Report: Lotka-Volterra Equations

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EP3227: Non-Linear Dynamics

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1 Introduction

The L-V Equations, or more commonly known as the Predator-Prey equations, are a set of Logistic Equations used to approximately map the populations of a number of species that are part of the same food cycle. For example, in the case of 2 species, the most common example, and one that was the first pair to be studied with these equations is the Lynx-Snowshow Hare. This predator-prey combination's 60-year data was plotted with the L-V equations to realise the correlation between their populations.

$$
\begin{aligned}
\frac{dx}{dt} &= \alpha x - \beta xy \\
\frac{dy}{dt} &= \delta xy - \gamma y
\end{aligned}
\tag{1}
$$

These equations are non-linear in their form due to the product terms (xy) that signify the inter-dependence of the predator-prey species.

Parameters

The parameters define whether the effect one species has on another's population is positive or negative. Since we have assumed a Predator-Prey model, the parameters are all positive: $\alpha, \beta, \gamma, \delta > 0$
The main idea behind the L-V equations is that the population of a species A (the prey) is dependent on two factors

- A's natural growth rate: α
- The effect of B's(the predator) population A's decay: β

... and for B

- B's natural decay rate: δ
- The effect of A's population on B - growth: γ

2 Stability Analysis

We will be using the following techniques to perform a stability analysis of the L-V equations along with its parameters:

- Fixed Point Analysis

Fixed Points

$$\dot{x} = \alpha x - \beta xy = 0$$
$$\dot{y} = \delta xy - \gamma y = 0$$

There are two solutions for the fixed point equations -

$$(x, y) = \{(0, 0), (\frac{\gamma}{\delta}, \frac{\alpha}{\beta})\} \tag{2}$$

Phase Portraits

Using the non-linear forms of these plots, and plotting their StreamPlot in Mathematica we obtain the following.

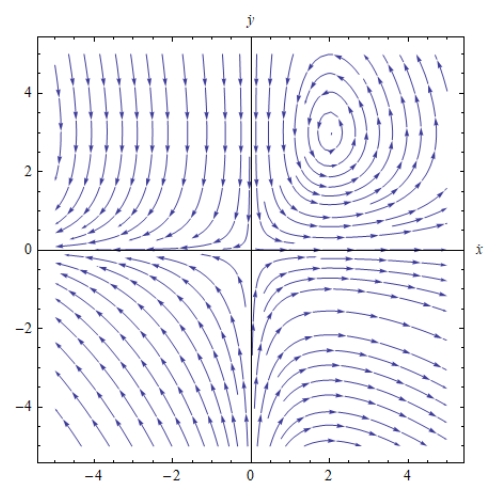

Figure 1: Phase Portrait Stream Plot

$$\dot{x} = 3.6x - 1.2xy \tag{3}$$
$$\dot{y} = 2xy - 4y \tag{4}$$

There are two kinds of nodes shown here, which will explained later in the Linearization.

Solving

Using equation (1), we can arrive at the following:

$$\begin{aligned}\frac{dx}{dy} &= \frac{x(\alpha-\beta y)}{y(\delta x-\gamma)} \\ &\implies (\frac{\alpha}{y}-\beta)dy = (\delta-\frac{\gamma}{x})dx \\ &\boxed{\implies C = \alpha ln(y)+\gamma ln(x)-\beta y-\delta x}\end{aligned} \tag{5}$$

Linearization of L-V Equation

The Jacobian of the set of equations is

$$J = \begin{pmatrix} \alpha-\beta y & -\beta x \\ \delta y & \delta x-\gamma \end{pmatrix} \tag{6}$$

Linearization about a fixed point demands that $\bar{f}_x = \bar{f}_{x_0} + J_{x_0}(x-x_0)$

$$J = \begin{cases} \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} & x_0 = (0,0) \\ \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{pmatrix} & x_0 = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta}) \end{cases}$$

$\therefore$ We may now write the LV equations in the linear form $\dot{X} = Ax$
At $x_0 = (0,0)$

$$\begin{aligned}\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ \dot{x} &= \alpha x \\ \dot{y} &= -\gamma y\end{aligned} \tag{7}$$

At $x_0 = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$

$$\begin{aligned}\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ \dot{x} &= -\frac{\beta\gamma}{\delta}y \\ \dot{y} &= \frac{\delta\alpha}{\beta}x\end{aligned} \tag{8}$$

For the above non-trivial fixed point:

$$\implies \ddot{x} = -\alpha\gamma x, \ddot{y} = -\alpha\gamma y$$

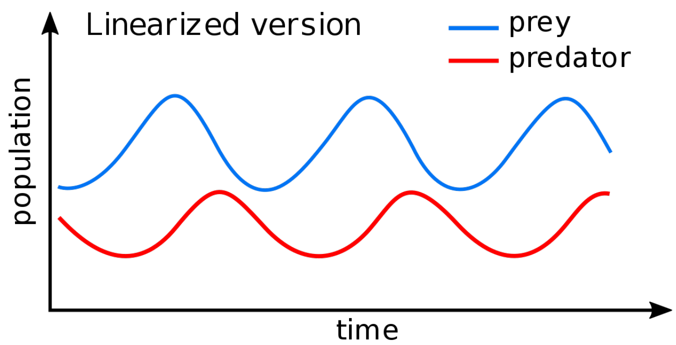

Figure 2: Predator-Prey Cycles

These are simple harmonic equations of motion, with the added constraint of the above phase-difference. The linearized time-dependent L-V equations are therefore Simple Harmonic, with one species' population trailing the others' by 90°, as shown by the above plot [1].

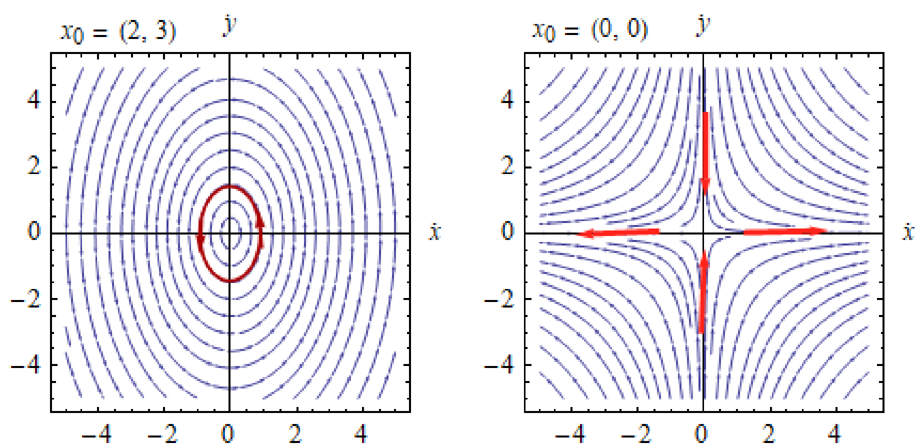

Figure 3: Phase Portrait at Two Critical Points using eqns 3,4
a. Center Point Node
b. Saddle Point Node

a Center-Point Node: The phase portrait has a central node, which signifies a 'neutrally stable' fixed point

b Saddle-Point Node: The phase portrait has a saddle-point node; it is stable in y and unstable in x.

[1]By AspidistraK - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=56484640

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Matrix Analysis

For the following Linearized equations (of 3,4):

$$\dot{x} = 3.6x - 1.2y \tag{9}$$
$$\dot{y} = 2x - 4y \tag{10}$$

a At $x_0 = (2, 3)$

$$J_{(2,3)} = \begin{pmatrix} 0 & -2.4 \\ 6 & 0 \end{pmatrix}$$
$$\text{Eigenvalues: } 3.79i, -3.79i$$
$$\text{Eigenvectors: } \begin{pmatrix} -0.53i \\ -0.84 \end{pmatrix}, \begin{pmatrix} 0.53i \\ 0.84 \end{pmatrix}$$
$$tr(J) = 0$$
$$\Delta(J) > 0$$
$$\implies \text{Center-Point Node}$$

b At $x_0 = (0, 0)$

$$J_{(0,0)} = \begin{pmatrix} 3.6 & 0 \\ 0 & -4 \end{pmatrix}$$
$$\text{Eigenvalues: } 3.6, -4$$
$$\text{Eigenvectors: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$tr(J) < 0$$
$$\Delta(J) < 0$$
$$\implies \text{Saddle-Point Node}$$

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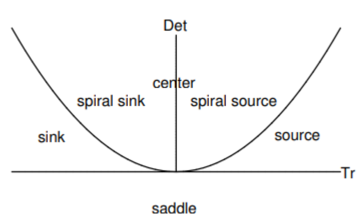

Figure 4: Det-Trace Diagram from Phase Plot Analysis

The above diagram[2] shows the different phase-plot structures one may expect after matrix analysis of a Linear set of ODEs.

[2] Source For Diagram