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Unit-2

Recurrence Relation:

$a_n = a+(n-1)d$ ⇒ Example of recurrence relation.

Ques. Let the number of bacteria in a colony doubles every hour. If the colony begins with 2 bacteria, how many bacteria will be there after n hours.

⇒ $a_0 = 2$

$a_1 = 2 \cdot a_0 = 2 \cdot 2 = 2^2$

$a_2 = 2 \cdot a_1 = 2 \cdot 2^2 = 2^3$

$\vdots$

$\boxed{a_n = 2 \cdot a_{n-1} = 2 \cdot 2^n = 2^{n+1}}$ Recurrence relation.

Defination:- A recurrence relation for the sequence $\{a_n\}$ is an equation that express a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a non negative integer.

Ex. $a_n = 2^n$

$a_n = a_{n-1} + a_{n-2}$

$a_n = a_{n-1} + 2$

- ## Modelling with Recurrence Relation :-

We can use recurrence relations to model a wide variety of problems such as finding compound interest, counting problems, no. of moves in Hanoi tower, and counting bit of strings with certain properties.

Ques. A young pair of rabbits (one of each gender) is placed on an islands. A pair of rabbits does not breed until they are 2 month old. After they are 2 month old, each pair of rabbits produces another pair each month. Find a recurrence relation for the no. of pairs of rabbits on the island after n months, assuming that rabbits never dies.

Solⁿ:

Reproducing pairs (at least 2 month old)	Young pair (less than two old month)	Month	Reproducing pairs	Young pairs	Total pairs
0	M F	1	0	1	$1 \to a_0$
0	M F	2	0	1	$1 \to a_1$
M F	M F	3	1	1	$2 \to a_2 = a_1 + a_0$
M F	M F M F	4	1	2	$3 \to a_3 = a_1 + a_2$
M F M F	M F M F M F	5	2	3	$5 \to a_4 = a_2 + a_3$
					⋮

Fibonacci Sequence $\leftarrow$ $a_n = a_{n-1} + a_{n-2}$

* A sequence is called a solution of recurrence relation if its terms satisfy the recurrence relation.

* The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Ques. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are the value of a_2 & a_3?

$\Rightarrow$ $a_2 = a_1 - a_0 = \cancel{3-5 = -2}$ $= 5-3 = 2$
$a_3 = a_2 - a_1 = 2-5 = -3$

Ques. Determine whether the sequence $\{a_n\}$, where $a_n = 1.\ 2^n$, $2.\ 3n$, $3.\ 5$ for non-negative integer n, is a sol^n of the recurrence relation
$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

$\Rightarrow$ 1. $a_n = 2^n$

$$a_n = 2a_{n-1} - a_{n-2}$$
$$2^n = 2(2^{n-1}) - (2^{n-2})$$
$$\neq 2^n - \frac{2^n}{4} = 2^n\left(\frac{3}{4}\right)$$

$a_n = 2^n$ is not a sol^n of recurrence relation.

1, 2, 3, 4, --- ⇒ sequence
1+2+3+4+--- ⇒ Series

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2. $a_n = 3n$

$$a_n = 2a_{n-1} - a_{n-2}$$
$$3n = 2(3(n-1)) - 3(n-2)$$
$$= 3n$$

$a_n = 3n$ is a solⁿ of recurrence relation.

3. $a_n = 5$

$$a_n = 2a_{n-1} - a_{n-2} = 2\cdot5 - 5 = 5$$

$a_n = 5$ is a solⁿ of recurrence relation

Ques: Find the first six terms of the following recurrence relation.

1. $a_n = -2a_{n-1}$, $a_0 = -1$

⇒ $a_1 = -2a_0 = -2(-1) = 2$ $\quad a_4 = -2(a_3) = -16$

$a_2 = -2(a_1) = -4$ $\quad a_5 = -2(a_4) = 32$

$a_3 = -2(a_2) = 8$ $\quad a_6 = -2(a_5) = -64$

2. $a_n = 2a_{n+2} + 3a_{n-1}$, $a_0 = 1, a_1 = -1$

⇒ ~~$a_1 =$~~

3. $a_n = 3a_{n-1}^2$ ~~$+ a_{n+1}$~~ , $a_0 = 3$

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Ques. Is the sequence $\{a_n\}$ a solution of recurrence relation
$$a_n = 8a_{n-1} - 16a_{n-2} \text{ if}$$

1. $a_n = 0$ 2. $a_n = -1$ 3. $a_n = 1$

- **Terminology:**

1. **Differential Operator (Δ):** It is denoted by Δ and defined as

$$\boxed{\Delta a_n = a_{n+1} - a_n} \quad \text{— (1)}$$

2. **Shift Operator:** The shift operator 'E' is defined as the operator that increases the argument of a function by one tabular interval.

Thus, $\boxed{Ea_n = a_{n+1}}$ — (2)

From (1) & (2),
$$\Delta a_n = Ea_n - a_n = (E-1)a_n$$

$$\boxed{\Delta = E - 1}$$

Relation b/w Differential & Shift Operator

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Remark :- The recurrence relation $a_n = a_{n-1} + a_{n-2}$ can be written as $a_{n+2} = a_{n+1} + a_n$

$$a_n = a_{n-1} + a_{n-2}$$

Applying shift operator 'E' two times both sides, we have =>

$$E^2 a_n = E^2 a_{n-1} + E^2 a_{n-2}$$

$$E(Ea_n) = E(Ea_{n-1}) + E(Ea_{n-2})$$

$$E(a_{n+1}) = E(a_n) + E(a_{n-1})$$

$$a_{n+2} = a_{n+1} + a_n$$

Que. Find a_{n+2} of the recurrence relation

$$a_n = 5a_{n-1} + 6a_{n-2} + 7^n$$

=> Applyin Shift operator 'E' two times both sides we have,

$$E^2(a_n) = 5E^2(a_{n-1}) + 6E^2(a_{n-2}) + E^2(7^n)$$

$$E[Ea_n] = 5E[Ea_{n-1}] + 6E(Ea_{n-2}) + E(E7^n)$$

$$E(a_{n+1}) = 5E[a_n] + 6E(a_{n-1}) + E(7^{n+1})$$

$$a_{n+2} = 5a_{n+1} + 6a_n + 7^{n+2}$$

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Q. Check whether the following sequence $a_n = 2$ is a sol^n of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$

$\Rightarrow$ $\{a_n\} = 2, \forall n$

$a_n = 2n^0 = 2$
$a_{n-1} = 2(n-1)^0 = 2$
$a_{n-2} = 2(n-2)^0 = 2$

Put values in eq^n,
$a_n = 2a_{n-1} - a_{n-2}$
$2 = 2(2) - 2$

$\therefore$ it is a sol^n.

- Linear Recurrence Relation with constant coefficients :-

The k^{th} order (or degree) linear recurrence relation of the form

$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = R(n), \quad n \geq 0$

where $C_0, C_1, C_2, \ldots, C_k \in R$
(i.e. $C_0, C_1, C_2, \ldots, C_k$ are constants) $C_k \neq 0$

Order or Degree = Higher subscript − Lower subscript.

Ex $a_n = a_{n-1} + a_{n-2}$
$= n - (n-2)$
$= 2$

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NOTE: (1) When $R(n) = 0$ then eqⁿ (*) is called homogenous linear recurrence relation with constant coefficient.

(2) When $R(n) \neq 0$ —— non homogneous.

- The general form of k^{th} order linear Recurrance relation is given by:

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = R(n), \quad n \geq 0$$

Ex:-
(1) $a_n = 2a_{n-1} + 3a_{n-2}$ Homo. LRR with CC
(2) $2a_{n-2} - 3a_{n-1} + 2a_n = 0$ Homo. LRR
(3) $2a_{n-2} + 4a_{n-1} + 5a_n = 2^n + 2$ Non Homo LRR
(4) $2a_{n-2} - a_{n-1} - a_n = n + 7^n$ Non Homo. LRR with CC
(5) $\frac{1}{n} a_n - 2a_{n-1} + a_{n-2} = 0$ Non Homo. LRR with VC
(6) $2^n a_n + 2a_{n-1} - n a_{n-2} = 0$ Non. Homo. LRR with VC
(7) $\frac{1}{n} a_{n-2} + a_{n-1} - a_n = 2^n + 7^n + 2$ Non Homo. LRR with VC
(8) $3^n a_{n-2} + 2a_{n-1} - \frac{1}{n} a_n = 2$ Non Homo. LRR with VC
(9) $2(a_{n-2})^2 + 3a_{n-1} + a_n = 0$ Homo. Non LRR with CC
(10) $2a_n a_{n-1} + a_{n-1} = 0$ Homo. Non LRR with CC
(11) $2(a_n)^2 - 3a_{n-1} + 7a_{n-2} = 3^n + n$ Non Homo. Non LRR with CC
(12) $\frac{1}{n}(a_n)^2 - 2a_{n-1} + 3a_{n-2} = 0$ Homo. Non LRR with VC
(13) $\frac{1}{n} a_n a_{n-2} + a_{n-1} + 2a_{n-2} = 2^n + 3^n + n + 2$ Non Homo. Non LRR with VC

(14) $7^n a_{n-2} + 2^n a_{n-1} + 3(a_n)^2 = 7^n + 2$

$\Downarrow$

Non. Homo. Non LRR with VC

* **Solution of Homogeneous Linear Recurrence Relation with constant coefficients:-**

The general form of k^{th} order (or degree) homogenous Linear Recurrance Relation with constant coefficients is given by:

$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_K a_{n-k} = 0$ — (1)

where $C_0, C_1, C_2, \dots, C_K$ are constants & $C_K \neq 0$.

Apply shift operator 'E' k times on eq^n (1), we have

$$C_0 E^K a_n + C_1 E^K a_{n-1} + C_2 E^K a_{n-2} + \dots + C_K E^K a_{n-k} = 0$$

or $$C_0 E^K a_n + C_1 E^{K-1} a_n + C_2 E^{K-2} a_n + \dots + C_K E^0 a_n = 0$$

$$\Rightarrow (C_0 E^K + C_1 E^{K-1} + C_2 E^{K-2} + \dots + C_K) a_n = 0$$

Since $a_n \neq 0$

$$\therefore C_0 E^K + C_1 E^{K-1} + C_2 E^{K-2} + \dots + C_K = 0 \quad \text{—} (*)$$

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Substitute $E = x$ in eq^n (*),
We have,

$$C_0 x^k + C_1 x^{k-1} + C_2 x^{k-2} + \dots + C_k = 0$$

(This is called characteristic polynomial of or auxillary eq^n of eq^n 1.)

- Possibilities of Roots of eq^n (*),

1. ~~Let $r_1, r_2, r_3, \dots, r_k$ are k distinct~~

1. Real & distinct Roots:

Let $r_1, r_2, r_3, \dots, r_k$ are k distinct roots of eq^n (*), then the General solution (GS) or (Complete sol^n (CS) / Complementary func^n (CF)) of HLRR is given by:

$$a_n = A_1 (r_1)^n + A_2 (r_2)^n + \dots + A_k (r_k)^n$$

where $A_1, A_2, \dots, A_k$ are constants.

2. Real & Equal Roots: n = multiplicity of roots

Let $r_1 = r = r_2 = \dots = r_k$ are k equal roots then the General sol^n is given by

$$a_n = [A_1 + A_2 n + A_3 n^2 + \dots + A_k (n)^{k-1}] (r)^n$$

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(3) Complex Roots :

Let $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2$, ..., $\alpha_k \pm i\beta_k$

K-complex roots then the GS is given by:

$$a_n = (\rho_1)^n [A_1 \cos n\theta_1 + B_1 \sin n\theta_1] +$$

$$(\rho_2)^n [A_2 \cos n\theta_2 + B_2 \sin n\theta_2] + \text{----}$$

$$\text{---} + (\rho_k)^n [A_k \cos n\theta_k + B_k \sin n\theta_k]$$

where, $\rho_1 = \sqrt{\alpha_1^2 + \beta_1^2}$, $\theta_1 = \tan^{-1}\left(\frac{\beta_1}{\alpha_1}\right)$

$\rho_2 = \sqrt{\alpha_2^2 + \beta_2^2}$, $\theta_2 = \tan^{-1}\left(\frac{\beta_2}{\alpha_2}\right)$

$\vdots$

$\rho_k = \sqrt{\alpha_k^2 + \beta_k^2}$, $\theta_k = \tan^{-1}\left(\frac{\beta_k}{\alpha_k}\right)$

Ex. $1 \pm i$

$\alpha_1 = 1$, $\beta_1 = 1$

$\rho_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta_1 = \tan^{-1}\left(\frac{\beta_1}{\alpha_1}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

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Ques. Solve the following recurrence relation

$$f_n = f_{n-1} + f_{n-2} \quad , f_0 = 1, f_1 = 1$$

Solⁿ ⇒

$$f_n - f_{n-1} - f_{n-2} = 0$$

Using shift operator, we have

$$E^2 f_n - E^2 f_{n-1} - E^2 f_{n-2} = 0$$

$$[E^2 - E - 1] f_n = 0$$

Auxillary Eqⁿ ⇒ $x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \quad , \quad r_2 = \frac{1-\sqrt{5}}{2}$$

The General Solⁿ is

$$f_n = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for $n=0$,

$$f_0 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = A_1 + A_2 \quad \text{—(1)}$$

for $n=1$,

$$f_1 = 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right)^1 \quad \text{—(2)}$$

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After solving eqn ① & ②, we have

$$A_1 = \frac{1+\sqrt5}{2\sqrt5}, \quad A_2 = \frac{\sqrt5-1}{2\sqrt5}$$

$$f_n = \left(\frac{1+\sqrt5}{2\sqrt5}\right)\left(\frac{1+\sqrt5}{2}\right)^n + \left(\frac{\sqrt5-1}{2\sqrt5}\right)\left(\frac{1-\sqrt5}{2}\right)^n$$

Q. Solve the following recurrence relation

$t_n = 0$, if $n=0$
$t_n = 5$, if $n=1$

$t_n = 3t_{n-1} + 4t_{n-2}$

Sol^n ⇒ Using shift operator, we have,

$$(E^2 - 3E - 4)t_n = 0$$

CP is $\rightarrow x^2 - 3x - 4 = 0$

$x^2 - 4x + x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4, -1$

$r_1 = -1, r_2 = 4$

GS is given by: $t_n = A_1(-1)^n + A_2(4)^n$

for $n=0$,

$t_0 = 0 = A_1 + A_2$ — ①

for $n=1$,

$t_1 = 5 = -A_1 + 4A_2$ — ②

From ① & ②, $A_1 = -1$, $A_2 = 1$

$t_n = (-1)(-1)^n + 4^n$

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Ques. Solve the following Recurrence Relation
$a_n + a_{n-2} = 0$, with $a_0 = 1$, $a_1 = -1$

$\Rightarrow$ $a_n + a_{n-2} = 0$

Using Shift Operator,

$$(1 \cdot E^2 + 0 \cdot E^1 + 1)\, a_n = 0$$

$$(E^2 + 1)\, a_n = 0$$

A.E.:- $x^2 + 1 = 0$

$$x = \pm i = 0 \pm i$$

$$\alpha = 0, \ \beta = 1$$

$$a_n = (\rho)^n (A_1 \cos n\theta + B_1 \sin n\theta)$$

$$\rho = \sqrt{\alpha^2 + \beta^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$a_n = (1)^n \left[A_1 \cos n\frac{\pi}{2} + B_1 \sin n\frac{\pi}{2}\right]$$

$$a_0 = 1 = 1\left[A_1 + B_1(0)\right]$$

$$A_1 = 1$$

$$a_1 = -1 = 1\left[\quad\right]$$

* Solution of NHLRR with constant coefficient :-

The general form of k^{th} order NHLRR with constant coefficients is given by

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = R(n)$$

$$(R(n) \neq 0)$$

where $C_0, C_1, C_2, \dots, C_k$ are constant and $C_k \neq 0$.

The G.S (or C.S) = CF + PI

CF → Complementary Function
PI → Particular Integral

Method to find PI :-

① ~~Select~~ Special / Inverse operator method
② Generating function

① Special/Inverse operator Method:

1. If $R(n) = a^n$, a is constant

then $PI = \frac{1}{F(E)} a^n = \frac{1}{F(a)} a^n$

where $F(E) = C_0 E^k + C_1 E^{k-1} + C_2 E^{k-2} + \dots + C_k$

Ques. Solve the following RR $a_n = 3a_{n-1} + 2a_{n-2} + 3^n$

$\Rightarrow \quad a_n - 3a_{n-1} - 2a_{n-2} = 3^n$

$$(E^2 - 3E - 2)a_n = 3^{n+2}$$

A.E. $\quad x^2 - 3x - 2 = 0$

$x^2 - 2x - x - 2 = 0$

$x = \frac{3 \pm \sqrt{17}}{2}$

~~$CF = A_1(1)^n + A_2(2)^n$~~

~~$PI = \frac{1}{F(E)} a^n$~~

$$CF = A_1\left(\frac{3+\sqrt{17}}{2}\right)^n + A_2\left(\frac{3-\sqrt{17}}{2}\right)^n$$

$$PI = \frac{1}{F(E)} a^n$$

$$= \frac{1}{E^2 - 3E - 2} \cdot 3^n \cdot 9$$

$$= \frac{9 \cdot 3^n}{(9-9-2)} = \frac{-9}{2} 3^n.$$

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Qu: Solve the following RR

$$a_{n+2} = -a_n + 7^n$$

$$\Rightarrow a_{n+2} + a_n = 7^n$$

$$E^2 a_n + a_n = 7^n$$

$$(E^2+1) a_n = 7^n$$

AE. $x^2+1=0$

$$CF = \left[A_1 \cos n\frac{\pi}{2} + B_1 \sin n\frac{\pi}{2}\right]$$

$$PI = \frac{1}{E^2+1} 7^n = \frac{7^n}{50}$$

$$GS = A_1 \cos n\frac{\pi}{2} + B_1 \sin n\frac{\pi}{2} + \frac{7^n}{50}$$

(2) If $R(n) = P(n)$, where $P(n)$ is a polynomial of order n.

then $PI = \frac{1}{F(E)} \cdot P(n)$

Replace E by $1+\Delta$

$$= \frac{1}{F(1+\Delta)} P(n)$$

$$= \frac{1}{(1+\Delta)} P(n) = n = (1+\Delta)^{-1} n$$

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$(1+\Delta)^{-1} = 1-\Delta+\Delta^2-\Delta^3+\Delta^4+\cdots$

$(1-\Delta)^{-1} = 1+\Delta+\Delta^2+\Delta^3+\Delta^4+\cdots$

- **Factorial Polynomial :-**

$n^{(m)} = n(n-1)(n-2)\cdots(n-(m-1))$

① $n^{(0)} = 1$

② $n^{(1)} = n$

③ $n^{(2)} = n(n-1) = n^2-n$

or $n^2 = n^{(2)}+n = n^{(2)}+n^{(1)}$ $(\because n^{(1)}=n$

④ $n^{(3)} = n(n-1)(n-2) = n^3-3n^2+2n$

$n^3 = n^{(3)}+3n^2-2n = n^{(3)}+3\left[n^{(2)}+n^{(1)}\right]-2n$

$n^3 = n^{(3)}+3n^{(2)}+n^{(1)}$

$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \cdots + {}^nC_n x^0 y^n$

- $\Delta n^{(m)} = m\, n^{(m-1)}$
- $\frac{1}{\Delta} n^{(m)} = \frac{n^{(m+1)}}{m+1}$

Ques. Find the General solution of the following Recurrence Relation

$$a_n = -2a_{n-1} + 3a_{n-2} + 2n^2 + n + 2$$

Sol^n: $a_n + 2a_{n-1} - 3a_{n-2} = 2n^2 + n + 2$.

Using shift operator, we have

$$(E^2 + 2E - 3)a_n = 2(n+2)^2 + (n+2) + 2$$

AE: $x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3, 1$

$CF = A_1(-3)^n + A_2$

$$PI = \frac{1}{F(E)} P(n) = \frac{2(n+2)^2 + (n+2) + 2}{E^2 + 2E - 3}$$

Replace E by $1+\Delta$,

$$= \frac{2(n+2)^2 + (n+2) + 2}{(1+\Delta)^2 + 2(1+\Delta) - 3} = \frac{2(n^2+4n+4) + (n+2) + 2}{1+\Delta^2 + 2\Delta + 2 + 2\Delta - 3}$$

$$= \frac{2n^2 + 9n + 12}{\Delta^2 + 4\Delta} = \frac{2n^2 + 9n + 12}{4\Delta\left(1 + \frac{\Delta}{4}\right)}$$

$$= \frac{1}{4}\frac{1}{\Delta}\left[1 + \frac{\Delta}{4}\right]^{-1}\left[2\left(n^{(2)} + n^{(1)}\right) + 9n^{(1)} + 12\right]$$

$$= \frac{1}{4}\left[1+\frac{\Delta}{4}\right]^{-1}\frac{1}{\Delta}\left[2n^{(2)}+11n^{(1)}+12\right]$$

$$= \frac{1}{4}\left[1+\frac{\Delta}{4}\right]^{-1}\left[2\frac{1}{\Delta}n^{(2)}+11\frac{1}{\Delta}n^{(1)}+\frac{1}{\Delta}12\cdot n^{(0)}\right]$$

$$= \frac{1}{4}\left[1+\frac{\Delta}{4}\right]^{-1}\left[2\cdot\frac{n^{(3)}}{3}+\frac{11}{2}n^{(2)}+12\cdot n^{(1)}\right]$$

$$= \frac{1}{4}\left[1-\left(\frac{\Delta}{4}\right)+\left(\frac{\Delta}{4}\right)^2-\left(\frac{\Delta}{4}\right)^3+\cdots\right]\left[\frac{2}{3}n^{(3)}+\frac{11}{2}n^{(2)}+12n^{(1)}\right]$$

$$= \frac{1}{4}\left[\frac{2\cdot n^{(3)}}{3}+\frac{11}{2}n^{(2)}+12n^{(1)}-\frac{1}{4}\left\{\frac{2}{3}\Delta n^{(3)}+\frac{11}{2}\Delta n^{(2)}+\right.\right.$$

$$\left.12\Delta n^{(1)}\right\}+\frac{1}{16}\left\{\frac{2}{3}\Delta^2 n^{(3)}+\frac{11}{2}\Delta^2 n^{(2)}+12\Delta^2 n^{(1)}\right\}$$

$$\left.-\frac{1}{64}\cdot 2\Delta^3 n^{(2)} - - - - -\right]$$

$$= \frac{1}{4}\left[\frac{2}{3}n^{(3)}+\frac{11}{2}n^{(2)}+12n^{(1)}\right]-\frac{1}{16}\left\{\frac{2}{3}2n^{(2)}+\frac{11}{2}n^{(1)}+12\right\}$$

$$+\frac{1}{64}\left\{\frac{2}{3}\Delta(2n^{(2)})+\frac{11}{2}\Delta(n^{(1)})+12\Delta(1)\right\}-\frac{1}{256}\left\{\frac{2}{3}\Delta^2(2n^{(3)})\right\}$$

$$- - -$$

$$= \frac{1}{4}\left[\frac{2}{3}n^{(3)}+\frac{11}{2}n^{(2)}+12n^{(1)}\right]-\frac{1}{16}\left\{\frac{4}{3}n^{(2)}+\frac{11}{2}n^{(1)}\right.$$

$$\left.+12\right\}+\frac{1}{64}\left\{\frac{4}{3}n^{(1)}+\frac{11}{2}\cdot 1+12\cdot 0\right\}-\frac{1}{256}\left\{\frac{4}{3}\Delta(n^{(1)})\right\}-$$

$$= \frac{1}{4}\left\{\frac{2}{3}n^{(3)} + \frac{11}{2}n^{(2)} + 12n^{(1)}\right\} - \frac{1}{16}\left\{\frac{4}{3}n^{(2)} + \frac{11}{2}n^{(1)} + 12\right\}$$

$$+ \frac{1}{64}\left\{\frac{4}{3}n^{(1)} + \frac{11}{2}\right\} - \frac{1}{256}\left\{\frac{4}{3}\right\}$$

$$= CS = CF + PI$$

$$= A_1(-3)^n + A_2 + \frac{1}{6}n^{(3)} + n^{(2)}\left\{\frac{11}{2} - \frac{1}{12}\right\}$$

$$+ n^{(1)}\left\{12 - \frac{11}{32} - \frac{11}{48}\right\} +$$

$$+ \left(\frac{11}{128} - \frac{1}{182} - \frac{3}{4}\right)$$

Q. Find G.S of the RR $a_n = 3a_{n-1} + 2n$

$\Rightarrow$ $a_n = 3a_{n-1} + 2n$

Using Shift Operator,

$$(E-3)\, a_n = 2(n+1)$$

AE. $x - 3 = 0$

$x = 3$.

$CF = A_1 (3)^n$

$PI = \frac{1}{F(E)} P(n) = \frac{1}{E-3} 2(n+1)$

Replace E by $1+\Delta$

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$$= \frac{1}{1+\Delta-3}\, 2(n+1) = \frac{2(n+1)}{\Delta-2} = \frac{2(n+1)}{-2\left(1-\frac{\Delta}{2}\right)}$$

$$= \frac{-(n+1)}{\left(1-\frac{\Delta}{2}\right)} = -\left(1-\frac{\Delta}{2}\right)^{-1}(n+1)$$

$$= -\left[1+\left(\frac{\Delta}{2}\right)+\left(\frac{\Delta}{2}\right)^2+\left(\frac{\Delta}{2}\right)^3+\cdots\right]\left(n^{(1)}+n^{(0)}\right)$$

$$= -\left[n^{(1)}+n^{(0)}+\frac{1}{2}\left\{\Delta(n^{(1)})+\Delta n^{(0)}\right\}+\frac{1}{4}\left\{\Delta^2 n^{(1)}+\Delta^2 n^{(0)}\right\} + \cdots\right]$$

$$= -\left[n+1+\frac{1}{2}\{1+0\}+\frac{1}{4}\cdot 0+0\cdots\right]$$

$$\Delta^2 n^{(1)} = \Delta(\Delta n^{(1)}) = 0$$

$$= -\left(n+\frac{3}{2}\right)$$

$$GS = A_1 (3)^n - \left(n+\frac{3}{2}\right)$$

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(3) If $R(n) = a^n P(n)$

then $PI = \frac{1}{F(E)} a^n P(n) = a^n \frac{1}{F(aE)} P(n)$

~~Ques. Find GS of the RR $a_{n+2} - 2a_{n+1} + a_n = n^2 2^n$~~

(4) If $R(n) = e^{\alpha n}$, $\alpha \rightarrow$ real no

then $PI = \frac{1}{F(E)} e^{\alpha n}$

Put $e^\alpha = b \rightarrow$ constant

$$PI = \frac{1}{F(E)} b^n$$

$$= \frac{1}{F(b)} b^n$$

(5) If $R(n) = \sin \alpha n$

then $PI = \frac{1}{F(E)} [\sin \alpha n]$

$$= \frac{1}{F(E)} \left[\frac{e^{i\alpha n} - e^{-\alpha n}}{2i} \right]$$

where $R(n) = \cos \alpha n$

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then $PI = \frac{1}{F(E)} \cos \alpha n$

$$= \frac{1}{F(E)} \left[\frac{e^{i\alpha n} + e^{-i\alpha n}}{2}\right]$$

Ques. Find GS of the RR $a_{n+2} - 2a_{n+1} + a_n = n^2 2^n$

Sol^n: $(E^2 - 2E + 1) a_n = 2^n \cdot n^2$

AE. $x^2 - 2x + 1 = 0$

$x = 1, 1$

$C.F = (c_1 + c_2 n) \cdot 1$

$PI = \frac{1}{(E-1)^2} 2^n n^2 = 2^n \frac{1}{(2E-1)^2} n^2$

Replace E by $1+\Delta$

$$2^n \frac{1}{(2(1+\Delta)-1)^2} n^2 = 2^n \frac{1}{(1+2\Delta)^2} n^2$$

$[1 + (4\Delta^2 + 4\Delta)]^{-1}$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots$

$PI = 2^n [1 + (4\Delta^2 + 4\Delta)]^{-1} n^2$

$= 2^n [1 - (4\Delta^2 + 4\Delta) + (4\Delta^2 + 4\Delta)^2 - \cdots] n^2$

$= 2^n [n^2 - (4\Delta^2 + 4\Delta) \cdot n^2 + (16\Delta^4 + 16\Delta^2 + 32\Delta^3) n^2 - \cdots]$

$\Delta^2 n^{(2)} = \Delta(\Delta n^{(2)})$
$\Delta^3 n^{(2)} = 0$

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$= 2^n [n^2 - 4\Delta^2 n^2 + 4\Delta n^2 + 16\Delta^4 n^2 + 16\Delta^2 n^2 + 32\Delta^3 n^2 + ---]$

$= 2^n [n^2 - 4\Delta^2(n^{(2)} + n^{(1)}) + 4\Delta(n^{(2)} + n^{(1)}) + 16\Delta^4(n^{(2)} + n^{(1)}) + 16\Delta^2(n^{(2)} + n^{(1)}) + 32\Delta^3(n^{(2)} + n^{(1)}) ----]$

$= 2^n [n^2 - 4(2+0) - 4(2n+1) + 16 \cdot 0 + 16 \cdot (2+0) + 32(0+0) + ---]$

$= 2^n [n^2 - 8 - 8n - 4 + 32]$

$= 2^n (n^2 - 8n + 20]$

$\underline{GS} \Rightarrow \quad a_n = (C_1 + C_2 n) + 2^n (n^2 - 8n + 20)$

Ques. Find GS of RR: $\frac{1}{E^2 - 5E + 4} e^{4n}$

$\Rightarrow$ $PI = \frac{1}{E^2 - 5E + 4} e^{4n}$

Put $e^4 = b \rightarrow$ const.

$= \frac{1}{E^2 - 5E + 4} b^n$

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$$= \frac{1}{b^2 - 5b + 4} b^n$$

$$= \frac{1}{e^8 - 5e^4 + 4} e^{4n}$$

Ques. Find GS of RR: $a_{n+2} + a_n = \sin 2n$

Sol^n. $(E^2+1) a_n = \sin 2n$

AE $x^2 + 1 = 0$

$x = \pm i$, $\rho = 1$, $\theta = \frac{\pi}{2}$

$$CF = \left(c_1 \cos n\frac{\pi}{2} + c_2 \sin n\frac{\pi}{2}\right)$$

$$PI = \frac{1}{(E^2+1)} \sin 2n$$

$$= \frac{1}{(E^2+1)} \left[\frac{e^{2in} - e^{-2in}}{2i}\right]$$

Put $e^{2i} = b$

$$= \frac{1}{2i} \left[\frac{b^n - (b^{-1})^n}{E^2+1}\right]$$

$$= \frac{1}{2i} \left[\frac{b^n}{E^2+1} - \frac{(b^{-1})^n}{E^2+1}\right]$$

$$= \frac{1}{2i} \left[\frac{b^n}{b^2+1} - \frac{(b^{-1})^n}{(b^{-1})^2+1}\right]$$

$b^{-1} = \frac{1}{b} = c$, constant

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$$= \frac{1}{2i}\left[\frac{e^{2in}}{e^{4i}+1} - \frac{e^{-2in}}{e^{-4i}+1}\right]$$

(6) If $R(n) = B\,a^n$
and $F(a) \neq 0$

then $PI = \frac{1}{F(a)} B\,a^n$

If $F(a) = 0$ and if a repeats r times
then let $F(E) = (E-a)^r\, g(E)$

$$PI = {}^nC_r \cdot \frac{B\,a^{n-r}}{g(E)}$$

Ques. $a_n - 3a_{n-1} + 2a_{n-2} = 2^n$

$\Rightarrow (E^2 - 3E + 2)\,a_n = 2^{n+2} = 2^n \cdot 4$

AE. $x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1, 2$

$CF = C_1 + C_2\, n\, (2)^n$

$AE = \frac{1}{(E-1)(E-2)}\, 2^n \cdot 4$

Case of failure

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$$= {}^nC_1 \frac{4 \cdot 2^{n-1}}{(E-1)} = 2 \cdot {}^nC_1 \frac{2^n}{E-1}$$

$$= 2n \cdot 2^n$$

$$= n\, 2^{n+1}.$$

~~Unit 3~~

Generating Function

Defn :- The generating function for the sequence

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \dots, a_n, \dots\} \text{ is}$$

defined and denoted by the series.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \quad , x \in [a, b] \quad — (*)$$

$$= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \quad — (1)$$

Ques. Find the generating function of $G(x) = \frac{1}{1-x}$

$\Rightarrow$ $[1-x]^{-1} = 1 + x + x^2 + x^3 + \dots + x^n + \dots$ — (2)

On comparing with (1) $\Rightarrow a_n = 1$

Comparing the coefficients of $1, x, x^2, x^3, \dots, x^n,$

$a_0 = 1, \; a_1 = 1, \; a_2 = 1, \dots \; a_n = 1,$

then $G(x) = \sum_{n=0}^{\infty} 1 \cdot x^n$

NOTE :- $(1+x)^{+n} = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 +$

$\dots + \frac{n(n-1)(n-2) \dots (n-(n-1))}{n!} x^n$

$+ \dots$

$$(1-x)^{+n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$\cdots + \frac{n(n-1)(n-2)\cdots(n-(n-1))}{n!}x^n + \cdots$$

Q. $G(x) = \frac{1}{(1-x)^2}$, Find generating function.

$$= G(x) = [1-x]^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$

$$G(x) = \sum_{n=0}^{\infty} (n+1)x^n$$

NOTE: (1). $G(x) = \frac{1}{(1-kx)}$

then G.F., $G(x) = \sum_{n=0}^{\infty} k^n x^n$

(2). $G(x) = \frac{1}{(1-kx)^2}$

then G.F., $G(x) = \sum_{n=0}^{\infty} k^n (n+1) x^n$

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- $$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$$G(x) - a_0 = a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$$G(x) - a_0 = x(a_1 + a_2 x + a_3 x^2 + \dots + a_n x^{n-1} + \dots a_{n+1} x^n + \dots)$$

$$\frac{G(x) - a_0}{x} = a_1 + a_2 x + a_3 x^2 + \dots + a_n x^{n-1} + a_{n+1} x^n + \dots \quad (3)$$

$$\frac{G(x) - a_0}{x} = \sum_{n=0}^{\infty} a_{n+1} x^n \rightarrow$$ Generating funcn of the sequence $\{a_{n+1}\}_{n=0}^{\infty}$

$$\frac{G(x) - a_0}{x} - a_1 = a_2 x + a_3 x^2 + a_4 x^3 + \dots + a_{n+1} x^n + a_{n+2} x^{n+1} + \dots$$

$$\frac{G(x) - a_0 - a_1 x}{x} = x(a_2 + a_3 x + a_4 x^2 + \dots + a_{n+1} x^{n-1} + a_{n+2} x^n + \dots)$$

$$\frac{G(x) - a_0 - a_1 x}{x^2} = a_2 + a_3 x + a_4 x^2 + \dots + a_{n+1} x^{n-1} + a_{n+2} x^n + \dots$$

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$$\frac{G(x)-a_0-a_1x}{x^2} = \sum_{n=0}^{\infty} a_{n+2}x^n \rightarrow \text{Generating func}^n$$

of the sequence $\{a_{n+2}\}_{n=0}^{\infty}$

$$\frac{G(x)-a_0-a_1x-a_2x^2-\dots-a_{k-1}x^{k-1}}{x^k} = \sum_{n=0}^{\infty} a_{n+k}x^n$$

$\downarrow$

Generating func^n of the sequence $\{a_{n+k}\}_{k=0}^{\infty}$

$$= a_k + a_{k+1}x + a_{k+2}x^2 + \dots$$

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- ## Procedure for solving Recurrance Relations

① Consider the RR

$$C_0 a_{n+k} + C_1 a_{n+k-1} + \cdots + C_n a_n = f(n)$$

where $C_0, C_1, C_2, \ldots, C_n$ are constants.

② Multiply on both sides of this relation by x^n and take summation from $n=0$ to ∞. Thus, we have

$$C_0 \sum_{n=0}^{\infty} a_{n+k} x^n + C_1 \sum_{n=0}^{\infty} a_{n+k-1} x^n + \cdots$$

$$\cdots + C_n \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} f(n) x^n .$$

③ Use generating func^n $G(x)$ as discussed above for all summation on LHS.

④ Collect the coefficents of $G(x)$, $a_0, a_1, a_2, \ldots,$ from LHS.

⑤ Simplify $\sum_{n=0}^{\infty} f(n) x^n$

⑥ Express the whole eq^n for $G(x)$ and then compare the coefficients of x^n on both sides to determine a_n which is the required sol^n of the RR.

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Ques. Find the GS of the RR using GF

$$a_n - 4a_{n-1} + 4a_{n-2} = 0, \quad n \geq 2$$

Sol^n. $a_n - 4a_{n-1} + 4a_{n-2} = 0$ —— (1)

Multiply x^n on both sides in eq^n (1) & take summation from $n=2$ to ∞.

$$\sum_{n=2}^{\infty} a_n x^n - 4\sum_{n=2}^{\infty} a_{n-1} x^n + 4\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow (a_2 x^2 + a_3 x^3 + \dots + a_{n+2} x^n + \dots) - 4(a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots)$$
$$+ 4(a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots) = 0$$

$$\Rightarrow G(x) - a_0 - a_1 x - 4x[G(x) - a_0] + 4x^2 G(x) = 0$$

Combine the coefficients of $G(x)$, $a_0(x)$, $a_1(x)$ we have,

$$G(x)[1 - 4x + 4x^2] + a_0(-1 + 4x) + a_1(-x) = 0$$

[Put $a_0 = 1$ & $a_1 = 2$]

$$G(x)[1 - 4x + 4x^2] + (-1 + 4x) - 2x = 0$$

$$G(x)[(1-2x)^2] + (2x - 1) = 0$$

$$G(x) = \frac{(1-2x)}{(1-2x)^2} = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n.$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$a_n = 2^n \quad \text{Ans.}$$

Ques. Find the GS of RR using GF

$$a_n - 5a_{n-1} + 6a_{n-2} = 3^n, \quad n \geq 2, \quad a_0 = 0, a_1 = 2$$

Solⁿ. $a_n - 5a_{n-1} + 6a_{n-2} = 3^n$

Multiply on both side x^n and take summation from $n=2$ to ∞.

$$= \sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1} x^n + 6\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2} 3^n x^n$$

$$= (a_2x^2 + a_3x^3 + \dots + a_n x^n + \dots) - 5(a_1x^2 + a_2x^3 + a_3x^4 \dots)$$
$$+ 6(a_0x^2 + a_1x^3 + a_2x^4 + \dots) = 3^2 x^2 \sum_{n=2}^{\infty} 3^{n-2} x^{n-2}$$

$$= G(x) - a_0 - a_1x - 5x[G(x) - a_0] + 6x^2 G(x) = 9x^2 \sum_{n\geq 2}^{\infty} 3^{n-2} x^{n-2}$$

Combine the coefficients of $G(x)$, $a_0(x)$, $a_1(x)$, we have.

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$$= G(x)[1-5x+6x^2] + a_0(-1+5x) + a_1(-x) = 9x^2 \sum_{k=0}^{\infty} 3^k x^k$$

(Put $a_0 = 0$ & $a_1 = 2$) $\qquad$ ($k = n-2$)

$$= G(x)[1-5x+6x^2] - 2x = 9x^2 \frac{1}{1-3x}$$

$$= G(x)[(1-2x)(1-3x)] = \frac{9x^2}{1-3x} + 2x$$

$$= G(x) = \frac{3x^2+2x}{(1-3x)^2(1-2x)} = \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{(1-3x)^2}$$

(Partial Fraction)

$$= 3x^2 + 2x = A(1-3x)^2 + B(1-3x)(1-2x) + C(1-2x)$$

Equating coefficients of x^2, x and 1 we have

$$3 = 9A + 6B, \quad 2 = -6A - 5B - 2C, \quad 0 = A + B + C$$

After solving, we have $A = 7$, $B = -10$, $C = 3$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{7}{1-2x} - \frac{10}{1-3x} + \frac{3}{(1-3x)^2}$$

$$= 7\sum_{n=0}^{\infty} 2^n x^n - 10\sum_{n=0}^{\infty} 3^n x^n + 3\sum_{n=0}^{\infty} (n+1) 3^n x^n$$

$$a_n = 7 \cdot 2^n - 10(3)^n + 3(n+1)(3)^n$$