Problem 02

moon lander with state $[h, v, m]^T$ to have the following dynamics

$$\dot{h}(t) = v(t)$$
$$\dot{v}(t) = -g + \frac{\alpha(t)}{m(t)} \quad \& \quad \dot{m}(t) = -K\alpha(t)$$

h is altitude, v is velocity & m is mass of moon lander, $\alpha(t) \in [0, 1]$

$$\min_{\alpha(t)} P(t) = \int_0^T \alpha(t)\,dt.$$

$$\min_{\alpha(t)} \int_0^T \alpha(t)\,dt = \frac{m_0 - m(T)}{K}$$

in term of general notations the state vector.

$$f = \begin{bmatrix} v \\ -g + \alpha/m \\ -K\alpha \end{bmatrix} \qquad \text{Thrust } l = \alpha$$

Hemiltonian:

$$H = -l + \lambda^T f$$
$$= -\alpha + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 + \lambda_3 \dot{x}_3$$
$$= -\alpha + \lambda_1 v + \lambda_2 \left(-g + \frac{\alpha}{m}\right) + \lambda_3(-K\alpha)$$

To find the optimal control policy we apply pontryagin's maximum principal

$$\alpha^* = \arg\max(H) \text{ w.r.t. } \alpha \in [0,1]$$

$$= \arg\max \left(-1 + \frac{\lambda_2}{m} - \lambda_3 k\right)\alpha + \lambda_1 v - \lambda_2 g.$$

$$\alpha^* = \begin{cases} 0 & b \leq 0 \\ 1 & b > 0 \end{cases}$$

where $b = -1 + \frac{\lambda_2}{m} - \lambda_3 K$.

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{bmatrix} = -\begin{bmatrix} 0 \\ \lambda_1 \\ \frac{\lambda_2 \alpha}{m^2} \end{bmatrix}$$

$$\dot{\lambda} = \begin{bmatrix} 0 \\ -\lambda_1 \\ \frac{\lambda_2 \alpha}{m^2} \end{bmatrix}$$

$$\dot{b} = \frac{\dot{\lambda}_2}{m} - \frac{\lambda_2 \dot{m}}{m^2} - \dot{\lambda}_3 K$$

now multiply $\dot{\lambda}$ in $\dot{b}$

$$\dot{b} = \frac{-\lambda_1}{m} \quad \frac{-\lambda_2}{m}(-K\alpha) - \left(\frac{\lambda_2 \alpha}{m}\right)K.$$

$$\dot{b} = \frac{-\lambda_1}{m}.$$

mass is always greater than 0 $\rightarrow m > 0$

The value of λ_1 is always constant.
$\dot{\lambda}_1 = 0$

If λ_1 is +ve then $\dot{b}$ is ~~greater~~ +ve
if λ_1 is -ve then $\dot{b}$ is -ve.

Thus b is monotonic because it's first derivat doesnot change sign.

SO first shutdown the engine & then turn on the engin at end to reach 0 velocity at the end.

with this we can say that optimal policy is

$$\alpha^* = \begin{cases} 0 & \text{if } b \leq 0 \quad @ \quad t \in [0, t^*] \\ 1 & \text{if } b > 0 \quad @ \quad t \in [t^*, T] \end{cases}$$