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Unit-1

Matrix Algebra

Matrix:- An $m \times n$ matrix is an object (need not be distinct) of the form:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m \times n}$$ → Rows of matrix

↓

columns of matrix.

Order:- If a matrix have m rows and n columns then the order of the matrix is $m \times n$.

$a_{11}, a_{12}, \cdots, a_{mn}$ ⇒ Elements of matrix

Real matrix:- When all the elements of the matrix is real then it is called the real matrix.

Complex matrix:- When one or more than one elements of the matrix A is complex.

Row vector:- Ex. $A = [a_{11} \;\; a_{12} \;\; \cdots \;\; a_{1n}]_{1 \times n}$

Column vector:- Ex $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m \times 1}$

Unit Vector:- $A = [a_{11}]_{1 \times 1}$ ~~X~~

Rectangular matrix:- $[A]_{m \times n}$ such that $m \neq n$.

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Square matrix: Let $[A]_{m \times n}$ such that $m = n$

Ex. $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}_{n \times n}$

- the elements $a_{11}, a_{22}, a_{33}, \ldots a_{nn}$ are called the diagonal elements.
- Other than these elements, the elements are called off-diagonal elements.

Trace: ⇒ sum of all diagonal elements.
⇒ $a_{11} + a_{22} + \ldots + a_{nn}$.

Diagonal matrix:- Ex. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}_{3 \times 3}$

Identity matrix:- Ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Equal matrix:-
- The order of both matrices must be same.
- Corresponding elements of both matrices must be same.

Ex. $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$

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Sub matrix :- A matrix obtained by omitting some rows or columns from a given matrix, then it is called sub matrix.

Ex Matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

Sub matrices : $[1]_{1\times1}$, $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}_{2\times2}$

MATRIX OPERATION :-

(1) **Addition / subtraction of matrix** :-

Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$ + $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$

$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$

(2) **Scalar multiplication** :-

For any scalar α,

$\alpha A = \alpha(a_{ij})_{n\times n} = (\alpha a_{ij})_{n\times n}$

Properties :-

1. $A+B = B+A$ (Commutative)
2. $A+(B+C) = (A+B)+C$ (Associative)
3. $\alpha(A+B) = \alpha A + \alpha B$
4. $(\alpha+\beta)A = \alpha A + \beta A$
5. $A + O_{n\times n} = A_{n\times n}$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $A + (-A) = O_{n\times n}$.

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③. Multiplication of Matrix:

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$ be matrices then,

$$C = A \times B = \sum_{k=1}^{n} (a_{ik})_{m\times n} (b_{kj})_{n\times p}$$

Row x column =
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Properties:-

Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$, $C = (c_{ij})_{p\times q}$

1. $(AB)C = A(BC)$
2. $A(B+C) = AB + AC$ {Left Distributive law}
3. $(A+B)C = AC + BC$ {Right Distributive law}
4. $\alpha(AB) = (\alpha A)B$

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* Types of Matrices :-

- **Transpose of Matrix :-** Row ↔ Column

$[A]_{m\times n}$ $\quad$ A' or $A^T = [A]_{n\times m}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m\times n}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n\times m}$$

Properties :- 1. If A is a row matrix then A^T is column matrix and vice versa.

2. $\det(A) = |A| = |A^T|$
3. $(A^T)^T = A$
4. $(AB)^T = B^T A^T$
5. $(A+B)^T = A^T + B^T$

- **Symmetric Matrix :-** $[A]_{n\times n}$ ⇒ A square matrix

A is symmetric if

$a_{ij} = a_{ji} \quad \forall\, i,j = 1,2,\ldots n$

$$\boxed{A = A^T}$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $\quad$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$A = A^T$, So A is symmetric.

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- **Skew Symmetric Matrix:-** $[A]_{n\times n}$

$a_{ij} = -a_{ji}$ $\boxed{A = -A^T}$

Ex.
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad -A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{A = -A^T}$$

So, A is skew symmetric matrix.

NOTE:- ⇒ In a skew symmetric matrix, the diagonal elements are zero.

⇒ $O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ⇒ null matrix
⇒ Both symmetric and skew symmetric

Properties:- For any real matrix A,
$A + A^T$ ⇒ Always symmetric
$A - A^T$ ⇒ Always skew symmetric

A = symmetric + skew-symmetric
$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

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- **Triangular Matrix** :- $[A]_{n\times n}$

⇒ Lower triangular matrix

Ex.

$$A=\begin{bmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ -1 & 1 & 2\end{bmatrix}$$

⇒ Upper triangular matrix

Ex.

$$A=\begin{bmatrix}1 & -1 & 6\\ 0 & 2 & 3\\ 0 & 0 & 1\end{bmatrix}$$

- **Conjugate matrix** :- $[A]_{n\times n}$

For conjugate matrix, $\overline{A}$

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \qquad \overline{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$$

$$A=\begin{bmatrix}1 & -i\\ i & 0\end{bmatrix} \qquad \overline{A}=\begin{bmatrix}1 & i\\ -i & 0\end{bmatrix}$$

- **Hermitian Matrix** :- $[A]_{n\times n}$

$$A=(\overline{A})^T=(\overline{A^T})=A^*$$

$$A=\begin{bmatrix}1 & i\\ i & 0\end{bmatrix} \qquad \overline{A}=\begin{bmatrix}1 & -i\\ -i & 0\end{bmatrix} \qquad (\overline{A})^T=\begin{bmatrix}1 & -i\\ -i & 0\end{bmatrix}$$

- Skew Hermitian Matrix:- $[A]_{n \times n}$

$$A = -(\bar{A})^T = -(\bar{A^T}) = -A^*$$

Properties ⇒ If A is Hermitian then the all diagonal elements are real.

⇒ If A is a real matrix then Hermitian matrix is symmetric & skew Hermitian matrix is skew symmetric

⇒ If A is skew Hermitian matrix then the diagonal elements of A either 0 or purely imaginary numbers.

⇒ If A is Hermitian then $A + A^*$ is also Hermitian.

⇒ If A is skew Hermitian then $A - A^*$ is also skew Hermitian.

⇒ If A is a complex matrix then A can always written in the form of Hermitian and skew Hermitian matrix

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$

- **Unitary Matrix :-** $[A]_{n\times n}$ be a complex matrix, then A is said to be unitary matrix if

$$\boxed{A \cdot A^{*} = A^{*}A = I}$$

- **Normal Matrix :-** $[A]_{n\times n}$ be a real matrix then A is said to be normal matrix if,

$$\boxed{AA^{T} = A^{T}A}$$

★ **Elementary Row/ Column Operation:-**

Let $[A]_{m\times n}$ then the following conditions are called elementary row operation:

1. Interchange of any two Rows
   ($R_i \sim R_j$)
   ith row   jth row

2. Multiplication/division of any non zero scalar (written αR_i).

3. Addition/ Subtraction of a scalar (non zero) multiple of any row. ~~(written as R_i~~ to another row
   → (i.e. $R_i \longleftrightarrow R_i \pm \alpha R_j$)

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- Echelon form of a matrix → Let $[A]_{m \times n}$ then the $[A]_{m \times n}$ is in echelon form if the following conditions are hold:

1. If the ith row of matrix is zero. (i.e. contains all elements will be zero) then any subsequent rows of matrix is also zero.

Ex
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \Rightarrow i^{th} \text{ row} \to 2^{nd} \\ \Rightarrow i^{th} \text{ row} \Rightarrow 3^{rd}. \end{array}$$

2. If a column contains a non zero entry of any row, then every subsequent entry in this column is zero.

Ex.
$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

3. Rows containing all zero occur only after all non zero rows.

Ex
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have to make upper triangular matrix.

Make first element 1. for finding rank

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Ques:- Reduce the following matrix in row echelon form.

(i). $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix}$

$\Rightarrow R_3 \rightarrow R_3 + 2R_2$

$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 3 & -3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 4 \end{bmatrix}$

Row Echelon form of matrix A

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* **Rank :-** The number of non zero rows in the row echelon of matrix A is called rank of A.

It is denoted by r or rank(A)

① $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

Echelon form of matrix A ⇒ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 4 \end{bmatrix}$

$r(A) = 3$

② $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix}$

Echelon form of matrix A ⇒ $\begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

$r(A) = 2$

Properties :-

1. Let $[A]_{n \times n}$ then the echleon form of A is an upper triangular matrix.

2. $[A]_{n \times n}$, $|A| \neq 0 \Rightarrow$ non singular
   $\Rightarrow r(A) = n$ = order of matrix.

3. $[A]_{n\times n}$, $|A| = 0 \to$ singular matrix

$r(A) < n$

4. $[A]_{m\times n}$, $m \neq n$ then $r(A) = \min(m,n)$

$(m, n \neq 0)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}_{\underset{m}{2}\times\underset{n}{3}}$$

$r(A) = 1$

Q $A = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}_{4\times 3}$ Find $r(A)$ using elementary row operation.

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 4 & -1 & 7 \\ 0 & -3 & -3 \end{bmatrix}$$

$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 0 & -9 & -9 \\ 0 & -3 & -3 \end{bmatrix}$$

$R_4 \to 3R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to 5R_3 - 9R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 2$

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* Solution of system of Linear Equations

Consider a system of equation with m knowns and n unknowns —

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$Ax = b, \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$\downarrow$ Coefficient matrix.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$\hookrightarrow$ solution vector

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$\hookrightarrow$ column matrix.

1. Unique solution, $x_1 \neq x_2 \neq \cdots x_n$
2. Infinite no. of solⁿ, $x_1 = \underbrace{\alpha}_{\text{scalar}} x_2, \; x_3 \neq \cdots x_n$
3. No solution

$$(A/b) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

(Augumented matrix).

Consistent :- If $r(A) = r(A/b)$ then the system ⊛ is called consistent.

1. Unique solⁿ :- $r(A) = r(A/b)$ = order of matrix
2. Infinite no. of solⁿ :- $r(A) = r(A/b) <$ order of matrix

When ⊛ is inconsistent i.e. $r(A) \neq r(A/b)$ then no solution.

Q. Solve the following system of linear equations.

$$2x_1 + x_2 - x_3 = 4$$
$$x_1 - x_2 + 2x_3 = -2$$
$$-x_1 + 2x_2 - x_3 = 2$$

$\Rightarrow$

$$Ax = b$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

$$(A/b) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

$$(A/b) = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right]$$

$R_2 \longleftrightarrow R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 2 & 1 & -1 & 4 \\ -1 & 2 & -1 & 2 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & -5 & 0 \\ -1 & 2 & -1 & 2 \end{array}\right]$$

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$R_3 \rightarrow R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & -5 & 8 \\ 0 & \cancel{0} & 1 & 0 \end{array}\right]$$

$$(A/b) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & -5 & 8 \\ 0 & 0 & 8 & -8 \end{array}\right]$$

$r(A) = 3$

$r(A/b) = 3$

As, $r(A) = r(A/b) = 3$

So, System is consistent.

$\therefore$ Unique sol^n exist.

$$Ax = b$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}$$

$x_3 = -1$

$3x_2 - 5x_3 = 8$

$x_2 = 1$

$x_1 - x_2 + 2x_3 = -2$

$x_1 - 1 - 2 = -2$

$x_1 = 1$

$$\boxed{x_1 = 1,\ x_2 = 1,\ x_3 = -1}$$

Ans

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Q. Solve the system of equations:

$$x_1 + 2x_2 + x_3 = -1$$
$$2x_1 + x_3 = -2$$
$$x_2 + 3x_3 = 1$$

$$\boxed{x_1 = \frac{-13}{11},\ x_2 = \frac{-1}{11},\ x_3 = \frac{4}{11}}$$

!!!

$$\boxed{\begin{array}{l|l} x_1 & = \frac{-26}{7} \\ x_2 & = \frac{10}{7} \\ x_3 & = \frac{-1}{7} \end{array}}$$

$$(A|b) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & 0 & 1 & -2 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & -4 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$R_3 \to 4R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & 11 & 4 \end{array}\right]$$

$r(A) = r(A|b) = \text{Order of matrix} = 3$

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

$11x_3 = 4$

$$\boxed{x_3 = \frac{4}{11}}$$

$-4x_2 - x_3 = 0$

$-4x_2 = \frac{4}{11}$

$$\boxed{x_2 = \frac{-1}{11}}$$

$x_1 - \frac{2}{11} + \frac{4}{11} = -1$

$x_1 = -1 - \frac{2}{11}$

$$\boxed{x_1 = \frac{-13}{11}}$$

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★ Eigen Values and Eigen Vectors

Let $[A]_{n \times n}$ be a square matrix of order n

$Ax = b$ → Non homogenous

$Ax = 0$ → Homogenous.

$$[A - \lambda I]x = 0$$

Let $[A - \lambda I]$ is singular

$$|A - \lambda I| = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (-1)^n \left[\lambda^n - C_1 \lambda^{n-1} + C_2 \lambda^{n-2} + \cdots \cdots + (-1)^n C_n \right]$$

$$= 0$$

$$P_n(\lambda) = \lambda^n - C_1\lambda^{n-1} + C_2\lambda^{n-2} + \dots + (-1)^n C_n = 0$$

$$\downarrow$$

$\lambda_1, \lambda_2, \dots, \lambda_n \rightarrow$ Eigen values

$$\boxed{\begin{array}{c}\lambda^n - C_1\lambda^{n-1} + C_2\lambda^{n-2} + \dots + (-1)^n C_n = 0 \\ \text{(Characteristic equation)}\end{array}}$$

$C_1, C_2, \dots C_n \Rightarrow$ are the coefficients of given polynomial.

Let $[A]_{n\times n}$ be a square matrix of order n

Suppose λ_1

$$[A - \lambda_1 I]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = (\alpha_1 \alpha_2 \dots \alpha_n)$$

Q. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ Find characteristics equation.

$\Rightarrow [A - \lambda I][x] = \begin{bmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= (1-\lambda)\left[(1-\lambda)^2 - 4\right] - 2\left[-(1-\lambda) - 2\right] = 0$

$= \boxed{-\lambda^3 + 3\lambda^2 - \lambda + 3 = 0}$

$-\lambda^2(\lambda - 3) - 1(\lambda - 3) = 0$

$(-\lambda^2 - 1)(\lambda - 3) = 0$

$\lambda = 3$

Q. Solve the following system of linear equations:

(1) $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$

$\Rightarrow (A|b) = \left[\begin{array}{ccc|c} 2 & -3 & 1 & -2 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & -3 & -2 \end{array}\right]$

$R_2 \leftrightarrow R_1$

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$$(A|b) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -3 & 1 & -2 \\ 2 & 1 & -3 & -2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$(A|b) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 3 & -7 & -8 \end{array}\right]$$

$R_3 \to R_3 + 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & -16 & -32 \end{array}\right]$$

A b

$r(A) = 3$
$r(A|b) = 3$

So, Unique solns.

$Ax = b$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -32 \end{bmatrix}$$

$$\begin{bmatrix} x - y + 2z \\ -y - 3z \\ -16z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -32 \end{bmatrix}$$

$z = 2$

$-y - 3z = -8$
$-y - 6 = -8$
$y = 2$

$x - y + 2z = 3$
$x - 2 + 4 = 3$
$x = 1$

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(2)

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow (A/b) = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 2 & 0 & -1 \\ 1 & 2 & 3 & 2 \end{array}\right]$$

$R_3 \to 3R_3 - R_1$. $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 4 & 8 & 5 \\ 0 & 2 & 0 & -1 \end{array}\right]$$

$R_3 \to 2R_3 - R_2$.

$$(A/b) = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 4 & 8 & 5 \\ 0 & 0 & -8 & -7 \end{array}\right]$$

$r(A) = r(A/b) = 3$

System is consistent.

$Ax = b$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$$

$-8z = -7$

$\boxed{z = \frac{7}{8}}$

$4y + 8z = 5$

$4y + 7 = 5$

$\boxed{y = -\frac{1}{2}}$

$3x + 2\left(-\frac{1}{2}\right) + \frac{7}{8} = 1$

$3x = 2 - \frac{7}{8} = \frac{9}{8}$

$\boxed{x = \frac{3}{8}}$

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Q. $A = \begin{bmatrix} 1 & 1 & i \\ 1 & 0 & i \\ -i & -i & 1 \end{bmatrix}$

Find Characteristic eq^n.

$\Rightarrow |A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & i \\ 1 & -\lambda & i \\ -i & -i & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(\lambda(\lambda-1)+(i)^2) - 1(1-\lambda+(i)^2) + i(-i-\lambda i) = 0$

$(1-\lambda)(\lambda^2-\lambda-1) + (-1+\lambda-1) + (1+\lambda) = 0$

$\lambda^2 - \cancel{\lambda} - \cancel{1} - \lambda^3 + \lambda^2 + \cancel{\lambda} + \lambda - \cancel{2} + \lambda + \cancel{1} = 0$

$-\lambda^3 + 2\lambda^2 + 2\lambda \cancel{-2} = 0$

$\lambda^3 - 2\lambda^2 - 2\lambda \cancel{+2} = 0$

$\lambda = 0 \quad \lambda^2 - 2\lambda - 2 = 0$

$\lambda = 0, 1+\sqrt{3}, 1-\sqrt{3}$

Put $\lambda = 0$ in $[A-\lambda I]x = 0$,

$\begin{bmatrix} 1 & 1 & i \\ 1 & 0 & i \\ -i & -i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x = y = z = 0$ Eigenvectors

Similarly Put $\lambda = 1 \pm \sqrt{3}$ one by one & find their corresponding Eigen vectors.

* Properties of Eign Values:-

Let $[A]_{n \times n}$ be an upper (or lower) diagonal matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Eign values of matrix A is diagonal elements.

$$\lambda = a_{11}, a_{22}, a_{33}$$

Sum of eign values $= \lambda_1 + \lambda_2 + \dots + \lambda_n$
$= \text{Trace}(A)$

Product of eign values $= |A|$
$= \lambda_1 \lambda_2 \dots \lambda_n$

① $\rightarrow$ $|A| = 0 \iff A$ is singular.

then at least one of eign value is zero and converse is also true.

② If λ is an eign value of matrix A then $\frac{1}{\lambda}$ is an eign value of A^{-1}.

③ If λ is an eign value of $\lambda^2, \lambda^3, \dots \lambda^m$ is an eigen value of $A^2, A^3, \dots A^m$.

Q. Find eign values and eign vector of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

$\Rightarrow$ $[A - \lambda I]\, x = 0$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (1-\lambda)x + 2y + 2z = 0$

$x - \lambda x + 2y + 2z = 0$

$\Rightarrow (2-\lambda)y + z = 0$

$2y - \lambda y + z = 0$

$\Rightarrow -x + 2y + z(2-\lambda) = 0$

$-x + 2y + 2z - z\lambda = 0$

So, $\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$

$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$ ⟹ Characteristics equation.

$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 2$ ⟹ Eign values

⟹ Put $\lambda = 1$ in $[A - \lambda I]x = 0$

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{3\times 3}{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}}\underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$2y + 2z = 0$ $\qquad$ $-x + 2y + z = 0$

$y = -z$ $\qquad$ $-x + 2y - y = 0$

$x = y$

Let $y = \alpha$ → scalar

$$x_1 = \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

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$\Rightarrow$ Put $\lambda = 2$, Similarly,

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Put $\lambda = 2$, Similarly,

* Caley-Hamilton Theorem:-

Statement:- Every square matrix satisfies its own characteristic equation.

$$\lambda^n - C_1\lambda^{n-1} + C_2\lambda^{n-2} + \dots + (-1)^n C_n = 0$$

$\downarrow$

$$\lambda = A, \quad \lambda^2 = A^2, \quad \dots \lambda^n = A^n$$

$$A^n - C_1 A^{n-1} + \dots + (-1)^n C_n = 0$$

Pre multiply by A^{-1},

$$A^{-1}\left[A^n - C_1 A^{n-1} + \dots + (-1)^n C_n\right] = A^{-1} \cdot 0$$

$$A^{n-1} - C_1 A^{n-2} + \dots + (-1)^n A^{-1} C_n = 0$$

Q Verify Caley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}$$ and find its inverse

Solⁿ:- $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 5 \\ 5 & 4-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(4-\lambda) - 25 = 0$$

$$\lambda^2 - 7\lambda - 13 = 0$$

Using Caley-Hamilton theorem, we have.

$A^2 - 7A - 13I = 0$ ⊛

$A^2 = A.A = \begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 34 & 35 \\ 35 & 41 \end{bmatrix}$

$= \begin{bmatrix} 34 & 35 \\ 35 & 41 \end{bmatrix} - \begin{bmatrix} 21 & 35 \\ 35 & 28 \end{bmatrix} - \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

⇒ For find A^{-1}, Pre multiply with A^{-1} in equation ⊛, we have

$A^{-1}[A^2 - 7A - 13I] = 0$

$= -7 A^1 \times A^{-1}$
$= -7 A^{1-1}$
$= -7 A^0$
$= -7$

$A^1 - 7I - 13A^{-1} = 0$

$A^{-1} = \frac{1}{13}[A - 7I]$

$= \frac{1}{13}\begin{bmatrix} -4 & 5 \\ 5 & -3 \end{bmatrix}$

Q. $A = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}$

$AA^{-1} = I$

$A^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 6 \\ 2 & -1 \end{bmatrix}$