Basic Stereo Reconstruction Example

Here is the complete example of the cube. Specifically the example shows two images of the cube with two different camera positions. Then the 3D location of the points are reconstructed using the point correspondences and the stereo reconstruction algorithm. (Note that I changed the example slightly from the one used in class)

1 Image 1 (left camera)

The calibration parameters are given as follows:

$$R = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad t = \begin{bmatrix} -3 \\ -0.5 \\ 3 \end{bmatrix}$$

.Therefore

$$^{c}T_{w} = \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Assume the intrinsic matrix is given by:

$$K = \begin{bmatrix} -100 & 0 & 200 \\ -0 & -100 & 200 \\ 0 & 0 & 1 \end{bmatrix}$$

To convert each vertex of the unit cube from world coordinates to pixel coordinates we need to multiply each vertex by $M = K(^{c}T_{w})$. The multiplication is done by augmenting the vertex to a 4×1 (i.e. a fourth component set to one is added to each 3D vertex). Also recall that the answer is a 2D point in homogeneous coordinates. To obtain the final answer we divide by the resultant 3-component vector by its third component. (Recall this division implements the perspective projection).

$$\begin{aligned} M &= \begin{bmatrix} -100 & 0 & 200 \\ -0 & -100 & 200 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -70.7 & -70.7 & 200 & 900 \\ 70.7 & -70.7 & 200 & 650.0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{aligned}$$

The vertices of the cube are transformed by multiplying by M and then dividing by the fourth component:

$$M\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}900\\650.0\\3\end{bmatrix}=\begin{bmatrix}900/3\\650.0/3\\1\end{bmatrix}=\begin{bmatrix}300\\216.67\\1\end{bmatrix}$$

$$M\begin{bmatrix}0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}829.3\\579.3\\3\end{bmatrix}=\begin{bmatrix}829.3/3\\579.3/3\\1\end{bmatrix}=\begin{bmatrix}276.43\\193.1\\1\end{bmatrix}$$

$$M\begin{bmatrix}1\\1\\0\\1\end{bmatrix}=\begin{bmatrix}758.6\\650.0\\3\end{bmatrix}=\begin{bmatrix}758.6/3\\650.0/3\\1\end{bmatrix}=\begin{bmatrix}252.87\\216.67\\1\end{bmatrix}$$

$$M\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=\begin{bmatrix}829.3\\720.7\\3\end{bmatrix}=\begin{bmatrix}829.3/3\\720.7/3\\1\end{bmatrix}=\begin{bmatrix}276.43\\240.23\\1\end{bmatrix}$$

$$M\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}1100\\850.0\\4\end{bmatrix}=\begin{bmatrix}1100/4\\850.0/4\\1\end{bmatrix}=\begin{bmatrix}275\\212.5\\1\end{bmatrix}$$

$$M\begin{bmatrix}0\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1029.3\\779.3\\4\end{bmatrix}=\begin{bmatrix}1029.3/4\\779.3/4\\1\end{bmatrix}=\begin{bmatrix}257.33\\194.83\\1\end{bmatrix}$$

$$M\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}958.6\\850.0\\4\end{bmatrix}=\begin{bmatrix}958.6/4\\850.0/4\\1\end{bmatrix}=\begin{bmatrix}239.65\\212.5\\1\end{bmatrix}$$

$$M\begin{bmatrix}1\\0\\1\\1\end{bmatrix}=\begin{bmatrix}1029.3\\920.7\\4\end{bmatrix}=\begin{bmatrix}1029.3/4\\920.7/4\\1\end{bmatrix}=\begin{bmatrix}257.33\\230.18\\1\end{bmatrix}$$

In matlab we can draw the cube by using the ”line” command to draw lines between the 2D vertices. The Figure 1 shows the image of the cube.

Here is the matlab code that generated this figure:

```
%setup points
p1 = [300 216.7]; p2 = [276.4 193.1]; p3 = [252.87 216.7]; p4 =
[276.43 240.23];
p5 = [275 212.5]; p6 = [257.33 194.83]; p7 = [239.65 212.5]; p8
= [257.33 230.18];
x = [p1(1) p2(1) p3(1) p4(1) p5(1) p6(1) p7(1) p8(1)];
y = [p1(2) p2(2) p3(2) p4(2) p5(2) p6(2) p7(2) p8(2)];
clf; figure(1)
% at the origin of the world coordinate system place an asterix
```

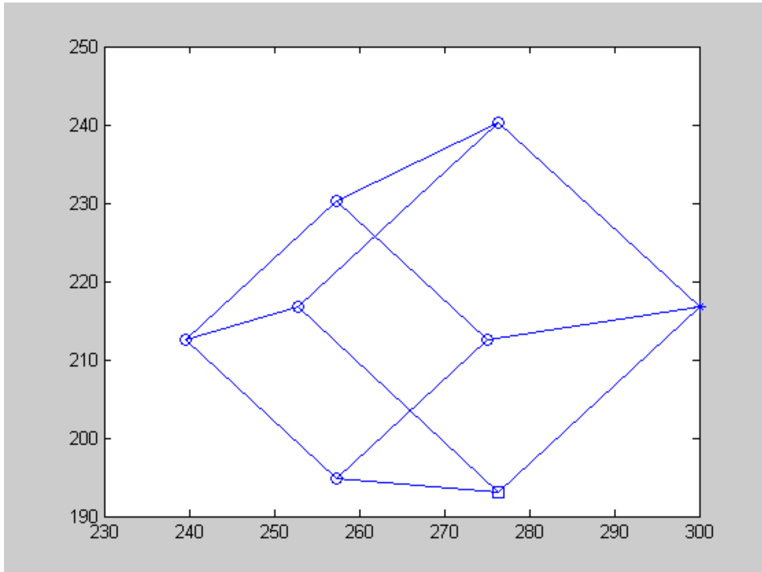

Figure 1: Image of the cube. Note that the point (0,0,0) is marked with an asterix (*) and the point (0,1,0) is marked with a square.

```
plot(x(1),y(1),'*'); hold on;
% at (0,1,0) in the world coordinate system place a square
plot(x(2),y(2),'s')
% mark all other vertices with a circle
plot(x(3:8),y(3:8),'o')
% connect vertices
line([p1(1) p2(1)],[p1(2) p2(2)])
line([p2(1) p3(1)],[p2(2) p3(2)])
line([p3(1) p4(1)],[p3(2) p4(2)])
line([p4(1) p1(1)],[p4(2) p1(2)])
line([p5(1) p6(1)],[p5(2) p6(2)])
line([p6(1) p7(1)],[p6(2) p7(2)])
line([p7(1) p8(1)],[p7(2) p8(2)])
line([p8(1) p5(1)],[p8(2) p5(2)])
line([p1(1) p5(1)],[p1(2) p5(2)])
line([p2(1) p6(1)],[p2(2) p6(2)])
line([p3(1) p7(1)],[p3(2) p7(2)])
line([p4(1) p8(1)],[p4(2) p8(2)])
```

Now let's move the camera and see a new image of the cube

2 Image 2 (right camera)

For the new camera position, we will orient the camera differently. This time the world coordinate system will be rotated by a positive 30 degrees about the z_c with respect to the camera coordinate frame. (In image 1 it was rotated by -45 degrees).

$$R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ t = \begin{bmatrix} -3 \\ -0.5 \\ 3 \end{bmatrix}$$

Therefore

$$^{c}T_w = \begin{bmatrix} 0.866 & -0.5 & 0 & -3 \\ 0.5 & 0.866 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Also

$$K = \begin{bmatrix} -100 & 0 & 200 \\ -0 & -100 & 200 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$M = \begin{bmatrix} -100 & 0 & 200 \\ -0 & -100 & 200 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 & -3 \\ 0.5 & 0.866 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$M = \begin{bmatrix} -86.6 & 50.0 & 200 & 900 \\ -50.0 & -86.6 & 200 & 650.0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Therefore, as in image 1, to convert each vertex of the unit cube from world coordinates to pixel coordinates we need to multiply each vertex by M. The vertices of the cube are transformed by multiplying by M and then dividing by the third component:

$$M \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 900 \\ 650.0 \\ 3 \end{bmatrix} = \begin{bmatrix} 900/3 \\ 650.0/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 300 \\ 216.67 \\ 1 \end{bmatrix}$$

$$M \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 950.0 \\ 563.4 \\ 3 \end{bmatrix} = \begin{bmatrix} 950.0/3 \\ 563.4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 316.67 \\ 187.8 \\ 1 \end{bmatrix}$$

$$M \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 863.4 \\ 513.4 \\ 3 \end{bmatrix} = \begin{bmatrix} 863.4/3 \\ 513.4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 287.8 \\ 171.13 \\ 1 \end{bmatrix}$$

$$M \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 813.4 \\ 600.0 \\ 3 \end{bmatrix} = \begin{bmatrix} 813.4/3 \\ 600.0/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 271.13 \\ 200.0 \\ 1 \end{bmatrix}$$

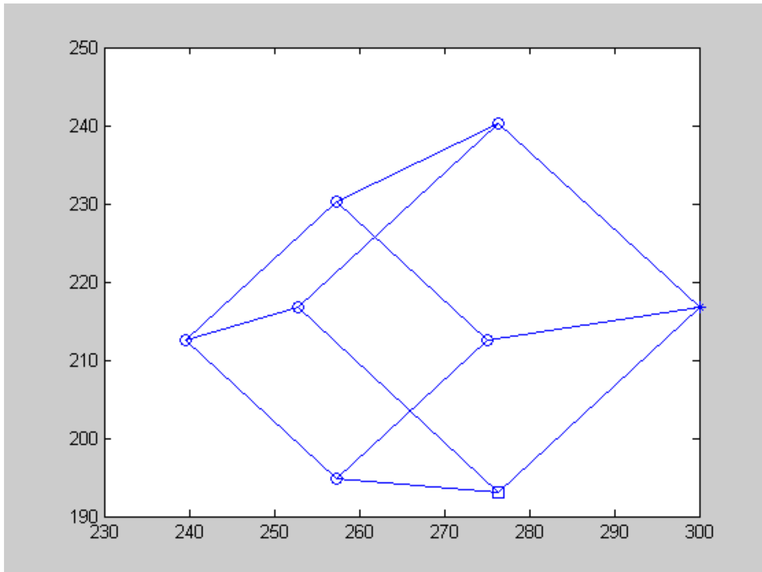

Figure 2: Image of the cube. Note that the point (0,0,0) is marked with an asterix (*) and the point (0,1,0) is marked with a square.

$$M\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}1100\\850.0\\4\end{bmatrix}=\begin{bmatrix}1100/4\\850.0/4\\1\end{bmatrix}=\begin{bmatrix}275\\212.5\\1\end{bmatrix}$$

$$M\begin{bmatrix}0\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1150.0\\763.4\\4\end{bmatrix}=\begin{bmatrix}1150.0/4\\763.4/4\\1\end{bmatrix}=\begin{bmatrix}287.5\\190.85\\1\end{bmatrix}$$

$$M\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1063.4\\713.4\\4\end{bmatrix}=\begin{bmatrix}1063.4/4\\713.4/4\\1\end{bmatrix}=\begin{bmatrix}265.85\\178.35\\1\end{bmatrix}$$

$$M\begin{bmatrix}1\\0\\1\\1\end{bmatrix}=\begin{bmatrix}1013.4\\800.0\\4\end{bmatrix}=\begin{bmatrix}1013.4/4\\800.0/4\\1\end{bmatrix}=\begin{bmatrix}253.35\\200.0\\1\end{bmatrix}$$

3 Stereo Reconstruction

Consider a corresponding point pair. For example, consider the world point ${}^{w}P = (0,0,0)$

In pixel coordinates this images to $(300, 216.7)$ in the left and the right camera. Suppose we only have the corresponding pixel coordinates and the camera parameters. We now show how to recover ${}^{w}P$. First convert from pixel coordinates to 2D camera coordinates. That is

$$((300 - o_x)(-s_x), (216.67 - o_y)(-sy)) = -100, -16.67$$

Therefore

$$p_l = (-100, -16.67)$$

Similarly,

$$p_r = (-100, -16.67)$$

(Note they are the same because the translation vectors are the same in each case).

Define $\tilde{p}_l$ as the 3D vector from the origin of the left camera to p_l. Define $\tilde{p}_r$ as the 3D vector from the origin of the right camera to p_r.Define S as the segment that joins the rays through $\tilde{p}_l$ and $\tilde{p}_r$. Note that if there were no noise in this system the rays would intersect exactly and S would be a point. But in practice it is a segment.

(In general $\tilde{p}_r$ and $\tilde{p}_l$ will not be the same. It is in this situation because the 3D world point is the origin of the world coordinate system and the translation vector for the two views is the same).

For stereo reconstruction define $a\ \tilde{p}_l$ as a point on the ray from the origin of the left camera passing through p_l the 2D point on the left image plane. Specifically this point is one endpoint on ray S. Define$\left(b\left({}^{l}R_r\right)\tilde{p}_r +^{l} t_r\right)$ as the point on the ray that passes through p_r and the origin of the right camera. Specifically this point is the other endpoint on ray S. **Important**: this ray must be expressed in the left camera coordinate system, that's why we need to use $\left({}^{l}R_r\right)$ and $\left({}^{l}t_r\right)$. After all, we want to express both endpoints of the segment in the same coordinate system ... we choose the left coordinate frame. Now the question remains, how do we get $\left({}^{l}R_r\right)$and $\left({}^{l}T_r\right)$?. The extrinsic camera parameters for the left and right image give us the following: For the left camera

$${}^{c}T_w = \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Denote l as the left camera coordinate frame, this matrix can be written as

$${}^{l}T_w = \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{l}R_w & {}^{l}t_w \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

.

For the right camera

$$ {}^{c}T_w = \begin{bmatrix} 0.866 & -0.5 & 0 & -3 \\ 0.5 & 0.866 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \end{bmatrix} $$

Denoting the right camera frame as r, this matrix can be written as

$$ {}^{r}T_w = \begin{bmatrix} 0.866 & -0.5 & 0 & -3 \\ 0.5 & 0.866 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} . = \begin{bmatrix} {}^{r}R_w & {}^{r}t_w \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} $$

We can get $({}^{w}T_r)$ by inverting. That is,

$$ {}^{w}T_r = \begin{bmatrix} ({}^{r}R_w)^T & -({}^{r}R_w)^T({}^{r}t_w) \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 & 0 & 2.85 \\ -0.50 & 0.866 & 0 & -1.07 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

We can concatenate tranformation matrix as follows:

$$ \begin{aligned} {}^{l}T_r &= \left({}^{l}T_w\right)\left({}^{w}T_r\right) \\ {}^{l}T_r &= \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & 0.5 & 0 & 2.85 \\ -0.50 & 0.866 & 0 & -1.07 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.259 & 0.966 & 0 & -1.74 \\ -0.966 & 0.259 & 0 & -3.27 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} $$

:

Therefore

$$ {}^{l}R_r = \begin{bmatrix} 0.259 & 0.966 & 0 \\ -0.966 & 0.259 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; {}^{l}t_r = \begin{bmatrix} -1.74 \\ -3.27 \\ 0 \end{bmatrix} $$

Now back to stereo reconstruction,
Note that

$$ \tilde{p}_l = \begin{bmatrix} -100 \\ -16.7 \\ f \end{bmatrix} = \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix} $$

$$\tilde{p}_r = \begin{bmatrix} -100 \\ -16.7 \\ f \end{bmatrix} = \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix}$$

Segment S is perpendicular to the ray through $\tilde{p}_r$ and the ray through $\tilde{p}_l$. Therefore it has the same direction as the following cross product q:

$$\begin{aligned} q &= \tilde{p}_l \times \left({}^l R_r \tilde{p}_r\right) \\ &= \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix} \times \left(\begin{bmatrix} 0.259 & 0.966 & 0 \\ -0.966 & 0.259 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix} \right) \\ &= \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix} \times \begin{bmatrix} -42.032 \\ 92.275 \\ 100 \end{bmatrix} = \begin{bmatrix} -10898 \\ 5796.8 \\ -9929.4 \end{bmatrix} \\ q/\left|q\right| &= \begin{bmatrix} -0.68792 \\ 0.36591 \\ -0.62678 \end{bmatrix} \end{aligned}$$

Note: the cross product has been normalized to a unit length vector for convenience.

From one endpoint of segment S, (given by $a\ \tilde{p}_l$), we can reach the other endpoint (given by ${}^l R_r\ b\ \tilde{p}_r + {}^l t_r$), by adding the appropriate vector $c\ (q/\left|q\right|)$ where c is a scalar.We can express this relationship as follows:

$$a\tilde{p}_l + c\left(\tilde{p}_l \times\ {}^l R_r \tilde{p}_r\right) = {}^l R_r b \tilde{p}_r + {}^l t_r$$

There are three unknowns (a, b, c) and three equations (one each for the x,y,z component of the 3D vectors).

For this example we have the following:

$$a \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix} + c \begin{bmatrix} -0.68792 \\ 0.36591 \\ -0.62678 \end{bmatrix} = b \begin{bmatrix} 0.259 & 0.966 & 0 \\ -0.966 & 0.259 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix} + \begin{bmatrix} -1.74 \\ -3.27 \\ 0 \end{bmatrix}.$$

which can becomes

$$\begin{bmatrix} -100a - 0.68792c \\ -16.7a + 0.36591c \\ 100a - 0.62678c \end{bmatrix} = \begin{bmatrix} -42.032b - 1.74 \\ 92.275b - 3.27 \\ 100b \end{bmatrix}.$$

In matrix form we have

$$\begin{bmatrix} -100 & 42.032 & -0.68792 \\ -16.7 & -92.275 & 0.36591 \\ 100 & -100 & -0.62678 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1.74 \\ -3.27 \\ 0 \end{bmatrix}.$$

This is solved exactly using matrix inversion (or any other method for solving systems of linear equations)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.030\,01 \\ 3.\,000\,8 \times 10^{-2} \\ 3.\,897 \times 10^{-4} \end{bmatrix}$$

The next step is to find ${}^{l}P$, the 3D coordinates of the point with respect to the left camera. We can see from the problem that the midpoint of the line segment is reached by starting from one endpoint ($a\ \tilde{p}_l$) and adding $c/2$ times $q/\left|q\right|$. Therefore we can get ${}^{l}P$ the 3D camera coordinate of the point

$$\begin{aligned} {}^{l}P &= 0.030\,01 \begin{bmatrix} -100 \\ -16.7 \\ 100 \end{bmatrix} + \frac{\left(3.\,897 \times 10^{-4}\right)}{2} \begin{bmatrix} -0.687\,92 \\ 0.365\,91 \\ -0.626\,78 \end{bmatrix} \\ &= \begin{bmatrix} -3.\,001\,1 \\ -0.501\,10 \\ 3.\,000\,9 \end{bmatrix} \end{aligned}$$

The final step is to find ${}^{w}P$. To get ${}^{w}P$ we simple do the matrix conversion,

$${}^{w}P = {}^{w}M_l \quad {}^{l}P$$

From the calibration parameters we have ${}^{l}M_w$. Simply invert to get ${}^{w}M_l$.That is,

$$\begin{aligned} {}^{w}P &= \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -3.\,001\,1 \\ -0.501\,10 \\ 3.\,000\,9 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2.\,5 \times 10^{-5} \\ -1.\,590\,9 \times 10^{-3} \\ 0.000\,9 \\ 1.0 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix} \end{aligned}$$

So indeed with the calibration parameters and the point correspondences we were able to determine that the point pair had the 3D world coordinate equal to $(0, 0, 0)$.

4 More Stereo Reconstruction

Let's try another set of point correspondences: We see that $(252.87, 216.67)$ from image 1 corresponds to $(287.8, 171.13)$ from image 2. Converting from pixel to 2D camera coordinates we see that

$$
\begin{aligned}
p_l &= (-(252.87-200), -(216.67-200)) \\
&= (-52.87, -16.67)
\end{aligned}
$$

$$
\begin{aligned}
p_r &= (-(287.8-200), -(171.13-200)) \\
&= (-87.8, 28.87)
\end{aligned}
$$

Now the 3D rays through the image plane can be expressed as

$$
\tilde{p}_l = \begin{bmatrix} -52.87 \\ -16.67 \\ f \end{bmatrix} = \begin{bmatrix} -52.87 \\ -16.67 \\ 100 \end{bmatrix}
$$

$$
\tilde{p}_r = \begin{bmatrix} -87.8 \\ 28.87 \\ f \end{bmatrix} = \begin{bmatrix} -87.8 \\ 28.87 \\ 100 \end{bmatrix}
$$

The direction of segment S is $q/|q|$

$$
\begin{aligned}
q &= \tilde{p}_l \times \left({}^l R_r \tilde{p}_r\right) \\
&= \begin{bmatrix} -52.87 \\ -16.67 \\ 100 \end{bmatrix} \times \left(\begin{bmatrix} 0.259 & 0.966 & 0 \\ -0.966 & 0.259 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -87.8 \\ 28.87 \\ 100 \end{bmatrix} \right) \\
&= \begin{bmatrix} -52.87 \\ -16.67 \\ 100 \end{bmatrix} \times \begin{bmatrix} 5.1482 \\ 92.292 \\ 100.0 \end{bmatrix} = \begin{bmatrix} -10896. \\ 5801.8 \\ -4793.7 \end{bmatrix} \\
q/|q| &= \begin{bmatrix} -0.82284 \\ 0.43814 \\ -0.36201 \end{bmatrix}
\end{aligned}
$$

For this example we have the following:

$$
a \begin{bmatrix} -52.87 \\ -16.67 \\ 100 \end{bmatrix} + c \begin{bmatrix} -0.82284 \\ 0.43814 \\ -0.36201 \end{bmatrix} = b \begin{bmatrix} 5.1482 \\ 92.292 \\ 100.0 \end{bmatrix} + \begin{bmatrix} -1.74 \\ -3.27 \\ 0 \end{bmatrix}.
$$

In matrix form:

$$
\begin{bmatrix} -52.87 & -5.1482 & -0.82284 \\ -16.67 & -92.292 & 0.43814 \\ 100 & -100.0 & -0.36201 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1.74 \\ -3.27 \\ 0 \end{bmatrix}
$$

This can be solved exactly using matrix inversion

$$
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3.0 \times 10^{-2} \\ 3.0 \times 10^{-2} \\ -9.5 \times 10^{-4} \end{bmatrix}
$$

Now solve for ${}^{l}P$.

$$
\begin{aligned}
{}^{l}P &= a\begin{bmatrix} -52.87 \\ -16.67 \\ 100 \end{bmatrix} + \frac{c}{2}\begin{bmatrix} -0.82284 \\ 0.43814 \\ -0.36201 \end{bmatrix} \\
&= (0.03)\begin{bmatrix} -52.87 \\ -16.67 \\ 100 \end{bmatrix} - \frac{0.00095}{2}\begin{bmatrix} -0.82284 \\ 0.43814 \\ -0.36201 \end{bmatrix} \\
&= \begin{bmatrix} -1.5857 \\ -0.50031 \\ 3.0002 \end{bmatrix}
\end{aligned}
$$

Finally solve for ${}^{w}P$

$$
\begin{aligned}
{}^{w}P &= \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1.5857 \\ -0.50031 \\ 3.0002 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1.0004 \\ 0.99995 \\ 0.0002 \\ 1.0 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1.0 \end{bmatrix}
\end{aligned}
$$

So indeed with the calibration parameters and the point correspondences we were able to determine that the point pair had the 3D world coordinate equal to $(1, 1, 0)$.

5 Comment about noise

The stereo reconstruction examples so far had no significant source of measurement noise since it was a synthetic example. This is why the values of c are so small. Let's repeat the previous reconstruction but first add some noise to the pixel values. Specifically, let the measured value of $(252.87, 216.67)$ be $(251, 218)$ and the measured value of $(287.8, 171.13)$ be $(289, 169)$.Converting from pixel to 2D camera coordinates we see that $((300 - o_x)(-s_x), (216.67 - o_y)(-sy)) = -100, -16.67$

$$
\begin{aligned}
p_l &= (-(251 - 200), -(218 - 200)) \\
&= (-51, -18)
\end{aligned}
$$

$$
\begin{aligned}
p_r &= (-(289 - 200), -(169 - 200)) \\
&= (-89, 31)
\end{aligned}
$$

Now the 3D rays through the image plane can be expressed as

$$\tilde{p}_l = \begin{bmatrix} -51 \\ -18 \\ 100 \end{bmatrix}$$

$$\tilde{p}_r = \begin{bmatrix} -89 \\ 31 \\ 100 \end{bmatrix}$$

The direction of segment S is $q/|q|$

$$\begin{aligned} q &= \tilde{p}_l \times \left({}^l R_r \tilde{p}_r\right) \\ &= \begin{bmatrix} -51 \\ -18 \\ 100 \end{bmatrix} \times \left(\begin{bmatrix} 0.259 & 0.966 & 0 \\ -0.966 & 0.259 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -89 \\ 31 \\ 100 \end{bmatrix} \right) = \begin{bmatrix} -11200 \\ 5789.5 \\ -4670 \end{bmatrix} \\ q/|q| &= \begin{bmatrix} -0.833\,02 \\ 0.430\,61 \\ -0.347\,34 \end{bmatrix} \end{aligned}$$

For this example we have the following:

$$a \begin{bmatrix} -51 \\ -18 \\ 100 \end{bmatrix} + c \begin{bmatrix} -0.833\,02 \\ 0.430\,61 \\ -0.347\,34 \end{bmatrix} = b \begin{bmatrix} 6.895 \\ 94.003 \\ 100.0 \end{bmatrix} + \begin{bmatrix} -1.74 \\ -3.27 \\ 0 \end{bmatrix}.$$

In matrix form:

$$\begin{bmatrix} -51 & -6.895 & -0.833\,02 \\ -18 & -94.003 & 0.430\,61 \\ 100 & -100.0 & -0.347\,34 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1.74 \\ -3.27 \\ 0 \end{bmatrix}$$

This can be solved exactly using matrix inversion

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2.947\,6 \times 10^{-2} \\ 2.933\,2 \times 10^{-2} \\ 0.041\,41 \end{bmatrix}$$

Now solve for ${}^l P$.

$$\begin{aligned} {}^l P &= a \begin{bmatrix} -51 \\ -18 \\ 100 \end{bmatrix} + \frac{c}{2} \begin{bmatrix} -0.833\,02 \\ 0.430\,61 \\ -0.347\,34 \end{bmatrix} \\ &= (0.029) \begin{bmatrix} -52.87 \\ -16.67 \\ 100 \end{bmatrix} + \frac{0.041\,41}{2} \begin{bmatrix} -0.833\,02 \\ 0.430\,61 \\ -0.347\,34 \end{bmatrix} \\ &= \begin{bmatrix} -1.550\,5 \\ -0.474\,51 \\ 2.892\,8 \end{bmatrix} \end{aligned}$$

:

Finally solve for ${}^{w}P$

$$
\begin{aligned}
{}^{w}P &= \begin{bmatrix} 0.707 & 0.707 & 0 & -3 \\ -0.707 & 0.707 & 0 & -0.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1.5505 \\ -0.47451 \\ 2.8928 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1.007 \\ 1.0431 \\ -0.1072 \\ 1.0 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1.0 \end{bmatrix}
\end{aligned}
$$

So the solution is not as accurate due to the measurement noise, but we were able to determine that the point pair had the 3D world coordinate close to $(1, 1, 0)$.