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Unit - IV.

Fourier Series

Periodic :- A function $f(x)$ is said to be periodic, if for some non zero constant p such that,

$$f(x+p) = f(x) \quad \forall x \in R$$

Ex. $\sin x$ and $\cos x$ are periodic with period 2π

Fourier Series :- Let $f(x)$ be a periodic function with period $2l$, $(l \neq 0)$ defined on $[-l, l]$ such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where a_0, a_n and b_n are called fourier coefficients and defined by

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

This is called Euler formula

NOTE :- When $l = \pi$,
then $f(x)$ is periodic with period 2π and defined on $[-\pi, \pi]$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Q. Find the Fourier series of the function
$f(x) = x, \quad -\pi \le x \le \pi \quad [-\pi, \pi]$
$f(x+2\pi) = f(x) \quad \forall\, x.$

$$= a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}(\pi^2 - \pi^2) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\ \cancel{\cos nx}\ \cos nx\,dx$$

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

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Integration by parts:

$$\int u \cdot v \, dx = \left[u \int v \, dx - \int \left(du \cdot \int v \, dx \right) dx \right]$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx$$

$$\int x \cos nx \, dx = \left[x \int \cos nx \, dx - \int \left(1 \cdot \int \cos nx \, dx \right) dx \right]$$

$$= \left[x \frac{\sin nx}{n} + \frac{1}{n^2} \cos nx \right]$$

$$\int_{-\pi}^{\pi} x \cos nx \, dx = \frac{1}{\pi} \left(\frac{\pi}{n} \left[\sin n\pi \right] - \left(-\frac{\pi}{n} \sin(-n\pi) \right) + \left[\frac{\cos nx}{\pi^2} \right]_{-\pi}^{\pi} \right)$$

$$a_n = \frac{1}{\pi n^2} \left[\cos n\pi - [\cos(-n\pi)] \right]$$

$$= \frac{1}{n^2 \pi} \left[\cos n\pi - \cos n\pi \right]$$

$$= 0$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$\int x \sin nx \, dx = x\int \sin nx \, dx - \int 1\left(\int \sin nx \, dx\right) dx$$

$$= -x\frac{\cos nx}{n} + \frac{1}{n}\int \cos nx \, dx$$

$$= -x\frac{\cos nx}{n} + \frac{1}{n^2}\sin nx$$

$$\int_{-\pi}^{\pi} x \sin nx \, dx = \left. -x\frac{\cos nx}{n} + \frac{1}{n^2}\sin nx \right]_{-\pi}^{\pi}$$

$$= -\pi\frac{\cos n\pi}{n} + \frac{1}{n^2}\sin n\pi - \left(\frac{\pi \cos(n\pi)}{n} + \frac{1}{n^2}\sin n(-\pi)\right)$$

$$= -\frac{\pi}{n}\cos n\pi + \cancel{\frac{1}{n^2}\sin n\pi} - \frac{\pi}{n}\cos n\pi + \cancel{\frac{1}{n^2}\sin n\pi}$$

$$= -\frac{2\pi}{n}\cos n\pi$$

$$b_n = -\frac{2}{n}\cos n\pi$$

n=1, ...

$-\frac{2\cancel{\pi}}{n\cancel{\pi}} \cos n\pi .$

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Put $n=1$

$$b_1 = -\frac{2}{1}\cos\pi = 2$$

Put $n=2$

$$b_2 = -\frac{2}{2}\cos 2\pi = -1$$

Put $n=3$

$$b_3 = -\frac{2}{3}\cos 3\pi = \frac{2}{3}$$

⋮

$$b_n = \frac{(-1)^{n+1}}{n} \cdot 2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot 2\sin(nx)$$

$$x = 2[\sin x] - \frac{1}{2}\cdot 2\sin 2x + \frac{2}{3}\sin 3x \ldots$$

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- **Even and odd function:-**

1. A function $f(x)$ is said to be even function if $f(-x) = f(x)\ \forall x$.

Ex. $f(x) = x^{2n}$ and $\cos(n\pi x)$ are even function, $n = 1, 2, 3, \ldots$

2. A function $f(x)$ is said to be odd function if $f(-x) = -f(x)\ \forall x$.

Ex. $f(x) = x^{2n+1}$ and $\sin(n\pi x)$ are odd function, $n = 1, 2, 3, \ldots$

Remark: 1) When the function $f(x)$ is an even function then

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx, \quad x \in [-l, l]$$

2) When $f(x)$ is an odd function then

$$\int_{-l}^{l} f(x)\,dx = 0, \quad x \in [-l, l]$$

3. When $f(x)$ is an even function then Fourier series of the function $f(x)$ is called cosine fourier series and the cosine fourier seirus is expansion is given by:

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$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \quad x \in [-l, l]$$

where, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ and

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

4. When $f(x)$ is an odd function then Fourier series of the function $f(x)$ is called ~~t~~ sine fourier series and the sine fourier series expansion is given by:

$$f(x) \simeq \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad x \in [-l, l]$$

where, $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$

Half - Range Fourier Series:-

$x \in [-l, l]$

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx$$

Theorem :- Fourier Cosine Series:- Let $f(x)$ be piecewise continous on $[0, l]$. Then the Fourier cosine series expansion of $f(x)$ on the half range $[0, l]$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Where, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ and

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx.$$

Fourier Sine series:- Let $f(x)$ be piecewise continous on $[0, l]$. Then the fourier sine series expansion of $f(x)$ on the half range $[0, l]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

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- Convergence of Fourier Series:-

1. Let $f(x)$ is piecewise continous on $[-l, l]$
i.e., $f(x)$ is defined and continous for all $x \in (-l, l)$ except, may be, at a finite number of points in $(-l, l)$.

2. At any point $x_0 \in (-l, l)$, where $f(x)$ is not continous, both the one-sided limits. $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exists and are finite.

$$f(x) = \frac{1}{2}\left[f(x_0 - l) + f(x_0 + l)\right]$$

Q. Find the fourier series expansion of the function.

$$f(x) = x^2 \quad , \quad -2 \leq x \leq 2$$

⇒ Since the function $f(x) = x^2$ is an even function, the fourier series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{2}\int_0^2 x^2\,dx = \int_0^2 x^2\,dx$

$= \left.\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$

→ $\int u \cdot v \cdot dx = u\int v\,dx - \int (du)\left(\int v\,dx\right) dx$

→ $\int \sin x\,dx = -\cos x$

→ $\int \cos x\,dx = \sin x$

→ $\int x^n\,dx = \frac{x^{n+1}}{n+1}$

→ $\sin(-\theta) = -\sin\theta$

→ $\cos(-\theta) = \cos\theta$

$\sin(n\pi) = 0$

$\cos(n\pi) = \begin{cases} 1, & \text{where } n \text{ is even} \\ -1, & \text{where } n \text{ is odd} \end{cases}$

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$$a_n = \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[x^2 \sin\left(\frac{n\pi x}{2}\right)\left(\frac{2}{n\pi}\right)\right]_0^2 - \int_0^2 2x \sin\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\, dx$$

$$= \left[\frac{2}{n\pi}\cdot 4\sin(n\pi) - 0\right] - \left[\frac{4}{n\pi} x\left(-\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\right)\right]_0^2 +$$

$$\int_0^2 \frac{4}{n^2\pi^2} 2.\cos\left(\frac{n\pi x}{2}\right) dx$$

$$= -\left[-\frac{8}{n^2\pi^2}(2\cos(n\pi) - 0)\right] + \frac{16}{n^3\pi^3}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{16}{n^2\pi^2}\cos n\pi = \frac{16}{n^2\pi^2}(-1)^n$$

$$x^2 = \frac{8}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}(-1)^n$$

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Formula :

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx \right]$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \cos bx + b \sin bx \right]$$

Q Find Fourier series expansion of the function

$$F(x) = e^{-x}, \quad 0 < x < 2\pi$$

$\Rightarrow$ The Fourier

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Q Find the fourier expansion of the following.

$$f(x) = \begin{cases} \pi + x & , -\pi < x \le 0 \\ 0 & , 0 < x < \pi \end{cases}$$

A) $a_0 = \frac{1}{l}\int_{-l}^{l} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} F(x)\,dx + \int_{0}^{\pi} F(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\,dx + \int_{0}^{\pi} 0\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi \cdot x + \frac{x^2}{2}\right]_{-\pi}^{0} = \frac{1}{\pi}\left[0 - \left(\pi(-\pi) + \frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = \frac{\pi}{2}$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\cos(nx)\,dx + \int_{0}^{\pi} 0 \cdot \cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[(\pi + x)\left[\frac{\sin nx}{x}\right]_{-\pi}^{0} - \int_{-\pi}^{0} 1 \cdot \frac{\sin nx}{n}\,dx\right] + 0$$

$$= \frac{1}{\pi}\left[0 + \cos\left(\frac{nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$= \frac{1}{n^2\pi}\left[1 - \cos n\pi\right]$$

$$= \begin{cases} \dfrac{2}{n^2\pi}, & \text{where n is odd} \\ 0, & \text{when n is even} \end{cases}$$

Q. Find the Fourier series expansion of the following periodic function of period 4.

$$f(x) = \begin{cases} 2+x, & -2 \le x < 0 \\ 2-x, & 0 < x \le 2 \end{cases}$$

$$f(x+4) = f(x)$$

$$\Rightarrow f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\,dx = \frac{1}{2} \int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2} \left[\int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx \right]$$

$$= \frac{1}{2} \left[\int_{-2}^{0} (2+x)\,dx + \int_{0}^{2} (2-x)\,dx \right]$$

$$= \frac{1}{2} \left[\left(2x + \frac{x^2}{2}\right)_{-2}^{0} + \left(2x - \frac{x^2}{2}\right)_{0}^{2} \right]$$

$$= \frac{1}{2} \left[-(-4+2) + 4 - 2 \right]$$

$$= \frac{1}{2} \left[2 + 2 \right] = \underline{\underline{2}}.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(2+x)\cos\left(\frac{n\pi}{2}x\right)dx + \int_{0}^{2}(2-x)\cos\left(\frac{n\pi x}{2}\right)dx\right.$$

$$= \cancel{\frac{1}{2}\left[\int_{-2}^{0}\right.}\ \int_{-2}^{0}\cancel{(2+x)\cos}$$

$$= \cancel{\frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{-2}^{0}\right.}$$

$$\int_{-2}^{0}(2+x)\cos\left(\frac{n\pi}{2}x\right)dx =$$