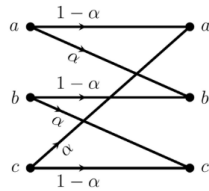


Answer

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Question:- The transition diagram of a discrete memoryless channel with three input symbols and three output symbols is shown in the figure. The transition probabilities are as marked.

The parameter α lies in the interval $[0.25, 1]$. The value of α for which the capacity of this channel is maximized, is **Solution:**



Variable	Description	Value
x_i	Input	x_0, x_1, x_2
y_i	Output	y_0, y_1, y_2
p_i	Input probability	p_0, p_1, p_2
q_i	Output probability	q_0, q_1, q_2
C	Channel Capacity	C
I	Mutual Information	I
H	Entropy	H

$$C = \sup_{p_X(x)} I(X, Y) \quad (1)$$

$$I(X, Y) = \sum_{x,y} p(x, y) \log_2 \frac{p(x, y)}{p(x)p(y)} \quad (2)$$

$$= \sum_{x,y} p(x, y) \log_2 \frac{p(y|x)}{p(y)} \quad (3)$$

$$= - \sum_{x,y} p(x, y) \log_2 p(y) + \sum_{x,y} p(x, y) \log_2 p(y|x) \quad (4)$$

$$= - \sum_y p(y) \log_2 p(y) - \left(- \sum_{x,y} p(x, y) \log_2 p(y|x) \right) \quad (5)$$

$$= H(Y) - H(Y|X) \quad (6)$$

Now,

$$\sum_{i=0}^2 p_i = 1 \quad (7)$$

$$\sum_{j=0}^2 q_j = 1 \quad (8)$$

$$H(\mathbf{q}) = - \sum_{i=0}^2 q_i \log_2 q_i \quad (9)$$

$$= - (q_0 \log_2 q_0 + q_1 \log_2 q_1 + q_2 \log_2 q_2) \quad (10)$$

$$H(Y|X) = - \sum_{i=0}^2 \sum_{j=0}^2 p_i p_{Y|X}(y_j|x_i) \log_2 (p_{Y|X}(y_j|x_i)) \quad (11)$$

$$= -p_0((1-\alpha) \log_2(1-\alpha) + \alpha \log_2 \alpha) \\ - p_1((1-\alpha) \log_2(1-\alpha) + \alpha \log_2 \alpha) \\ - p_2((1-\alpha) \log_2(1-\alpha) + \alpha \log_2 \alpha) \quad (12)$$

Using (10) and (12) in (6)

$$I(X, Y) = - (q_0 \log_2 q_0 + q_1 \log_2 q_1 + q_2 \log_2 q_2) \\ + p_0((1-\alpha) \log_2(1-\alpha) + \alpha \log_2 \alpha) \\ + p_1((1-\alpha) \log_2(1-\alpha) + \alpha \log_2 \alpha) \\ + p_2((1-\alpha) \log_2(1-\alpha) + \alpha \log_2 \alpha) \quad (13)$$

$$\Rightarrow \frac{d}{d\alpha} I(X, Y) = p_0 \log_2 \left(\frac{\alpha}{1-\alpha} \right) + p_1 \log_2 \left(\frac{\alpha}{1-\alpha} \right) \\ + p_2 \log_2 \left(\frac{\alpha}{1-\alpha} \right) \quad (14)$$

For Maxima or minima $\frac{d}{d\alpha} I(X, Y) = 0$

$$\log_2 \left(\frac{\alpha}{1-\alpha} \right) (p_0 + p_1 + p_2) = 0 \quad (15)$$

$$\Rightarrow \alpha = \frac{1}{2} \quad (16)$$