MEAN FIELD GAMES

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MEAN FIELD GAMES-

CLASSICAL MODEL

Mean field game (MFG) theory is a mathematical framework that studies strategic interactions among a large number of agents or players who are making decisions based on the aggregate behaviour of the entire population. This theory is useful when the number of players is very large such that the contributions of individual interactions are negligible.

The term "mean field" refers to the average state of the population, which influences each agent's strategy without requiring knowledge of specific interactions with other agents.

The mathematical equations involve typically two key partial differential equations-

Hamilton-Jacobi-Bellman (HJB) Equation: This describes the optimal control problem for an individual agent.

The general form can be represented as:

$$-\partial_t u - v\Delta u + H(x,m,Du) = 0$$

where u is the value function, H is the Hamiltonian representing the cost structure, m is the distribution of agents, and Du is the gradient of the value function.

Fokker-Planck Equation: This models the evolution of the distribution of agents over time, reflecting how the aggregate behaviour changes as agents make decisions.

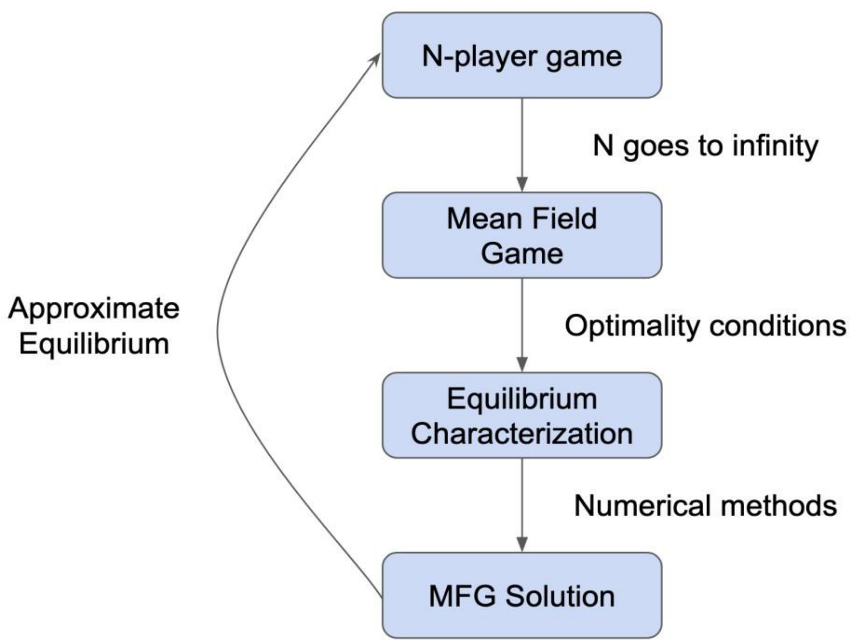
Its form generally looks like:

$$\partial_t m - v\Delta m - \mathrm{div}(D_pH(x,m,Du)m) = 0$$

where DpH denotes the derivative of the Hamiltonian with respect to its momentum variable.

Key factors influencing the classical model

- **STATE VARIABLES**- These are the conditions affecting the players in the games. It includes position (x) and time (t). These variables define how agents perceive their environment.
- **CONTROL VARIABLES**-These are the strategies that players choose to influence their future states. It is Denoted as α(t,x), these are the decisions made by an agent at time t while in state x. The choice of action directly impacts the agent's trajectory and outcomes within the game. Its aim is to maximise the payoff based on their current state and the distribution of other agents.
- **PAYOFF FUNCTION**- The payoff function quantifies the benefits or costs associated with specific actions taken by agents.
  General Form: $L(s,X_s,\alpha_s,m(s))$
  S: Time
  X_s: State of the agent at time s
  α_s: Control action taken by the agent at time
  $m(s)$: The population density at time s



Application of Classical Model in Traffic Flow Management

In urban environments, MFGs can model how drivers choose their routes based on the average traffic conditions rather than individual interactions. Each driver aims to minimize their travel time, leading to an equilibrium state where traffic distribution is optimized. The model incorporates:

- State Variables: Position and velocity of vehicles.
- Control Variables: Route and speed choices.
- Payoff Function: Minimization of travel time, which helps in predicting traffic patterns and reducing congestion.

HJB equation governs the optimal strategy for each driver:

$$-\partial tu(t,x)-v\Delta u(t,x)+H(x,m(t,x),Du(t,x))=0$$

where :

$u(t,x)$: Value function representing the minimum expected travel time starting from position x at time t.

$H(x,m(t,x),p)$: Hamiltonian function that captures the cost associated with actions taken by drivers, often formulated as:

$$H(x,m,p) = [p2+2c(m)]/2$$

where c(m) represents congestion costs depending on the average density of vehicles m(t,x).

The evolution of vehicle density over time is described by a Fokker-Planck equation.

$$\partial tm(t,x)-v\Delta m(t,x)-\mathrm{div}(DpH(x,m(t,x),Du(t,x))m(t,x))=0$$

This equation captures how changes in individual strategies affect the overall distribution of vehicles on the road.

The interplay between the HJB and Fokker-Planck equations allows for deriving optimal control strategies for drivers. Each driver adjusts their speed and route based on both their individual state and the average traffic conditions predicted by others' choices. This leads to a Nash equilibrium where no driver can unilaterally improve their travel time.

SECOND ORDER GAMES

Second order mean field games are a type of differential game that describes the strategic interactions between a large number of small, indistinguishable players. Second-order MFGs typically involve agents whose dynamics are governed not just by first-order differential equations but by second-order dynamics, which can capture more complex behaviors . These systems are particularly relevant in modeling scenarios involving large populations of indistinguishable agents whose interactions can be described through their collective behavior.

This system consists of two PDEs, The Hamilton-Jacobi-Bellman (HJB) equation and the Fokker-Planck(FP) equation, the two equations are coupled which in turn reflect the interdependencies between individual agent strategies and the overall population dynamics.

Key Features of Second Order MFGs

Degenerate Diffusion- The case where the diffusion coefficient can approach zero in certain regions. This complicates the analysis and solution of the associated PDEs because it may lead to non-uniform parabolic behavior. The distinguishing features of this model considered are

- It is not uniformly parabolic, including the first order case as a possibility.
- The coupling is a local operator on the density.

Local Coupling- In this model local coupling operators which depend on density are used most often. This helps in a more realistic representation of interactions when working with dense populations.

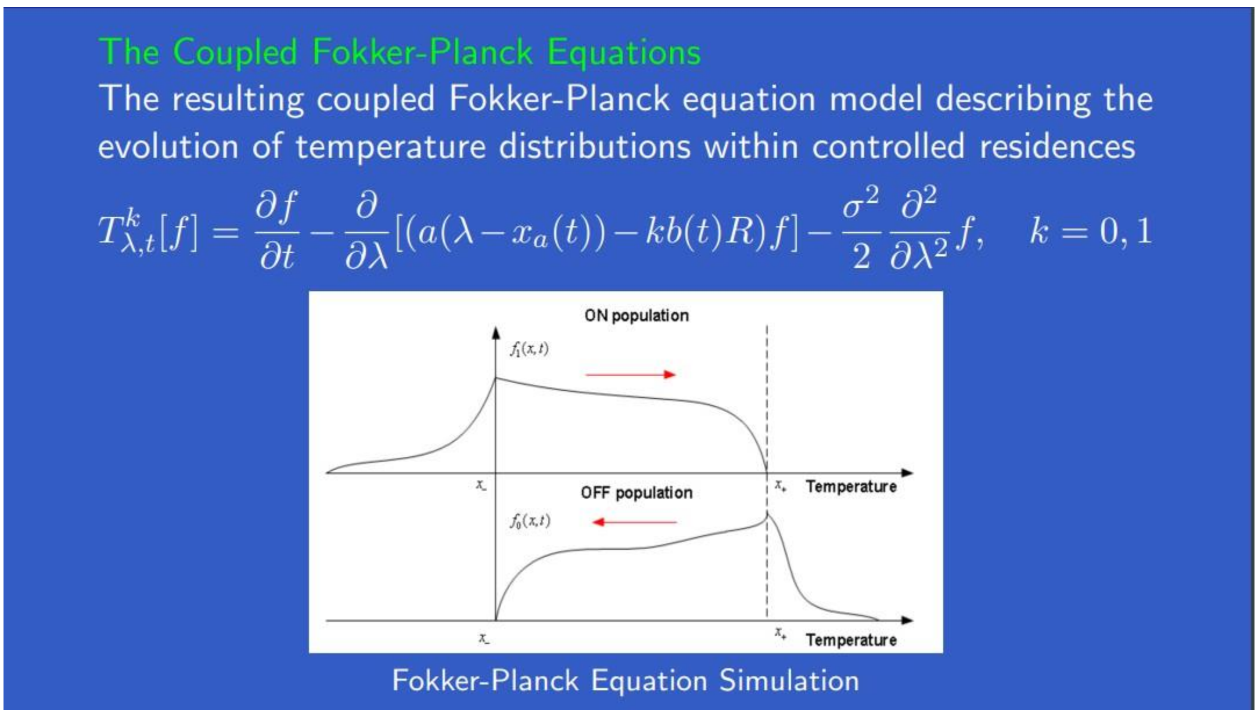
Weak Solutions- A differential equation may have solutions that are not differentiable, a weak solution is a concept of solution to a partial differential equation that may not be differentiable but satisfies the equation in an integral sense.

Application of Second Order Mean Field Theory in Energy systems and grid managements

In energy management systems MFGs can be used to adjust their acceleration in energy consumption or production to optimize grid stability and efficiency based on the supply and demand of the item.

- State Variables: Energy consumption, energy production
- Control Variables: Acceleration or rate of change of energy consumption or product.
- Payoff Function: Helps in adjusting to the changes in energy usage and maintaining a stable energy grid.

Diffusion model of heating/cooling loads-



Results:

- The optimal control problem becomes one of controlling PDEs using on-off signals.
- A fraction of customers is inevitably penalized.
- The smaller this fraction, the less effective the control is.

Some challenges in applying mean field games to energy grid management are-

- **Complexity in modelling interactions** -Considering the stochastic behaviour of a vast number of interacting agents is complex, MFGs rely on a correct statistical description of these interactions; any inaccuracies can lead to suboptimal control strategies.
- **Scalability issues** - As the number of agents increases, computational demands grow significantly. MFGs aim to provide decentralized solutions, but ensuring that these solutions scale effectively while maintaining performance is a critical challenge
- **Heterogeneity among agents** - Agents in an energy system may have diverse characteristics and objectives, leading to heterogeneous behaviors. Designing MFG frameworks that can accommodate this diversity while still achieving collective goals poses significant difficulties.
- **Communication constraints** - Effective implementation of MFGs often requires significant data exchange between agents and a central authority or aggregator. Minimizing communication while ensuring that agents can still make informed decisions is crucial. Reducing communication overhead without sacrificing performance is a persistent challenge in large-scale applications.
- **Dynamic and uncertain environment** - The energy grid operates in an environment characterized by uncertainty, particularly with the integration of renewable energy sources like solar and wind power. The intermittency and variability of these sources complicate the modeling and control processes within MFG frameworks
- **Local constraints VS Global objectives** - Balancing local constraints (such as individual comfort or operational limits) with glbal objectives (such as overall grid stability or efficiency) is challenging. MFGs must be designed to account for these competing interests without compromising system performance.

Hence there was a need to develop robust strategies that can adapt to dynamic environments and diverse agent behaviors. Developing methods that ensure reliable performance under varying conditions is essential for practical applications.

Infinite horizon

The MFGs where the game continues eternally into the future are referred to as having an infinite horizon. This makes it difficult to define strategies and rewards.

Key Differences between Infinite and Finite MFGs

Time frame, mathematical formulation, solution properties, and applications are some of the main distinctions between finite and infinite MFGs.

- **Time horizon**

  Games in finite horizon MFGs are defined over a predetermined, time-limited period, and players base their choices on this time constraint. In contrast, games that span an infinite amount of time into the future are included in infinite horizon MFGs, where players' strategies are assessed based on long-term performance rather than a particular goal.

- **Mathematical formulation**

  Finite horizon MFGs often utilize optimal control theory, leading to systems of equations that include Hamilton-Jacobi-Bellman (HJB) equations and Fokker- Planck equations. The solutions are usually derived under the assumptions about dynamics and payoffs being over a finite period of time. In infinite horizon MFGs players aim to minimize long-term average costs or maximize long-term rewards and this often involves discounted costs or ergodic formulations. The probabilistic weak formulation is also often employed to handle complexities arising from infinite time frames.

- **Features of the solution**

  Existence and uniqueness: While infinite horizon games face difficulties like singular measures and lack of compactness in solution spaces, and necessitate innovative methods to establish comparable results, finite horizon games typically have well-established results regarding the existence and uniqueness of solutions.

Convergence properties: Solutions for a specified period can be determined immediately in situations with finite horizons. Infinite horizon MFGs, on the other hand, examine how finite horizon game solutions converge to infinite horizon game solutions over time.

- **Applications**

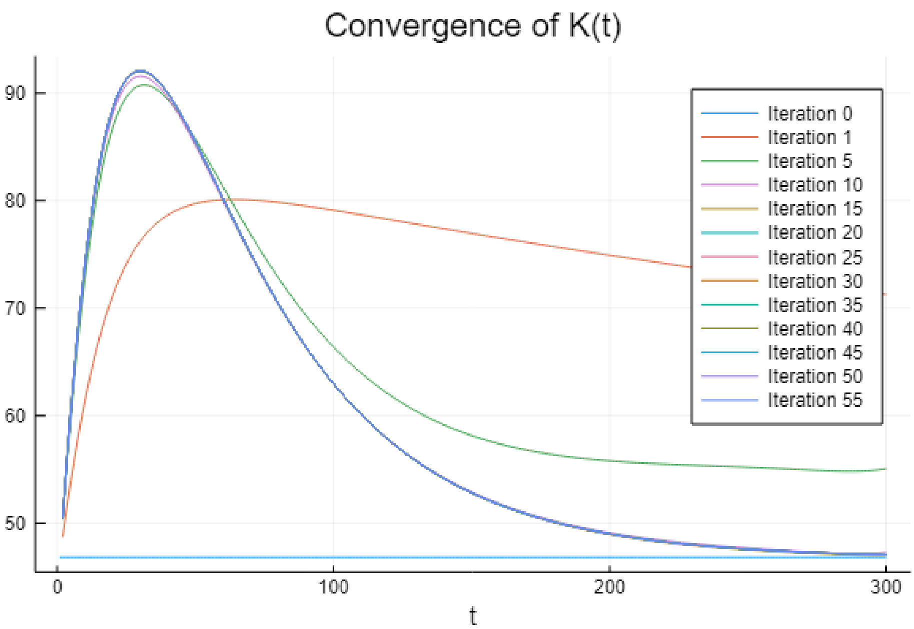
  While infinite horizon games have broader applications in domains like economics and finance where long-term strategies are essential, finite horizon games are frequently used in situations where decisions must be made within a given timeframe, such as project management or resource allocation over a limited period. They are employed to represent situations that don't have a predefined endpoint, like the best way to extract resources or invest.

Application of Infinite Horizon MFG in Macroeconomics

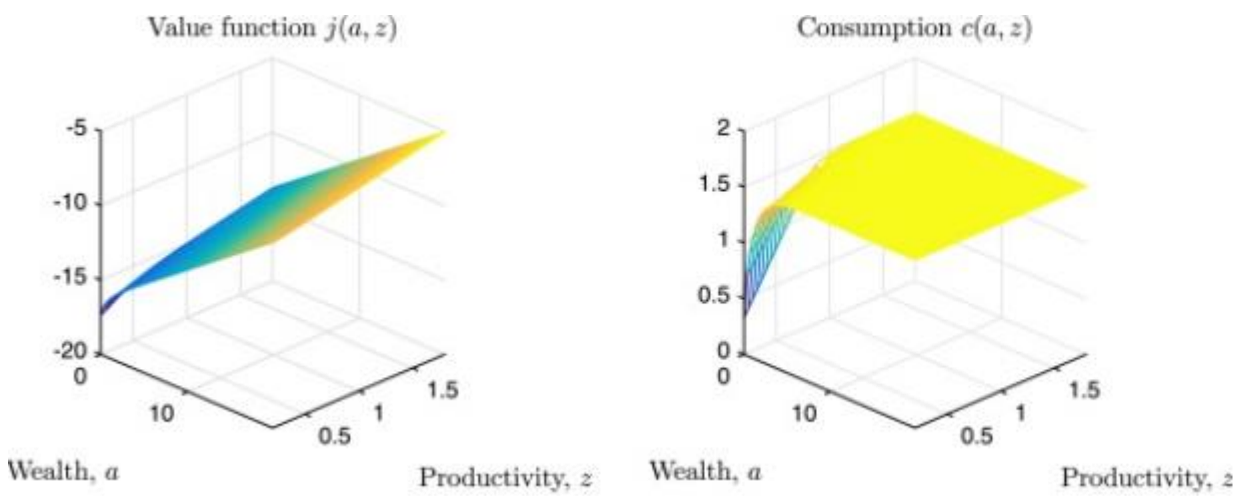
Infinite horizon MFGs are particularly useful in macroeconomic models which involve agents making decisions about consumption, savings, and investment over an indefinite time frame.

- **Heterogeneous Agent Models**: These models depict how several agents, such as employees or investors, behave when interacting with one another. Policymakers can gain a better understanding of the dynamics of wealth distribution and economic growth by examining how individual decisions add up to impact the economy as a whole.

- **Aiyagari Model**: This model examines how people save money in an economy with incomplete markets. Here, agents must decide how much to save for future spending in the face of idiosyncratic income shocks. Studying long-term effects on capital accumulation and income distribution is made possible by the infinite horizon approach.



Iterations of Aiyagari model with aggregate uncertainty



Social optima in economies with heterogenous agents

The equations used here are:

- Hamilton-Jacobi-Bellman (HJB) Equation

The HJB equation is used to determine the optimal control strategies for agents in MFGs. It can be expressed as:

$$-\partial V/\partial t + H(x,\nabla V)=0,$$

where:

$V(t,x)$ is the value function,

H is the Hamiltonian, which expresses the dynamics of the game and the costs associated with actions.

- Fokker-Planck Equation

This equation describes the evolution of the distribution of states among agents. It is coupled with the HJB equation and takes the form:

$$\partial \rho/\partial t+\nabla\cdot(\rho b(x))=0,$$

where:

$\rho(t,x)$ is the density of agents in state space,

$b(x)$ represents the drift or flow of agents.

- Forward-Backward Stochastic Differential Equations (FBSDEs)

FBSDEs are often used to model the dynamics of agents in infinite horizon MFGs.
A typical formulation is:

$dX_t=b(t,X_t,Y_t,L(X_t,Y_t))dt+\sigma W_t,$

$dY_t=-f(t,X_t,Y_t,L(X_t,Y_t))dt+Z_t dW_t,$

where:

Xt denotes the state process (eg:- wealth or consumption),

Yt is the adjoint process (represents value or utility),

$L(Xt,Yt)$ is a function representing the law of the state process,

Wt is a Brownian motion.

- Cost Functional

The cost functional that agents aim to minimize and can be represented as:

$$J(u)=E[\int_0^{+\infty} e^{-\beta t} g(X_t,u(t))dt],$$

where:

$g(Xt,u(t))$ is a running cost function depending on the state and control action,

$\beta>0$ is a discount factor reflecting time preference.

- Invariant Mean Field Game Equations

In an invariant mean field game equation marginal distributions of state variables remain constant over time. This is expressed as

$$V(S,D)=\psi\max(u(\psi,S)+E[V(S',D')|S,D])$$

D represents the distribution of agents across states,

$V(S,D)$ is defined on an infinite-dimensional state space.

- Stationary Mean Field Game Framework

$$V(S,D)=u(S)+E[V(S',D')|S,D]$$

This equation helps analyze how agents' decisions affect long-term distribution without fixing initial conditions.

By integrating FBSDEs, HJB equations, Fokker-Planck equations, and cost functional, interactions among heterogeneous agents over an indefinite time period can be analysed.

Robustness in Mean Field Games

Refers to the resilience of mean field equilibria against model uncertainties. This includes uncertainties in state dynamics and payoff structures, which can significantly affect the outcomes of the game.

Key differences between continuous time and discrete time mean field games in terms of robustness:

Parameter	Continuous- time	Discrete- time
Modelling framework	Involves stochastic DE’s where agent dynamics is influenced by continuous motion like brownian motion. Robustness in this framework focuses on how equilibria can withstand perturbations in these stochastic dynamics.	In contrast, discrete-time MFGs operate on a sequence of time steps where agents make decisions based on the current state and the average behavior of others. The robustness of equilibria in this setting is often evaluated through algorithms like value iteration, which must converge despite potential model misspecifications.
Convergence conditions	The convergence of solutions and equilibria is often guaranteed under certain regularity conditions related to the underlying stochastic processes. The literature emphasizes the importance of Lipschitz continuity and other smoothness conditions to ensure robustness against perturbations	Recent findings highlight that discrete-time MFGs can achieve robust equilibria through well-defined convergence conditions for value iteration algorithms. These algorithms have been shown to yield stable results even when the underlying model is not perfectly specified, provided that the state space is finely quantized
Impact of uncertainty	Uncertainties typically arise from continuous disturbances in states or payoffs, and robustness is analyzed using tools from stochastic control	The impact of uncertainty is examined through a more structured approach, where agents' decisions are made at discrete intervals. This allows

	theory. The focus is on how agents adapt their strategies in response to these uncertainties over continuous time	for a clearer analysis of how model approximations affect the stability of equilibria.
Applications	Robustness in continuous - time MFGs is crucial where agents continuously adapt to changing environment Example - Finance and resource management.	Robustness in discrete-time models are relevant where decisions are made at specific intervals.The robustness findings here support decentralized control strategies that can effectively manage large populations under uncertainty Example- Network systems, game - based learning environment.

Banach fixed point theorem

It provides conditions under which a mapping has a unique fixed point and guarantees convergence to that fixed point through iterative processes.

Let (X,d) be a complete metric space, and let $T:X\to X$ be a contraction mapping. This means there exists a constant $0<k<1$ such that for all $x,y\in X$

$$d(T(x),T(y))\leq k\cdot d(x,y)$$

Then, the Banach Fixed Point Theorem states:

1. Existence of a Fixed Point: There exists a unique point $x*\in X$ such that $T(x*)=x*$

2. Convergence: For any initial point $x_0\in X$, the sequence defined by $x_{n+1}=T(x_n)$ converges to the fixed point $x*$.

Importance of Theorem

- Existence and Uniqueness: It guarantees not only that a fixed point exists but also that it is unique, which is crucial in many applications.
- Iterative Methods: The theorem provides a constructive method to find fixed points through iteration, making it useful in numerical methods and algorithms.

Problem statement - Let $X=[0,1]$ with the standard metric $d(x,y)=|x-y|$. Define the mapping $T:X\rightarrow X$ by:

$$T(x)= x/2$$

- Verifying contraction:

  For any $x,y\in[0,1]$:

  $d(T(x),T(y))=|T(x)-T(y)| = |x/2-y/2| = ½|x-y| = k|x-y|$

  here $k=½ <1$,hence T is a contraction

- Apply Banach's Fixed Point Theorem:
  Since (X,d) is complete and T is a contraction, there exists a unique fixed point $x*$.
- Finding the Fixed Point:

  Solve $T(x*)=x*$

  $x^*=x^*/2$

  $x^*=0$

- Convergence of iterates:

  Starting from any initial point $x0=1$:

  1. $x1=T(1)=0.5$
  2. $x2=T(2)=0.25$
  3. $x3=T(3)=0.125$
  4. Continuing this process shows that the sequence converges to the fixed point $x^*=0$.

Contraction Mapping

The contraction mapping property is a fundamental concept in mathematical analysis that plays a crucial role in establishing convergence in mean-field games (MFGs).

This property is particularly useful when analyzing algorithms designed to find equilibria in MFGs, especially under conditions of uncertainty and dynamic interactions among agents.

Definition - A mapping T is said to be a contraction if there exists a constant $0<k<1$ such that for any two points x and y in its domain, the following holds:

$$\|T(x)-T(y)\| \leq k\|x-y\|$$

This means that the mapping brings points closer together, which is essential for ensuring that repeated applications of the mapping will converge to a single point.

For instance, when establishing a mean-field equilibrium (MFE), one can define an operator H that maps a mean-field state to another mean-field state based on the optimal policies of agents.

Existence of fixed points

By the Banach Fixed Point Theorem, if His a contraction, it has a unique fixed point. This fixed point corresponds to the mean-field equilibrium, meaning that there exists a stable distribution of strategies among agents as the population size approaches infinity.

Effect on robustness

The contraction property enhances robustness against disturbances or uncertainties in system dynamics. When operators are contractive, small perturbations in initial conditions or model parameters do not significantly affect the final outcome, ensuring stability in equilibrium solutions even under variable conditions.

Problem setup- Consider a mean-field game involving a large number of identical agents who aim to minimize their costs over a finite time horizon. Each agent's cost function depends on its own action and the average behavior of the population.

Model

1. Stochastic Differential equation - $dXt = utdt + \sigma dWt$

where ut is the control action chosen by the agent, σ is the volatility, and Wt is a standard Wiener process.

2. Cost Function: J(u) = $[\int_0^T ((Xt - Xd)^2$ + R(Ut)^2)dt]

where Xd is a desired state, and R>0 is a weighing factor for the control effort.

3. Mean Field Interaction: The mean field m(t) represents average state of all agents at time t, which influences each agent's dynamics and cost.

Solution using contraction mapping

1. Define the operator: H(m(t)) = u*(t,m(t))

   u*(t,m(t)) is derived from solving the HJB equation associated with the cost function.

   Operator H maps the current mean field m(t) to a new mean field based on optimal controls

2. Establishing Contraction: For two different mean fields $m1(t)$ and $m2(t)$ we demonstrate $||H(m1(t)) - H(m2(t))|| \leq k||m1(t)-m2(t)||$

   For some k<1.

3. Iterative scheme: Starting from an initial guess for the mean field m0(t), iterate using:

$$mn+1(t) = H(mn(t))$$

This iterative scheme converges to a fixed point due to the contraction property, which corresponds to the mean-field equilibrium. Using the Euler method for stochastic Differential equations we can compute the dynamics and update the mean field iteratively until convergence is achieved.

4. Convergence check: Monitor the difference between successive iterations

$$||m_{n+1}(t) - m_n(t)|| < \epsilon$$

where $\epsilon > 0$ is a small threshold indicating convergence.

Results:

- The optimal control policy for each agent.
- The resulting mean field that describes the average behavior of all agents over time.
- Verification that this policy leads to minimizing costs as per the defined cost function.

Conclusion

A strong framework for examining strategic interactions between a large number of agents, where each agent's choices are impacted by the group's overall behavior, is provided by mean-field game (MFG) theory. The study of continuous-time dynamics and optimal control issues is made easier by this method, which successfully reduces the complexity of classical game theory by approximating the behavior of a representative agent in a large population.

Mean field games (MFG) research is thriving and diverse right now, with a number of important areas of concentration. There is a notable integration of reinforcement learning with MFGs to effectively learn equilibria in complex multi-agent systems, alongside explorations of mean field Stackelberg games to understand hierarchical decision-making between leaders and large groups of followers. Additionally, researchers are analyzing agent behavior in MFG situations using probabilistic techniques such as forward-backward stochastic differential equations. Additionally, MFG theory is being applied to cyber security, optimizing defense strategies against widespread threats by modeling interactions among agents. In order to handle more complicated situations, such as time-dependent policies and non-homogeneous populations, efforts are being made to generalize MFG frameworks. Finally, interdisciplinary collaboration through workshops is enhancing numerical methods and theoretical formulations across various fields. Collectively, these investigations highlight the growing significance of mean field games in addressing contemporary challenges across diverse sectors.

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