Answer

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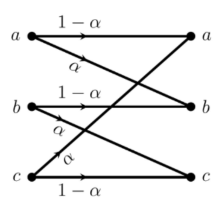
Question:- The transition diagram of a discrete memoryless channel with three input symbols and three output symbols is shown in the figure. The transition probabilities are as marked.
The parameter α lies in the interval $[0.25, 1]$. The value of α for which the capacity of this channel is maximized, is **Solution:**



Variable	Description	Value
x_i	Input	x_0, x_1, x_2
y_i	Output	y_0, y_1, y_2
p_i	Input probability	p_0, p_1, p_2
q_i	Output probability	q_0, q_1, q_2
C	Channel Capacity	C
I	Mutual Information	I
H	Entropy	H

$$C = \max_{p_X(x)} I(X, Y) \tag{1}$$

$$I(X, Y) = \sum_{x,y} p(x, y) \log_2 \frac{p(x, y)}{p(x)\,p(y)} \tag{2}$$

$$= \sum_{x,y} p(x, y) \log_2 \frac{p(y|\, x)}{p(y)} \tag{3}$$

$$= -\sum_{x,y} p(x, y) \log_2 p(y) + \sum_{x,y} p(x, y) \log_2 p(y|\, x) \tag{4}$$

$$= -\sum_{y} p(y) \log_2 p(y) - \left(-\sum_{x,y} p(x, y) \log_2 p(y|\, x)\right) \tag{5}$$

$$= H(Y) - H(Y|\, X) \tag{6}$$

Now,

$$\sum_{i=0}^{2} p_i = 1 \tag{7}$$

$$\sum_{j=0}^{2} q_i = 1 \tag{8}$$

$$H(\mathbf{q}) = -\sum_{i=0}^{2} q_i \log_2 q_i \tag{9}$$

$$= -\left(q_0 \log_2 q_0 + q_1 \log_2 q_1 + q_2 \log_2 q_2\right) \tag{10}$$

$$H(Y|\, X) = -\sum_{i=0}^{2} \sum_{j=0}^{2} p_i p_{Y|X}\left(y_j|\, x_i\right) \log_2 \left(p_{Y|X}\left(y_j|\, x_i\right)\right) \tag{11}$$

$$\begin{aligned} &= -p_0 \left((1-\alpha) \log_2 (1-\alpha) + \alpha \log_2 \alpha\right) \\ &\quad - p_1 \left((1-\alpha) \log_2 (1-\alpha) + \alpha \log_2 \alpha\right) \\ &\quad - p_2 \left((1-\alpha) \log_2 (1-\alpha) + \alpha \log_2 \alpha\right) \end{aligned} \tag{12}$$

Using (10) and (12) in (6)

$$\begin{aligned} I(X, Y) &= -\left(q_0 \log_2 q_0 + q_1 \log_2 q_1 + q_2 \log_2 q_2\right) \\ &\quad + p_0 \left((1-\alpha) \log_2 (1-\alpha) + \alpha \log_2 \alpha\right) \\ &\quad + p_1 \left((1-\alpha) \log_2 (1-\alpha) + \alpha \log_2 \alpha\right) \\ &\quad + p_2 \left((1-\alpha) \log_2 (1-\alpha) + \alpha \log_2 \alpha\right) \end{aligned} \tag{13}$$

$$\begin{aligned} \implies \frac{d}{d\alpha} I(X, Y) &= p_0 \log_2 \left(\frac{\alpha}{1-\alpha}\right) + p_1 \log_2 \left(\frac{\alpha}{1-\alpha}\right) \\ &\quad + p_2 \log_2 \left(\frac{\alpha}{1-\alpha}\right) \end{aligned} \tag{14}$$

For Maxima or minima $\frac{d}{d\alpha} I(X, Y) = 0$

$$\log_2 \left(\frac{\alpha}{1-\alpha}\right)(p_0 + p_1 + p_2) = 0 \tag{15}$$

$$\implies \alpha = \frac{1}{2} \tag{16}$$