Rotational Motion

Quarry Lane Physics Club

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Rotational Motion

Rotational Kinematics

Radians

Definition: An angle in radians is defined as $\frac{l}{r}$, where l is the arclength of the arc that some angle, θ, subtends.

Units: rad (unitless) $\rightarrow 1\,\mathrm{rad}$

Using the definition given, one full revolution is defined as $2\pi\,\mathrm{rad}$. This is because the circumference of a circle is $2\pi r$, for a radius, r. With cancellation of r, the quotient $\frac{l}{r}$ defining the angle in radians is independent of the radius of the arc subtended, making it a valid angle measure.

Angular Velocity

Definition: The rate at which an angle changes over time.

Units: rad/s $\rightarrow 1\,\mathrm{rad/s}$

Equation:

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}$$

Or, taking the limit as Δt tends to zero:

$$\omega = \frac{d\theta}{dt}$$

Which gives the instantaneous angular velocity. Note that every particle of a body has the same angular speed, but this is not the case for linear speed.

Angular Acceleration

Definition: The rate of change of angular velocity

Units: rad/s² $\rightarrow 1\,\mathrm{rad/s^2}$

Equations:

$$\alpha = \frac{d\omega}{dt}$$

As with angular velocity, the position of a particle in an object does not determine its speed: all the particles that compose an object will have the same angular acceleration.

Linear Velocity and Acceleration in Circular Motion

An angle in radians may be written in the form: $\theta = \frac{l}{r}$. This means that $l = \theta r$ (θ must be in radians). Since l is a linear quantity, both sides of this last equation may be differentiated to get:

$$v = \omega r$$

Once more, differentiating gets:

$$a_T = \alpha r$$

In this case, a is the tangential acceleration pointing along the span of the velocity vector. Hence, for clarity, the subscript T is used. Just as tangential acceleration, centripetal acceleration may also be put in terms of angular quantities:

$$a_R = \omega^2 r$$

Keep in mind that the dependence on r of these linear quantities indicates that they will vary in different parts of the object, whereas the angular quantities will not.

Rotational Dynamics

Torque

Definition: The rotational analogue to force.

Units: Nm $\rightarrow$ $1\,\mathrm{N\,m}$

Equation:

$$\tau = \vec{r} \times \vec{F} = \|\vec{r}\|\|\vec{F}\| \sin\theta$$

Given this definition, it is easy to see that the torque on an object is only dependent on the component of the force perpendicular to the radius.

Angular Momentum

Definition: The rotational analogue to angular momentum.

Units: kgm²/s $\rightarrow$ $\mathrm{kg\,m^2/s}$

Equation:

As was the case for torque, angular momentum is computed by the cross product of $\vec{r}$ and $\vec{p}$:

$$l = \vec{r} \times \vec{p} = \|\vec{r}\|\|\vec{p}\| \sin\theta$$

Rotational Kinetic Energy and Rotational Inertia

Using the $1/2mv^2$ definition of kinetic energy, it may also be defined in terms of angular quantities.

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 r^2$$

Since r^2 varies for the particles constituting the body, the total kinetic energy may be expressed in the following way:

$$K = \frac{1}{2}(\sum_{i=1}^{n} m_i r_i^2)\omega^2$$

The sum in this equation is represented by I, known as the rotational inertia. In this way, kinetic energy of a rotating body is:

$$K = \frac{1}{2}I\omega^2$$

Conservation of Angular Momentum

As was done with linear momentum, given that $\vec{\tau_{\text{ext}}} = \frac{d\vec{L}}{dt}$, if $\vec{\tau}$ is constant, then:

$$\Delta\vec{L} = 0$$

Then, for motion about a fixed axis or one through the center of mass that remains parallel, we say that $\vec{L} = I\vec{\omega}$. Qualitatively, this means that if there is a change in I, say, it grows, then $\vec{\omega}$ must decrease for $\vec{L}$ to remain constant.