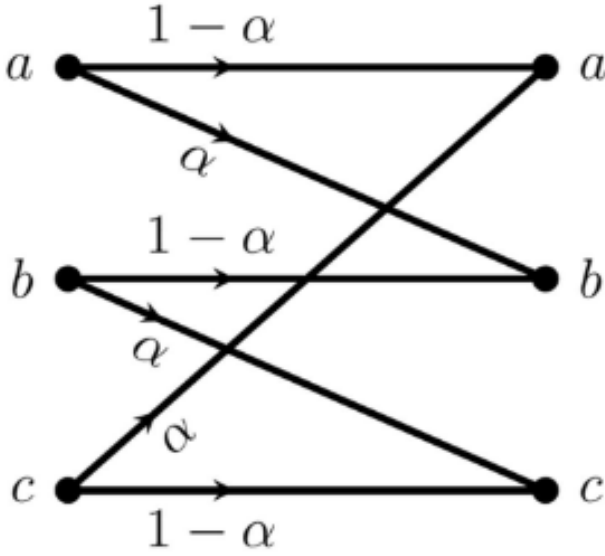


# Assignment

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Question:- The transition diagram of a discrete memoryless channel with three input symbols and three output symbols is shown in the figure. The transition probabilities are as marked. The parameter  $\alpha$  lies in the interval  $[0.25, 1]$ . The value of  $\alpha$  for which the capacity of this channel is maximized, is

**Solution:**



Variable	Description	Value
$X$	Input variable	$X = \{0, 1, 2\}$
$Y$	Output variable	$Y = \{0, 1, 2\}$
$C$	Channel Capacity	$C$
$I$	Mutual Information	$I$
$H$	Entropy	$H$

$$C = \max_{p(X,Y)} I(X, Y)$$

$$\begin{aligned}
 I(X, Y) &= \sum_{x,y} p(x, y) \log_2 \frac{p(x, y)}{p(x) p(y)} \\
 &= \sum_{x,y} p(x, y) \log_2 \frac{p(y|x)}{p(y)} \\
 &= - \sum_{x,y} p(x, y) \log_2 p(y) + \sum_{x,y} p(x, y) \log_2 p(y|x) \\
 &= - \sum_y p(y) \log_2 p(y) - \left( - \sum_{x,y} p(x, y) \log_2 p(y|x) \right) \\
 &= H(Y) - H(Y|X)
 \end{aligned}$$

Now,

$$p_X(0) = p_X(1) = p_X(2) = \frac{1}{3} \quad (7)$$

For  $X = 0$

$$p_Y(0) = \sum_{y=0}^2 p_{Y|X}(y|0) p_X(0) \quad (8)$$

$$= \frac{1}{3} \sum_{y=0}^2 p_{Y|X}(y|0) \quad (9)$$

$$= \frac{1}{3} \quad (10)$$

Similarly for  $X = 1, 2$

$$p_Y(0) = p_Y(1) = p_Y(2) = \frac{1}{3} \quad (11)$$

$$H(Y) = - \sum_{y=0}^2 p_Y(y) \log_2 p_Y(y) \quad (12)$$

$$= - \frac{1}{3} \log_2 \frac{1}{3} \times 3 \quad (13)$$

$$= \log_2 3 \quad (14)$$

$$H(Y|X) = - \sum_{x=0}^2 \sum_{y=0}^2 p_X(x) p_{Y|X}(y|x) \log_2 (p_{Y|X}(y|x)) \quad (15)$$

$$= - \sum_{x=0}^2 p_X(x) \sum_{y=0}^2 p_Y(y|X) \log_2 (p_Y(y|X)) \quad (16)$$

$$= -3 \times \frac{1}{3} (\alpha \log_2 \alpha + (1 - \alpha) \log_2 (1 - \alpha)) \quad (17)$$

Using (14) and (17) in (6)

$$I(X, Y) = \log_2 3 + \alpha \log_2 \alpha + (1 - \alpha) \log_2 (1 - \alpha) \quad (18)$$

$$\Rightarrow \frac{d}{d\alpha} I(X, Y) = (\log_2 \alpha - \log_2 (1 - \alpha)) \quad (19)$$

$$(1) \text{ For maxima or minima } \frac{d}{d\alpha} I(X, Y) = 0$$

$$(2) \text{ At } \alpha = \frac{1}{2}, I(X, Y) \text{ is minimum}$$

$$(2) \text{ As } I(X, Y) \text{ is maximum at } \alpha = 0, 1$$

$$\text{But } \alpha \in [0.25, 1],$$

$$(3) \quad \alpha = 1 \quad (20)$$

