Delhi Technological University

Department of Applied Physics

Lagrange Points

Mid-term Evaluation Project Report

(EP-205) Classical Mechanics

A Project by:-
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Her continued support & valuable criticism have been huge contributions towards the successful completion of this project.

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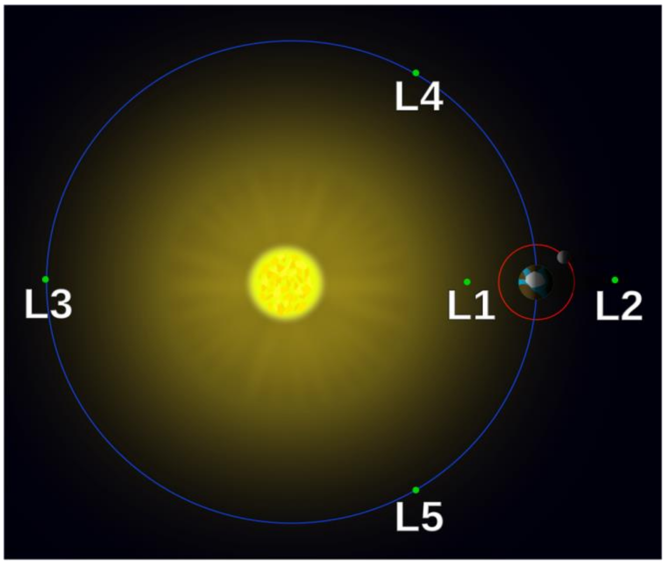
An Introduction to Lagrange Points

Lagrange Points are orbital points of stability formed in a system of two large co-orbiting bodies. At Lagrange points the gravitational forces of the two large bodies cancel out in such a way that a small object placed in orbit there retains just enough centripetal acceleration to move with them.

In general, there exist 5 Lagrange Points for any system of 2 large co-orbiting bodies in space. L_1, L_1 & L_3 correspond to the set of collinear points, all lying on the line through the centres of the two bodies. While Points L_4 and L_5 each act as the third vertex of an equilateral triangle formed with the centers of the two large bodies.



https://en.wikipedia.org/wiki/Lagrange_point#/media/File:Lagrange_points_simple.svg

The Lagrange points are named in honor of the Italian-French mathematician Josephy-Louis Lagrange who in 1772 published an "Essay on the three-body problem". In the first chapter he considered the general three-body problem. From that, in the second chapter, he demonstrated two special constant-pattern solutions, the collinear and the equilateral, for any three masses, with circular orbits.

Calculating the Lagrange Points

Let us begin by setting up the system relative to which Lagrange Points will be discussed:

We consider two masses in circular orbits that are rotating around their center of mass. The Lagrangian points are those points where a negligible mass would experience no force, as the forces of gravitational pull of the two bodies would effectively cancel each other out.

Let M_1 and M_2 be the masses of the two larger bodies. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors, where the origin is set at the center of mass. We thus have

$$M_1\mathbf{r}_1 + M_2\mathbf{r}_2 = 0\,.$$

We assume that the masses rotate in circular orbits. The angular velocity ω can be calculated using Kepler's law, or by considering the centripetal force and the gravitational force. In any case, one finds

$$\omega^2 = \frac{G(M_1+M_2)}{|\mathbf{r}_1-\mathbf{r}_2|^3} = \frac{GM_2}{|\mathbf{r}_1-\mathbf{r}_2|^2|\mathbf{r}_1|}\,,$$

Where G is the gravitational constant and the last term uses $M_1r_1 + M_2r_2 = 0$.

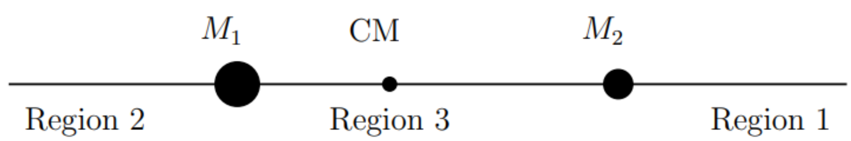
Apart from the choice of origin, we make another consideration that is: We consider a coordinate system that is co-rotating, thus one in which the masses are at fixed positions. Since this is a non-inertial frame, two additional forces show up: the 1 centrifugal force and the Coriolis force.

The forces $\boldsymbol{F_m}$ on a mass m at position r are now given as:-

$$\mathbf{F}_m = -\frac{GM_1m}{|\mathbf{r}-\mathbf{r}_1|^3}(\mathbf{r}-\mathbf{r}_1) - \frac{GM_2m}{|\mathbf{r}-\mathbf{r}_2|^3}(\mathbf{r}-\mathbf{r}_2) + m\omega^2\mathbf{r} - 2m\omega\times\dot{\mathbf{r}}\,, \qquad (1)$$

Where ω is the angular velocity vector; ω^2 is as above the square of its norm.

For stationary points the Coriolis force is irrelevant, as we want points with zero velocity $\dot{r}$. We first consider those points that lie on the line through the masses M_1 and M_2. We divide this line into three parts: Region 1 is to the right of M_2, region 3 is between M_1 and M_2 and region 2 is to the left of M_1 – see the figure below, where CM stands for center of mass. We will in general not be able to solve exactly for the positions where $Fm = 0$, but we will find good approximations if $M_1 >> M_2$.



The 1st Region and 2nd region have the same equation for $Fm = 0$, but with M_1 and M_2 swapped. However, if we assume $M_2 << M_1$ the approx. equations are different.

1st Region

forces are balanced at this point, that is $Fm = 0$. Balance of forces in the third region is determined by the following equation:

$$\frac{GM_1}{(x+r_1)^2} + \frac{GM_2}{(x-r_2)^2} = \frac{G(M_1+M_2)}{(r_1+r_2)^3}x\,.$$

This is a quintic equation, so solving it for general solutions manually on paper is a rigorous & lengthy calculation. However software's like **MATLAB** or **GNU Octave** can carry out the computation of a quintic equation quite easily using inbuilt functions such as "roots()" . Since this is region "2" according to the figure shown above, solving this equation for x would give us the Lagrange Point in space known as **"L_2"**. This equation will be used in the explanations offered in the "Simulation" section ahead.

However if one does not want make use of software's to ease the computation, this equation may also alternatively be solved through the following procedure:

Manual Procedure

We take new parameters $\alpha = \frac{M2}{M1}$ and $z = \frac{x}{r2}$ and obtain the following equation

$$(z-1)^2 + \alpha(z+\alpha)^2 = \frac{1}{(1+\alpha)^2} z(z-1)^2(z+\alpha)^2 \,. \qquad (2)$$

We now want to use perturbation theory. The perturbation parameter is α. We would like to find a perturbative expansion in α for a solution. The zeroth order problem is obtained by setting $\alpha = 0$. For $\alpha = 0$ eqn. (2) reduces to

$$(z-1)^2 = z^3(z-1)^2$$

Which is solved for by $z = 1$. Conventional perturbation theory would now proceed as: expand the original equation to 1st order α and consider a solution of the form $z = 1 + \lambda\alpha$ and put it into the equation, discard all higher order terms α^2 , α^3 and so on and then determine λ. This however would not be a successful approach as demonstrated below:

$$\frac{1}{(1+\alpha)^2} = 1 - 2\alpha + 3\alpha^2 - 4\alpha^3 - \ldots$$

We find that the function

$$P(z) = (z-1)^2 + \alpha(z+\alpha)^2 - \frac{1}{(1+\alpha)^2} z(z-1)^2(z+\alpha)^2$$

Is to first in α given by

$$P(z) = (z-1)^2(1-z^3) + \alpha z^2(1 + 2(z-1)^3) + O(\alpha^2) \,.$$

To find an approx. solution, Let us take $z = 1+(\alpha\lambda)$

$$0 = \alpha + O(\alpha^2) \,.$$

The symbol $O(\alpha^2)$ stands for terms that are at least quadratic in α. The variable λ is no longer present in the equation.

The reason for this bad behavior is that $z = 1$ is not an ordinary zero of $P(z)$, but a third order zero, indeed $P(z) = -(z - 1)^3 (1 + z + z^2) + O(\alpha)$. To study these kind of problems this would be the appropriate way to proceed:

we consider the following problem: Let $P(x(\epsilon);\epsilon)$ be a polynomial in x that is given as an expansion in ϵ:

$$P(x;\epsilon) = P_0(x) + \epsilon P_1(x) + \epsilon^2 P_2(x) + \dots$$

Where all the $P_i(x)$ are polynomials – which is not really a necessary requirement yet. We assume $P_0(0) = 0$ and want a perturbative expansion for this zero in ϵ. We thus want an expression $x(\epsilon) = \epsilon x_1 + \epsilon^2 x_2 + \dots$ such that $P(x(\epsilon);\epsilon) = 0$ for all orders in ϵ. Using the first order Taylor expansion $P_0(\epsilon x_1) = \epsilon x_1 P'(0)$, we find that the first order equation is

$$\epsilon x_1 P_0'(0) + \epsilon P_1(0) = 0\,.$$

Now we clearly see the structure; this equation can only be solved for x_1 if $x = 0$ is an ordinary zero of P_0. If it is a double zero, then $P'_0(0) = 0$ and one cannot solve for x_1.

Let us assume that $P_0(0) = P'_0(0) = P_0''(0)$ but $P_0'''(0) \neq 0$.
Then $P_0(x) = \frac{1}{6} P_0'''(0)x^3$ + higher order terms. This makes it clear that x^3 should be of order ϵ, and consequently we can expand in powers of $\epsilon^{1/3}$, thus $x = \epsilon^{1/3}a + \epsilon^{2/3}b + \dots$.

We thus consider the problem to find approximate solutions for the zero of $P(x;\epsilon) = x^3Q(x) + \epsilon R(x) + \dots$, where $Q(0) \neq 0$ and the ellipsis contains terms of higher order in ϵ. Putting in $x(\epsilon) = a\epsilon^{1/3} + b\epsilon^{2/3} + \dots$ one finds

$$P(x(\epsilon);\epsilon) = \epsilon(a^3Q(0) + R(0)) + \epsilon^{4/3}(3a^2bQ(0) + a^3Q'(0) + aR'(0)) + \dots\,.$$

Requiring $P(x(\epsilon);\epsilon) = 0$ to all orders in ϵ thus results in $a = -\sqrt[3]{\frac{R(0)}{Q(0)}}$. Now we return to our original problem and consider the transformation $z = 1 + y$, so that the zero lies at $y = 0$. We then find

$$P(y) = -y^3(3+3y+y^2) + \alpha(1+\alpha)^2(1+2y^3) + \alpha^2(1+y)(2-y^2)(2+6y+3y^2) + \ldots .$$

Identifying $Q(y) = -(3 + 3y + y^2)$ and $R(y) = (1 + y)^2(1 + 2y^2)$, we find $Q(0) = -3$ and $R(0) = 1$. Thus we have

$$y(\epsilon) = \epsilon^{1/3}\sqrt[3]{\frac{1}{3}} + \ldots = \sqrt[3]{\frac{\epsilon}{3}} + \ldots .$$

Retracing back the definitions we find

$$x = r_2\left(1 + \sqrt[3]{\frac{M_2}{3M_1}}\right).$$

This is widely known as Lagrange Point L_2.
To determine if the result we have obtained is in accordance with actual coordinates at which these points are present let us check using some experimental values:
For the system earth-sun we have:
M_1 = M of Sun $\approx 2 \times 10^{30}$ kg,
M2 = M of Earth $\approx 6 \times 10^{24}$ kg and
$r_2 \approx 150 \times 10^6$ km.
Thus $x \approx r_2 + 1.5 \cdot 10^6$ km (approx). That is, L_2 lies at 1.5 Million km from the earth, away from the sun.

2nd Region

To calculate the Lagrange point for region 2, let x be the distance between the negligible mass and the center of mass. Distance between CM and M_1 is r_1 & Distance between CM and M_2 is r_2. Balance of four forces in 3rd region is then given by the equation

$$\omega^2 x = \frac{GM_1}{(x-r_1)^2} + \frac{GM_2}{(x+r_2)^2}\,. \tag{3}$$

Again, a general solution is extremely difficult to find, so we assume $M_1 >> M_2$, and thus $r_1 << r_2$. We put $x = zr_2$, $M_2 = \epsilon M_1$ and thus $r_1 = \epsilon r_2$. Together with $\omega^2 = \frac{G(M1+M2)}{(r1+r2)^{\wedge}3}$ this turns eqn.(3) into

$$\frac{1}{(1+\epsilon)^2} z(z-\epsilon)^2(z+1)^2 = (z+1)^2 + \epsilon(z-\epsilon)^2\,. \tag{4}$$

For convenience we multiply through by $(1 + \epsilon)^2$. Then, to first order in ϵ eqn.(4) reduces to

$$0 = (z^3-1)(z+1)^2 - \epsilon(2z^4 + 4z^3 + 5z^2 + 4z + 2)\,. \tag{5}$$

The relevant solution for $\epsilon = 0$ is $z = 1$. Since $z^3 - 1 = (z - 1)(1 + z + z^2)$ we recast eqn.(5) in the form

$$0 = (z-1)Q(z) + \epsilon R(z)\,.$$

Now $z = 1$ is an ordinary zero and we thus can safely take $z(\epsilon) = 1 + \lambda\epsilon$ and find to first order $\lambda = \frac{R(1)}{Q(1)}$. In our case $Q(1) = 12$ and $R(1) = 17$. If we now retrace the definitions we find $x = r_2(1+ \frac{17}{12}\frac{M2}{M1})$; the distance between this point and the mass M_1 is thus

$$x - r_1 = r_2\Big(1 + \frac{5}{12}\frac{M_2}{M_1}\Big)$$

Meanwhile the distance between M_1 and M_2 is given as $(r_1 + r_2) = r_2(1+\frac{M2}{M1})$. This Lagrange point is known as L3 & is found in region 2. For the sun-earth system we find that L3 lies around 600 km outside the orbit of the earth.

3rd Region

Balance of forces is now given by

$$\frac{GM_1}{(x+r_1)^2} - \frac{GM_2}{(x-r_2)^2} = \frac{G(M_1+M_2)}{(r_1+r_2)^3}x\,. \qquad (6)$$

We put $r_1 + r_2 = a$, $z = \frac{x}{a}$, $s = \frac{r2}{r2+r1}$, $t = \frac{r1}{r1+r2} = (1 - s)$, so that $M_2 = t(M_1 + M_2)$ and $M_1 = s(M_1 + M_2)$. This reduces equation(6) to

$$s(z-s)^2 - t(z+t)^2 = z(z+t)^2(z-s)^2\,.$$

If M_1 is much greater than M_2 then s tends to 1 & t tends to 0. We thus put $s = (1 - \beta)$ & $t = \beta$ and find the following equation for z up to 1st order in β

$$0 = (z-1)^3(1+z+z^2) + \beta(4z^4 - 6z^3 + 4z^2 - 4z + 3)\,.$$

This eqn is solved by $z = 1+\lambda(\beta^{1/3})$ and we find $\lambda = -\sqrt[3]{\frac{1}{3}}$. Hence we have

$$x = a\left(1 - \sqrt[3]{\frac{M_2}{3M_1}}\right),$$

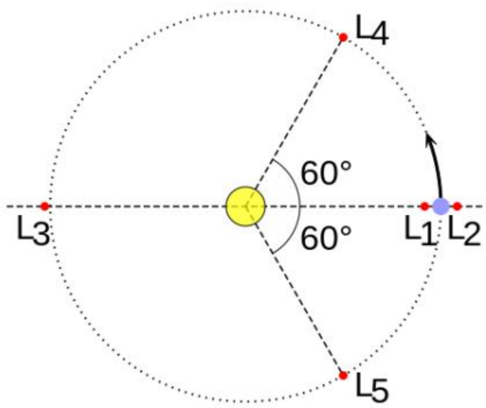
so that for the system sun-earth we find that this Lagrangian points lies approx. $1.5\text{x}10^6$ km from the earth. This Lagrange point is called L_1.

Lagrange Points L_4 & L_5

Lagrange Points L_4 and L_5 each act as the third vertex of an equilateral triangle formed with the centres of the two large bodies.

The reason these points are in balance is that, at L_4 and L_5, the distances to the two masses are equal.

Accordingly, the gravitational forces from the two massive bodies are in the same ratio as the masses of the two bodies, and so the resultant force acts through the barycentre (centre of mass of 2 or more bodies) of the system.



https://jfuchs.hotell.kau.se/kurs/amek/prst/15_lapo.pdf

As these points lie at the vertex of the equilateral triangle formed with centres of the two bodies it is relatively easy to calculate the location of these points.

Using the geometry of an equilateral triangle the locations of Lagrange Points L_4 & L_5 may be given as:-

$$L_4 = \left(\frac{R}{2}\left[\frac{M1-M2}{M1+M2}\right], \frac{\sqrt{3}}{2}R\right)$$

$$L_5 = \left(\frac{R}{2}\left[\frac{M1-M2}{M1+M2}\right], -\frac{\sqrt{3}}{2}R\right)$$

It is important to note that the $\left[\frac{M1-M2}{M1+M2}\right]$ terms are present in the x coordinates of these points to make up for the fact that the center of mass of the system won't lie exactly at the centre of the sun but in front of it by a factor of $\left[\frac{M1-M2}{M1+M2}\right]$ since the Earth also has a gravitational influence on the sun.

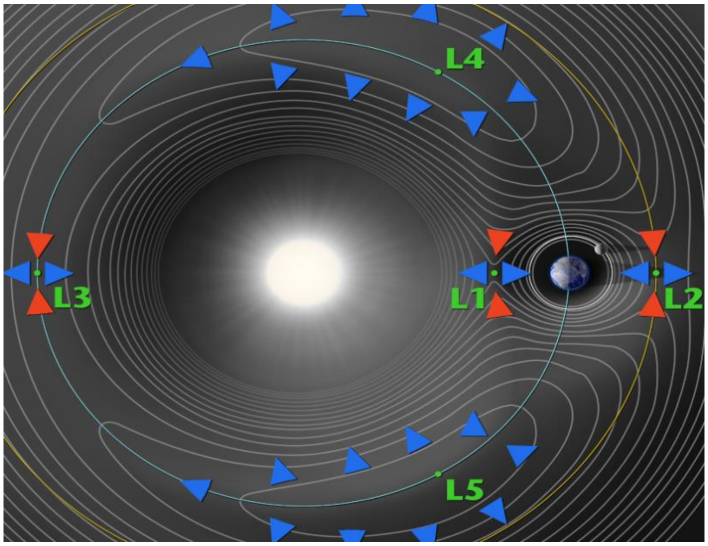
Stability of Lagrange Points

Now that we have obtained the Lagrange points for the restricted three body problem, let us briefly discuss their stability.

Of the five Lagrange points, three are unstable and two are stable. The unstable Lagrange points - labeled L_1, L_2 and L_3 - lie along the line connecting the two large masses. Meanwhile L_4 & L_5 are naturally stable points.

L_1, L_2 & L_3 are nominally unstable points due to being present on saddle points (i.e. points where the potential is curving up in one direction and down in the other) however when a satellite placed at these points begins to drift out of orbit the Coriolis force comes into play & slings it back into orbit about its original position.

Meanwhile L_4 & L_5 are naturally stable points and correspond to hilltops. (i.e. points where the potential is curving only outwards)



A figure illustrating the stability of Lagrange Points source: https://solarsystem.nasa.gov/resources/754/what-is-a-lagrange-point/

The figure shown above gives a clearer picture of the concept of saddle points & hilltops in a pseudo-potential between two large co-orbiting bodies.

The stability of the Lagrange Points is also demonstrated in the simulation aspect of this project by plotting contour lines of the surface potential similar to as seen above

Simulation

For the simulation aspect of this project I have developed a program in GNU Octave that computes Lagrange Points for a highly simplified system for which a few approximations have been made to reduce the complexity of calculating the location of these points.

This program is dependent on two function files, namely:-

(i) `lagrangePoints.m`
(ii) `SurfPotential.m`

These are discussed & illustrated below:-

Function File: lagrangePoints.m

Since computing the locations of Lagrange Points L_1, L_2 & L_3 require solving quintic functions for specific values a function file to compute them has been implemented. This function file, given a certain value of mu which is the mass ratio of M_2 & total mass of the 2 bodies ($mu=M_2/(M_1+M_2)$) will return coordinates of the Lagrange points as a matrix in R^3 space with x, y & z coordinates.

The quintic equations have been re-written in terms of 'mu' & 'l' (where l=1-mu) for the highly idealized case that R=1 (the distance between the two bodies=1).

To solve the quintic equations, the inbuilt function 'roots' is used. The required solutions are obtained by considering solutions only within appropriate bounds (based on which region the point is present in)

Points L_4 & L_5 are computed using the expressions discussed earlier.

Note that this solves the following quintic equations with highly idealized assumptions.

The code for the function file is displayed below:-

```
function LP = lagrangePoints(mu);



%The first row corresponds to L1,
%the second L2, and so on to the last which is L5.

%Computing the location of the lagrange points
    l=1-mu;

    LP = zeros(5,3);



     %L1
     p_L1=[1, 2*(mu-l), l^2-4*l*mu+mu^2, 2*mu*l*(l-mu)+mu-l,
    mu^2*l^2+2*(l^2+mu^2), mu^3-l^3];
     L1roots=roots(p_L1);
     %initialize L1 for loop
     L1=0;
     for i=1:5
         if (L1roots(i) > -mu) & (L1roots(i) < l)
          L1=L1roots(i);
         end
     end
     LP(1,1) = L1;

     %L2
     p_L2=[1, 2*(mu-l), l^2-4*l*mu+mu^2, 2*mu*l*(l-mu)-(mu+l),
    mu^2*l^2+2*(l^2-mu^2), -(mu^3+l^3)];
     L2roots=roots(p_L2);
     %initialize L2 for loop
     L2=0;
     for i=1:5
         if (L2roots(i) > -mu) & (L2roots(i) > l)
             L2=L2roots(i);
         end
     end
     LP(2,1) = L2;


     %L3
     p_L3=[1, 2*(mu-l), l^2-4*mu*l+mu^2, 2*mu*l*(l-mu)+(l+mu),
    mu^2*l^2+2*(mu^2-l^2), l^3+mu^3];
     L3roots=roots(p_L3);
     %initialize L3 for loop
     L3=0;
     for i=1:5
         if L3roots(i) < -mu
             L3=L3roots(i);
         end
     end
     LP(3,1) = L3;

     %L4
     LP(4,1) = 0.5 - mu;
```

```
    LP(4,2) = sqrt(3)/2;

    %L5
    LP(5,1) = 0.5 - mu;
    LP(5,2) = -sqrt(3)/2;
```

Function File: SurfPotential.m

Apart from computing the Lagrange Points I also decided to plot lines of equipotentials to demonstrate the different stabilities of the unstable collinear Lagrange Points (L_1, L_2 & L_3) and naturally stable points L_4 & L_5.

In order to do so I wrote a function file called SurfPotential.m

To grasp the working of this function file however we must first understand what Surface-Potential of a system of really is.

Surface-Potentials

When the motion of the test mass is confined to the plane containing the massive bodies, the acceleration in the rotating reference frame may be written in the form:

$$\bar{a}_{rotating} = -2\bar{\omega} \times \bar{v}_{rotating} - \bar{\nabla} U$$

where U is defined as the Surface potential or pseudo-potential. For convention, the potential is defined as negative in an isolated gravitational well.

$$U = -\frac{\omega^2 r^2}{2} - \frac{Gm_1}{|\bar{r} - \bar{r}_1|} - \frac{Gm_2}{|\bar{r} - \bar{r}_2|}$$

The first term on the right-hand-side generates the centrifugal force. The second and third terms are the gravitational potentials for masses m_1 and m_2.

The code for this function file is displayed below:-

```
% SurfPotential.m
%
% Calculates the potential for the circular, restricted 3-body problem
evaluated in
% the rotating reference frame tied to m1 & m2
% Assumes G = 1 and R = 1, where R  is the distance between masses m1 & m2
% The origin of the coordinate system is placed at the center of mass point
%
% passed parameters:
% m1 = mass of object 1 (typically set m1 + m2 = 1)
% m2 = mass of object 2
% (x, y) = coordinates of position to evaluate potential. x and y may be
% single values, or arrays.
%
% returned value = pseudo-potential at position (x,y)
% if x & y are arrays, crtbpPotential will return an array
%
%

function U = SurfPotential(m1, m2, x, y)


    omega = sqrt(m1+m2);              % angular velocity of massive bodies

    x1 = -m2/(m1+m2);                 % x coordinate of m1
    x2 = 1 + x1;                      % x coordinate of m2

    r1 = sqrt((x-x1).^2+y.^2);        % distance from m1 to (x,y)
    r2 = sqrt((x-x2).^2+y.^2);        % distance from m2 to (x,y)

    U = -omega^2/2*(x.^2+y.^2) - m1./r1 - m2./r2;
```

This function file when used with a short piece of code can be used to plot the potential represented as a surface plot for a mass ratio of m_2/m_1. An example to illustrate this:-

```
% A program to plot the surface plot of the potential
% between two bodies to demonstrate usage
% of crtbpPotential.m function file

%Parameters and Initialization

M1 = 1;         % mass 1
M2 = 0.1;       % mass 2
M = M1 + M2;    % total mass


P = 2*pi * sqrt(1 / M);      % period from Kepler's 3rd law
```

```
omega0 = 2*pi/P;             % angular velocity of massive bodies

[X,Y] = meshgrid(-1.5:0.05:1.5);     % grid of (x,y) coordinates
U = SurfPotential(M1, M2, X, Y);    % calculate potential for grid points



% define a custom, gray color map to render surface plot
map = ones(2 , 3)*.7;
colormap(map);


% Left figure will show potential from directly above

figure 1

surf(X,Y,U,'FaceColor','interp','EdgeColor','none','FaceLighting','phong')

title(sprintf('Surface Plot of Potential between m_1 = %.1f & m_2 = %.1f',M1,
M2), "fontsize", 14);
xlabel('x axis', "fontsize", 14)
ylabel('y axis', "fontsize", 14)
zlabel('z axis', "fontsize", 14)
zlim([-3 -1]);
view(90,90)      % viewing angle straight overhead
camlight left



% Right figure will show potential tilted 30 degrees

figure 2

colormap(map);
surf(X,Y,U,'FaceColor','interp','EdgeColor','none','FaceLighting','phong')

title(sprintf('Surface Plot of Potential between m_1 = %.1f & m_2 = %.1f from
a different angle',M1, M2), "fontsize", 16);
xlabel('x axis', "fontsize", 14)
ylabel('y axis', "fontsize", 14)
zlabel('z axis', "fontsize", 14)
zlim([-3 -1]);
view(90,30)      % viewing angle inclined by 30 degrees
camlight left
```

Note: - Ocatve.gui also provides in built graphical interface on Figure windows that allows the user to rotate or move the 3-d Plot for different viewing angles. This has been used to obtain the output shown below:

Screenshots of the Output windows:

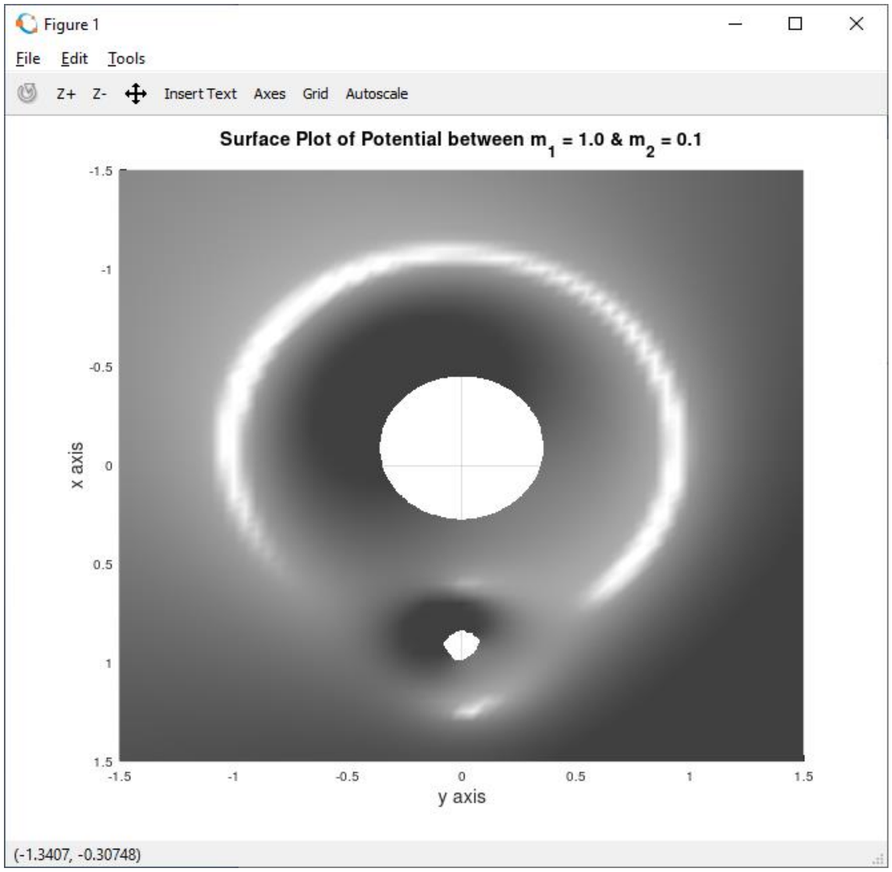


Figure 1
File Edit Tools
Z+ Z- Insert Text Axes Grid Autoscale
Surface Plot of Potential between m1 = 1.0 & m2 = 0.1
x axis
y axis
-1.5
-1
-0.5
0
0.5
1
1.5
(-1.3407, -0.30748)

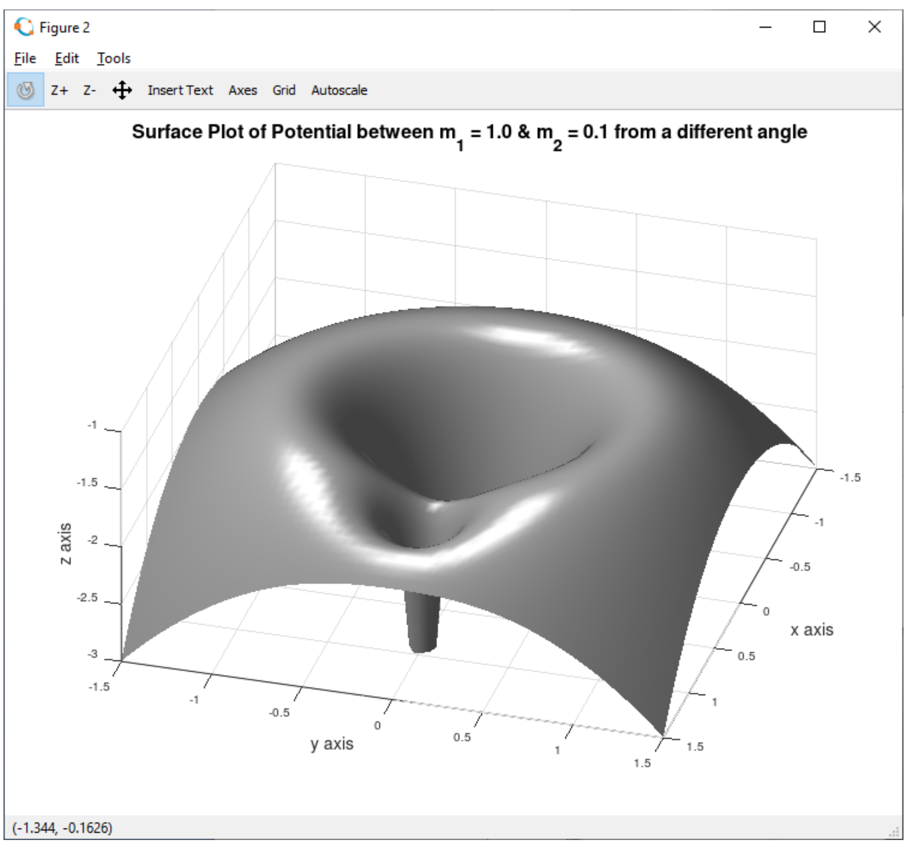


Figure 2
File Edit Tools
Z+ Z- Insert Text Axes Grid Autoscale
Surface Plot of Potential between m1 = 1.0 & m2 = 0.1 from a different angle
z axis
x axis
y axis
-1
-1.5
-2
-2.5
-3
-0.5
0
0.5
1
1.5
(-1.344, -0.1626)

Now that both the function files required for the actual program have been discussed & their individual functionality has been shown, let us have a look at the actual program used to plot the Lagrange Points.

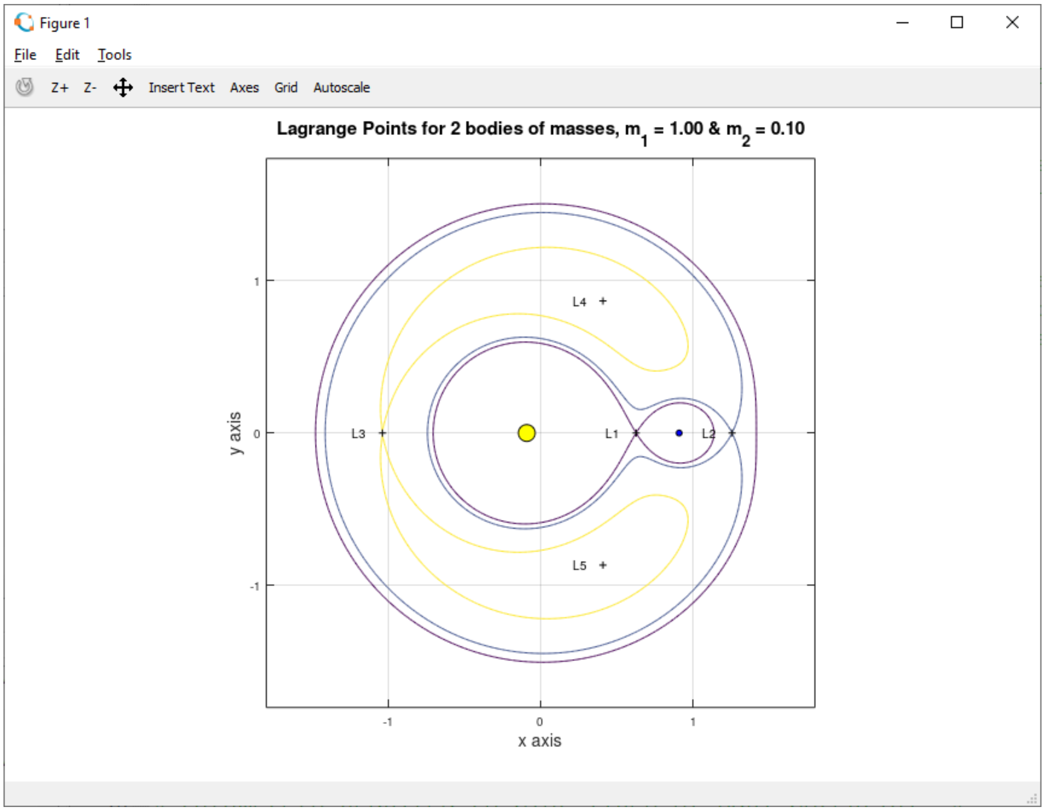
Lagrange Points Simulation

The code for the program that computes the Lagrange Points & the contour lines of the surface potential for 2 masses with a mass ratio of 0.1 (not a realistic mass ratio) is shown below:

```
% LagrangePointsProject.m
%
% Dhruv Tyagi 2K19/EP/032
%
% Plots Lagrange points and contour lines of Surface potential that pass
through them
% for the circular, restricted 3-body problem.
%
% Assumes G = 1 and R = 1, where R  is the distance between masses m1 & m2.
% The origin of the coordinate system is placed at the center-of-mass point
%
% Function Files used:
%    SurfPotential.m - returns pseudo-potential (SOURCE: https://www.matlab-
monkey.com/celestialMechanics/CRTBP/LagrangePoints/crtbpPotential.m)
%    lagrangePoints.m - returns 5x3 array containing (x,y,z) coordinates
%                       of the Lagrange points for given values of m1, m2
%

clc;
clear all;

% Parameters and Initialization  %

M1 = 1;       % mass 1
M2 = 0.1;     % mass 2
M = M1 + M2;  % total mass

% finding Lagrange points %

LP = lagrangePoints(M2/(M1+M2))

% Parameters Required to plot lines of Equi-potential  %

R = 1;        % distance between M1 and M2 set to 1
G = 1;        % Gravitational Constant set to 1
mu = G*M;
mu1 = G*M1;
mu2 = G*M2;
```

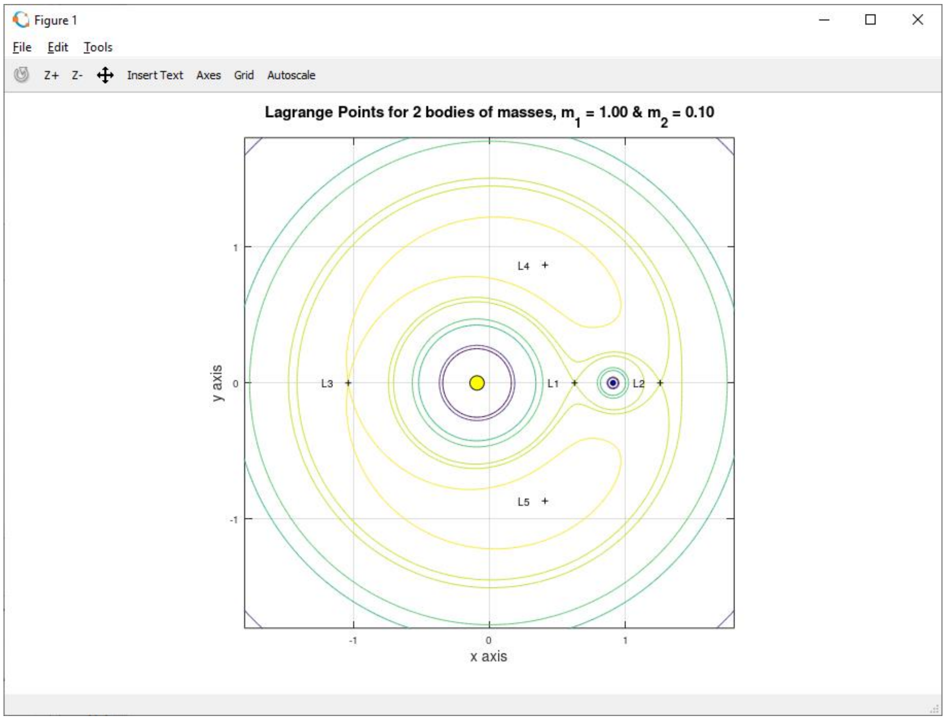
```
[X,Y] = meshgrid(-2:0.01:2);
U = SurfPotential(mu1, mu2, X, Y);

%P = 2*pi * sqrt(R^3 / mu);  % period from Kepler's 3rd law
%omega0 = 2*pi/P;            % angular velocity of massive bodies

% finding Pseudo-potentials %

LP1_level = SurfPotential(mu1, mu2, LP(1,1), LP(1,2));
LP2_level = SurfPotential(mu1, mu2, LP(2,1), LP(2,2));
LP3_level = SurfPotential(mu1, mu2, LP(3,1), LP(3,2));

%%%%% Plotting  %%%%%
figure
% plotting zero-velocity curves that run through L1, L2, L3
contour(X,Y,U,[LP1_level LP2_level LP3_level])
hold on
grid on

% plotting bodies M1 & M2
plot(-M2/(M1+M2),0,'ko','MarkerSize',14,'MarkerFaceColor','y')
plot(M1/(M1+M2),0,'ko','MarkerSize',5,'MarkerFaceColor','b')

% plotting Lagrange points and labels
plot(LP(:,1),LP(:,2),'k+')
labels = {'L1', 'L2', 'L3', 'L4', 'L5'}
text(LP(:,1)-.2,LP(:,2),labels)

% plotting title, limits, etc.
title(sprintf('Lagrange Points for 2 bodies of masses, m_1 = %.2f & m_2 =
%.2f',M1, M2), "fontsize", 15);
xlabel('x axis', "fontsize", 14)
ylabel('y axis', "fontsize", 14)
axis square
axis equal
xlim([-1.8 1.8])
ylim([-1.8 1.8])
```

The output of this program is shown below:



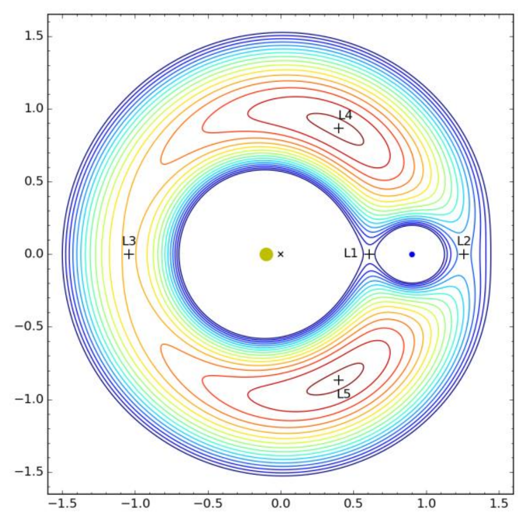
As seen above, the Lagrange Points have been successfully plotted for the two masses represented by the yellow and the blue markers on the plot. Also 3 lines of potential have been plotted that correspond to points L_1, L_2 & L_3.

One may increase the number of lines of potential in order to get a clearer idea of the way the potential curves around these points by simply adding more contour plot arguments. The output would then look like:-



From the plot above one can clearly infer that points L_1, L_2 & L_3 are present at saddle points in the potential and are hence unstable. Meanwhile L_4 & L_5 are present at hilltop regions as previous discussed in the 'stability of Lagrange Points' section.

For a more 'realistic' system one may expect to obtain a plot like this:



Source: - https://leancrew.com/all-this/2016/08/lagrange-points-redux/

Applications of Lagrange points

As points L_1, L_2 & L_3 are points of unstable equilibrium. Satellites and observatories present at these points require a need for control thrusters & orbital corrections to hold them in orbit. While asteroids, planetary dust, & other unknown objects have accumulated at points L_4 and L_5.

Why points L_1 & L_2 are preferred over other points:

It is important to note that point L_3 is unlikely to offer any real utility since it remains hidden behind the Sun at all times & is difficult to reach in practice. However, in theory a satellite sent there may be used to study sun-spots & solar flares formed at the surface of the Sun.

In the natural world, groups of asteroids – called the Trojan asteroids – have been observed to cluster at the L_4 and L_5 Lagrange points due to their natural stabilities, thus rendering them unusable regions for observatories & space telescopes. However a study of the foreign debris that these points trap would allow us to examine composition of debris from outer space.

Due to these reasons points L_3, L_4 & L_5 are not in common use; hence mainly the uses of L_1 & L_2 have been discussed.

Sun–Earth

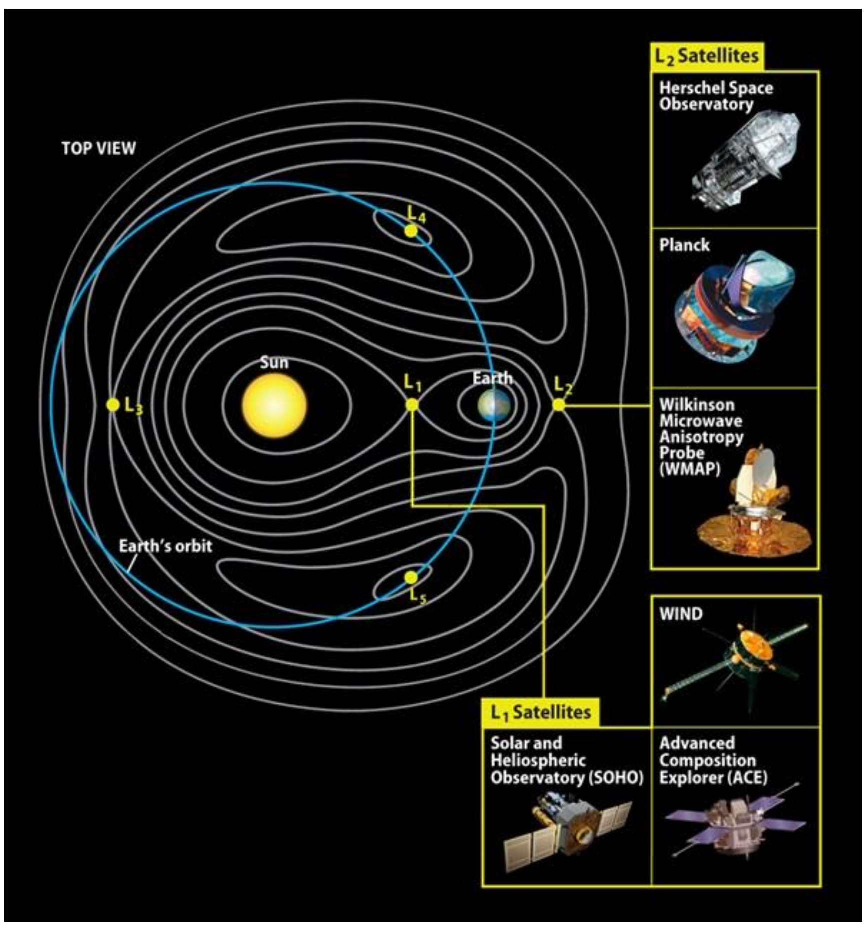
Sun–Earth L_1 is aptly suited for making observations of the Sun–Earth system since objects here are never shadowed by the Earth or the Moon.

Sun–Earth L_2 is a good spot for space-based observatories. Because an object around L_2 will maintain the same relative position with respect to the Sun and Earth, shielding and calibration of satellites is much simpler.

As such there have been several space flight missions & there are currently several satellites at different Lagrange points in the Sun-Earth system as illustrated in the table below:

Since Lagrange Points are considered to be points of stability, as one may imagine they constitute a plethora of applications in modern spaceflight as well as astronomy. A few of them are listed below:-

- Solar and Heliospheric Observatory (SOHO) and the Advanced Composition Explorer (ACE) are a few spacecrafts currently in orbit around L_1.
- Herschel Space Telescope (HSO), Planck Space Observatory, & NASA's Wilkinson Microwave Anisotropy Probe (WMAP), WIND spacecraft are some of the satellites situated at Sun–Earth L_2 Lagrange Points used for a wide range of research and exploration.



Future missions at Sun-Earth Lagrange Points:-

- In the 2021, ISRO also plans to send its Aditya-L_1 to the Sun–Earth L_1 from where it will observe the Sun constantly and study the solar corona, the region around the Sun's surface.
- Currently slated for launch in late 2024, the Interstellar Mapping and Acceleration Probe will be placed near L_1.

The following missions are all planned for Lagrange point L2 for the Sun-Earth sytem:

- 2021: James Webb Space Telescope will use a halo orbit
- 2022: Euclid Space Telescope
- 2024: Nancy Grace Roman Space Telescope (WFIRST) will use a halo orbit
- 2031: Advanced Telescope for High Energy Astrophysics (ATHENA) will use a halo orbit

At points L_4 & L_5 a future space mission that may be considered:-

- Space colonization and manufacturing at points L_4 & L_5 - First proposed in 1974 by Gerard K. O'Neill and subsequently advocated by the L_5 Society.

Conclusion

To conclude, through this project I was able to learn about the solutions to the circular restricted three body problem known as Lagrange Points. I was able to understand how these are derived using complex concepts such as perturbation theory.

Creating a program to compute Lagrange points also gave me an insight into the utility and resourcefulness of software's such as MATLAB and GNU Octave. I was able to learn & improve my knowledge of coding and gain an insight into the world of simulating real world problems in virtual environments.

Once again I would like to thank our professor Dr. Rinku Sharma who gave me the opportunity to work on this project & further my knowledge on the topic to its current state.

References

The references & resources used in this project have been cited below:

https://solarsystem.nasa.gov/resources/754/what-is-a-lagrange-point/

https://en.wikipedia.org/wiki/Lagrange_point

https://www.spaceacademy.net.au/library/notes/lagrangp.htm

https://descanso.jpl.nasa.gov/monograph/series12/LunarTraj--08AppendixALocatingtheLagrangePoints.pdf

https://gereshes.com/2018/12/03/an-introduction-to-lagrange-points-the-3-body-problem/

https://www.phys.uconn.edu/~rozman/Courses/P2200_13F/downloads/restricted-three-body.pdf

https://www.macmillanlearning.com/studentresources/college/physics/tiplermodernphysics6e/classial_concept_review/chapter_13_ccr_21_lagrangian_points.pdf

The Calculation involved were based off the following resources:-

An Introduction to modern Physics, Second Edition – Bradley W. Carroll, Dale A Ostlie
"Close Star Binary Systems" pg [773] – [781]

http://wray.eas.gatech.edu/physicsplanets2014/LectureNotes/LagrangePointDerivation_MontanaStateU.pdf

https://www.sciencedirect.com/topics/physics-and-astronomy/lagrangian-points

https://www.mat.univie.ac.at/~westra/lagrangepoints.pdf

https://jfuchs.hotell.kau.se/kurs/amek/prst/15_lapo.pdf

While researching on the topic I discovered this remarkable page, which has a wealth of graphics related to comets and asteroids:
http://sajri.astronomy.cz/asteroidgroups/groups.htm
The animations on this site showcase how foreign objects from outer space can get stranded & accumulate to form trojan fields at Lagrange Points L_4 & L_5.