MTH302.

Date.
Page.

Probability and Statistics

* Random Experiment
  - → An experiment
  - → More than one outcome.
  - → Not possible to predict exact outcome in advance

* Set of all possible outcomes of a random experiment is called Sample Space.
  Ex: Throwing a dice ; $S = \{1, 2, 3, 4, 5, 6\}$

* Event : Subset of sample space
  Throwing a dice ;-
  Getting Even no. $E = \{2, 4, 6\}$
  Getting odd no. $E = \{1, 3, 5\}$

* Calculate Probability :
  If a random experiment has n possible outcomes out of which m outcomes are in favour of occurance of an event E.

then $P(E) = \dfrac{\text{No. of favourable cases for occurance of an event } E}{\text{No. of total outcomes.}}$

$= \dfrac{n(E)}{n(S)}$

• Ex. $S = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \{2, 4, 6\}$ $E_2 = \{1, 3, 5\}$ E_3 = No. greater than 6.

E_4 = No. less than 7.

$\Rightarrow P(E_1) = \frac{3}{6}$ $P(E_2) = \frac{3}{6}$ $P(E_3) = \frac{0}{6}$ $P(E_4) = \frac{6}{6}$

Date. |
Page. |

NOTE :- $0 \leq P(E) = \frac{n(E)}{n(S)} \leq 1$

$0 \leq P(E) \leq 1$

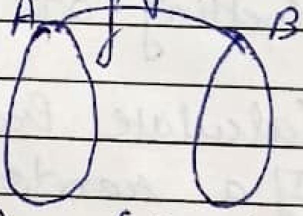
For any event E of a random experiment,

$0 \leq P(E) \leq 1$

$P(E) = 0$ Impossible event

$P(E) = 1$ Sure event.

* Function: A function f from a set A to B is rule that maps every element of set A to unique element of set B.



$A = \{1, 2, 3\}$

$B = \{a, b, c\}$

$A \times B = \{(1,a)\ (1,b)\ (1,c)\ (2,a)\ (2,b)\ (2,c)\ (3,a)\ (3,b)\ (3,c)\}$

Relation $R_1 = \{(1,a)\ (2,b)\ (3,c)\}$

↳ Also a function

Relation $R_2 = \{(1,a)\ (1,b)\ (1,c)\}$

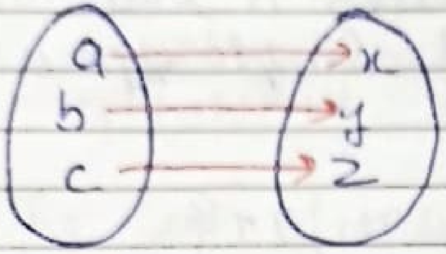
↳ Not a function.

→ All functions are relations but not vice versa

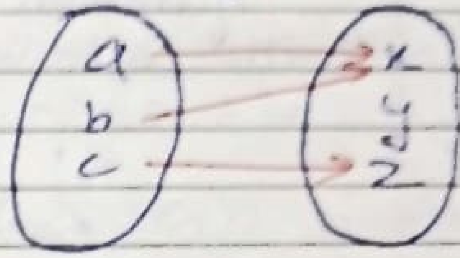
→ $P(E^c) = 1 - P(E)$

Function :

1 to 1

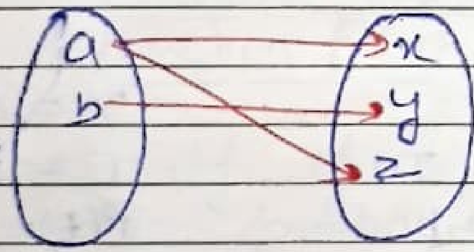


Many to 1

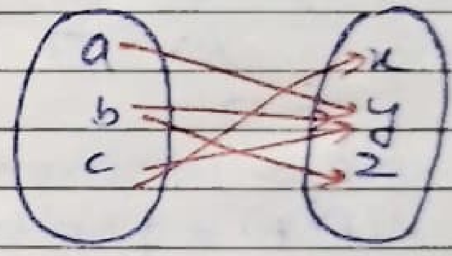


Non Function:

1 to many



Many to Many.



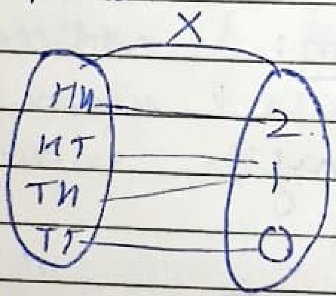
Unit-1
Random Variable and probability distribution

- **Random Variable:** A random variable is a function that associates a real number with each element of sample space

Ex: Two coins are tossed,

$$S = \{HH, HT, TH, TT\}.$$

Let X: no. of Heads



X: 0 1 2

$X = 0$
$E = TT$
$P(E) = \frac{1}{4}$

$X = 1$
$E = HT, TH$
$P(E) = \frac{1}{2}$

$X = 2$
$E = HH$
$P(E) = \frac{1}{4}$

$P(X=0) = \frac{1}{4}$ $\quad P(X=1) = \frac{1}{2}$ $\quad P(X=2) = \frac{1}{4}$

X:	0	1	2
P(X=x):	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\Rightarrow$ Probability distribution

- **Discrete Sample Space:** If a sample space contains finite or countably infinite number of possibilities

Date. |
Page. |

- **Discrete Random Variable:** A random variable X is called discrete random variable if its set of possible outcomes is finite or countably infinite.

- **Probability function / Probability mass function / Probability distribution function :-**

The set of ordered pairs $(x, P(x))$ is pmf or Probability distribution function of a discrete random variable X, if for each outcome x:

(i) $P(x) \geq 0$

(ii). $\sum_{x} P(x) = 1$

- **Commulative distribution (cdf) or distribution function F(x) :**

The cdf of a discrete random variable X with probability function $(x, P(x))$ is defined as

$$F(x) = P(X \leq x)$$

Date. |
Page. |

$x =$	X	0	1	2
$P(x) =$	$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
Cdf	$F(x)$	$\frac{1}{4}$	$\frac{3}{4}$	1

$\Rightarrow$ Pmf

$F(0) = P(X \leq 0) = P(X=0) \Rightarrow \frac{1}{4}$

$F(1) = P(X \leq 1)$
$= P(X=0) + P(X=1)$
$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$F(2) = P(X \leq 2)$
$= P(X=0) + P(X=1) + P(X=2)$

$\Rightarrow F(1) + P(X=2)$
$= \frac{3}{4} + \frac{1}{4} = 1$

Numerical: A random variable X has following distribution:

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2k$	$2k$	$3K$	K^2	$2K^2$	$7K^2+K$

① Evaluate K.
② Evaluate $P(X<6)$ & $P(X \geq 6)$
$P(0<X<5)$
③ Determine the cdf of X.
④ If $P(X \leq a) > \frac{1}{2}$

Find the minimum value of a.

Date. |
Page. |

(5) If $F(4) = \frac{8}{10}$, $F(3) = \frac{5}{10}$, Find $P(X=4)$

Solⁿ: (1) $\sum_{x} P(x) = 1$

$$0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0$$

$$K = \cancel{-1}, \frac{1}{10} \checkmark$$

(2) $\rightarrow P(X<6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}$$

$\rightarrow P(X \geq 6) = 1 - P(X<6)$

$\rightarrow P(0<X<5) \Rightarrow P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$\Rightarrow \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10}$$

(3)

	x	0	1	2	3	4	5	6	7
pdf	$P(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$
cdf		0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	$\frac{100}{100}$

(4) $P(X \leq a) > \frac{1}{2}$

(5) $P(X=4) = \frac{8}{10} - \frac{5}{10} = \frac{3}{10}$

Date.
Page.

Q. ☐ ☐ ☐

Three electronic components.

N – nondefective , D – defective.

$S = \{NNN, NND, NDN, DNN, DND, DDN, NDD, DDD\}$

X : no. of defected items.

X	0	1	2	3
$P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
cdf	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8}$

← Pdf

* **Continous Random Variable** : A random variable X is said to be continous random variable if it takes all values in the given interval.

Continous Sample Space : If it contains infinite number of possibilities equal to number of points on a line segment

In case of continous random variable, the probability is calculated for the interval.

$P(a < x < b) = P(a \leq x \leq b) = P(a \leq x \leq b) =$

$= P(a < x < b) = \int_a^b f(x)\, dx$

Date. |
Page. |

where $f(x)$ is probability density function.

* **Probability Density function**:- The function $f(x)$ is a pdf for continous random variable over the set of real numbers if

(i). $f(x) \geq 0 \quad \forall x$

(ii). $\int_{-\infty}^{\infty} f(x)\, dx = 1$

NOTE:- When X is continous random variable then,

$$P(X=x) = 0 \quad \text{for all values of } x.$$

Proof:-
$$P(a \leq x \leq b) = \int_a^b f(x)\, dx$$

now, both a & b are equal,

So, $P(X=x) = 0$

* **Commulative distribution function**: $(F(x))$ (cdf)
The cdf $F(x)$ of a continous random variable X with pdf $f(x)$ is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \quad -\infty < x < \infty$$

NOTE:- $P(a \leq x \leq b) = F(b) - F(a)$

and $\frac{dF(x)}{dx} = f(x)$, if derivative exists.

Proof :- LHS.

$$P(a \leq x \leq b)$$

$$= \int_a^b f(x)\,dx = \int_a^{-\infty} f(x)\,dx + \int_{-\infty}^{b} f(x)\,dx$$

$$= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= F(b) - F(a)$$

Ex. Let X be a continous random variable with pdf,

$$f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

(1) Verify $f(x)$ is pdf

(2) Find $P(0 < x < 1)$

(3) Find the cdf.

(1) $\Rightarrow$ (i) $f(x) \geq 0 \quad \forall\, x$

(ii) $$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\int_{-\infty}^{-1} f(x)\,dx}^{\,0} + \int_{-1}^{2} f(x)\,dx + \cancel{\int_{2}^{\infty} f(x)\,dx}^{\,0}$$

$$= \int_{-1}^{2} \frac{x^2}{3}\,dx = \left.\frac{x^3}{9}\right|_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1$$

Date. |
Page. |

(2) $\Rightarrow$ $P(0 < x < 1)$

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x^2}{3}\,dx = \frac{1}{3\times 3} x^3 \Big]_0^1$$

$$= \frac{1}{9} x^3 \Big]_0^1 = \frac{1}{9}.$$

(3) $\Rightarrow$ $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \underbrace{\cancel{\int_{-\infty}^{-1} f(t)\,dt}}_{0} + \int_{-1}^{x} f(t)\,dt$$

$$F(x) = \int_{-1}^{x} f(t)\,dt$$

$$= \int_{-1}^{x} \frac{t^2}{3}\,dt = \frac{1}{9}\left[t^3\right]_{-1}^{x}$$

$$F(x) = \frac{1}{9}(x^3+1).$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x^3+1}{9} & -1 < x < 2 \\ 1 & x > 2 \end{cases}$$

$F(2) = 1$
$F(3) = F(2) + \underbrace{P(2 < x < 3)}_{0}$
$= 1$

Date. |
Page. |

Ex. Let X be cont R.V.,

$$f(x) = 6x(1-x) \quad 0 \le x \le 1$$

① Verify pdf $\Rightarrow$ (i) $f(x) \ge 0$

(ii) $= \int_0^1 f(x)\,dx$

$$= \int_0^1 6x(1-x)\,dx$$

$$= 6\int_0^1 (x - x^2)\,dx$$

$$= 6\left(\frac{x^2}{2}\right)_0^1 - 6\left(\frac{x^3}{3}\right)_0^1$$

$$= 6\left(\frac{1}{2}\right) - 6\left(\frac{1}{3}\right) = 1$$

② Find cdf $\Rightarrow$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{x} 6t(1-t)\,dt$$

$$= 6\left[\int_{-\infty}^{x} (t - t^2)\,dt\right] = 6\left[\frac{t^2}{2}\right]_{-\infty}^{x}$$

Date. |
Page. |

* Joint Probability distribution

$f(x,y)$
X & Y

① $f(x,y) = P(X=x, Y=y)$
② $f(x,y) \geq 0$
③ $\sum_x \sum_y f(x,y) = 1$

$f(x,y)$

① $f(x,y) \geq 0$
② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

⇒ Let X and Y be two discrete random variables then the function $f(x,y)$ is said to be joint probability distribution if

(i) $f(x,y) = P(X=x, Y=y)$

(ii) $f(x,y) \geq 0 \ \forall \ (x,y)$

(iii) $\sum_x \sum_y f(x,y) = 1$

⇒ If X and Y are continous random variable then $f(x,y)$ in Joint distribution density function if

(i) $P[(x,y) \in A] = \iint_A f(x,y)\, dx\, dy$

where A is any region in XY plane.

(ii) $f(x,y) \geq 0 \quad \forall \ (x,y)$

(iii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$.

Date. |
Page. |

* Marginal distribution:

Let X and Y be discrete random variable; then marginal distribution of random variable X is given,

$$g(x) = \sum_{y} f(x,y)$$

and marginal distribution of random variable Y is given by

$$h(y) = \sum_{x} f(x,y)$$

If X and Y are continous random variable then marginal distribution of random variable X is given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

and marginal distribution of random variable Y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Date. |
Page. |

conditional Probability ⇒

A, B

NOTE: $P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$

$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$

* Conditional distribution:

Let X and Y be two discrete or continous random variable then the conditional distribution of Y given $X = x$,

$$f(y/x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0$$

and conditional distribution of X given $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \quad h(y) > 0$$

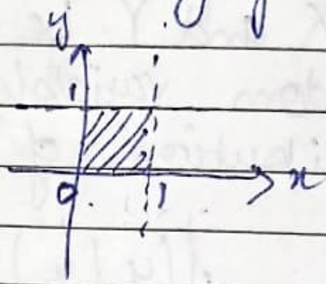
*

Example: Let X and Y be random variable with Joint density function.

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(1) Verify $f(x,y)$ is Joint density function.

Sol^n: (i) Clearly, $f(x,y) \geq 0$
$\forall (x,y)$.



(ii) $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$

$= \int_0^1 \int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$

$= \frac{2}{5}\int_0^1 \left[x^2+3yx\right]_0^1 dy$

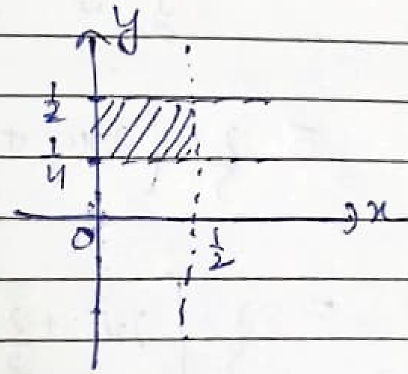
$= \frac{2}{5}\int_0^1 (1+3y)\,dy$

$= \frac{2}{5}\left[y+\frac{3y^2}{2}\right]_0^1 = \frac{2}{5}\times\frac{5}{2} = 1$

Date. |
Page. |

② Find $P[(x,y) \in A]$ where

$$A = \left\{ (x,y) : 0 < x < \frac{1}{2},\ \frac{1}{4} < y < \frac{1}{2} \right.$$



Sol^n: $P[(X,Y) \in A]$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{0}^{\frac{1}{2}} \frac{2}{5}(2x+3y)\,dx\,dy$$

$$= \frac{2}{5} \int_{\frac{1}{4}}^{\frac{1}{2}} \left[x^2 + 3xy\right]_0^{\frac{1}{2}} dy$$

$$= \frac{2}{5} \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{4} + \frac{3}{2}y\right) dy = \frac{2}{5}\left[\frac{1}{4}y + \frac{3}{4}y^2\right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{2}{5} \times \frac{1}{4}\left[\frac{1}{2} + \frac{3}{4} - \frac{1}{4} - \frac{3}{16}\right]$$

$$= \frac{1}{10}\left[1 - \frac{3}{16}\right] = \frac{13}{160}$$

Date.

Page.

(3) Find the marginal distribution of X.

Soln ⇒ $g(x) = \int_0^1 f(x,y)\,dy$

$= \int_0^1 \frac{2}{5}(2x+3y)\,dy$

$= \frac{2}{5}\left(2xy + \frac{3y^2}{2}\right)_0^1$

$= \frac{2}{5}\left[2x + \frac{3}{2}\right] = \frac{2}{5}\left[\frac{4x+3}{2}\right]$

$g(x) = \frac{4x+3}{5}$

$h(y) = \int_0^1 f(x,y)\,dx$

$= \int_0^1 \frac{2}{5}(2x+3y)\,dx$

$= \frac{2}{5}\left[\frac{2x^2}{2} + 3yx\right]_0^1$

$h(y) = \frac{2}{5}\left[1+3y\right] = \frac{2+6y}{5}$

Date. |
Page. |

* mean / Expectation of random variable :-

Let X be a random variable with probability distribution $f(x)$, then Mean / expected value of random variable X is defined as:

$$\mu = E(X) = \sum_{x} x f(x)$$

if X is discrete random variable.

and $$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continous random variable.

Ex. Toss two coins:

$S = \{HH, HT, TH, TT\}$

X = no. of heads.

X:	0	1	2
$P(X=x)$:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$f(0) = P(X=0) = \frac{1}{4}$

$f(1) = P(X=1) = \frac{1}{2}$

$f(2) = P(X=2) = \frac{1}{4}$

$$E(X) = \sum_{x} x f(x)$$

$$= \sum_{x=0}^{2} x f(x)$$

$$= 0 \times f(0) + 1 \times f(1) + 2 f(2)$$

$$= 0 + \frac{1}{2} + 2 \times \frac{1}{4} = 1$$

Date. |
Page. |

Ex. Let X be r.v. that denotes life of an electronic device in mass pdf :

$$f(x) = \begin{cases} \frac{20000}{x^3} & , x > 100 \\ 0 & , \text{elsewhere} \end{cases}$$

Find the expected life of this device.

$\Rightarrow$
$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \times \frac{20000}{x^3}\, dx$$

$$= \int_{100}^{\infty} \frac{20000}{x^2}\, dx$$

$$E(x) = 20000 \int_{100}^{\infty} x^{-2}\, dx$$

$$= -20000 \left[\frac{1}{x}\right]_{100}^{\infty}$$

$$= 200 \text{ hrs}$$

Expected life of device.

Date.
Page.

Defⁿ: Let X be a random variable with probability distribution $f(x)$ the expected value of random variable.

$$\mu_{g(x)} = E(g(X)) = \sum_{x} g(x) f(x)$$

if X is discrete random variable

and

$$\mu_{g(x)} = E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

if X is continous random variable

Ex:

X:	0	1	2
$f(x) = P(X=x)$:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Find the expected value of $g(X) = 4X-1$

$$\Rightarrow E[g(X)] = \sum_{x} g(x) f(x)$$

$$= \sum_{x=0}^{2} (4x-1) f(x)$$

$$= (4\times 0-1) f(0) + (4\times 1-1) f(1) + (4\times 2-1) f(2)$$

$$= 3$$

Data. | Page. |

Ex: Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find expected value of $g(X) = 4X+3$

Sol: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

$$= \int_{-1}^{2} (4x+3) \frac{x^2}{3}\, dx$$

$$= 8.$$

* Properties of Mean:-

1. If a and b are constants then,

$$E[aX+b] = a\,E(X) + b$$

NOTE: (i) If $a=0$ then, $E(b) = b$

(ii) If $b=0$ then, $E[aX] = a\,E[X]$

Date. |
Page. |

2. $E[g(x) \pm h(X)] = E[g(x)] \pm E[h(x)]$

Ex $g(x) = 4X+3$

$E[g(x)] = E[4X+3]$
$= 4E(X)+3$

Ex $Y = (X-1)^2$

$E[Y] = E[(X-1)^2]$

$= E[X^2+1-2X]$

$= E(X^2) + E(1) - 2E(X)$

NOTE.

$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) f(x,y)$

$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) \, dx \, dy$

Date |
Page |

Qn. Find $E(Y/X)$, for the joint density function:

$$f(x, y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0<x<2,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

Soln. Here $g(X, Y) = \frac{Y}{X}$

$\therefore E[g(X, Y)] = E\left(\frac{Y}{X}\right)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\, f(x, y)\, dx\, dy$$

$$= \int_0^1\int_0^2 \frac{y}{x} \times \frac{x(1+3y^2)}{4}\, dx\, dy$$

$$E\left(\frac{Y}{X}\right) = \frac{1}{4}\int_0^1\int_0^2 (y+3y^3)\, dx\, dy$$

$$= \frac{1}{4}\int_0^1 \left[yx + 3y^3 x\right]_0^2 dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} + \frac{3}{4}y^4\right]_0^1 = \frac{1}{2}\left[\frac{1}{2} + \frac{3}{4}\right] = \frac{5}{8}$$

Date. |
Page. |

Ex. $g(X,Y) = X^2 + Y$

$E[g(X,Y)] = ?$ (Use Previous Ex. for pdf)

$\Rightarrow g(X,Y) = X^2 + Y$

$$E[g(X,Y)] = \int_0^1 \int_0^2 (x^2+y)\left(\frac{x(1+3y^2)}{4}\right) dx\, dy$$

$$= \frac{1}{4}\int_0^1 \int_0^2 (x^2+y)(x+3y^2x)\, dx\, dy$$

$$= \frac{1}{4}\int_0^1 \int_0^2 (x^3 + yx + 3y^2x^3 + 3y^3x)\, dx\, dy$$

$$= \frac{1}{4}\int_0^1 \left[\frac{x^4}{4} + y\frac{x^2}{2} + \frac{3y^2x^4}{4} + 3y^3\frac{x^2}{2}\right]_0^2 dy$$

$$= \frac{1}{4}\int_0^1 (4 + 2y + 12y^2 + 6y^3)\, dy$$

$$= \frac{1}{4}\left[4y + \frac{2y^2}{2} + \frac{12y^3}{3} + \frac{6y^4}{4}\right]_0^1$$

$$= \frac{1}{4}\left[4 + 1 + 4 + \frac{3}{2}\right] = \frac{1}{4}\left[9 + \frac{3}{2}\right]$$

$$= \frac{21}{8}$$

* **Variance of a Random Variable :** Let X be a random variable with probability distribution $f(x)$, then variance of X, is defined as (with mean μ)

$$\sigma^2 = E[(X-\mu)^2]$$

If X is discrete r.v. then,

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

$$\left(\text{as } E(X) = \sum x f(x)\right)$$

If X is continous R.V. then,

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

NOTE.- The variance of R.V. X is

$$\sigma^2 = E[X^2] - [E[X]]^2$$

$$= E[X^2] - [\mu]^2$$

NOTE:- The positive square root of variance is called standard deviation.

Ex:- Let X be a R.V. with prob. distribution as

X	0	1	2	3
$f(x) = P(X=x)$	0.51	0.38	0.10	0.01

Calculate the variance of X.

Date.
Page.

Sol^n $\mu = E(X) = \sum_{x=0}^{3} x f(x) \Rightarrow$

$= 0 \times f(0) + 1 \times f(1) + 2 \times f(2) + 3 \times f(3)$

$= 0 + 1 \times 0.38 + 2 \times 0.10 + 3 \times 0.01$

$= 0.61$

Now, $E[X^2] = \sum_{x=0}^{3} x^2 f(x)$

$= 0.38 + 4 \times 0.10 + 9 \times 0.01$

$= 0.87$

$\sigma^2 = E[X^2] - [\mu]^2$

$= 0.87 - 0.3721 = 0.4979$

Ex Let X be a R.V. with density function

$$f(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find mean & variance.

Sol^n: $\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \Rightarrow \int_{-\infty}^{\infty} 2x(x-1)\, dx$

$= \int_1^2 (2x^2 - 2x)\, dx$

$= \Big[\frac{2x^3}{3} - x^2 \Big]_1^2 \Rightarrow \frac{2}{3}[8-1] - [4-1]$

$= \frac{2}{3}(7) - 3 \Rightarrow \frac{14-9}{3}$

$\Rightarrow \frac{5}{3}$

$$E[X^2] = \int_1^2 2x^2(x-1)\,dx$$

$$\int_1^2 (2x^3 - 2x^2)\,dx \Rightarrow \int_1^2 \frac{2x^4}{4} - \frac{2x^3}{3}$$

$$\Rightarrow \frac{1}{2}[16-1] - \frac{2}{3}[7]$$

$$= \frac{15}{2} - \frac{14}{3} \Rightarrow \frac{17}{6}$$

$$\sigma^2 \rightarrow \frac{17}{6} - \frac{25}{9} \Rightarrow \frac{153-150}{54} \Rightarrow \frac{3}{54}$$

* $\sigma^2_{g(x)} = E[(g(x) - \mu_{g(x)})^2]$

If X is ~~discrete R.v.~~, discrete R.V.

$$\sigma^2_{g(x)} = \sum_x (g(x) - \mu_{g(x)})^2 f(x)$$

If X is continous R.V.

$$\sigma^2_{g(x)} = \int_{-\infty}^{\infty} [g(x) - \mu_{g(x)}]^2 f(x)\,dx$$

Date.
Page.

Ex. Calculate the variance of $g(x) = 2x+3$ where X is R.V. with prob. distribution

X	0	1	2	3
$f(x) = P(X=x)$	1/4	1/8	1/2	1/8

$\Rightarrow \mu_{g(x)} = E[g(x)]$

$E(X) = \sum_{x=0}^{3} x f(x)$

$\Rightarrow 0\times\frac{1}{4} + 1\times\frac{1}{8} + 2\times\frac{1}{2} + 3\times\frac{1}{8} = \frac{3}{2}$

$\mu_{g(x)} = E[2x+3] \Rightarrow 2E[x]+3 \Rightarrow 6$

$E[x^2] = \sum_{x=0}^{3} x^2 f(x)$

$= 0\times\frac{1}{4} + 1\times\frac{1}{8} + 4\times\frac{1}{2} + 9\times\frac{1}{8}$

$= \frac{10}{8}+2 = \frac{26}{8} = \frac{13}{4}$

$\sigma^2_{g(x)} = E[((g(x)) - \mu_{g(x)})^2]$

$= E[(2x+3-6)^2]$

$= E[(2x-3)^2] = E[4x^2+9-12x]$

$= \cancel{E[4x^2}$

$= 4E[(X)^2]+9-12E[X]$

$= 4\times\frac{13}{4} + 9 - 12\times\frac{3}{2} = 13+9-18 = 4$

Date. | Page. |

Ex $f(x) = \begin{cases} 2(x-1) & , 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

$g(x) = 2x+1$

$E(x) = 2\int_1^2 x(x-1)\,dx$

Solⁿ: $\sigma^2_{g(x)} \Rightarrow$

$= 2\left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^2$

$\mu_{g(x)} = 2E[x] + 1$

$= \frac{5}{3}$

$= 2 \times \frac{5}{3} + 1 = \frac{10}{3} + 1 = \frac{13}{3}$

$= E\left[\left\{2(x)+1-\frac{13}{3}\right\}^2\right]$

$E(x^2) = 2\int_1^2 x^2(x-1)$

$= E\left[\left(2x - \frac{10}{3}\right)^2\right]$

$= 2\left(\frac{x^4}{4} - \frac{x^3}{3}\right)_1^2$

$= E\left[4x^2 + \frac{100}{9} - \frac{40}{3}x\right]$

$= \frac{17}{6}$

$= 4E(x)^2 + \frac{100}{9} - \frac{40}{3}E(x)$

$= 4 \times \frac{17}{6} + \frac{100}{9} - \frac{40}{3} \times \frac{5}{3}$

$= \frac{68}{6} + \frac{100}{9} - \frac{200}{9}$

$= \frac{68}{6} - \frac{100}{9}$

$= \frac{2}{9}$

Date. |
Page. |

Q1 If X and Y are R.V. with joint prob. dist. $f(X, Y)$ then $E[g(X, Y)] =$

$$\sum_x \sum_y g(x, y) f(x, y) \text{ and}$$

$$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(X, Y) f(x, y)\, dx\, dy$$

$\Rightarrow$ If we have $f(X, Y)$ given, find marginal dist.

$$g(x) = \sum_y f(x, y) \quad \text{or} \quad \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$E(X) = \sum_x x\, g(x) \quad \text{or} \quad \int_{-\infty}^{\infty} x\, g(x)\, dx$$

Also, $h(y) = \sum_x f(x, y)$ or $\int_{-\infty}^{\infty} f(x, y)\, dx$

$$E(y) = \sum_y y\, h(y) \quad \text{or} \quad \int_{-\infty}^{\infty} y\, h(y)\, dy.$$

Date.
Page.

* Co Variance :-

NOTE: If X and Y are two random variables with joint probability distribution $f(x,y)$ then,

$$E[X] = \sum_x x g(x)$$

and $$E[Y] = \sum_y y h(y)$$

where $g(x)$ and $h(y)$ are marginal distribution of X and Y.

If X and Y are continous random variable then,

$$E[X] = \int_{-\infty}^{\infty} x g(x) dx$$

and

$$E[Y] = \int_{-\infty}^{\infty} y h(y) dy$$

Covariance: Let X and Y be two random variables with joint probability dist. $f(x, y)$ then covariance of X & Y is given by

$$Cov(X\ Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

Date. |
Page. |

If X and Y are discrete random variables then,

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= \sum_x \sum_y (x-\mu_x)(y-\mu_y) f(x,y)$$

If X and Y are continous random variables. then,

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y)\, dx\, dy.$$

NOTE: When X & Y are independent then $\sigma_{XY} = 0$ but converse is not true.

If X and Y are random variables with joint probability distribution $f(x,y)$ then,

$$\sigma_{XY} = E[XY] - E[X][Y]$$

$$= E[XY] - \mu_X \mu_Y$$

Date. |
Page. |

Ex.:- Two random variables X and Y with joint probability density function.

$$f(x\,y) = \begin{cases} 2-x-y & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find covariance of X & Y.

Solⁿ: $\sigma_{XY} = E[XY] - E[X]\,E[Y]$

marginal distribution of X is given by

$$g(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 (2-x-y)\,dy = \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$g(x) = \frac{3-2x}{2}$$

Also, marginal distribution of Y,

$$h(y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 (2-x-y)\,dx$$

$$h(y) = \frac{3-2y}{2}$$

Date :
Page :

$$E[X] = \int_0^1 x\, g(x)\, dx = \int_0^1 \frac{x\,(3-2x)}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{2}{3}\right] = \frac{5}{12}$$

$$E[Y] = \int_0^1 y\, h(y)\, dy = \int_0^1 y\left(\frac{3-2y}{2}\right) dy = \frac{5}{12}$$

~~$E[g(x,y)] =$~~

$$E[XY] = \int_0^1\int_0^1 xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy\,(2-x-y)\, dx\, dy$$

$$= \int_0^1 \left[\frac{2x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left(y - \frac{y}{3} - \frac{y^2}{2}\right) dy$$

$$= \left(\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6}\right)_0^1 = \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{6}$$

$$\sigma_{XY} = E[XY] - E[X]E[Y]$$

$$= \frac{1}{6} - \frac{5}{12} \times \frac{5}{12}$$

$$= \frac{-1}{144}$$

-ve sign
(means if X is taking large value Y will take small value
$\therefore X \propto \frac{1}{Y}$)

* Chebyshev's Theorem:

Let X be a random variable with mean μ and standard deviation σ, then for any positive number k, the probability that a value of X lie in interval $[\mu - k\sigma, \mu + k\sigma]$ is at least $1 - \frac{1}{k^2}$

that is,

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

Ex. A random variable X has $\mu = 8$ and $\sigma^2 = 9$ then find (1) $P(-4 < x < 20)$

$\Rightarrow$

$\mu - k\sigma = -4$ | $\mu + k\sigma = 20$

$8 - k(3) = -4$

+ve value

$k = 4$

Ans: $1 - \frac{1}{k^2} = 1 - \frac{1}{4^2} = \frac{15}{16}$

2Eg Find: $P(|X-8| \geq 6)$

$\Rightarrow = 1 - P(|X-8| < 6)$

$= 1 - P(-6 < X-8 < 6)$

$= 1 - P(8-6 < X < 8+6)$

$= 1 - P(2 < X < 14)$

NOTE:
$|x| < a$
$-a < x < a$

Formulas : Unit 1

Date. |
Page. |

(1)

pdf $\begin{cases} P(x) \geq 0 \\ \sum_x P(x) = 1 \end{cases}$ cdf $\left[F(x) = P(X \leq x) \right.$

(2) $P(a < x < b) = P(a \leq x < b) = P(a < x \leq b) = P(a \leq x \leq b) =$

$$= \int_a^b f(x)\,dx$$

(3) $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

(4) $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$P(a \leq x \leq b) = F(b) - F(a)$

(5) $f(x,y) = P(X = x, Y = y)$

$f(x,y) \geq 0 \quad \forall (x,y)$

$\sum_x \sum_y f(x,y) = 1$

(6) $f(x,y) \geq 0 \quad \forall (x,y)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Date.
Page.

(7) marginal distribution

$g(x) = \sum_{y} f(x,y)$ $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$h(y) = \sum_{x} f(x,y)$ $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

(8) Conditional distribution

$f(y|x) = \frac{f(x,y)}{g(x)}$, $g(x) > 0$

$f(x|y) = \frac{f(x,y)}{h(y)}$, $h(y) > 0$

(9) Mean | Expectation

$\mu = E(X) = \sum_{x} x f(x)$

$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

(10)

$\mu_{g(x)} = E(g(x)) = \sum_{x} g(x) f(x)$

$\mu_{g(x)} = E(g(x)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$

Date. |
Page. |

(11) $E[ax+b] = aE(X) + b$

where a & $b \Rightarrow$ const.

$$E[g(x) \pm h(x)] = E[g(x)] \pm E[h(x)]$$

(12) $E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) f(x,y)$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

(13) Variance

$$\sigma^2 = E[(X-\mu)^2]$$

$$\sigma^2 = E[(X-\mu)^2] = \sum_{x} (x-\mu)^2 f(x)$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$\sigma^2 = E[X^2] - [E[X]]^2$$

$$= E[X^2] - [\mu]^2$$

$+\sigma \longrightarrow$ Standard deviation

(14) $$\sigma^2_{g(x)} = E\left[\left(g(x) - \mu_{g(x)}\right)^2\right]$$

$$\sigma^2_{g(x)} = \sum_x \left[g(x) - \mu_{g(x)}\right]^2 f(x)$$

$$\sigma^2_{g(x)} = \int_{-\infty}^{\infty} \left[g(x) - \mu_{g(x)}\right]^2 f(x)\, dx$$

(15) $$E[X] = \sum_x x\, g(x) \qquad E[X] = \int_{-\infty}^{\infty} x\, g(x)\, dx$$

$$E[Y] = \sum_y y\, h(y) \qquad E[Y] = \int_{-\infty}^{\infty} y\, h(y)\, dy$$

(16) Covariance

$$Cov(X\, Y) = \sigma_{XY} = E\left[(X - \mu_X)(Y - \mu_Y)\right]$$

$$\sigma_{XY} = \sum_x \sum_y (x - \mu_x)(y - \mu_y) f(x, y)$$

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y) f(x, y)\, dx\, dy.$$

Date.

Page.

(17) $\sigma_{XY} = E[XY] - E[X][Y]$

$= E[XY] - \mu_X \mu_Y$

(18) Chebyshev's Theorem

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$