sum_{k_1=0}^2 \sum_{k_2=0}^2 \Pr(X = k_1) \Pr(Y = k_2|X) \log_2 (\Pr(Y = k_2|X)) \quad (7)$$

$$= - \sum_{k_1=0}^2 \Pr(X = k_1) \sum_{k_2=0}^2 \Pr(Y = k_2|X) \log_2 (\Pr(Y = k_2|X)) \quad (8)$$

$$= -3 \times \frac{1}{3} (\alpha \log_2 \alpha + (1 - \alpha) \log_2 (1 - \alpha)) \quad (9)$$

Using (6) and (9) in (2)

$$I(X, Y) = \log_2 3 + \alpha \log_2 \alpha + (1 - \alpha) \log_2 (1 - \alpha) \quad (10)$$

$$\Rightarrow \frac{d}{d\alpha} I(X, Y) = (\log_2 \alpha - \log_2 (1 - \alpha)) \quad (11)$$

For maxima or minima $\frac{d}{d\alpha} I(X, Y) = 0$

$$\log_2 \alpha - \log_2 (1 - \alpha) = 0 \quad (12)$$

$$\alpha = \frac{1}{2} \quad (13)$$

$$\frac{d^2}{d\alpha^2} I(X, Y) = \frac{1}{\log 2} \left(\frac{1}{\alpha} + \frac{1}{1 - \alpha} \right) \quad (14)$$

$$\frac{d^2}{d\alpha^2} I(X, Y)|_{\alpha=\frac{1}{2}} = \frac{4}{\log 2} \quad (15)$$

$$\geq 0 \quad (16)$$

At $\alpha = \frac{1}{2}$, $I(X, Y)$ is minimum

As $I(X, Y)$ is maximum at $\alpha = 0, 1$ But $\alpha \in [0.25, 1]$,

$$\alpha = 1 \quad (17)$$

$$C = \max_{p(X,Y)} I(X, Y) \quad (1)$$

$$I(X, Y) = H(Y) - H(Y|X) \quad (2)$$

