

h, Recurrence Transience and invariant distributions

Dhruva Sambrani

7 September

$$h_i^{j(n)} = P_i(T_j = n) = f_{ij}^n$$

$$\text{To Prove: } p_{ij}^n = \sum f_{ij}^m p_{jj}^{(n-m)}$$

Proof:

$$\begin{aligned} p_{ij}^n &= P_i(X_n = j) = P(X_n = j | X_0 = i) \\ &= \sum_{m=1}^n P_i(X_n = j, T_j = m) \\ &= \sum_{m=1}^n P_i(X_n = j | T_j = m) P(T_j = m) \\ &= \sum_{m=1}^n f_{ij}^m P(T_j = m) \\ &= \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)} \} \text{ from SMP} \end{aligned}$$

Define $N_i = \sum_{k=1}^{\infty} \mathbf{I}_{X_n=i}$ = no of visits to state i

Proposition:

$$P_i(N_j = k) = \begin{cases} 1 - f_{ij} & k = 0 \\ f_{ij}(1 - f_{jj})f_{jj}^k - 1 & k > 0 \end{cases}$$

Where $f_{ij} = P_i(T_j < \infty)$.

For $k = 0$,

$$\begin{aligned} P_i(N_j = 0) &= P_i(T_j = \infty) \\ &= 1 - P_i(T_j < \infty) \\ &= 1 - f_{ij} \end{aligned}$$

For $k > 0$

Proof by induction:

$$\begin{aligned} P_i(N_j = k) &= P_i(T_j^{k+1} - T_j^k = \infty, T_j < \infty) \\ &= P_i(T_j^{k+1} - T_j = \infty | T_j^k < \infty) P_i(T_j^\infty < \infty) \\ &= P_j(T_j = \infty) P_i(T_j^k < \infty) \\ &= (1 - f_{jj}) P_i(T_j^k < \infty) \\ &= (1 - f_{jj}) P_i(N_j \geq k) \\ &= (1 - f_{jj}) (1 - \sum_{r=0}^{k-1} P_i(N_j = r)) \\ &= (1 - f_{jj}) (1 - [1 - f_{ij} + \sum_{r=0}^{k-1} f_{ij}(1 - f_{jj})f_{jj}^{r-1}]) \text{ from induction hypothesis} \\ &= (1 - f_{jj}) (f_{ij} - f_{ij}(f_{jj}^0 - f_{jj}^{k-1})) \end{aligned}$$

$$= (1 - f_{jj})(f_{ij}f_{jj}^{k-1})$$

$$P_i(N_i \geq k) \stackrel{?}{=}$$

Recurrence and Transience of a Markov Chain

$i \in S$ is called recurrent if $P(X_n = i \text{ for infinitely many } n) = 1$

it is called transient if $P(X_n = i \text{ for infinitely many } n) = 0$

Theorem: TFAE

1. $i \in S$ is recurrent
2. $f_{ii} = 1$
3. $P_i(N_i = \infty) = 1$
4. $E_i[N_i] = \infty$
5. $\sum_n p_{ii}^{(n)} = \infty$

Proof:

1. $1 \iff 3$ by defn
2. $3 \iff 4$
 - a. $3 \implies 4$ is obvious
 - b. $4 \implies 3$ requires the fact that $N_i \text{ Geom}(f_{ii}) \implies P_i(N_i = \infty) = 1$
3. $1 \iff 2$ - $f_{ii} = P_i(T_i < \infty) = 1$; so if $P(X_n = i) = 1$ for some n , it must hit i again in finite time.
4. $4 \iff 5$ - $E_i[\sum_{n=1}^{\infty} I_{X_n=i}] = \sum_n P_i(X_n = i) = \sum_n p_{ii}^n$

Theorem: TFAE

1. $i \in S$ is transient
2. $f_{ii} < 1$
3. $P_i(N_i < \infty) = 1$
4. $E_i[N_i] < \infty$
5. $\sum_n p_{ii}^{(n)} < \infty$

In particular, i transient $\implies \lim_{n \rightarrow \infty} p_{ii}^{(n)} = 0$

Proposition: If j is transient then $p_{ij}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$.

Recall

$$\begin{aligned} p_{ij}^n &= \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)} \\ \sum_n p_{ij}^n &= \sum_n \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)} \\ &= \sum_m f_{ij}^{(m)} \sum_{n=m}^{\infty} p_{jj}^{(n-m)} \\ &= (1 - E_j[N_j]) \sum_{m=1}^{\infty} f_{ij}^{(m)} \\ &= (1 - E_j[N_j]) P_i(T_j < \infty) \\ &< \infty \\ &\implies p_{ij}^n \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Transience is a class property

Let $i \in S$ be transient. C be communicating class of i and take $j \in C$.

$$\exists n, m \text{ st } p_{ij}^{(n)}, p_{ji}^{(m)} > 0$$

$$\begin{aligned} \forall r \geq 0; p_{ii}^{(n+r+m)} &\geq p_{ij}^n p_{jj}^r p_{ji}^m \\ \sum_r^\infty p_{jj}^{(r)} &\leq \frac{1}{p_{ii}^n p_{ji}^m} \sum_{r=0}^\infty p_{ii}^{(n+r+m)} < \infty \text{ because } i \text{ is transient.} \\ \implies p_{jj}^{(r)} &\rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned}$$

Hence, j is also transient.

Recurrent classes are closed

Let $i \in C$; C not closed.

Then $\exists j$ st $i \rightarrow j$; $j \nrightarrow i$;

Then $\exists m$ st $p_{ij}^{(m)} > 0$;

$$\begin{aligned} P_i(X_n = i, X_m = j) &= P_i(X_n = i | X_m = j) P_i(X_m = j) = 0 \\ P_i(X_n = i \text{ for infinitely many } n) &= P_i(A_n) \\ &= P_i(X_m = j \cap A_n) + P_i(X_m \neq j \cap A_n) \\ &= P_i(X_m \neq j \cap A_n) \\ &\leq P_i(X_m \neq j) = 1 - P_i(X_m = j) < 1 \end{aligned}$$

Therefore i is not recurrent, which is a contradiction.

Positive Recurrence and Null Recurrence

$i \in S$, i recurrent, is positive recurrent if $E_i[T_i] < \infty$

$i \in S$, i recurrent, is null recurrent if $E_i[T_i] = \infty$

Connection with the stationary distribution

Aim:

1. Irreducibility + Recurrence \implies balance equation is satisfied
2. Irreducibility + Positive Recurrence \implies normalization is satisfied

Harmonic Functions

$h : S \rightarrow \mathbf{R}$ is a harmonic wrt to P where P is row stochastic if $h(x) = \sum_{y \in S} p_{xy} h(y)$

If P is irreducible, h is a constant function.

Since S is finite, h attains maximum at some point say x_0

$$h(x_0) \geq h(x) \forall x \in S$$

Let $z \in S$, st $p_{x_0 z} > 0$ and suppose $h(z) < h(x_0)$.

$$h(x_0) = \sum_{x \in S} p_{x_0 x} h(x) = p_{x_0 z} h(z) + \sum_{x \neq z} p_{x_0 x} h(x) < h(x_0) \sum_{x \in S} p_{x_0 x} = h(x_0)$$

That means, $h(z) \geq h(x_0) \implies h(z) = h(x_0)$

Now show that any $z' \in S$, Then exists $PATH(x_0 \rightarrow z')$, and $h(k) = h(x_0)$ for each successive k .

If $Ph = h$

$$(P - I)h = 0$$

$$h = cI$$

$$\implies \dim(\text{Ker}(P - I)) = 1$$

$$\begin{aligned}
&\implies \dim(\text{Ker}(P^T - I)) = 1 \\
&\implies (P^T - I)v^T = 0 \text{ has 1 dimensional solution} \\
&\implies vP = v \text{ has 1 dimensional solution.}
\end{aligned}$$

Hence any $\pi P = \pi$ has to a constant multiple of v .

But due to normalization there is only one invariant probability distribution.

Existance and Uniqueness of invariant measure

Define:

$$Y_i^a = E \left[\sum_{n=0}^{T_a-1} I_{X_n=i} \right]$$

Theorem:

Let $MC(P)$ be irreducible and recurrent, the following hold

1. $Y_a^a = 1$
2. $Y^a P = Y^a$, where $Y^a = (Y_i^a)$
3. $0 < Y_i^a < \infty \forall i \in S$

Proof:

1 is obvious.

$$\begin{aligned}
\mathbf{2} - Y_i^a &= E_a[\sum_{n=0}^{T_a-1} I_{X_n=i}] \\
&= E_a[\sum_{m=1}^{\infty} \sum_{n=0}^{m-1} I_{X_n=i, T_a=m}] \\
&= E_a[\sum_{n=1}^{\infty} I_{X_n=i, T_a \geq n}] - (1) \\
&= \sum_{n=1}^{\infty} \sum_j P_a(X_n = i, X_{n-1} = j, T_a \geq n) \\
&= \sum_{n=1}^{\infty} \sum_j P_a(X_n = i | X_{n-1} = j, T_a \geq n) P_a(X_m = j, T_a \geq n) \\
&= \sum_{n=1}^{\infty} \sum_j p_{ji} P_a(X_{n-1} = j, T_a \geq n) \\
&= \sum_{n=1}^{\infty} P(X_{n-1} = j, T_a \geq n) \\
&= E_a[\sum_{n=1}^{\infty} I_{X_{n-1}=j, T_a \geq n}] \\
&= E_a[\sum_{n=0}^{\infty} I_{X_n=j, T_a \geq (n+1)}] \\
&= Y_j^a
\end{aligned}$$

Hence, **2** holds

3 - For $i \in S \exists n_1, n_2 > 0$ st

$$p_{ia}^{(n_1)}, p_{ai}^{(n_2)} > 0$$

$$Y^a P^n = Y^a$$

$$\implies Y_i^a \geq p_{ai}^{(n_2)} Y_a^a > 0$$

Theorem: If λ is invariant measure of irreducible MC, $\lambda_a = 1$ for some a . Then $\lambda \geq Y^a$. If chain is also recurrent,

$$\lambda = Y^a$$

$$\lambda P = \lambda$$

$$\begin{aligned}\lambda_j &= \sum p_{i_0j} \lambda_{i_0} = \sum_{i_0 \neq a} p_{i_0j} + p_{aj} \lambda_a \\ &= \sum_{i_0 \neq a} p_{i_0j} \sum_{i_1} p_{i_1 i_0} \lambda_{i_1} + p_{aj} \lambda_a \\ &= \sum_{i_0 \neq a} p_{i_0j} \sum_{i_1 \neq a} p_{i_1 i_0} \lambda_{i_1} + \sum_{i_0 \neq a} p_{i_0j} p_{ai_0} \lambda_a + p_{aj} \lambda_a \\ &= \sum_{i_0 \neq a} p_{i_0j} \sum_{i_1 \neq a} p_{i_1 i_0} \lambda_{i_1} + \sum_{i_0 \neq a} p_{ai_0} p_{i_0j} \lambda_a + \sum_{i_1 \neq a} \sum_{i_0 \neq a} p_{ai_1} p_{i_1 i_0} p_{i_0j} \lambda_a + \dots\end{aligned}$$

The first term ≥ 0 , and hence,

$$\begin{aligned}\lambda_j &\geq \lambda_a [p_{aj} + \sum_{i_0 \neq a} p_{ai_0} p_{i_0j} \dots] \\ &= P_a(X_1 = j, T_a \geq 1) + P_a(X_2 = j, T_a \geq 2) + \dots \\ &= Y_j^a \text{ --- from (1)}\end{aligned}$$

If chain is recurrent,

$$\text{Define } \mu_j = \lambda_j - \lambda_a Y_j^a$$

$$0 = \mu_a = \sum_{i \in S} \mu_i p_{ia}^{(n)}$$

$$n \text{ st } p_{ja}^{(n)} > 0$$

$$\sum_{i \in S} \mu_i p_{ia}^{(n)} \geq \mu_i p_{ja}^{(n)}$$

$$\implies \mu_j = 0$$

This can be done for any $k \in S$ since the chain is irreducible. $\implies \mu = 0$

Invariant Probability Distribution

Theorem: Consider an irreducible MC(S, P)

1. Some state $i \in S$ is positive recurrent
2. All states are positive recurrent
3. The chain has an invariant probability distribution λ .

If the above holds, $\lambda = 1/E_i[T_i] \forall i \in S$

Proof:

2 \implies 1 is obvious

1 \implies 3 -

$i \in S$ is recurrent, then MC is recurrent. $\implies Y^i$ is an invariant measure.

$$\begin{aligned}\sum_{k \in S} Y_k^i &= \sum_k E_i[\sum_{n=0}^{T_a-1} I_{X_n=k}] = E_i[\sum_n^{T_a-1} \sum_k \mathbf{1}_{X_n=k}] \\ &= E_i[T_i]\end{aligned}$$

If positive recurrent, $E_i[T_i] < \infty$.

Hence $\lambda_j = Y_j^i / E_i[T_i]$ is an invariant probability distribution.

3 \implies 2

Since λ is pdist, $\exists j \text{ st } \lambda_j > 0$.

$$\implies \lambda_k = \sum p_{ik}^{(n)} \lambda_i \forall n, j.$$

Pick an n such that $p_{jk}^{(n)} > 0$.

This $\implies \lambda_k > 0 \forall k$.

Fix $a \in S$,

$$\pi_i = \lambda_i / \lambda_a$$

From before,

$$Y_i^a \geq \pi_i = \lambda_i / \lambda_a$$

Summing over all $i \in S$ we get,

$$E_a[T_a] \geq 1/\lambda_a < \infty$$

implies a is positive recurrent.

But we can fix any a , since $\lambda_i > 0$ for all i .

Hence, all $a \in S$ are positive recurrent.

Law of large numbers

Consider irreducible $MC(\alpha, P)$.

Define $N_i(n) = \sum_{k=0}^{n-1} I_{X_k=i}$

1. If the chain is transient or Null recurrent, $N_i(n)/n \rightarrow 0$ as $n \rightarrow \infty$
2. If the chain is positive recurrent with invariant pdist λ .

$$N_i(n)/n \rightarrow 1/E_i[T_i]$$

$$T_i^{(r)} = \inf\{n > T_i^{r-1}; X_n = i\}$$

$$S_1 = T_i^{(1)}$$

$$S_2 = T_i^{(2)} - T_i^{(1)}$$

$\{S_j\}$ are i.i.d.