University of Groningen

Slow-roll inflation and the Hamilton-Jacobi Formalism

Jelmer Levenga (S3134407)

Supervisor:
D.Roest & A. Mazumdar

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Abstract

In this paper the cosmological theory of inflation are explored. The initial problems that gave rise to the idea are discussed as well as the solution inflation provides. In particular this paper looks at the single field slow-roll model of inflation and its consequences and predictions. Slow-roll inflation is a field theory in which the inflaton field ϕ drives inflation. The precise evolution of this field is determined in large part by the connected potential, which dominates over the kinetic terms of the action. For the slow-roll model two specific parameters are introduced, ϵ and η. These parameters will guarantee that the model considered follows slow-roll conventions and inflation lasts long enough to solve the problems it was introduced for. Perturbations of single-field theory are discussed and worked through. These perturbations on a quantum scale are the reason for primordial fluctuations, which are scaled up due to inflation to allow for observables like large-scale structure and temperature anisotropies in the CMB. Higher order perturbations and non-Gaussian effects are also explored, which place other limitations on the models. These ideas are presented and worked out explicitly in an example model in a case study. Finally the Hamilton-Jacobi formalism is considered. This formalism makes the same predictions to first order but is inherently more exact and gives rise to attractor behavior of the fields. The differences between regular slow-roll and the Hamilton-Jacobi formalism are discussed regarding non-Gaussian terms and the plausibility of measurable differences.

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1 Introduction

The Hot Big Bang model, first considered by Lêmaitre in 1927 is currently the most widely accepted theory of the early universe [11]. In this model the universe is traced back to an incredibly dense and hot state at early times. From that state the universe expanded and cooled, to evolve eventually to the universe we live in. While this model made some predictions that turned out to be correct, such as the existence of a cosmic microwave background, it also introduced a few problems that required a solution. The most prevalent at the time of writing is the theory of inflation. However, inflation is not without its own problems, such as requiring specific conditions to be met and the existence of an initial singularity at which physics fails, limiting its predictive power at the very earliest times of the universe. A few different solutions have been proposed, such as string gas cosmology and matter bounce [5]. The early universe is still an active area of research and hence an interesting topic to investigate. In this paper I will explore a model of inflation, the single field-slow roll model, and how it solves these problems, as well as the predictions it makes and the triumphs it has, to explain why it is now the most prevalent theory of the early universe. I will be considering perturbations in the field and how they can be measured, including non-Gaussian terms. Then I will look into a specific form of the model, the Hamilton-Jacobi Formalism to discuss how these non-Gaussianities might translate to this formalism and what differences might pop up, as well as the size of these differences and whether this formalism would be the next breakthrough in the field.

2 Background

2.1 CMB

In 1965 Arno Penzias and Robert Wilson discovered a background signal in on their antenna telescope. This background was consistent in all directions and whatever they tried, it would not go away. This background was the first reading of the Cosmic Microwave Background (CMB). This background radiation was measured and it was found to be fitted by an almost perfect blackbody spectrum with a temperature of around 2.73K [11]. This is the radiation left over from the very first moments of the universe. The almost perfect blackbody distribution implies that the source of this radiation was almost certainly thermal radiation from a blackbody. In the Big Bang model, early universe all the matter would have been compressed in a much smaller volume and consequently incredibly hot. This hot and dense matter produced blackbody radiation from its heat. This would explain the existence of this radiation. The other prevalent theory at the time, the steady state universe, which claimed that the universe existed forever and has always been expanding could not adequately explain the radiation. This discovery made the Big Bang theory as prevalent as it is now. Even though the radiation would have been present since the very early universe, that is not the era from which it can be detected. In the early phases of the universe, it was too hot for the electrons and the atoms, or before that the quarks to bond. This means the universe was not electrically neutral and since photons interact with charges, they were scattered continuously. The universe was essentially opaque, no light could pass through. As the universe cooled the electrons could bind to the nuclei, in a process known as recombination. This made the universe neutral and transparent for photons. This is the earliest moment we can look back to, around 250.000 years after the the beginning of the universe [11], but it still carries the fingerprint of the earliest periods of the universe when it was emitted, making it incredibly valuable for research into early-universe behaviour like inflation. The Universe at the time the CMB was emitted was certainly much hotter than 2.73K, even though that is the temperature measured today. Because the expansion of the universe the photons become red-shifted, their wavelength increases as the universe expands. This causes the photons to lose energy and shifts the spectrum, such that it looks as if the original blackbody that emitted the radiation was much cooler.
As it turns out, the radiation was not exactly isotropic, there were small temperature differences on the order of a few mK. A measurement of these anisotropies was done in 1992 by NASA [6] and later improved by WMAP [7]. A skymap made using the data of the WMAP programme showing the temperature anisotropies of the CMB is presented in Figure 1. The importace and possible cause of these anisotropies will be adressed later in this paper.

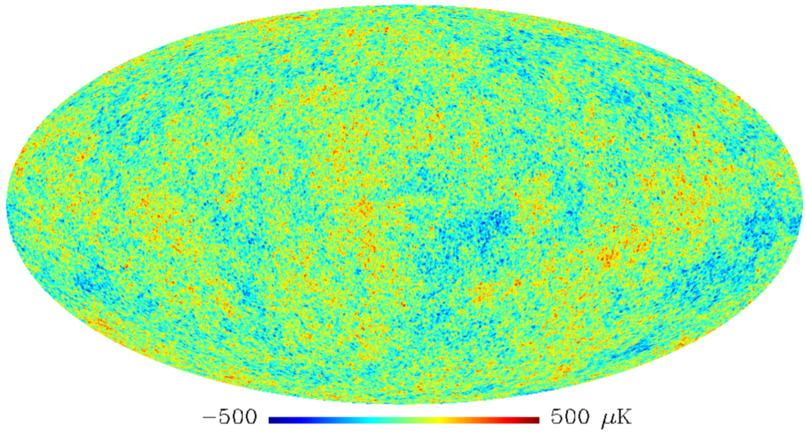

Figure 1: A skymap of the temperature anisotropies in the CMB based on the WMAP and PLANCK PR2 data,

2.2 Technical background

Before treating inflation, first a basis of cosmic dynamics should be laid down. This is a technical section containing the equations that will be used throughout the rest of the paper, hence a refresher of these principles will be provided here. In all equations, the convention $c = 1$ will be used and for some functions, on subsequent uses, their dependence on variables will be suppressed. four very important equations are the Friedmann equations, the Friedmann-Robertson-Walker (FWR) metric and the fluid equation. These require general relativity to be derived in their most complete form, however once done their simple form easily allow for their using without the knowledge of general relativity. The FWR metric is defined as

$$ds^2 = -dt^2 + a(t)^2(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin\theta^2 d\phi^2)) \tag{1}$$

Which means that in a flat space-time all distances are multiplied by a factor a(t). This is called a comoving frame, in which spatial components are scaled with the expansion rate of the universe. Such that even as the universe expands, the variables in this frame stay scaled properly. The metric can also be written more concisely as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2}$$

g is the metric, defined by (-1,a^2,a^2,a^2) and The Friedmann equations are given by [11]

$$H(t)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{R_0^2 a(t)^2} \tag{3}$$

where k is the curvature of space, $\rho(t)$ is the energy density of the universe, R_0 is the radius of curvature of the universe, $a(t) \equiv 1$ at $t = t_0$ and H is the Hubble constant $H \equiv \frac{\dot{a}}{a}$. Usually this equation is normalized with $8\pi G = R_0 = 1$
Written in terms of the mass density ρ the Friedmann equation becomes:

$$\rho = \frac{1}{8\pi G} 3H^2 + \frac{3k}{R_0^2 a(t)^2}$$

The second equation is

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3p) = \frac{-1}{6}(\rho + 3p) \tag{4}$$

Where p is the pressure exerted by the contents of the universe. This pressure can be represented as $P = \omega\epsilon$, where ω is a dimensionless number and depends on the contents of the universe.[11]. $\omega = 0$ for matter, $\frac{1}{3}$ for radiation and -1 for dark matter. This means that all the pressure generated by dark matter is exactly opposite to its energy density. Equation 3 and 4 imply that

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{5}$$

Can be rewritten as [11]

$$\frac{1}{\rho} d\rho = -3(1 + \omega)\frac{1}{a} da$$

Which leads to

$$\rho(a) = \rho_0 a^{-3(1+\omega)} \tag{6}$$

Substituting this into equation 3 and using $H = \frac{\dot{a}}{a}$ it follows that

$$\dot{a}^2 \propto a^{-(1+3\omega)} \tag{7}$$

Then assuming a(t) follows a power law: $a \propto t^x$, $\dot{a} \propto t^{2q-2}$. This can be solved for x using equation 7 to get

$$a(t) \propto t^{2/(3+3\omega)} \tag{8}$$

The fluid equation can be derived by considering a sphere of radius R, expanding with the universe, such that $R = a(t)r$, where r is some initial constant radius[11]. The volume of this sphere is given by

$$V = \frac{4\pi}{3} r^3 a^3$$

As the universe expands, the volume of the sphere changes as

$$\dot{V} = 4\pi r^3 \dot{a} a^2 = 3V\frac{\dot{a}}{a} = 3VH$$

The energy content of the sphere is given as $E = V\rho$. The change in the energy density is then

$$\dot{E} = V(\dot{\rho} + 3H\rho)$$

The first law of thermodynamics is

$$dQ = dE + pdV$$

In a homogeneous universe, there is no heat flow, $dQ = 0$ this is called adiabatic expansion. Dividing by dt, the first law can also be written as

$$\dot{Q} = \dot{E} + P\dot{V} = 0$$

Substituting the previous equation yields

$$V(\dot{\rho} + 3H\rho + 3HP) = 0$$

Or

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{9}$$

This is the fluid equation, which looks exactly the same in general relativity as well. Inflation is inherently subject to general relativity and is therefore a field theory. In a field theory the action and Lagrangian determine the dynamics of the system, with

$$S = \int d^4x\mathcal{L}$$

The integral is over d^4x, which indicates 3+1 dimensional space. Most variables have indices, either Greek or Roman indices. An index up indicates all positive values for spatial components and an index below indicates negative spatial components. Greek indices like μ or ν range from $0 - 3$, where 0 is reserved for the time value. Roman indices like i or j range from $1 - 3$, so run over spatial values only. Indices on either side of an equation should match and contracted indices, with one up and one down create a Lorentz-invariant variable, that is consistent with special relativity.

3 Reasons for inflation

The idea of inflation was initially coined as a solution to a few lingering problems in cosmology, that were difficult to explain and required significant fine-tuning of free parameters [8]. The new theory of inflation solved these problems without requiring any fine-tuning of parameters. A universe that started out with a general set of initial conditions could grow into the universe as we measure it today. These problems that had to be solved by inflation are known as the horizon problem, the flatness problem and the monopole problem.

3.1 The horizon problem

One of the foundations that modern astronomy rests on is the idea that the universe is homogeneous and isotropic, the idea that there is no preferred location in space and no preferred direction to look in. A good example of this is the cosmic microwave background, the CMB. This radiation has a certain temperature, which is measured to be approximately constant at 2.73K, with some very small anisotropies [15]. However, that could not be explained using the models at the time. Estimating the distance the photons we can observe now travelled can be done using the proper distance, the distance light travels in a certain time, but where the expansion of the universe is also taken into account. The proper distance takes the form

$$d_p(t_0) = \int_{t_{ls}}^{t_0} \frac{1}{a(t)} dt = \int_{t_{ls}}^{t_0} d\tau \tag{10}$$

Where t_0 is the current time and t_{ls} is the time at which the photons were emitted and a(t) is the expansion rate of the universe at time t and τ is the conformal time. These values can be computed and gives a proper distance, of $0.98 * d_{hor}$. d_{hor} is the horizon distance, the distance at which light is too far away to have reached us. The horizon distance is the distance at which two points were in causal contact between the time t and a later time t'. The horizon distance is also referred to as the comoving horizon. This implies that two opposite points in the sky, are at a distance of 1.96 horizon distances away and are unable to exchange any information with each other. They should not necessarily have the same temperature if they were not in contact with each other. It can be shown that this would be true for points at an angular separation of only 1.1 degrees for a total of $10^{83} patches$[8]. This should mean that every patch of sky should have a different temperature to a patch approximately 1 degree away. The CMB measurements have shown that this is not the case, not just one or two patches have the same temperature, but the whole sky. At some point in time, these had to have been in contact and were separated across the universe to allow for this.

3.2 The Flatness problem

In Einstein's theory of general relativity, space-time itself can have an intrinsic geometry. That geometry determines properties such as the expansion and evolution of the universe. There are three main types of geometry space can have: closed, flat or open. These are purely geometric, but in 4-dimensional space time, the metric includes time, which leads to the three different space-times: De-sitter, Minkowski and anti-De-Sitter (ADS) space. When measured, the curvature of space-time seems to be almost non-existent. That is not a problem in itself, but if traced back in time, it becomes even more strange. in equation 3 we can define the critical density as density at which the curvature is 0. The equation then becomes

$$H(t)^2 = 8\pi G \rho_{crit} \tag{11}$$

$$\rho_{crit} = \frac{3H(t)^2}{8\pi G}$$

Then we define a parameter Ω, the density parameter as

$$\Omega = \frac{\rho}{\rho_{crit}} = \frac{3H^2 + \frac{3k}{R_0^2 a(t)^2}}{3H^2}$$

$$|1 - \Omega| = -\frac{k}{R_0^2 a(t)^2 H^2} \tag{12}$$

$|1 - \Omega|$ is a measure of the curvature of space, with 1 being very curved and 0 being flat. It is measured to be [11]

$$|1 - \Omega| \leq 0.005$$

The curvature of space is quite small now, but it can easily be traced back in time using these equations. $(aH)^{-1} = \frac{1}{\dot{a}}$. The time dependence of a is given in equation 8. For matter $a \propto t^{1/3}$ and for radiation $a \propto t^{1/2}$. Taking time derivatives shows that for matter $\dot{a} \propto t^{-2/3}$ and for radiation $\dot{a} \propto t^{-1/2}$. This means that equation 12 increases in time. As we run time backwards, this value must have been even smaller than it is now. In the most extreme case, at the Planck time $|1 - \Omega| \leq 10^{-62}$. While it is possible that this could be merely a coincidence, a mechanism that would make any arbitrary geometry flat as time passes would be much more likely.

3.3 The Monopole problem

When inflation was initially proposed it was mainly meant as a solution to the horizon and flatness problem [8]. However, inflation turns out to be a solution to the monopole problem as well. In the early moments of the universe energies were very high, so high that the fundamental forces could unite. The weak and electromagnetic force unite at energies of approximately 1 TeV and the strong and now united electroweak force would unite at energies of 10^{12} TeV. When they decouple at lower temperatures, there is a sudden drop of symmetries in the system, leading to topological defects. It is theorized that zero-dimensional defects would be magnetic monopoles [11]. These monopoles would have energy densities of 10^{94} TeV m^{-3} and would dominate the energy density of the universe after only 10^{-16} seconds. That would imply magnetic monopoles would be very abundant throughout the universe, but none have been found yet. The upper bound on their density parameter Ω is $\Omega < 5 * 10^{-16}$. Some mechaninism was needed to suppress the density of these monopoles during the evolution of the universe, to the point where they are now virtually undetectable. It should be noted that altough several theories require magnetic monopoles, that is the only basis for their possible existence and it is possible they do not exist, though that does not invalidate inflation.

4 Inflation solutions

Inflation requires a positive acceleration in the expansion rate of the universe, $\ddot{a} > 0$. Substituting that into equation 4 means that a negative pressure is required, $P < \frac{-\rho}{3}$. Combining equation 3 and equation 4 and integrating gives the expansion rate of the universe

$$a(t) \propto \begin{cases} t^{2/3(1+\omega)} & \omega \neq -1 \\ e^{H(t-t_i)} & \omega = 1 \end{cases}$$

or

$$a(t) \propto \begin{cases} t^{2/3(1+\omega)} & t < t_i \\ e^{H(t-t_i)} & t_i < t < t_f \\ e^N t^{2/3(1+\omega)} & t_f < t \end{cases}$$

Where inflation started at time t_i and ended at time t_f. Assuming that the expansion rate is rather large, the ratio is given by

$$\frac{a(t_f)}{a(t_i)} = e^N \tag{13}$$

where N is the number of e-folds which is given as $N = H(t_f - t_i)$.

4.1 The horizon problem

The proper distance in equation 10 can be modified to give the horizon distance, by scaling it with the expansion rate of the universe

$$d_{hor} = a(t) \int_0^t \frac{1}{a(t)} dt \tag{14}$$

Which can also be written as

$$d_{hor} = \int_0^a \frac{1}{Ha^2} da = \int_0^a d \ln a \frac{1}{aH} \tag{15}$$

Expressing the horizon distance in an integral over $\frac{1}{aH}$, which is known as the comoving Hubble radius. The early universe was radiation dominated [11] with a time dependence of $a(t) \propto t^{1/2}$. This means the horizon distance at the start of inflation can be written as

$$d_{hor}(t_i) = a(t_i) \int_0^{ti} \frac{1}{a(\frac{t}{t_i})^{1/2}} dt = 2t_i$$

Assuming a_i and t_i are constant in this period of time. at the end of inflation this was

$$d_{hor}(t_f) = a(t_i) e^N (\int_0^{t_i} \frac{1}{a(\frac{t}{t_i})^{1/2}} dt + \int_{t_i}^{t_f} \frac{1}{a e^{H_i(t-t_f)}} dt)$$

The first term has been evaluated already. The second terms evaluates to

$$\int_{t_i}^{t_f} \frac{1}{ae^{H_i(t-t_f)}}dt = \frac{1-e^{-N}}{a_i H_i}$$

assuming am explicit time dependence only.e^N is presumed to be quite large, $N \approx 60$ to solve the flatness problem [9], so this simplifies to

$$d_{hor} = e^N(2t_i + \frac{1}{H_i}) \tag{16}$$

If we make a few very rough estimates, that $N \approx 60$, and $\frac{1}{H_i} \approx t_i$ this means that the horizon grew from 10^{-27}m at $t_i = 10^{-}36$s to 15m [11]. This shows that two points that were very closely packed and in causal contact could be blown up to sizes many orders of magnitude larger, solving the horizon problem. In Figure 2 a visual representation of this is given.

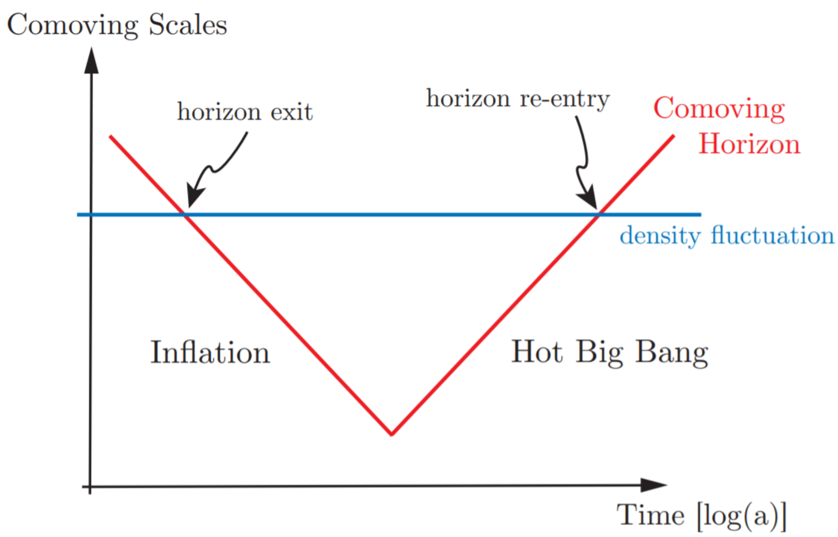

Figure 2: A visual representation of the solution of the horizon problem. Points above the blue line are causally connected. Inflation rapidly pulled them apart, after which due to the passing of time, they are only causally connected again much later.

4.2 The flatness problem

To solve the flatness problem consider equation 12. If the Hubble constant is assumed to be constant the equation takes the form

$$|1-\Omega| \propto \frac{1}{a(t)^2}$$

During inflation the expansion rate of the universe is exponential, this equation becomes

$$|1 - \Omega| \propto \frac{1}{e^{2N}}$$

Whatever the curvature was at the beginning of the universe, it drops exponentially, causing it to approach a value of zero very quickly. This mechanism would make any arbitrary curvature flat after inflation, explaining our observations, without the need to fix initial condition.

4.3 The monopole problem

The expansion rate of the universe is given by $a(t)$. This is the radial expansion rate. Hence, the volume of the universe expands at a rate of $a(t)^3$. During inflation the volume grows as e^{3N}. If magnetic monopoles cannot be created or destroyed the number density and energy density of magnetic monopoles evolves as $n_i e^{-3N}$. Assuming an initial number density of $10^{82} m^{-3}$ and 65 e-folds, the number density drops to $0.002 m^{-3}$ and including the expansion of the universe after inflation this drops further too $10^{-83} m^{-3}$ [11]. This density is much smaller than the upper bound on the density measured, thus explaining why magnetic monopoles have not been detected.

5 The slow roll model

The slow roll model of inflation is a field theory with a single scalar field ϕ. ϕ is called the inflaton field and is the field that drives inflation. This can be used in Einsteins equations to find the action

$$S = \int d^4x\sqrt{-g}(\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)) \tag{17}$$

This action can be written as the sum of two terms

$$S = S_{GR} + S_\phi$$

Where R represents the Ricci tensor, a tensor often used in General relativity. It will not be used in the following equations. With the action, the Energy-Momentum tensor for the field ϕ can be constructed, using the action for ϕ. The explicit derivation can be found in the Appendix. The results of these calculations are equations detailing the pressure P and energy density ρ as a function of ϕ.

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{18}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{19}$$

The equation of state corresponding to these parameters is

$$\omega_\phi = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{20}$$

The equation of state ω has been used a few times before. This has been measured for various different components of the universe. $\omega = 0$ for matter, $\frac{1}{3}$ for radiation and -1 for dark matter. This can eliminate the dependence of either P or ρ from an equation by substitution at the end, simplifying some equations.

5.1 Equations of dynamics

The values of ρ and P can be substituted into equation 4. Inflation requires a positive acceleration, $\rho + 3P < 0$. This gives the relation

$$2\dot{\phi}^2 < V(\phi)$$

To achieve inflation, the potential has to dominate over to kinetic term of the field. This also leads to the relation between ρ and P that $\rho \approx -P$.

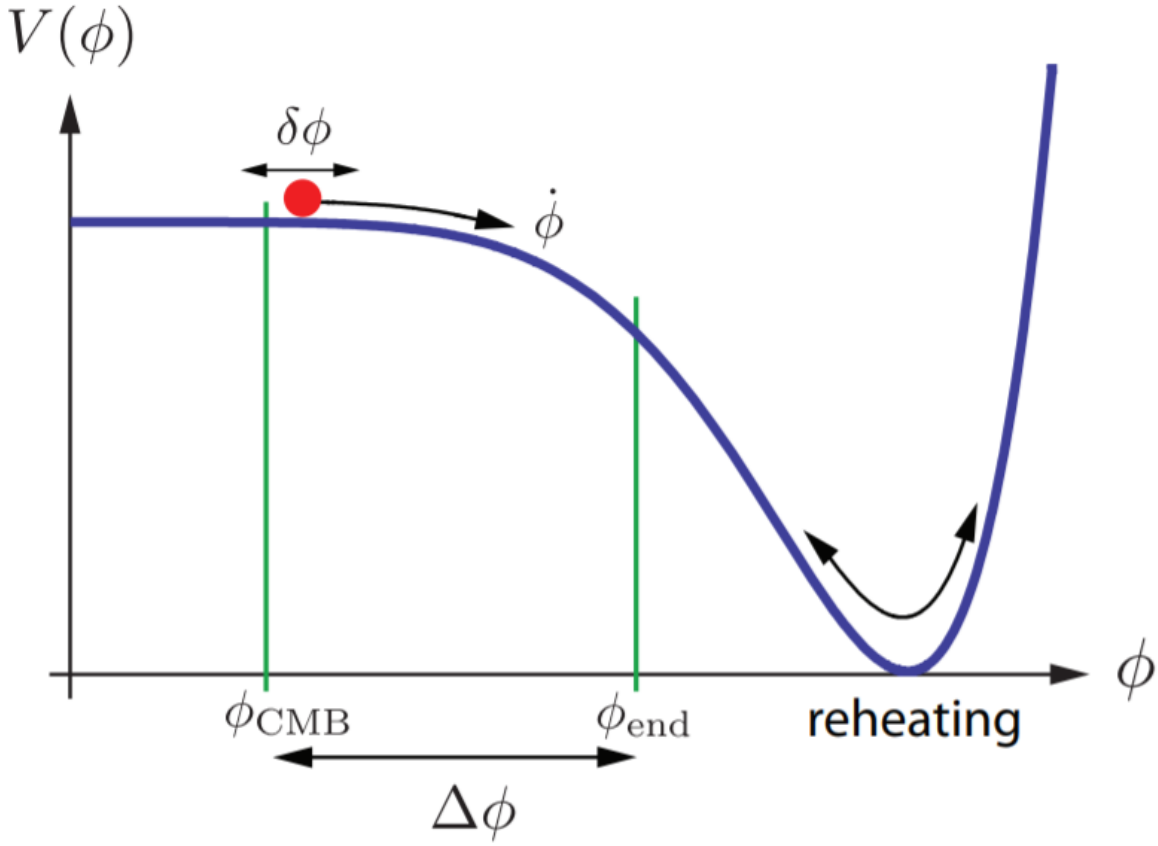

Figure 3: An example of of the potential. The field rolls into the well created by the potential, which needs to dominate over the kinetic term to make inflation possible.

The fluid equation, equation 9 can be worked out explicitely using

$$\dot{\rho} = \ddot{\phi}\dot{\phi} + \dot{V}$$

Substituting gives

$$\ddot{\phi}\frac{d\phi}{dt} + \frac{dV}{dt} + 3H\frac{d^2\phi}{dt^2} = 0$$

Dividing out $\frac{d\phi}{dt}$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{21}$$

This equation governs the evolution of the field. It looks remarkably similar to the equation of motion in regular Newtonian mechanics for the variable ϕ. The term proportional to $\dot{\phi}$ can be seen as the friction term, which is usually proportional to the velocity. The term $3H$ is therefore known as the Hubble friction [9] Another important equation comes from the flatness problem, where it was shown that

$$H^2 = 3\rho_{crit}$$

Which leads to the relation

$$H^2 = 3(\frac{1}{2}\dot{\phi}^2 + V(\phi)) \tag{22}$$

This is not a groundbreaking equation like some of the previous ones, but this is a useful equation for its simplicity. It relates the Hubble parameter to the energy density directly, and thus to $\dot{\phi}$. This allows for some useful substitutions in some cases.

5.2 Slow-roll parameters

The second Friedman equation, equation 4 can be written differently, introducing a new variable ϵ. Using $H = \frac{\dot{a}}{a}$, $\dot{H} = \frac{\ddot{a}}{a^2} - H^2$. or

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon) \tag{23}$$

giving

$$\epsilon = \frac{-\dot{H}}{H^2} \tag{24}$$

Inflation requires a positive acceleration, meaning $\epsilon < 1$. That is the first slow-roll parameter. In De-Sitter space $\rho \approx -p$ means $\epsilon \to 0$, which gives a stronger $\epsilon \ll 1$. With the definition earlier, $N = Ht$, so $dN = Hdt$, equation 24 can be written as

$$\epsilon = -\frac{dln(H)}{dN} \tag{25}$$

Inflation can last as long as this is satisfied and stops at the moment $\epsilon = 1$. Inflation needs to last long enough if it is able to solve the flatness, horizon or monopole problem, around 60 e-folds, requiring ϵ to be small during that time. To ensure that holds, a second slow-roll parameter has to be introduced: η. Because inflation requires that $\phi < V(\phi)$. Sustaining this equation requires the field ϕ to evolve very slowly, to catch up with the potential relatively late. This implies that, using equation 21, that the contribution from the acceleration must be very small: $\left|\ddot{\phi} \ll H\dot{\phi}, V(\phi)\right|$. The slow roll parameter η fixes this condition:

$$\eta = \frac{-\ddot{\phi}}{H\dot{\phi}} < 1 \tag{26}$$

Often, the slow-roll parameters are expressed in terms of the potential [9]:

$$\epsilon_v = \frac{M_{pl}^2}{2}(\frac{V_{,\phi}}{V})^2 \tag{27}$$

$$\eta_v = M_{pl}^2(\frac{V_{,\phi\phi}}{V})^2 \tag{28}$$

Where M_{pl}^2 is the reduced Planck mass, also written as $M_{pl} = \sqrt{\frac{m_p}{8\pi}}$ which is usually normalized to 1 and the comma in the subscript denotes a partial derivative, with respect to the other subscript. These are not exactly the same as the previous slow roll parameters, but are related via

$$\epsilon_v \approx \epsilon \tag{29}$$

$$\eta \approx \eta_v - \epsilon_v \tag{30}$$

Using these definitions it is possible to express the number of e-folds N. N is expressed as

$$dN = Hdt$$

$$N = \int_{t_0}^{t} Hdt \tag{31}$$

using the slow-roll assumptions $H^2 \approx \frac{1}{3}V(\phi)$ and $\dot{\phi} \approx \frac{-V_{,\phi}}{3H}$ equation 31 can be rewritten as

$$N \approx \int_{\phi_f}^{\phi_i} \frac{V}{V_{,\phi}} d\phi$$

$$N \approx \int_{\phi_f}^{\phi_i} \frac{1}{\sqrt{2\epsilon_v}} d\phi \tag{32}$$

The number of e-folds is measured to be around 60 to 65 [11]. This is this number necessary to solve the problems posed earlier. Any model for inflation should produce around this number of e-folds.

6 Power spectrum & perturbations

6.1 Power spectra

In the previous section the field ϕ was assumed to be constant in space and only evolve in time. This need not necessarily be true however. What happens if small perturbations to the field ϕ are introduced? In an equation this can be written out as

$$\phi = \phi(t) + \delta\phi(x,t)$$

To be able to connect these models to something that can be measured a power spectrum has to be constructed. A power spectrum gives the scale, the size of a certain variable and can be measured. The power spectrum can be connected to a two-point correlation function. Such a function measures the correlations between two values, for example the temperature between two point on the sky in the CMB. The power spectrum of a function can be seen as a Fourier decomposition for any function living in higher dimensional spaces and in single-variable space simplifies to a Fourier decomposition. The two-point correlation function is defined as the expectation value of a field between two points

$$\xi_A =< A(x)A(x+r) > \tag{33}$$

where we assume that due to isotropy, the function does not depend on angle, but on the radius r only [9]. Any function A can be written as a Fourier series defined as

$$A_k = A_0 \int d^3x A(x) e^{-ik\cdot x}$$

$$A(x) = A_1 \int d^3x A_k e^{+ik\cdot x}$$

The Dirac-delta function is defined as[9]

$$\delta(k) = A_0 A_1 \int d^3x e^{\pm ik\cdot x}$$

The integral evaluates to $(2\pi)^3$, so normalization gives $A_0 A_1 = \frac{1}{(2\pi)^3}$. If we wish to evaluate the function $< A_k A_{k'} >$ we substitute the previous equations to find

$$< A_0 A_0 \int d^3x A(x) e^{-ik\cdot x} d^3x' A(x') e^{+ik\cdot x'} >=$$

We can write x' as a perturbation from x, $x + r$

$$< A_0^2 \int d^3x d^3r e^{-i(k+k')\cdot x} A(x)A(x+r) e^{-ikx} >=$$

$$\frac{A_0}{A_1}\delta(k+k') \int d^3r \xi_A(r) e^{-ikx} \tag{34}$$

Which looks a lot like a Fourier transform in itself. The power spectrum can be defined as the Fourier transform

$$P_A(k) = A_0 \int d^3 r \xi_A(r) e^{-ikx}$$

Then the equation simplifies to

$$< A_k A_{k'} >= \frac{1}{A_1} \delta(k + k') P_A(k) \tag{35}$$

Where A_1 is a free variable. By convention it is often chosen that $\frac{1}{A_1} = (2\pi)^3$ [9]. The variance in the power spectrum is usually defined as

$$\sigma_A^2 = \int d\ln(k) \Delta_A^2(k)$$

Where $\Delta_A^2(k) = \frac{k^3}{2\pi^2} P_A(k)$, relating the variance with the power spectrum itself. The Fourier transform allows us to go from position space to momentum space. k signifies the comoving momentum, so it is scaled with the expansion rate of the universe. This ensures that k remains constant in time, with a corresponding wavelength $\lambda\ \frac{1}{k}$. The evolution of k can be represented graphically with the comoving scale. k is scaled with the comoving horizon. During inflation it decreases very rapidly and after inflation it increases quite quickly. A plot of this is given in Figure 4

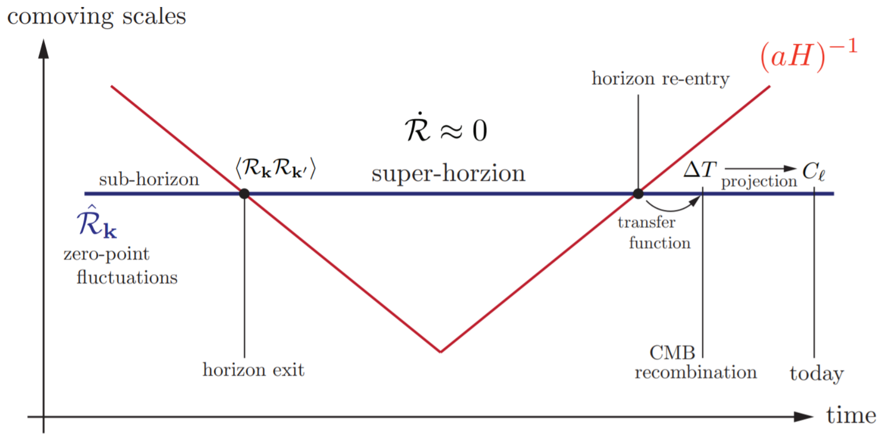

Figure 4: A visual representation of the evolution of k. The horizontal line is the wavelength λ. The first crossing is called the Horizon crossing, and the second crossing is called the re-entry. Whenever the wavelength is larger than the comoving horizon it is the super-horizon regime, otherwise it is the sub-horizon regime.

There are three distinct regions that can be distinguished, based on the wavelength. The wavelength starts out smaller than the comoving horizon, in the sub-horizon regime. As the universe evolves, the wavelength stays constant, but the horizon becomes smaller rather quickly due to inflation. Their crossing is aptly known as the Horizon crossing, and the sub-horizon regime makes way for the super-horizon regime. When matter and radiation take over, the horizon increases again, until it crosses the wavelength a final time, which is called re-entry. In the super-horizon regime $\frac{k}{aH}$ is very small much less then unity, while in the sub-horizon regime it is much larger than unity. At the moment of horizon crossing $k = aH$. The dependence of the power spectrum on the scale is given by the so-called spectral index or tilt. The spectral index is defined as

$$n_s - 1 = \frac{d\Delta_A^2}{d\ln k} \tag{36}$$

If there is no dependence on the scale, Δ_A^2 does not depend on k and $n_s = 1$ The spectral index can be rewritten as

$$n_s - 1 = \frac{d\Delta_A^2}{d\ln k} = \frac{d\Delta_A^2}{dN}\frac{dN}{d\ln k} \tag{37}$$

This is a more useful form for some calculations. The spectral index can be measured in the CMB, this gives a restriction on the power spectrum, which gives the model some predictive power and is a very useful quantity in defining the dependence of the power spectrum and is thus invaluable for measuring and defining perturbations in the power spectrum.

6.2 Perturbations

To look at perturbations the first place to find them is in the metric itself:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Four perturbations variables for g can be defined such that

$$g_{\mu\nu,perturbed} = g_{\mu\nu} + \delta g_{\mu\nu} \tag{38}$$

: [39]

$$\delta g_{00} = -2a^2\Phi \tag{39}$$

$$\delta g_{i0} = a^2 B_i \tag{40}$$

$$\delta g_{ij} = -2a^2(\Psi\delta_{ij} - E_{ij}) \tag{41}$$

Where $B_i = \partial_i B + \hat{B}_i$ and $E_{ij} = E_{ij}^s + E_{ij}^v + E_{ij}^t$, a scalar, vector and tensor component. all the introduced perturbation variables without indices and E_{ij}^s are scalar perturbations.

In first order, scalar, vector and tensor perturbations can be considered separately. This gives a new perturbed metric

$$ds^2 = a^2(-(1+2\Phi)d\tau^2 + 2B_i dx^i d\tau + ((1-2\Psi)\delta_{ij} + 2E_{ij})dx^i dx^j) \tag{42}$$

Here we run into a problem. By performing a coordinate transformation, perturbation can be induced or even removed. It is not physically reasonable that a change of origin would create completely different measurements, so this problem has to be resolved. An example of a coordinate transformation inducing perturbations can be seen by considering the unperturbed metric

$$ds^2 = a^2(-d\tau^2 + \delta_{ij} dx^i dx^j)$$

and taking the coordinate transformation

$$\vec{x} \to \vec{\tilde{x}} = \vec{x} - \epsilon(\tau, \vec{x})$$

$$dx^i = d\tilde{x}^i + \frac{d\epsilon^i}{d\tau} d\tau + \partial_k \epsilon^i d\tilde{x}^k \tag{43}$$

then

$$ds^2 = a^2(-d\tau^2 + 2\frac{d\epsilon^i}{d\tau} d\tau dx^i + (\delta_{ij} + 2\partial_{ji}\epsilon_j)d\tilde{x}^i d\tilde{x}^j \tag{44}$$

Which means that $\frac{d\epsilon^i}{d\tau} = B_i$ and $\hat{E}_i = \epsilon_i$. By a simple coordinate transformation, two perturbation have been introduced that were not present before. This is known as the Gauge problem, where it is currently impossible to tell if any perturbations are physical or fictitious, due to a coordinate transform. Performing a coordinate transformation on the scalar perturbation gives [39]

$$\tilde{\Phi} = \Phi + \epsilon^{0\prime} + \mathcal{H}\epsilon^0$$

$$\tilde{B} = B - \epsilon^0 + \epsilon'$$

$$\tilde{\Psi} = \Psi - \mathcal{H}\epsilon^0$$

$$\tilde{E} = E + \epsilon$$

Where a prime denotes a derivative with respect to conformal time and $\mathcal{H}$ is defined as $\mathcal{H} = aH$.

Any arbitrary matter scalar, such as ϕ or ρ transforms as

$$\delta\tilde{\sigma} = \delta\sigma + \sigma'\epsilon^0$$

There are two possible solutions to the Gauge problem, that can both be applied to solve this problem. The first is to fix the Gauge. That means choosing particular values of epsilon, such that certain perturbations become 0. An example of this is the uniform density Gauge. In this Gauge $\delta\tilde{\rho} = 0$, so $\delta\rho = -\rho'\epsilon^0$, and $\epsilon^0 = -\frac{\delta\rho}{\rho'}$ This gives slices of uniform density, but doesn't completely fix the Gauge. ϵ^i can still be fixed, using any of the other equations, so this still leaves a degree of freedom available. Another example is the spatially flat Gauge. In this Gauge $\tilde{\Psi} = 0$ and $\tilde{E} = 0$, by setting $\epsilon = -E$ and $\epsilon^0 = \frac{\Psi}{\mathcal{H}}$. This simplifies the metric to

$$ds^2 = a^2((1 + 2\Phi)d\tau^2 + 2\partial_i B dx^i d\tau + \delta_{ij} dx^i dx^j)$$

The other option is to change variables in such a way that they remain invariant under coordinate transformations. These are Gauge invariant variables. Two important variables are ξ and Q [39]

$$-\xi = \Psi + \mathcal{H}\frac{\delta\rho}{\rho'} \tag{45}$$

The interpretation of the variable ξ is the curvature of a hypersurface, a surface of more than three dimensions, on slices of constant density. The minus sign is not necessary. Notations in literature differ and sometimes do or do not include it. When working in the uniform density Gauge, this can be reduced to

$$-\xi = \Psi \tag{46}$$

and in the spatially flat Gauge to

$$-\xi = \mathcal{H}\frac{\delta\rho}{\rho'} \tag{47}$$

The variable Q is defined as

$$Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\Psi \tag{48}$$

Q can be reduced to

$$Q = \delta\phi$$

in the spatially flat Gauge.
The choice of Gauge is completely arbitrary, any choice is fine. the spatially flat Gauge is quite convenient, so that is the one that will be used throughout. Tracking the evolution of these invariant variables is non-trivial, but it can be shown from the Einstein equations that in momentum space, ξ evolves as : [9]

$$\dot{\xi} = \frac{H}{\rho + p}\delta P_{non-adiabatic} + \mathcal{O}\frac{k^2}{(aH)^2}$$

Where $\delta P_{non-adiabatic} = \delta p - \frac{\dot{p}}{\dot{\rho}}\delta p$. Which is a Gauge invariant quantity in itself. For adiabative expansion, $\delta p_{non-adiabatic}$ is by definition 0, so it reduces to

$$\dot{\xi} = \mathcal{O}\frac{k^2}{(aH)^2}$$

In the super-horizon regime, $\frac{k}{aH} \ll 1$, so in effect, on super-horizon scales $\dot{\xi} = 0$. From the moment of horizon exit, to horizon re-entry, the value of ξ remains constant. If it's value at horizon exit is known, it is known until it re-enters it again, making it a very useful variable to work with.

6.3 Power spectrum

A perturbation in the field ϕ in any variable can be written as $\frac{\partial}{\partial\phi}\sigma\delta\phi$. Applying this to the energy density ρ, in equation 18, we find that in the slow roll regime, where $\dot{\phi}^2 \ll V(\phi)$

$$\delta\rho = \frac{\partial V}{\partial\phi}\delta\phi \tag{49}$$

Using equation 21 it follows that

$$\frac{\partial V}{\partial\phi} = -3H\dot{\phi} \tag{50}$$

Combining these equations gives

$$\delta\rho = -3H\dot{\phi}\delta\phi \tag{51}$$

Substituting that into equation 47 ,using equations 9, 18 and 19 gives

$$-\xi = \frac{H}{\dot{\phi}}\delta\phi \tag{52}$$

This can be related to the slow-roll parameter ϵ.

$$\epsilon = \frac{\dot{\phi}^2}{\sqrt{2}H^2}$$

$$\xi = \frac{1}{\sqrt{2\epsilon}}\delta\phi \tag{53}$$

This relates the variable ξ to perturbations in the field, which is very convenient and allows the power spectrum of ξ to be connected to that of $\delta\phi$.

$$<\xi\,\xi> = \frac{1}{2\epsilon} < \delta\phi\,\delta\phi > \tag{54}$$

To be able to find any perturbations, the action has to be expanded, into a quadratic term. Otherwise the perturbations are non-existent. It can be shown that the action to second order in ξ is given by[9]

$$S_2 = \int d^4x\epsilon^2(\xi^2 - a^-2(\partial_i\xi)) \tag{55}$$

Now we can define a new variable v, known as the Mukhanov variable, defined as

$$v = z\xi \tag{56}$$

$$z = 2a^2\epsilon \tag{57}$$

It is important to note that v is not a scalar, but a scalar field, like ξ is. using integration by parts on equation 55, switching to conformal time and substituting in the Mukhanov variable gets the new resulting action

$$S_2 = \frac{1}{2}\int d\tau d^3x(v'^2 + (\partial_i v)^2 + \frac{z''}{z}v^2) \tag{58}$$

The field v can be expanded, using the the Fourier definition

$$v(\tau, x) = \int \frac{d^3k}{(2\pi)^3} v_k(\tau)e^{ik\cdot x} \tag{59}$$

Substituting in this definition of v and varying the action in equation 58 with respect to the field v_k, using the Euler-Lagrange equation

$$(\frac{\partial \mathcal{L}}{\partial v'})' + \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i v)} - \frac{\partial \mathcal{L}}{\partial v} = 0 \tag{60}$$

this works out to

$$\frac{e^{ik\cdot x}}{(2\pi)^3}(v_k'' + k^2 v_k - \frac{z''}{z}v_k) = 0 \tag{61}$$

There is only a small difference in varying the action with respect to v or v_k, only the factor k^2 from the gradient appears extra. Equation 61 is known as the Mukhanov-Sasaki equation and effectively governs the evolution of the field ξ. To solve this equation the fields have to be quantized.

6.4 Quantization

In order to quantize the field v, it has to be promoted to a quantum mechanical operator $\hat{v}$. This operator should behave exactly like a quantum mechanical harmonic oscillator, such that

$$\hat{v} = \int \frac{d^3k}{(2\pi)^3}(v_k(\tau)\hat{a}_k e^{ik\cdot x} + v_k^* \hat{a}_k^\dagger e^{-ik\cdot x}) \tag{62}$$

Here a_k and $a_k^\dagger$ are annihilation and creation operators respectively. They take a the harmonic oscillator an energy level lower or higher by acting them as an operator on an energy state. Because of this the annihilation operator has the constraint that when acting on a vacuum state, the resulting energy becomes 0, to allow for minimum energy solutions. In an equation this means

$$\hat{a}_k |0\rangle = 0$$

Additionally, these operators are normalized by the following commutation relation

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = (2\pi)^3 \delta(k - k') \tag{63}$$

This is how the creation and annihilation operators are defined, but this requires the additional constraint

$$< v_k v_{k'} >= \frac{i}{\hbar}(v_k^* v_k' - v_k^{*\prime} v_k) = 1$$

This is one of the boundary conditions necessary in order to solve equation 61, but another boundary condition is necessary to solve it completely. This can be done by choosing the vacuum state $|0\rangle$. A convenient choice is picking a time in the past, when $\tau \to -\infty$. This means that $k \gg aH$ and $a(\tau)0 \to 0$. This simplifies equation 61 to

$$v_k'' + k^2 v_k = 0 \tag{64}$$

This is a simple harmonic oscillator with the solution

$$v_k = \frac{1}{\sqrt{2\omega}} e^{-i\omega t} \tag{65}$$

Which is the positive frequency result for a quantum mechanical harmonic oscillator [18]. With a frequency $\omega = k$, this gives the result

$$v_k = \frac{1}{\sqrt{2k}} e^{-ikt} \tag{66}$$

With this result the behaviour of v_k is fixed for any value of k and because it is related to the field ξ, the behaviour of $\delta\phi$ is now also quantized. This is a very powerful result, dictating the behaviour of ξ_k, which can directly be worked out to the variance and thus the power spectrum of ξ and thus $\delta\phi$, it also allows for the computation of the tilt n_s. These are observable in the CMB. From the power temperature map, the power spectrum can be constructed. with the power spectrum it is possible to work through these calculations and work out the tilt. These predictions are incredibly powerful in either confirming the theory up to the measurable accuracy or dismissing it.

6.5 Example

In de-Sitter space, the slow-roll parameter approaches 0. This implies

$$\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}$$

. The solution to this equation is given by

$$v_k = \alpha \frac{1}{\sqrt{2k}} e^{-ikt}(1 - \frac{i}{kt}) + \beta \frac{1}{\sqrt{2k}} e^{ikt}(1 + \frac{i}{kt}) \tag{67}$$

evaluating the equation in the approximation $\frac{1}{k\tau} \to 0$ allows the use of the boundary condition 66. This gives

$$\alpha \frac{1}{\sqrt{2k}} e^{-ikt} + \beta \frac{1}{\sqrt{2k}} e^{ikt} = \frac{1}{\sqrt{2k}} e^{-ikt}$$

Which gives the solution $\alpha = 1$, $\beta = 0$. The solution are the called the Bunch-Davies modes and given by

$$v_k = \frac{1}{\sqrt{2k}} e^{-ikt}(1 - \frac{i}{kt}) \tag{68}$$

now we can compute the power spectrum of the field $\hat{\psi}_k = \frac{1}{a} v_k$ Here the field ψ_k plays the role of the perturbations $\delta\phi$. using equation 35 we find

$$< \hat{\psi}_k \hat{\psi}_{k'} >= (2\pi)^3 \delta(k + k') \frac{|v_k|}{a^2} \tag{69}$$

$$|v_k| = \frac{1}{\sqrt{2k}} \sqrt{1 + \frac{1}{k^2\tau^2}}$$

$$|v_k|^2 = \frac{1}{k^3\tau^2}(1 + k^2\tau^2)$$

Thus equation 69 can be written as

$$< \hat{\psi}_k \hat{\psi}_{k'} >= (2\pi)^3 \delta(k + k') \frac{H^2}{2k^3}(1 + k^2\tau^2)$$

On superhorizon scales, where $k\tau \ll 1$, this becomes a constant,

$$< \hat{\psi}_k \hat{\psi}_{k'} >= (2\pi)^3 \delta(k + k') \frac{H^2}{2k^3} \tag{70}$$

and equivalently

$$\Delta_\psi^2 = \frac{H^2}{(2\pi)^2}$$

Using equation 52 this can now be related to the power spectrum of ξ:

$$< \xi_k \xi_{k'} >= (2\pi)^3 \delta(k + k') \frac{H_*^2}{2k^3} \frac{H_*^2}{\dot{\phi}_*^2}$$

With the corresponding variance

$$\Delta_\xi^2 = \frac{H_*^2}{(2\pi)^2} \frac{H_*^2}{\dot{\phi}_*^2} = \frac{H_*^2}{8\pi^2} \frac{1}{\epsilon} \tag{71}$$

Where the star denotes that these values are taken at horizon crossing, since ξ remains constant throughout. Δ_ξ^2 is also referred to as Δ_s^2 as a reminder that this is for scalar perturbations only. Now that the power spectrum of ξ is known and is connected to that of $\delta\phi$, this can be measured [15]. The interpretation of these calculations is that there were small quantum fluctuations in in the early universe. These quantum fluctuations were subsequently subjected to the inflationary period. Though they were small initially they were now increased to relatively large sizes. This would be the source of measurable fluctuations, like the temperature and density fluctuations that can be observed in for example the CMB. These fluctuations are important because regions with slightly higher densities would exert a larger gravitational force, attracting other matter. This would allow for the formation of large structures and eventually the universe as we know it. Even though the fluctuations considered here are very small, their effect is very large. [20]

The spectral index can be computed using the power spectrum. The spectral index for this field is called the scalar spectral index n_s, because it relates specifically to a scalar field and is defined in equation 37.

$$n_s - 1 = \frac{d \ln \Delta^2 - \xi}{dN} \frac{dN}{d \ln k} \tag{72}$$

Evaluating the first term gives

$$\frac{d \ln \Delta^2 - \xi}{dN} = \frac{d \ln \frac{H_*^2}{8\pi^2} \frac{1}{\epsilon}}{dN} =$$

$$\frac{d \ln \frac{1}{8\pi^2}}{dN} + 2 \frac{d \ln H}{dN} - \frac{d \ln \epsilon}{dN}$$

The first term is a constant so evaluates to 0. The second term is the definition of ϵ in equation 25. For the final term, it can be shown that [9]

$$\frac{d \ln \epsilon}{dN} = 2(\epsilon - \eta)$$

For solving the term

$$\frac{dN}{d \ln k}$$

N should be expressed in terms of k or vice versa. Because the equation is considered at horizon crossing, $k = aH$, so $\ln K = \ln a + \ln H$. Using the definition of the number of e-folds in equation 13, $\ln a = N$, such that

$$\frac{dN}{d \ln k} = \frac{1}{\frac{d \ln k}{dN}} = \frac{1}{\frac{dN + \ln H}{dN}} =$$

$$\frac{1}{1 - \epsilon}$$

Assuming that ϵ is small in this regime, it approximates to

$$\frac{dN}{d \ln k} \approx 1 + \epsilon \tag{73}$$

This gives the result for the spectral index that

$$n_s - 1 = (2\eta - 4\epsilon)(1 + \epsilon) \approx 2\eta - 4\epsilon$$

Because the slow-roll parameters are already assumed to be small, any second order squared term is negligible. In the slow-roll approximation, these parameters can be related to the potential, as in equations 29 and 30. This gives the constraint on any model with a potential:

$$n_s - 1 = 2\eta_v - 6\epsilon_v \tag{74}$$

The spectral index can be measured and is observed to have a value of $n_s = 0.9626 \pm 0.0057$ [17]. This means that the power spectrum is almost completely independent on the scale. This is a powerful point in favor of slow roll. In slow roll, both ϵ and η are very small, so $n_s - 1 \approx 0$ and $n_s \approx 1$. This is exactly what the measurements show, so this is a good indication that slow-roll inflation is a good theory with powerful predictions. How close the value of n_s is to the predicted value is one of the big triumphs of slow-roll inflation.

6.6 Tensor perturbations

Using a similar argument as before, a second-order action can be constructed with perturbations in the tensor terms only. This can be done, because to first order, these perturbations are separate. The corresponding action in terms of the field v_k reads [9]

$$S = \sum_s \int \frac{1}{2} d\tau d^3k((v_k^{s'})^2 - (k^2 - \frac{a''}{a})v_k^{s\,2}) \tag{75}$$

Where in De-Sitter space $\frac{a''}{a} = \frac{2}{\tau^2}$, just as in the case for the scalar perturbations. This is essentially two copies of the earlier action in equation 58.

$$\Delta_h^2(k) = 2\Delta_\psi^2 = 4\frac{H_*^2}{(2\pi)^2}$$

Here h is a measure of the tensor perturbations and is responsible for the gravitational waves and their polarizations. Scalar perturbations do not contribute to to the formation of gravitational waves, at least not in first order. Higher order derivative terms of scalar perturbations could lead to the formation of waves. Care should be taken in normalizations, where factors of 2 have been inserted and Plack masses should be added for the right units. Then the power spectrum for tensor perturbations is given by

$$\Delta_t^2 = \frac{2}{\pi^2} H_*^2 \tag{76}$$

The ratio of Δ_S^2 to Δ_t^2 is known as the tensor-to-scalar ratio. This ratio works out to

$$r = \frac{\Delta_t^2}{\Delta_s^2} = 16\epsilon$$

This quantity can also be measured and it is observed that $r < 0.064$. These constraints combined give upper bounds on the (approximate) slow-roll parameters [17]

$$\epsilon_v < 0.0097 \tag{77}$$

$$\eta_v = -0.010 \tag{78}$$

Just as in the case of the scalar perturbations, these measurements place restrictions on any proposed model, narrowing down the possibilities.

7 Case study

To make use of these ideas and equations, a case study can provide a clear example. In slow-roll inflation the dominant force is the potential $V(\phi)$. This potential should contain a minimum and then be rise. For this example I will consider a potential of the form

$$V = V_0(\cos \frac{d}{\phi + b}) \tag{79}$$

Where a, b and V_0 are variable parameters.The variable d controls the "steepness" of both the flat first part of the potential and the well. The variable b controls where the origin of the plot is located. This potential is shown in figures 5 and 6. The potential looks like the potential for natural inflation

$$V = V_0(\cos \frac{\phi}{f} + 1) \tag{80}$$

The potential has a slightly different dependence on ϕ, and an extra variable is introduced for more control over the potential.

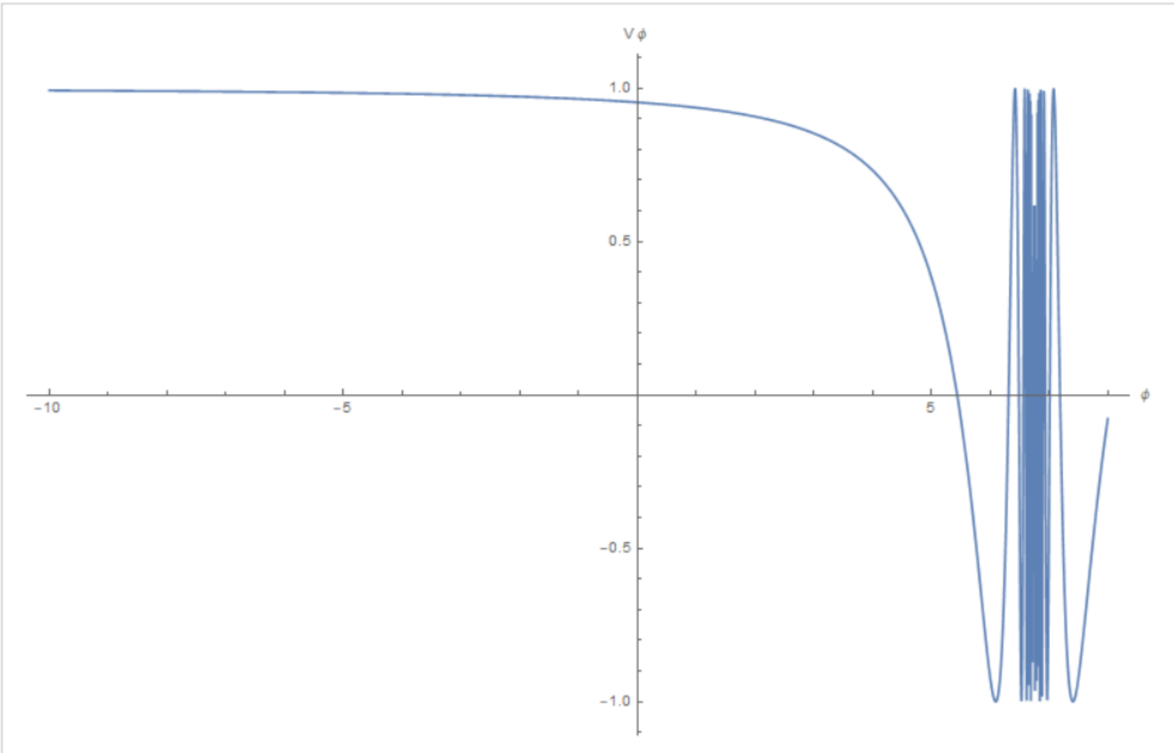

Figure 5: The potential used in this case study with a small value of d

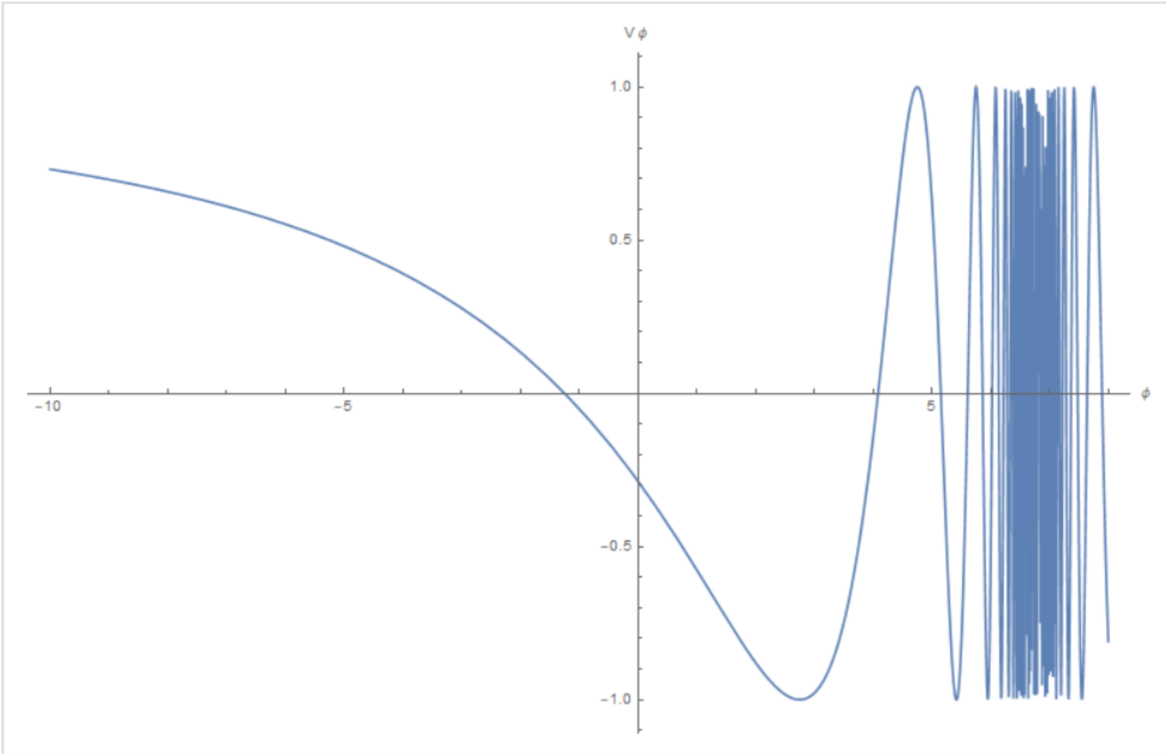

Figure 6: The potential used in this case study with a large value of d. The potential is much less steep in the potential well, but rolls into it more quickly.

From the potential, the corresponding slow-roll parameters can be computed. The first slow-roll parameter works out to be

$$\epsilon_v = \frac{M_{pl}^2}{2}\frac{V_{,\phi}^2}{V^2} = \frac{M_{pl}^2}{2}\frac{d^2\sin^2\frac{d}{\phi+b}}{\cos^2\frac{d}{\phi+b}(b+\phi)^4} \tag{81}$$

Defining the new dimensionless variable $\Phi = \frac{a}{b+\phi}$ simplifies the formula to

$$\epsilon_v = \frac{M_{pl}^2}{2(b+\phi)^2}\Phi^2\tan^2\Phi \tag{82}$$

Or alternately, by introducing d instead of $b+\phi$

$$\epsilon_v = \frac{M_{pl}^2}{2d^2}\Phi^4\tan\Phi$$

The second slow roll parameter is given by

$$\eta_v = M_{pl}^2 \frac{V_{,\phi\phi}}{V} = M_{pl}^2 \frac{d^2 \frac{\cos\Phi}{(b+\phi)^4} - 2d \frac{\sin\Phi}{(b+\phi)^3}}{\cos\Phi}$$

$$\eta_v = M_{pl}^2 (\Phi^2 \frac{1}{(b+\phi)^2} - 2\Phi \frac{\tan\Phi}{(b+\phi)^2})$$

$$\eta_v = \frac{\Phi M_{pl}^2}{(b+\phi)^2} (\Phi - 2\tan\Phi) \tag{83}$$

This can also be written alternatively as

$$\eta_v = \frac{M_{pl}^2}{d^2} \Phi^3 (\Phi - 2\tan\Phi) \tag{84}$$

It should be noted that these parameters have to be dimensionless. Making the substitution to the dimensionless variable makes that more clear. Both b and d have the same units as ϕ, such that $\frac{M_{pl}}{b+\phi}$ is also dimensionless. Secondly, in these calculation, two asymptotes appear, for both $\frac{a}{b+\phi}$, which implies $b \neq -\phi$ and for $\tan \frac{a}{b+\phi} = \tan\Phi$, which implies that $\Phi =\neq \pm\frac{\pi}{2}$.
The two slow roll parameters should be small, the first parameter ϵ gives

$$\frac{M_{pl}^2}{2(b+\phi)^2} \Phi^2 \tan^2\Phi < 1$$

or in the alternate form

$$\frac{M_{pl}^2}{2d^2} \Phi^4 \tan^2\Phi < 1$$

$$\frac{\Phi^2}{d} \tan\Phi > \frac{M_{pl}}{\sqrt{2}} \tag{85}$$

And from the parameter η we can see

$$\frac{\Phi M_{pl}^2}{(b+\phi)^2} (\Phi - 2\tan\Phi) < 1$$

Or in the alternate form

$$\frac{\Phi^4 - 2\Phi^3 \tan\Phi}{d^2} > M_{pl}^2$$

Solving for these inequalities is not trivial. Substituting the relation in equation 85 gives

$$\frac{\Phi^4}{d^2} - \frac{\sqrt{2}\Phi}{d} M_{pl} > M_{pl}^2$$

Solving this system of equations first requires transforming the inequility into an equality. There is a single solution in which all variables are real and $\Phi = d \neq 0$:

$$d = 0.732$$

$$\Phi = 0.0017$$

. This also fixes the value for $b+\phi$:

$$b+\phi = 441.2$$

By looking at the inequality, it is clear that for this to hold, either Φ can become larger, by making $b+\phi$ smaller or d can become smaller, so the real relations become

$$d < 0.732 \tag{86}$$

$$b+\phi < 441.2$$

Finally, the number of e-folds N can be computed, to provide an extra constraint.

$$N(\phi) = \frac{1}{M_{pl}} \int_{\phi_{end}}^{\phi_{in}} \frac{d\phi}{\sqrt{2\epsilon_v}}$$

This integral can be simplified by making it dimensionless. That can be done by reintroducing the dimensionless variable Φ. That way, the number of variables under the integral will be reduced from 3 to 1, while all the variables that carry dimensions can be pulled out of the integral. It also makes it more clear to see if the final quantity is indeed dimensionless, since the number of e-folds should not carry any units.

$$\Phi = \frac{d}{\phi + b}$$

The integral is over ϕ, but this can be substituted to integrate over Φ. However, the variable $d\phi$ will require adjusting as well, because we want to go from a variable with units to one without. Writing $d\phi$ in terms of $d\Phi$ gives the result.

$$d\Phi = \frac{-d}{(\phi+b)^2} d\phi$$

$$d\phi = \frac{-(\phi+b)^2}{d} d\Phi = \frac{-d}{\Phi^2} d\Phi$$

Now the equation can be simplified, written only in terms of Φ and d,

$$N(\phi) = \frac{d}{M_{pl}^2} \int_{\Phi_{end}}^{\Phi_{in}} \frac{1}{\Phi^2 \tan\Phi} \frac{-d}{\Phi^2} d\Phi =$$

$$\frac{-d^2}{M_{pl}^2} \int_{\Phi_{end}}^{\Phi_{in}} \frac{1}{\Phi^4 \tan\Phi} d\Phi$$

Even such a simple-looking equation cannot be solved analytically. Inflation requires that $N \approx 60$, so using the value for d obtained earlier, a bound on the integral can be placed.

$$-110 < \int_{\Phi_{end}}^{\Phi_{in}} \frac{1}{\Phi^4 \tan\Phi} d\Phi < 0$$

Fluctuations in the CMB have found a value for ϕ at the initial time when the CMB was formed. $\phi_{in} \approx 15M_{pl}$. Assuming that at that time $\phi \approx \Phi$, the end value of Φ can be found. Normalizing the Planck mass to 1, it works out that $\Phi_{end} < 0.22$. Inflation ends when $\epsilon_v = 1$. using this a definitive value if d can be found: $d_{end} \approx 0.008$, which is indeed smaller than the value computed earlier.

8 Non-Gaussianities in single-field slow-roll

8.1 ADM model

Non-Gaussianities are any deviations from a Gaussian distribution. All observables are locked in a statistical configuration, the power spectrum or correlation functions. A Gaussian distribution only allows for even-point correlation functions. In principle there would only be a correlation between two points and a 4-point correlation function is the product of two two-point functions. Any Non-Gaussian term would be due to the non-linearity of gravity and would lead to an odd-number correlation function. To find the three-point correlation function for a scalar perturbation, the action has to be expanded into quadratic and cubic terms. The action can be expressed very conveniently in a specific formalism, the ADM (named after Richard Arnowitt, Stanley Deser and Charles W. Misner) formalism. While the quadratic terms in the Lagrangian have already been looked at in the perturbation section, redoing it in this formalism is still useful to get a better working of the formalism and use that as a springboard to go to cubic Lagrangian terms. In the ADM formalism the metric is written as:

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{87}$$

Here g_{ij} are three-dimensional slices of the metric at constant time. $N(\vec{x}$ is the lapse function and $N_i(\vec{x}$ is called the shift function. They act as Lagrange multipliers in the Lagrangian and are the equivalent of the perturbations Φ and B [24]. The action of a scalar field ϕ minimally coupled to gravity is given by

$$S = \frac{1}{2}\int d^4x\sqrt{-g}(R - (\nabla\phi)^2 - 2V)$$

As follows from equation 17. Another variable is E_{ij}, which is related to the extrensic curvature of spatial slices K_{ij}, via the relation $K_{ij} = N^{-1}E_{ij}$. with

$$E_{ij} = \frac{1}{2}(g_{ij} - \nabla_i N_i \nabla_j N_j)$$

and

$$E_i^i = E$$

In the ADM formalism, this action becomes [10]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}(NR - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^{-1}\partial_i\phi)^2 - Ng^{ij}\partial_i\phi\partial_j\phi - 2V) \tag{88}$$

In order to solve the Gauge problem, The gauge should be fixed and a specific gauge invariant under coordinate transformations should be chosen to make sure any perturbations in the action are physical and not introduced by the transformations. A convenient Gauge is the comoving Gauge, where the Gauge-invariant variables are

$$\delta\psi = 0 \tag{89}$$

$$g_{ij} = a^2((1-2\xi)\delta_{ij} + h_{ij}) \tag{90}$$

$$\partial_i h_{ij} = h_i^i = 0 \tag{91}$$

The difference in this particular choice is that the field is not perturbed, but all perturbations are captured in the metric perturbation ξ. From the action the constraint equation on the Lagrange multipliers N and N_i can be found. The constraint for N can be found by varying the Lagrangian with respect to N:

$$\frac{\partial \mathcal{L}}{\partial N} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{N}}$$

The Lagrangian depends only on N, so the second term is 0. taking the derivative gives

$$R^3 - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}(\dot{\phi}^2 - N^i\partial_i\phi)^2 - g^{ij}\partial_i\phi\partial_j\phi$$

But because the Gauge is fixed with $\delta\phi = 0$, the terms involving $\partial_i\phi$ are 0. The constraint for N is thus

$$R - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 = 0$$

and the constraint for N^i can be found in a similar way and works out to

$$\nabla_i(N^{-1}(E_j^i - \delta_j^i E)) = 0 \tag{92}$$

These equations can be solved by splitting N^i into two parts:

$$N^i = \partial_i\psi_i + \bar{N}^i \tag{93}$$

with the constraint that $\partial_i\bar{N}^i = 0$ The deviation in N can be written as

$$N = 1 + \alpha$$

Where α can be written as a power expansion series. To first order

$$\alpha_1 = \frac{\dot{\xi}}{H} \tag{94}$$

$$\partial_i^2 N^i = 0$$

and

$$\psi_1 = \frac{-1}{a^2}\frac{\xi}{H} + \partial^{-2}\epsilon_v\dot{\xi} = \tag{95}$$

$$\psi_1 = -\frac{\xi}{H} + \frac{\xi}{H}\epsilon_v\dot{\xi} \tag{96}$$

Where ∂^{-2} is an anti-derivate, such that $\partial^{-2}\partial^2\xi = \xi$.[9] Substituting these solutions back into the action in equation 88 gives a perturbed action

$$S = \frac{1}{2}\int a^2 e^{\xi}(1+\frac{\dot{\xi}}{H})(-4\partial^2\xi - 2(\partial\xi)^2 - 2V(a^2e^{2\xi})) + a^3e^{3\xi}\frac{1}{1+\frac{\dot{\xi}}{H}}(-6(H+\dot{\xi})^2 + \dot{\phi}^2) \tag{97}$$

Where derivatives linear in ψ have been left out and where a can also be written as $a = e^H$. Using integration by parts and the equations 18,21 and 22. The action can be written as [10]

$$S = \frac{1}{2}\int d^4x \frac{\dot{\phi}^2}{H^2}(a^3\dot{\xi}^2 - a(\partial\xi)^2) \tag{98}$$

or in terms of the slow roll parameter ϵ_v

$$S = \int \epsilon_v(a^3\dot{\xi}^2 - a(\partial\xi)^2) \tag{99}$$

This is the action for the free field ξ. Like in the discussion about perturbations, switching over to k-space allows for easier manipulation of the action. The field ξ can be written as a set of infinite harmonic oscillators with differing spring and mass constants each.

$$\xi = \int \frac{d^3k}{(2\pi)^3}\xi_k e^{i\vec{k}\cdot\vec{x}} \tag{100}$$

Then the equation of motion in the action can be computed.

$$\frac{\delta\mathcal{L}}{\delta\xi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\xi}} - \frac{\partial\mathcal{L}}{\partial\xi} = 0$$

The first term can be computed directly, since $\dot{\xi}$ appears in the Lagrangian explicitely and gives

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\xi}} = \frac{d}{dt}(\frac{\dot{\phi}^2}{H^2}a^3\dot{\xi})$$

The second term does not appear explicitely in the Lagrangian, but it does when transforming to k-space. The derivatives then give a factor $(ik)^2$. The equation of motion is thus

$$\frac{\delta\mathcal{L}}{\delta\xi} = \frac{d}{dt}(\frac{\dot{\phi}^2}{H^2}a^3\dot{\xi_k}) + \frac{\dot{\phi}^2}{H^2}ak^2\xi_k = 0$$

This field has to be quantized in the same way as the earlier field, such that we can write ξ_k as the sum of two independent solutions ξ_k^{cl} and ξ_k^{cl*}

$$\xi_k = \xi_k^{cl}a^\dagger_{\vec{k}} + \xi_k^{cl*}a_{-\vec{k}}$$

Where $a_{-\vec{k}}$ and $a^\dagger_{-\vec{k}}$ are the standard creation and annihilation operators with the standard commutation relations. This gives the same equation as before

$$v_k'' + k^2 v_k = 0$$

which produces the same result as before using the same arguments. Thus lending credence to this way of tackling the problem. While this method may appear to be more involved, it can be handled step by step in a similar fashion for most problems, which is very useful and when using it, the results do not differ at all.

8.2 Cubic Lagrangian terms

In order to compute the Cubic Lagrangian terms the ADM formalism will be adhered to. The Gauge chosen will be [10]

$$\delta\phi = 0$$
$$g_{ij} = e^{2H+2\xi}\hat{g_{ij}}$$
$$det(\hat{g}) = 1 \tag{101}$$
$$\hat{g_{ij}} = (\delta_{ij} + h_{ij} + \frac{1}{2}h_{il}h_{lj} + ...)$$

Higher order terms do not contribute to the action, so first and second order is enough to define the cubic terms. Using this Gauge and substituting this into the action to find [10]

$$S = \int e^{H+\xi}(1+\frac{\dot{\xi}}{H})(-2\partial^2\xi - (\partial\xi)^2) + e^{3H+3\xi}\frac{1}{2}\frac{\dot{\phi}}{H^2}\dot{\xi}^2(1-\frac{\dot{\xi}}{H}) +$$
$$+ e^{3H+3\xi}(\frac{1}{2}((\partial_i\partial_j\psi)^2 - (\partial^2\psi)^2)(1-\frac{\dot{\xi}}{H}) - 2\partial_i\psi\partial_i\xi\partial^2\psi)$$

Where only terms up to ξ^3 and $\dot{\xi}^3$ are included, since ψ is defined in equation 95 and is linear in ξ and $\dot{\xi}$. It can be shown that the action is of the order ϵ^2 [10], which requires a lot a lot of integration and another choice of Gauge and hence will not be repeated. A dependence of ϵ^2 means that the three-point correlation function will be heavily suppressed and is unlikely to be measured. To be able to get the three-point correlation function the field ξ can be expanded using a new variable ξ_c

$$\xi = \xi_c + \frac{1}{2}\frac{\ddot{\phi}}{\dot{\phi}H}\xi_c^2 + \frac{1}{2}\frac{\dot{\phi}^2}{H^2}\xi_c^2 + \frac{1}{4}\frac{\dot{\phi}^2}{H^2}\partial^{-2}(\xi_c\partial^2\xi_c) + ... \tag{102}$$

Where the dots indicate those terms vanish outside the horizon, or are higher order in ϵ. Since ξ does not evolve outside the horizon, those terms are negligible and do not need to be included. When written in this form, the action simplifies to

$$S_3 = \int \frac{\dot{\phi}^4}{H^4}e^{5H}\xi_c^2\partial^{-2}\dot{\xi} + ... \tag{103}$$

Where the dots are terms of higher order in ϵ. This action is of higher order, so when quantized will naturally lead to the higher order terms present in the three-point correlation function, which can be computed from this point.

8.3 Three point correlation function

The three-point correlation function can be expressed in what is called the interaction picture. This picture is a mixture of the Schrödinger picture and the Heisenberg picture. Both vectors and operators carry a time dependence that lead to time dependence in the observables. This is especially convenient because it allows for interactions to be more easily computed. In this picture the three-point correlation function can be represented as [10]

$$< \xi^3(t) >=< U_{int}\xi^3(t)U_{int}^{-1} >$$

Where U_{int} is an operator and given by

$$U_{int} = Te^{-i\int_{t_0}^{t} H_{int}(t')dt'}$$

Where T gives a time ordering, t_0 is an arbitrary early time and H_{int} is the interaction Hamiltonian. For the system we are considering, the cubic terms of the Lagrangian, the Hamiltonian is related to the Lagrangian as

$$H_{int} = -L_{int}$$

Substituting this in and taking the first order approximation of the exponential

$$e^x \approx 1 + x$$

This introduces a commutation relation between ξ^3 adn H_{int} :

$$< \xi^3(t) >= -i\int_{t_0}^{t} < [\xi^3(t), H_{int}(t')]dt' \tag{104}$$

Where care should be taking when integrating, ξ does not depend on the integration variable t'. An important variable that has not been addressed until now is the type of vacuum that is worked in. The type of vacuum used is related to the type of space-time that is being considered. For this discussion the interacting vacuum will be used, which in the De-Sitter space-time it is the Hartle-Hawking vacuum [26]. The impact of choosing the type of vacuum is the region over which the integrations in equation 104 take place. In this case, it is split into three distinct regions: The region outside the horizon, the region around horizon crossing and the region (deep) inside the horizon. The benefit of this is that space-time near horizon crossing is very close to De-Sitter spacetime and the region deep inside the horizon does not contribute due to rapid field oscillations [10]. If we have a field of the form

$$\xi = \xi_c + \lambda\xi_c^2$$

The corresponding three-point function would look like

$$< \xi(x_1)\xi(x_2)\xi(x_3) >=< \xi_c(x_1)\xi_c(x_2)\xi_c(x_3) > +2\lambda(< \xi(x_1)\xi(x_2) >< \xi(x_1)\xi(x_3) > +\text{cyclic}) \tag{105}$$

The cyclic denotes cyclic permutations, such that every possible double combination of $\xi(x_1)$,,$\xi(x_2)$ and $\xi(x_3)$ will appear. This can be achieved by cyclic permutation where $1 \to 2 \to 3 \to 1$. The 2λ appears because of the combination of two two-point correlation functions. The reason for the redefinition of the field becomes apparent in this way of writing the correlation function. The first term can be directly computed from the action of equation 103. This term is to be evaluated near horizon crossing, after which ξ is constant, which means this computation will be done in De-Sitter space. The first term evaluates to [10]

$$< \xi_c^3 >= (2\pi)^3\delta^3(\sum(\vec{k}_i))\frac{i}{\prod(2k_i^3)}\frac{H^6*}{\dot{\phi}^2*}\int_{-\infty}^{0} d\tau k_1^2 k_2^2 e^{ik_t\tau} + \text{cyclic}$$

where $k_t = k_1 + k_2 = k_3$, the sum of all spatial k values. The cyclic permutations on $k_i = k_1$ and $k_j = k_2$ mean that every combination is possible where $i \neq j$. for every combination there is a symmetric one, which gives a factor 4. This means the integral can be rewritten as

$$\int_{-\infty}^{0} d\tau k_1^2 k_2^2 e^{ik_t\tau} + \text{cyclic} = 4\frac{1}{ik_t}\sum_{i>j} k_i^2 k_j^2$$

The first term is hence given by

$$< \xi_c^3 >= (2\pi)^3\delta^3(\sum(\vec{k}_i))\frac{}{\prod(2k_i^3)}\frac{H^6*}{\dot{\phi}^2*}4\frac{1}{k_t}\sum_{i>j} k_i^2 k_j^2$$

The second term comes from the field redefinition in equation 102. The ξ_c^2 terms will translate to summation over $k^3 : \sum k_i^3$. Not forgetting the factor 2, adding these terms to the previous result gives the final correlation function

$$< \xi_{\vec{k}_1}\xi_{\vec{k}_2}\xi_{\vec{k}_3} >= (2\pi)^3\delta^3(\sum \vec{k}_i)\frac{H*8*}{\dot{\phi}^4*}\frac{1}{\prod_i 2k_i^3}\mathcal{A}* \tag{106}$$

Where

$$\mathcal{A}* = 2\frac{\ddot{\phi}*}{\dot{\phi}*H*}\sum_i k_i^3 + \frac{\dot{\phi}^2*}{H^2*}(\frac{1}{2}\sum_i(k_i^3) + \frac{1}{2}\sum_{i\neq j}(k_i k_j^2) + 4\frac{1}{k_t}\sum_{i>j} k_i k_j \tag{107}$$

While it looks as if $\mathcal{A}$ is just a constant, contracting so many terms into it, it actually contains most of the interesting physics. It contains all the dependence of the spectrum on the momentum k, at the time of horizon crossing. In writing these equations it is assumed that all k's are the same order of magnitude, so no term is negligible or overpowering. It is also possible that one k, say k_3 is much smaller than the other two, so that it crosses the horizon much sooner. The difference in that scenario versus the one with all same order k's is that k_1 and k_2 cross the horizon slightly earlier too, which only leads to small perturbation in the final result. In that case

$$< \xi_{\vec{k}_1}\xi_{\vec{k}_2}\xi_{\vec{k}_3} >\approx (2\pi)^3\delta^3(\sum_i k_i^3)\frac{H^4*}{\dot{\phi}^2}\frac{H_3^4*}{\dot{\phi}_3^2*}\frac{1}{2k_1^2k_2^2}(\frac{2\ddot{\phi}*}{H*\dot{\phi}*} + \frac{2\dot{\phi}^2*}{H^2*})$$

Where the subscript 3 denotes the value at the time k_3 crosses the horizon. The final variable to define is F_{NL}. f_{NL} is a measure of the Non-Gaussianity of the spectrum and can be measured. f_{NL} is defined as [9]

$$F_{NL} = \frac{5}{18}\frac{B_R}{P_R^2}$$

where B_R is the bi-spectrum or three-point correlation function. For this discussion

$$f_{NL} \approx \frac{5}{18}\frac{\mathcal{A}}{\sum_i k_i^3}$$

. This value has been measured and an upper limit has been placed, using the CMB and Large Scale Structures [35]. A specific value has not been found and it is unlikely the CMB will be able to provide anything better than an upper bound of $f_{NL} < 3$ [32]. These Non-gaussianities are computed for three scalar fields, ξ. It can also be done for 1,2 or 3 graviton fields instead of scalars. These graviton perturbations are the tensor fluctuations responsible for gravitational waves. Calculations for these fluctuations are similar, but more involved. For that reason, they will not be reproduced here. These calculations can be found in the works of Maldacena, where all possible combinations are worked through [10].

9 Hamilton-Jacobi formalism

9.1 introduction

In previous discussions the main variable of interest was the field ϕ and the related field ξ. These variables give rise to fluctuations which can be measured and place bounds on inflation. However, it is not the only option of variable choice. Another choice is the Hubble parameter H or $\mathcal{H}$, where

$$\mathcal{H}(\phi) = aH(\phi) = \dot{a}$$

Since H is the main variable, taking its derivative would be very helpful

$$\frac{\partial H}{\partial \phi} = \frac{H'}{\phi'} = \frac{-\mathcal{H}' - \mathcal{H}^2}{a\phi'}$$

The primes are derivatives with respect to conformal time τ, which is a comoving variable. Using the second Friedman equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p)$$

Where it should be noted that

$$\mathcal{H}' = \frac{\ddot{a}}{a}$$

Remembering

$$\rho = \frac{1}{2}\phi^{2\prime} + V(\phi) = 3H^2$$

$$p = \frac{1}{2}\dot{\phi}^{2\prime} - V(\phi)$$

It follows that

$$\mathcal{H}^2 - \mathcal{H}' = \dot{a}^2 - \ddot{a}\frac{a^2}{2}(\rho + p)$$

$$\frac{\partial H}{\partial \phi} = -\frac{\phi'}{2} \tag{108}$$

$$\frac{d\phi}{dt} = \frac{\phi'}{a} = -2H_{,\phi} \tag{109}$$

Where different notation can be used, by introducing the planck mass. With it introduced the equation becomes

$$\frac{d\phi}{dt} = -\frac{m_p^2}{4\pi}H_{,\phi}$$

For any derivations, this factor will be normalized to 1, but for calculations in specific models, it will be used, to get all the factors correct. These equations of motion are first order differential equations only. That is the power of the Hamilton-Jacobi formalism. The equations of motion of regular slow-roll as outlined in equation 21 are of second order. This makes working in this formalism in principle easier and requires less computing power. It is very possible that because of this some predictions can only be computed in this formalism, highlighting its potential usefulness and power. Equation 109 can be integrated to

$$\int dt = -\frac{1}{2}\int \frac{d\phi}{H'} \tag{110}$$

The time t can be related to the Hubble constant and from there the expansion rate a. From here the Friedmann equation

$$H^2 = \frac{1}{3}(\frac{1}{2}\dot{\phi}^2 + V(\phi))$$

can be rewritten as

$$H^2 = \frac{1}{3}(2H_{,\phi})^2 + V(\phi))$$

$$(H_{,\phi})^2 - \frac{3}{2}H^2 = -\frac{1}{2}V(\phi) \tag{111}$$

This is an exact equation, if the Hubble parameter is obtained from equation 110, the corresponding potential can be computed. In other words, every Hubble parameter has a corresponding exact potential that can generate that parameter. Explicitly, the potential as a function of the Hubble parameter is

$$V(\phi) = 3H^2 - 2(H_{,\phi})^2 \tag{112}$$

and with the Planck mass introduced

$$V(\phi) = \frac{3M_p^2}{8\pi}(H^2 - \frac{m_p^2}{4\pi}(H_{,\phi})^2)$$

Here these are written down explicitly, but all following variables can be written in this way. Using the normalization $\frac{m_p^2}{8\pi} = 1$ and dimensional analysis, these factors are to be added whenever the dimensions require.

9.2 Slow-roll

The slow-roll parameter ϵ has actually already been expressed in terms of H. In equation 25

$$\epsilon = -\frac{d\ln H}{dN} = -\frac{d\ln H}{Hdt}$$

This is a derivative with respect to time, but from equation 109 this can be rewritten as

$$dt = -\frac{1}{2H_{,\phi}}d\phi$$

It should be noticed that there is a subtle difference here with regular Slow-roll. In regular slow-roll $\dot{\phi} \approx V_{,\phi}$, where in the Hamilton-Jacobi formalism, this equation shows $\dot{\phi} \approx H_{,\phi}$

$$-\frac{d\ln H}{Hdt} = -\frac{d\ln H}{(-\frac{H}{2H_{,\phi}})d\phi} =$$

$$\frac{2H_{,\phi}}{H}\frac{d\ln H}{d\phi} =$$

$$\epsilon_h = 2\frac{H_{,\phi}^2}{H^2}$$

Here the subscript h denotes that this is the slow-roll parameter in the Hamilton-Jacobi formalism. The parameter η has also already been defined as

$$\eta = -\frac{d\ln H_{,\phi}}{dN}$$

Using a similar calculation as before

$$\eta_h = 2\frac{H_{,\phi\phi}}{H}$$

It should be noted that equation 109 implies that η can also be written as

$$\eta_h = \frac{d\ln\dot{\phi}}{dN} = \frac{\ddot{\phi}}{H\dot{\phi}}$$

This is the exact same equation for η as calculated in equation 26, confirming that indeed the Hamilton-Jacobi formalism is consistent with all previous equations.

9.3 Attractor

If we consider some force that would cause the Hubble parameter to drift slightly from it's initial value, what would be the result? Consider a small perturbation

$$H = \bar{H} + \delta H$$

Where the bar signifies background level. Substituting this parameter into equation 111 we find

$$\bar{H}_{,\phi}{}^2 + 2\bar{H}_{,\phi}\delta H_{,\phi} + \delta H_{,\phi}^2 - \frac{3}{2}(\bar{H}^2 + 2\bar{H}\delta H + \delta H^2) = -\frac{1}{2}V(\phi)$$

The original terms in equation 111 are still present and can thus be removed

$$2\bar{H}_{,\phi}\delta H_{,\phi} + \delta H_{,\phi}^2 - \frac{3}{2}(2\bar{H}\delta H + \delta H^2) = 0$$

Since the perturbation δH is assumed to be small, the squares of the perturbations are negligible. This leads finally to the relation

$$2\bar{H}_{,\phi}\delta H_{,\phi} = \frac{3}{2}(2\bar{H}\delta H \tag{113}$$

Which can also be written as

$$\frac{d\ln\delta H}{d\phi} = \frac{3}{2}\frac{\bar{H}}{\bar{H}_{,\phi}}$$

Integrating over ϕ to get

$$\delta H = \delta H(\phi_i)exp(\frac{3}{2}\int_{\phi_i}^{\phi_f}\frac{\bar{H}}{\bar{H}_{,\phi}}d\phi)$$

Using $dN = Hdt = -\frac{1}{2}\frac{H}{H_{,\phi}}d\phi$ the integral can be solved to find

$$\delta H = \delta H(\phi_i)e^{3(N_i - N)} \tag{114}$$

Any potential perturbation from the background will evolve as an inverse exponential as N increases. This has the effect that since H depends on $\dot{\phi}$ explicitly and ϕ in the potential any deviations in these parameters will necessarily also smooth out over time. Whatever the value of these initial conditions, they are "attracted" towards a single value by minimizing the deviation from that solution as the number of e-folds increases. This makes ruling out models very difficult since all initial conditions would evolve towards the same final values with only very small differences.

9.4 Example model

One model for Hamilton-Jacobi inflation is called quasi-exponential inflation [19] where the Hubble constant is proposed as

$$H(\phi) = H_{inf}exp(\frac{\frac{\phi}{m_p}}{p(1+\frac{\phi}{m_p})}) \tag{115}$$

Where p is a free, dimensionless scaling factor and H_{inf} is a scaling factor with units of Planck mass. Substituting this Hubble constant into equation 109 gives the result

$$exp(\frac{\frac{\phi}{m_p}}{p(1+\frac{\phi}{m_p})})p(1+\frac{\phi}{m_p})(1+p+2p^2+p(1+4p)\phi+2p^2\phi^2)+e^{\frac{1}{p}}\int_{-x}^{\infty}\frac{1}{ze^z}dz = -\frac{3p^2}{2\pi}H_{inf}t$$

Where $x = \frac{1}{p(1+\frac{\phi}{m_{pl}})}$. In the Hamilton-Jacobi formalism the potential has an exact solution in terms of the Hubble parameter, as seen in equation 110. In this case

$$V = \frac{3H^2_{infm^2_p}}{32\pi^2p^2}\frac{exp(\frac{\frac{2\phi}{m_p}}{p(1+\frac{\phi}{m_{pl}})})}{(1+\frac{\phi}{m_p})^4}((4\pi p^2-1)+16\pi p^2\frac{\phi}{m_p}+24\pi p^2\frac{\phi^2}{m_p^2}+16\pi p^2\frac{\phi^3}{m_p^3}+4\pi p^2\frac{\phi^4}{m_p^4})$$

All the higher order terms in ϕ are assumed to be small. That reduces the potential to

$$V(\phi) \approx \frac{3H_{inf}^2 M_p^2}{8\pi} exp(\frac{\frac{2\phi}{m_p}}{p(1+\frac{\phi}{m_{pl}})})$$

The slow-roll parameters corresponding to this model are

$$\epsilon_h = \frac{1}{4\pi p^2(1+\frac{\phi}{m_p})^4}$$

$$\eta_h = \frac{1-2p-2p\frac{\phi}{m_p}}{4\pi p^2(1+\frac{\phi}{m_p})^4} = 1-2p-2p\frac{\phi}{m_p}\epsilon_h$$

Slow-roll inflation ends when $\epsilon_h = 1$, this constraint can be connected to the value of ϕ at the end of inflation:

$$\phi_{end} = m_p(\sqrt{\frac{1}{2p\sqrt{\pi}}} - 1)$$

The power spectrum in this model can also be calculated. The two-point correlation function in the Hamilton-Jacobi formalism does not differ form the correlation function calculated earlier in equation 71. Substituting in the values calculated [19]

$$\Delta_A^2 = \frac{4H_{inf}^2 p^2}{m_p^2} exp(\frac{\frac{2\phi}{m_p}}{p(1+\frac{\phi}{m_{pl}})})(1+\frac{\phi}{m_p})^4 \tag{116}$$

With the corresponding spectral index

$$n_s = 1 - \frac{1+2p+2p\frac{\phi}{m_p}}{2\pi p^2(1+\frac{\phi}{m_p})^4}$$

Working through the model and equations allowed us to find these values, which can be measured. This example model shows how the equations can be performed to end up at these final observable variables.

10 Discussion on the three-point correlation function in the Hamilton-Jacobi formalism

The two-point correlation function is the same in the Hamilton-Jacobi formalism as it is in the regular single-field slow-roll formalism. The three-point correlation function has not been worked out for the Hamilton-Jacobi formalism however. This formalism seems to be mainly suited for calculating potentials and slow-roll parameters[38]. In specific instances it has been computed, in multi-field theory for example [36] [37]. The bispectrum has its origin in Non-Gaussian terms which originate from the non-linearity in gravity. That makes computations inherently difficult, since they require extensive use of general relativity. The difficulty in using this formulation is that the main variable H or $\mathcal{H}$ are not fields, hence they cannot be redefined and compute a correlation function with. Such a function is possible to define with ϕ and the corresponding redefinition ξ. The relation between ϕ and ξ is more direct, whereas the dependence of $\mathcal{H}$ on ϕ depends on the specific model. The direct connection in regular slow-roll makes it in my opinion conceptually easier to work on non-Gaussian correlation functions.

Differences in the potential spectra of H and ϕ might show up due to a different coupling to the potential $V(\phi)$. In the Hamilton-Jacobi formalism H is related directly to V, with a sightly different dependence of $\dot{\phi}$, being either approximately equal to V in regular slow-roll or $\mathcal{H}$ in Hamilton-Jacobi. In regular slow-roll, the inflaton field is related via an approximation or an inequality. This more approximate dependence on the potential could hide smaller terms, perhaps on the order of ϵ or higher powers. These more exact properties makes the Hamilton-Jacobi formalism appealing for computations of Non-Gaussianities, though the uncertain relation of H on the field ϕ makes this practically challenging. Picking a Hubble constant to work with is a guess, where a lot of different options are possible [36], [38]. While the regular method is less exact, this approximate nature allows for more general statements which can be compared to measurement more easily [29] [30] [35]. In my opinion, while the exactness is very appealing, the necessity to choose a specific form of the Hubble parameter makes computing Non-Gaussianities in the Hamilton-Jacobi formalism less useful than the regular method in its present state. However, if the proposed Hubble parameters considered are more general, like an arbitrary polynomial or periodic function, the predictive power of this type of model should also increase.

The difference in coupling to the potential is at most on the order of ϵ, due to the nature of the approximation. Thus any threepoint correlation function will most likely also have a differing factor of up to ϵ. This is not insignificant as it could be the difference between detectable and invisible, which does create incentive to work in this field.

In conclusion, I believe that in the current state the predictive capabilities of the Hamilton-Jacobi formalism are limited, due to needing a specific form of the Hubble parameter. However, the pay-off is potentially very large. Additional work would potentially be able to give an upper bound on the difference between the two formalisms, which would be a great breaktrough and a great step towards more accurate models.

11 Appendix

Here the energy-momentum tensor will be constructed and manipulated to find the pressure and energy density. Starting from the action

$$S = \int d^4x\sqrt{-g}(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi))$$

The Energy-Momentum tensor is given by the equation [9]

$$T^{\mu\nu} = \frac{1}{2\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}}{\delta g_{\mu\nu}}$$

Where δ means varying the Lagrangian, in this case with respect to $g_{\mu\nu}$. Varying with respect to the metric requires a bit of care. Two important definitions are

$$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\delta g_{\mu\nu}$$

Using the Jacobi formula

$$d\,Det(g_{\mu\nu} = tr(adj(g_{\mu\nu})dg_{\mu\nu})$$

Using $g = Det(g_{\mu\nu}$, it follows that

$$\delta g = g\, g^{\mu\nu}\delta g_{\mu\nu}$$

Substituting this into equation 11 gives the result:

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$$

The second important formula is[16]

$$\delta g^{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g_{\alpha\beta}$$

Using the chain rule, varying the Lagrangian becomes

$$T^{\mu\nu} = \frac{1}{2\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}(\frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\mathcal{L} + \sqrt{-g}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}})$$

Substituting the Lagrangian $\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$ back in gives

$$T^{\mu\nu} = g^{\mu\nu}(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi) + \partial^\mu\phi\partial^\nu\phi$$

Where care should be taken to make sure the same indices are not reused. The diagonal entries of the energy-momentum tensor represent the energy density and the pressure of the field: $T_{00} = \rho$ and $T_{ij} = P_\phi g_{ij}$.

$$T_{00} = g_{00}(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi) + \partial^0\phi\partial^0\phi = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi) + \dot{\phi}^2$$

$$T_{ij} = g_{ij}(-\frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi - V(\phi)) + \partial_i \phi \partial_j \phi$$

The pressure can be extracted from the equation using $g^{ij}T_{ij} = 3P$. The equations for the energy density and pressure of the field are given as

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a(t)^2} + V(\phi)$$

$$P = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\frac{(\nabla\phi)^2}{a(t)^2} - V(\phi)$$

Where the metric in equation 1 was used to scale the space components with the factor $\frac{1}{a}$. If we impose homogeneity and isotropy on the field ϕ, ϕ does not depend on spatial position or on angle, $\phi = \phi(t)$. This simplifies the equations further to find

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

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