# AN INTRODUCTION TO BASICS OF GAME THEORY

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# OUTLINE

- The Basics
- An illustrative example
- Exposition of the “Normal Form”
- Pareto-Optimality and Nash-equilibrium

# Game Theory: what it is all about

- Game Theory is the study of interactions between *self-interested agents*
- What does it mean to say that agents are self-interested?
- It does not mean that they care only about themselves
- It also does not mean that they want to cause harm to each other, or to the environment
- *Instead, it means that each agent has his own description of the states of the world he likes*
- … and this could also include altruistic motivations of doing good to others
- *and he acts in a manner to bring about these states of the world.*

# Game Theory: what it is all about

- An Agent’s preference for different states of the world is quantified by expressing them as numbers
- An ‘utility function’ maps the different identified states into these numbers
- In a dynamic environment when there is uncertainty about prevalent states, this utility becomes a function of a probability distribution of the prevalence of different states
- A *rational* agent always acts to maximize his utility, or “payoff” in a given environment.

# Game Theory: what it is all about

- With the outcome of an action, or the probability of the outcome of an action, known to an agent and represented numerically using an utility function, prediction of the optimal responses of a rational agent in a given situation is conceptually straightforward
- … but only apparently
- because in the world other agents are also acting that concurrently change the states of the world (effectively transforming this into a nonlinear problem -:)
- and this really gives rise to a range of possibilities and what we call as *Game Theory*.

# A simple illustrative example

- Imagine that You and your colleague are the only people using the internet from a location, and traffic is governed by the TCP protocol
- One of the features of this protocol is the backoff mechanism, which reduces the transfer rates of users for a while when there is congestion, till the latter subsides
- suppose you & your colleague can manipulate this mechanism so that you can choose to go for this if you wish, or else continue as normal
- let us call the former strategy as C (i.e. following the protocol) and the latter (faulty representation) as D

# A simple illustrative example

Payoff Matrix (Utility function)

| **You ⇓ Collg⇒** | **C** | **D** |
|---|---|---|
| C | -1, -1 | -4, 0 |
| D | 0, -4 | -3, -3 |

- "Payoffs" to you & your colleague (first and second numbers in each cell) are listed in the table above as delays denoted by –(ve) signs
- Notice that either 'player' is better of 'playing' D, irrespective of what the other player is playing. Hence the "equilibrium" settles at (D, D).
- Notice also that the net delay faced by the *set of players* is highest when both play D, and the least when both play C (Co-operative & non-cooperative games)

# A simple illustrative example

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- Would your responses change if you knew more about the nature of your colleague and his most likely behavior?
- Would your responses be different if you & coll could communicate and possibly make mutual agreements before your actions started ?
- Would your responses change if you (both) knew that this game will be played out only once, a few times, or a very large no. of times?
- Under what changes to the users' payoffs' will their decisions remain unchanged?

- All addressed in different chapters of Game Theory.

# The Normal Form (baseline) representation of a Game

- The normal form is the most familiar representation of strategic interactions in game theory
- It amounts to a representation of every player's utility for every state of the world, in the special case where the states depend only on the player's combined actions
- There are settings where the states of the world also depend on the randomness of the environment (Bayesian Games) and /or are time-dependant (Extensive form Games).
- These can also be reduced to (larger expressions of) normal form games. Thus the normal-form representation is the most fundamental in GT.

# The Normal Form (baseline) representation of a Game

A Normal Form Game is a tuple (N, A, u):

- *N is a finite set of n players, indexed by i;*
- $A = A_1 \times A_2 \times \ldots. \times A_n$ *, where* $A_i$ *is a finite set of actions available to player i.*

  *Each vector* $a = (a_1 \times a_2 \times \ldots. \times a_n) \in A$ *is called an action profile;*
- $u = (u_1 \times u_2 \times \ldots. \times u_n)$ *where* $u_i : a \rightarrow \Re$ *is a real valued utility (payoff) function for player i.*

- The matrix shown in example is a natural way to represent games. The matrix will be n-dimensioned, and each cell will have n-elements.

# Strategies and Solution Concepts

- We have just seen that $a_i \in \{A_i\}$ is a possible action of agent i
- Game Theory deals conceptually with more than actions $a_i$ *per se* but with strategies $s_i$, where $s_i$*'s* represent probabilistic combinations across $a_i$*'s* for agent *i*
- $s_i$ is called a pure-form strategy if it is fixed on a particular $a_i$ to the exclusion of all other possible actions
- In this Introduction we will avoid these complications, still talk of $s_i$'s rather than $a_i$'s for the sake of some standard definitions, **but think of each $s_i$ as just an $a_i$, i.e. a strategy $\Leftrightarrow$ an action**, for any agent
- Note that every strategy of every agent is geared towards maximizing its own utility, i.e. payoff, in the given environment
- The complexity arises from the fact that every agent is trying to do the same concurrently, introducing different kinds of dynamics into the process
- So some baseline Solution Concepts are defined which enable easier analysis of this dynamics.

# The first Solution Concept: Pareto Optimality

- Note that for the Normal Form definition we had an action profile a, $a = (a_1 \; x \; a_2 \; x \; \ldots . x \; a_n) \in A$ , across all agents
- In an analogous manner we can define a strategy profile $s \in S$
- For a moment let us look at a Game from the perspective of an unattached, unbiased external observer
- … and ask that, can one strategy profile be better for the agents playing the game than another profile, and by corollary, can there be an optimal profile? This brings us to certain definitions:
- <u>Pareto Domination</u>: Strategy profile *s Pareto dominates* strategy profile *s'* if for all $i \in N$, $u_i(s) \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$.
- <u>Pareto Optimality</u>: Strategy profile s is Pareto optimal if there does not exist another strategy profile $s' \in S$ which Pareto dominates it.

# The second Solution Concept: Nash Equilibrium

- Now let us reset our perspective back from that of an external observer to that of an individual agent *i*
- …. and ask, what could be the best strategy $s_i$ when the others are playing a strategy profile $s_{-i}$, where
- $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1, \ldots,} s_{n-1}, s_n)$, i.e. $s = (s_i, s_{-i})$
- Best Response: *Player i's best response to the strategy profile* $s_{-i}$ *is a strategy* $s_i^* \in S_i$ *such that* $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ *for all strategies* $s_i \in S_i$.
- and this immediately brings us to the concept of the Nash Equilibrium:
- Nash Equilibrium: A Strategy profile $s = (s_1, s_2, \ldots, s_n)$ is a *Nash Equilibrium* if, for all agents i, $s_i$ is a best response to $s_{-i}$.

# THANK YOU