Unit ⇒ 2

(Linear Differential Equations - 1st type)

• **Differential Equations:-**

A differential equation can be defined ~~...~~ as an equation containing derivatives of various order and the derivatives.

• **Ordinary Differential Equation (ODE):-**

A differential equation which involve one independent variable is called ordinary differential equation.

$x \rightarrow$ independent variable
$y \rightarrow$ dependent variable.
$\frac{dy}{dx} \rightarrow$ rate of change.

$$y = y(x)$$

Ex (1) $\frac{dy}{dx} + 4y = 0$

$$y' = \frac{dy}{dx}$$

$$y'' = \frac{d^2y}{dx^2}$$

$$\vdots$$

$$y^n = \frac{d^ny}{dx^n}$$

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Order :- The order of a DE is the order of highest derivative occuring in the equation.

Ex. ① $y'' + (y')^2 + 4y = 0$ Order = 2
Degree = 1

② $x^2 y'' + x(y')^4 + 3y^2 = 0$ order = 2
Degree = 1

③ $(y''')^2 + 2y' + 4y = 3x$ Order = 3
Degree = 2

Degree :- The degree of a DE is the degree (or power) of the highest order occuring in the equation after the equation has been made free of radicals and fractions in its derivative.

④ $1 + (y'')^2 = (y')^4$. Degree = 2

⑤ $\left(1 + (y'')^2\right)^{\frac{1}{3}} = (y')^2$

$1 + (y'')^2 = (y')^6$ Degree = 2

⑥ $1 + \frac{y}{(y'')^2} = 2x$

$(y'')^2 + y = 2x(y'')^2$ Degree = 2

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- ## Linear Differential Equation (LDE) :-

Defⁿ :- A differential equation is linear when the dependent variable and its derivative occur only in the first degree and no products of the dependent variable and its derivative or of various order derivative occur.

Ex.
1. $(y'') + 2y' + 3yx = \sin x$ — Linear
2. $x^2 y'' + xy' + 2y = 3x$ — Linear
3. $(y'')^2 + 2y' + \sin x \, y = 0$ — Non linear
4. $yy' + 2y = \cos x$ — Non linear
5. $\cos y \, y' + 2y = 0$ — Non linear
6. $\sin x \, y' + y = 2x$ — linear

★ Solution ← $y = f(x)$ → explicit form

$$y' + y = x$$
$$\downarrow$$
$$f(y, y', x) = 0$$

$\boxed{y^2 = xy + x + 1}$ → implicit form

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The form of linear differential equation:-

$a_0(x)y^n + a_1(x)y^{n-1} + \dots + a_{n-1}(x)y' + a_n(x)y = r(x)$ — ⊛

↳ $n=2$

2nd order LDE

$a_0(x)y^2 + a_1(x)y + a_2(x)y = r(x)$

$a_0(x), a_1(x), \dots a_n(x)$ & $r(x)$ are called coefficients of LDE

When $n=1$, 1st order LDE

$a_0(x)y' + a_1(x)y = r(x)$

1. When $a_0(x), a_1(x), \dots, a_n(x)$ are constants then eqn ⊛ is called LDE with constant coefficients.

Ex. ① $y''' + 2y' + y = 3x$. ② $y''' + y'' + 2y' + 4y = c$

2. When $a_0(x), a_1(x), a_2(x) \dots, a_n(x)$ are variables then eqn ⊛ is called LDE with variable coefficients.

Ex. • $x^2y'' + xy' + 2y = 0$
• $(x^2-1)y'' + x^2y' + xy = 2x$

Theorem :- If the function $a_0(x)$, $a_1(x)$, $a_2(x)$, ----, $a_n(x)$ and $r(x)$ are continous over the interval I and $a_0(x) \neq 0$ on interval I, then there exists a unique solution to the initial value problem

$$a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x)y = r(x)$$

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2, \ldots,\ y^n(x_0) = y_n.$$

→ $y'' + y' + 2y = 0$ on $[0, 1]$

$\boxed{y(0) = 0,\ y(1) = 1} \Rightarrow$ Boundary condⁿ.

$y'(0) = 0,\ y'(1) = 0$

1) solⁿ exist
2) Infinite solⁿ
3) no solⁿ
4) unique solⁿ ✓

normal interval on $(-\infty, \infty)$

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$(x^2-10)y'' + y' + 2y' = 2x$ $\quad I = (-\infty$

$x^2 - 10 = 0$

$x^2 = 10 = \sqrt{10}$

$x = \pm\sqrt{10}$ $\quad (-\infty, \infty) - \{\pm\sqrt{10}\}$

$(-\infty, -\sqrt{10}) \cup (-\sqrt{10}, \sqrt{10}) \cup (\sqrt{10}, \infty)$

Let $y_1(x), y_2(x), y_3(x), \text{———}, y_n(x)$ are function
then LC of these f^n are given by

$$c_1y_1 + c_2y_2 + c_3y_3 + \text{———} + c_ny_n$$

where $c_1, c_2, c_3, \text{———} c_n$ are arbitrary const.

Linear Independence (LI)

$$c_1y_1 + c_2y_2 + c_3y_3 + \text{———} + c_ny_n = 0$$

after comparing both sides, we have

$$c_1 = 0 = c_2 = c_3 = \text{———} = c_n$$

LI

Ex: $y_1 = 1, \quad y_2 = 2-x^2, \quad y_3 = 3+x$

$$c_1y_1 + c_2y_2 + c_3y_3 = 0$$

$$c_1 \cdot 1 + c_2 \cdot (2-x^2) + c_3(3+x) = 0$$

$$(c_1 + 2c_2 + 3c_3) + (3c_3) \cdot x - 2c_2x^2 = 0$$

compare both sides,

$$-2c_2 = 0 \Rightarrow c_2 = 0$$

$$c_3 = 0$$

$$c_1 = 0$$

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Linear. Dependence (LD).

$c_1 y_1 + c_2 y_2 + c_3 y_3 + \text{———} + c_n y_n = 0$

There exist $c_i \neq 0$ $\forall$, $i = 1, 2, \text{—} n$

$a_0(x) y^n + a_1(x) y^{n-1} + \text{———} + a_n(x) \cdot y$

• **Wronskian** :- Let $f_1(x), f_2(x), \dots, f_n(x)$ are the functions. Then the wronskian of these function is given by :

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

1) If $W(x) \neq 0$ for all $x \in I$ then $f_1(x), f_2(x), \dots, f_n(x)$ are linearly independent.

2) There exists $x \in I$ such that $W(x) = 0$ then $f_1(x), f_2(x), \dots f_n(x)$ are linearly dependent

Ex $f_1(x) = 1$, $f_2(x) = 1 - x$, $f_3(x) = x^2 - x$ on $I = (-\infty, \infty)$

$$W(f_1, f_2, f_3) = \begin{vmatrix} 1 & 1-x & x^2-x \\ 0 & -1 & 2x-1 \\ 0 & 0 & 2 \end{vmatrix} = -2 \neq 0$$

$\therefore f_1(x), f_2(x), f_3(x)$ are L.I.

Ex. $f_1(x) = 1$, $f_2(x) = \sin x$, $f_3(x) = \cos x$ on $I = (-\infty, \infty)$

$$\Rightarrow W(f_1, f_2, f_3) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -1 \neq 0$$

$\therefore f_1(x), f_2(x), f_3(x)$ are LI on $(-\infty, \infty)$.

* Method of solution of LDE :-

1. Differential operator (D)

$$D = \frac{d}{dx}$$

$$D(f(x)) = \frac{df(x)}{dx}, \quad D(Df(x)) = D\left(\frac{df(x)}{dx}\right)$$

$$= \frac{d^2 f(x)}{dx^2}$$

$$= \cdots D^n f(x) = \frac{d^n f(x)}{dx^{n-1}}$$

The general form of LDE:-

$$a_0(x) y^n + a_1(x) y^{n-1} + \cdots + a_n(x) y = r(x)$$

$$[a_0(x) D^n y + a_1(x) D^{n-1} y + \cdots + a_n(x)] y = r(x) \quad \text{--- (1)}$$

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Method of solution of LDE:-

Put $y = e^x$, $y' = e^x$, ..., $y^n = e^x$ in eq^n (1), we have.

$a_0(x) D^n + a_1(x) D^{n-1} + \cdots + a_n(x) = 0$ — (2)

Put $D = m$, $D^2 = m^2$, ... $D^n = m^n$ in eq^n (2), we have

$a_0(x) m^n + a_1(x) m^{n-1} + \cdots + a_n(x) = 0$

$\hookrightarrow$ This is called auxilliary eq^n.

$m = m_1, m_2, \ldots, m_m$

Put $n = 2$ in eq^n (2),

$a_0(x) D^2 + a_1(x) D + a_2(x) = 0$

$\Rightarrow a_0(x) m^2 + a_1(x) m + a_2(x) = 0$

$m = m_1, m_2$

1.) Real and distinct, $m = m_1, m_2$

$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

where c_1 & c_2 are arbitrary constant.

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2) Real and equal : $m = m_1, m_1$

$$y(x) = (c_1 + c_2 x) e^{m_1 x}$$

3) Complex conjugate roots:

$$m = m_1 + i m_2$$

$$y(x) = (c_1 \cos m_2 x + c_2 \sin m_2 x) e^{m_1 x}$$

Q Find the solution of the following differential equation:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \qquad y(0) = 0, \quad y'(0) = 1$$

$$\Rightarrow (D^2 + 5D + 6)y = 0$$

$$m^2 + 5m + 6 = 0$$

$$m = -2, -3$$

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} \quad \text{—— (*)}$$

$$0 = y(0) = c_1 + c_2 \quad \text{—— (1)}$$

$$y(x) = -2c_1 e^{-2x} - 3c_2 e^{-3x}$$

$$y'(0) = 1 = -2c_1 - 3c_2 \quad \text{—— (2)}$$

From (1) & (2),

$$c_1 = 1, \; c_2 = -1$$

$$y(x) = e^{-2x} - e^{-3x}$$

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Q. $4y'' + 4y' + y = 0$

$\Rightarrow (4D^2 + 4D + 1)y = 0$

Auxilliary Eqn:- $4m^2 + 4m + 1 = 0$

$(2m+1)^2 = 0$

$m = -\frac{1}{2}, -\frac{1}{2}$

$y(x) = (c_1 + c_2 x)e^{-\frac{x}{2}}$

Q. $y'' + 4y' + 13y = 0$, $y(0) = 0$, $y'(0) = 1$

$(D^2 + 4D + 13)y = 0$

A.E. $\Rightarrow$ $m^2 + 4m + 13 = 0$

$m = -2 \pm 3i$

$y(x) = [c_1 \cos 3x + c_2 \sin 3x]e^{-2x}$ — ⊛

$y(0) = 0 = [c_1 + 0]e^0$

$c_1 = 0$

$y(x) = c_2 \sin 3x \, e^{-2x}$

$y'(x) = c_2[\cos 3x \, 3e^{-2x} - 2\sin 3x \, e^{-2x}]$

$y'(0) = 1$

$c_2(3) = 1$

$\boxed{c_2 = \frac{1}{3}}$

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Put c_1 and c_2 in eq^n (*), we have

$$y(x) = \frac{1}{3}\sin 3x\, e^{-2x}$$

* Higher order LDE with constant coefficients

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_n y = 0$$

$$[a_0 D^n + a_1 D^{n-1} + \cdots + a_n] y = 0$$

$$AE \Rightarrow a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

1) Roots: Real & Distinct :-

⇒ Let $m = m_1, m_2, \ldots m_n$ are real and distinct roots.

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

where $c_1, c_2, \ldots c_n$ are arbitrary constants

2.) Real and Equal roots :-

$$m = m_1 = m_2 \cdots = m_n$$

$$y(x) = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \cdots + c_n x^{n-1}) e^{m_1 x}$$

3) Complex conjugate roots :-

$$m = m_1 + i m_2, m_3 + i m_4, m_5 + i m_6, \ldots m_{n-1} + i m_n$$

$$y(x) = (C_1 \cos m_2 x + C_2 \sin m_2 x) e^{m_1 x} +$$
$$(C_3 \cos m_4 x + C_4 \sin m_4 x) e^{m_3 x} + \cdots\cdots +$$
$$(C_{n-1} \cos m_n x + C_n \sin m_n x) e^{m_{n-1} x}$$

Q. Find the solution of the following differential equations:

(1) $y''' - 2y'' - 5y' + 6y = 0$

$$(D^3 - 2D^2 - 5D + 6) y = 0$$

AE $\Rightarrow m^3 - 2m^2 - 5m + 6 = 0$

$$(m-1)(m^2-4) = 0$$
$$m = 1, -2, 3$$

$$y(x) = C_1 e^{x} + C_2 e^{-2x} + C_3 e^{+3x}$$

where C_1, C_2 and C_3 are arbitrary constants.

(2) $y^{IV} - 5y'' + 4y = 0$

$$(D^4 - 5D^2 + 4) y = 0$$

AE $\Rightarrow m^4 - 5m^2 + 4 = 0$

$$m^4 - 4m^2 - m^2 + 4 = 0$$
$$m^2(m^2-4) - 1(m^2-4) = 0$$
$$(m^2-1)(m^2-4) = 0$$
$$m = \pm 1, \pm 2$$

$y(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + c_4 e^{2x}$

where c_1, c_2, c_3 & c_4 are arbitrary consts.

(3) $\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = 0$

$(D^4 + 4D^2)y = 0$

AE $\cdot\ m^4 + 4m^2 = 0$

$m^2(m^2+4) = 0$

$m = 0, \pm 2i$

$y(x) = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x)$

where c_1, c_2, c_3 & c_4 are arbitrary constants

* LDE with Variable coefficients :-

$L(y) = a_0(x) y^n + a_1(x) y^{n-1} + \dots + a_n(x) y = 0$

$a_0(x) x^n y^n + a_1(x) x^{n-1} y^{n-1} + \dots + a_n(x) y = 0$

Put $y = x^m$, m is a positive integer.

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$\vdots$

$y^n = m(m-1)(m-2) \dots (m-(n-1)) x^{m-n}$

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$$x\mathcal{D} = m$$
$$x^2\mathcal{D}^2 = m(m-1)$$
$$x^3\mathcal{D}^3 = m(m-1)(m-2)$$
$$\vdots$$
$$x^n\mathcal{D}^n = m(m-1)(m-2)\dots(m-(n-1))$$

Cauchy Euler Equation $\left[a_0(x)x^n\mathcal{D}^n + a_1(x)x^{n-1}\mathcal{D}^{n-1} + \dots + a_n(x)\right]y = 0$

For second order LDE with variable coefficient

$$\left[a_0(x)\,x^2y'' + a_1(x)\,xy' + a_2(x)\,y\right] = 0 \quad \text{--- (1)}$$

$$\left[a_0(x)\,x^2\mathcal{D}^2 + a_1(x)\,x\mathcal{D} + a_2(x)\right]y = 0$$

A.E. :- $\left[a_0(x)\,m(m-1) + a_1(x)\cdot m + a_2(x)\right] = 0 \quad \text{--- (2)}$

Let $m = m_1, m_2$ are the roots of eq^n (1),

(1) **Real and distinct** :-

$$m = m_1, m_2 \quad (m_1 \neq m_2)$$

$$y(x) = c_1x^{m_1} + c_2x^{m_2}$$

(2) **Real and equal Roots** :- $m = m_1, m_1,$

$$y(x) = (c_1 + c_2\ln x)\,x^{m_1}$$

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③ Complex conjugate roots:-

$$m = m_1 + i m_2$$

$$y(x) = [c_1 \cos m_2 \ln x + c_2 \sin m_2 \ln x] x^{m_1}$$

Q. Find the solution of the following differential equations:

1. $x^2 y'' + 2xy' - 2y = 0$

$\Rightarrow [x^2 D^2 + 2xD - 2] y = 0$

A.E. $\Rightarrow [m(m-1) + 2m - 2] = 0$

$m^2 - m + 2m - 2 = 0$

$m^2 + m - 1 = 0$

$m = 1, -2$

$y(x) = c_1 x + c_2 x^{-2}$

2. $2x^2 y'' + xy' - 6y = 0$

$[2x^2 D^2 + xD - 6] y = 0$

$2m(m-1) + m - 6 = 0$

$2m^2 - 2m + m - 6 = 0$

$2m^2 - m - 6 = 0$

$m = \frac{1 \pm \sqrt{1+48}}{4} = \frac{1 \pm 7}{4} = \frac{8}{4}, \frac{-6}{4} = 2, -\frac{3}{2}$

$y(x) = c_1 x^2 + c_2 x^{-\frac{3}{2}}$

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Q. $4x^2y'' + y = 0$

$\Rightarrow$ $4x^2D^2 + y = 0$

$4m(m-1) + 1 = 0$

$(2m-1)^2 = 0$

$m = \frac{1}{2}, \frac{1}{2}$

$y(x) = (c_1 + c_2 \ln x)x^{\frac{1}{2}}$

Q. $4x^2y'' + 8xy' + 17y = 0$

$= (4x^2D^2 + 8xD + 17)y = 0$

$= 4m(m-1) + 8m + 17 = 0$

$= 4m^2 - 4m + 8m + 17 = 0$

$= 4m^2 + 4m + 17 = 0$

$m = \frac{-4 \pm \sqrt{16 - 16\cdot17}}{8} = \frac{-4 \pm 4\sqrt{-16}}{8}$

$= \frac{-4 \pm 16i}{8} = \frac{-1 \pm 4i}{2}$

$m = \frac{-1+4i}{2}, \frac{-1-4i}{2}$

$y(x) = (c_1 \cos 2\ln x + c_2 \sin 2\ln x)x^{-\frac{1}{2}}$

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Q. Find the general solution of the following differential equation.

1. $y'' + y' - 2y = 0$, $y(0) = 0$, $y'(0) = 2$

2. $25y'' - 20y' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$

3. $(D^2 + 9D)y = 0$

4. $y^{iv} + 50y'' + 625y = 0$

5. $y^{iv} + 2y'' + y = 0$

6. $x^2y'' + xy' - y = 0$

7. $x^2y'' - xy' + y = 0$

(1) $y'' + y' - 2y = 0$

$\Rightarrow (D^2 + D - 2)y = 0$ $\qquad m^2 + m - 2 = 0$

$m = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$

$\underline{m = 1, -2}$

$y(x) = c_1 e^x + c_2 e^{-2x}$

$c_1 = -c_2$

$y(0) = c_1 + c_2 = 0$

$y_1'(0) = c_1 + c_2(-2) = 2$

$-c_2 - 2c_2 = 2 \Rightarrow -3c_2 = 2$

$\boxed{c_1 = \frac{2}{3}}$ $\qquad \boxed{c_2 = -\frac{2}{3}}$

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$$y(x) = c_1 e^{x} + c_2 e^{-2x}$$

$$y = \frac{2}{3} e^{x} - \frac{2}{3} e^{-2x}$$

(2) $25y'' - 20y' + 4y = 0$

$(25D^2 - 20D + 4)y = 0$

AE $\Rightarrow 25m^2 - 20m + 4 = 0$

$$m = \frac{20 \pm \sqrt{400 - 4 \cdot 25 \cdot 4}}{50}$$

$$= \frac{20}{50} = \frac{2}{5} \qquad m = \frac{2}{5}, \frac{2}{5}$$

$$y(x) = (c_1 + c_2 x) e^{\frac{2}{5}x} = c_1 e^{\frac{2}{5}x} + c_2 x e^{\frac{2}{5}x}$$

$y(0) = 1 \qquad y'(0) = 2$

$(c_1) e^{0} = \boxed{c_1 = 1}$

$$\frac{2}{5} c_1 + c_2 \left(e^{\frac{2}{5}(0)} + \frac{2}{5}(0) \right) = 2$$

$$\frac{2}{5}(1) + c_2(1) = 2$$

$$c_2 = 2 - \frac{2}{5} = \frac{8}{5}$$

$$\boxed{c_2 = \frac{8}{5}}$$

$$y(x) = \left(1 + \frac{8}{5} x\right) e^{\frac{2}{5}x}$$

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(3) $(D^2+9D)y=0$

$m^2+9m=0$

$m(m+9)=0$

$m=0,-9$

$\cancel{y(x)=(C_1\cos m_2 x + C_2 \sin m_2 x)e^{m_1 x}}$

$y(x)=C_1e^{0x}+C_2e^{-9x}$

$y(x)=C_1+C_2e^{-9x}.$

(4) $y^{iv}+50y''+625y=0$

AE ⇒ $(m^4+50m^2+625)=0$

$m^4+25m^2+25m^2+625=0$

$m^2(m^2+25)+25(m^2+25)=0$

$(m^2+25)^2=0$

$\cancel{m=\pm5,\pm5}$ $m=\pm5i,\pm5i$

$\cancel{y=(C_1+C_2x)e^{-5x}+(C_3+C_4x)e^{5x}}$

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(5) $y^{iv} + 2y'' + y = 0$

$(m^4 + 2m^2 + 1) = 0$

$m^4 + m^2 + m^2 + 1 = 0$

$m^2(m^2+1) + 1(m^2+1) = 0$

$(m^2+1)^2 = 0$

$m = \pm i, \pm i$

(6) $x^2 y'' + xy' - y = 0$

$(x^2 D^2 + xD - 1)y = 0$

$m(m-1) + m - 1 = 0$

$m^2 - m + m - 1 = 0$

$m^2 - 1 = 0$

$m = 1, -1$

$y(x) = c_1 x + c_2 x^{-1}$

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★ Higher order LDE with variable coefficients (Cauchy - Euler DE)

$a_0(x)\, x^n y^n + a_1(x)\, x^{n-1} y^{n-1} + \dots\dots +$

$+ a_{n-1}(x)\, xy' + a_n(x) y = r(x)$, $a_0(x) \neq 0$

where $r(x) = 0$.

$a_0(x)\, x^n y^n + a_1(x)\, x^{n-1} y^{n-1} + \dots\dots + a_n(x) y = 0$ —— (1)

Put $xD = m$

$x^2 D^2 = m(m-1)$

$x^3 D^3 = m(m-1)(m-2)$

$\vdots$

$x^n D^n = m(m-1)(m-2) \dots\dots (m-(n-1))$

Auxiliary eq^n $\Rightarrow$ $a_0(x)\left[m(m-1)(m-2) \dots (m-(n-1))\right] +$

$+ a_1(x)\left[m(m-1)(m-2) \dots\dots (m-(n-2))\right]$

$+ \dots\dots\dots + a_n(x) = 0$

1. Real and distinct roots:-

$m = m_1, m_2, \dots\dots, m_n$ are distinct roots.

then the soln of eq^n (1), is given by:-

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3} + \cdots + c_n x^{m_n}$$

where $c_1, c_2, c_3, \cdots c_n$ are arbitrary constants.

2. **Real and Equal roots :-**

$m = m_1, m_1, \cdots, m_1$
then sol^n,

$$y(x) = \left[c_1 + c_2 \ln x + c_3 (\ln x)^2 + c_4 (\ln x)^3 + \cdots + c_n (\ln x)^{n-1}\right] x^{m_1}$$

3. **Complex roots:-**

$m = m_1 \pm i m_2, m_3 \pm i m_4, \cdots, m_{n-1} \pm i m_n$

are complex roots then,

$$y(x) = \left[c_1 \cos m_2 (\ln x) + c_2 \sin m_2 (\ln x)\right] x^{m_1} + \left[c_3 \cos m_4 (\ln x) + c_4 \sin m_4 (\ln x)\right] x^{m_3} + \cdots + x^{m_{n-1}} \left[c_{n-1} \cos m_n (\ln x) + c_n \sin m_n (\ln x)\right]$$

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Q. Find the general sol^n of the following differential eq^n :-

(1) $x^2y'' + 2xy' - 2y = 0$

$\Rightarrow [x^2D^2 + 2xD - 2]y = 0$

A.E. $\Rightarrow m(m-1) + 2m - 2 = 0$

$m^2 + m - 2 = 0$

$m = 1, -2$

$y(x) = c_1 x^1 + c_2 x^{-2} = c_1 x + \frac{c_2}{x^2}$

(2) $2x^2y'' + xy' - 6y = 0$

$\Rightarrow [2x^2D^2 + xD - 6] = 0$

A.E. $\Rightarrow 2m(m-1) + m - 6 = 0$

$2m^2 - \cancel{2}m + \cancel{2}m - 6 = 0$

$\cancel{2m^2 = 6}$ $\quad 2m^2 + m - 6 = 0$

$\cancel{m^2 = 3}$ $\quad 2m^2 + 4m - 3m - 6 = 0$

$\cancel{m = \pm\sqrt{3}}$ $\quad 2m(m+2) - 3(m+2) = 0$

$\cancel{y(x) = c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}}$ $\quad (2m-3)(m+2) = 0$

$m = \frac{3}{2}, -2$

$y(x) = c_1 x^{\frac{3}{2}} + c_2 x^{-2}$

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(3) $4x^2y'' + y = 0$

$\Rightarrow (4x^2D^2 + 1)y = 0$

AE $\Rightarrow 4m(m-1) + 1 = 0$

$4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0$

$m = \frac{1}{2}, \frac{1}{2}$

$y(x) = [C_1 + C_2(\ln x)]\, x^{\frac{1}{2}}$

(4) $4x^2y'' + 8xy' + 17y = 0$

$(4x^2D^2 + 8xD + 17)y = 0$

$4m(m-1) + 8m + 17 = 0$

$4m^2 - 4m + 8m + 17 = 0$

$4m^2 + 4m + 17 = 0$

$m = \frac{-4 \pm \sqrt{16 - 4.4.17}}{8} = \frac{-4 \pm 4\sqrt{-16}}{8} = \frac{-4 \pm 16i}{8}$

$= \frac{-1 \pm 4i}{2}$

$y(x) = [C_1 \cos 2 \ln x + C_2 \sin 2 \ln x]\, x^{-\frac{1}{2}}$

(5) $2x^2y'' + 3xy' - 3y = 0$

$\Rightarrow (2x^2D^2 + 3xD - 3)y = 0$

$AE \Rightarrow 2m(m-1) + 3m - 3 = 0$

$2m^2 + m - 3 = 0$

$2m^2 + 3m - 2m - 3 = 0$

$m(2m+3) - 1(2m+3) = 0$

$(m-1)(2m+3) = 0$

$m = 1, \frac{-3}{2}$

$y(x) = c_1 x^1 + c_2 x^{-\frac{3}{2}}$

(6) $x^3y''' + 5x^2y'' + 5xy' + y = 0$

$\Rightarrow (x^3D^3 + 5x^2D^2 + 5xD + 1)y = 0$

$AE \Rightarrow m(m-1)(m-2) + 5m(m-1) + 5m + 1 = 0$

$m^3 + 2m^2 + 2m + 1 = 0$

$(m+1)(m^2 + m + 1) = 0$

$m = -1$ | $m^2 + m + 1 = 0$

$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

$y(x) = c_1 x^{-1} + \left[c_2 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] x^{-\frac{1}{2}}$