h, Recurrence Transience and invariant distributions

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$h_i^{j(n)} = P_i(T_j = n) = f_{ij}^n$

To Prove: $p_{ij}^n = \sum f_{ij}^m p_{jj}^{(n-m)}$

Proof:

$p_{ij}^n = P_i(X_n = j) = P(X_n = j | X_0 = i)$

$= sum_{m=1}^n P_i(X_n = j, T_j = m)$

$= sum_{m=1}^n P_i(X_n = j | T_j = m) P(T_j = m)$

$= sum_{m=1}^n f_{ij}^m P(T_j = m)$

$= sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)}$ } from SMP

Define $N_i = \sum_{k=1}^{\infty} \mathbf{I}_{X_n = i}$ = no of visits to state i

Proposition:

$$P_i(N_j = k) = \begin{cases} 1 - f_{ij} & k = 0 \\ f_{ij}(1 - f_{jj}) f_{jj}^{k-1} & k > 0 \end{cases}$$

Where $f_{ij} = P_i(T_j < \infty)$.

For $k = 0$,

$P_i(N_j = 0) = P_i(T_j = \infty)$

$= 1 - P_i(T_j < \infty)$

$= 1 - f_{ij}$

For $k > 0$

Proof by induction:

$P_i(N_j = k) = P_i(T_j^{k+1} - T_j^k = \infty, T_j < \infty)$

$= P_i(T_j^{k+1} - T_j = \infty | T_j^k < \infty) P_i(T_j^{\infty} < \infty)$

$= P_j(T_j = \infty) P_i(T_j^k < \infty)$

$= (1 - f_{jj}) P_i(T_j^k < \infty)$

$= (1 - f_{jj}) P_i(N_j >= k)$

$= (1 - f_{jj})(1 - \sum_{r=0}^{k-1} P_i(N_j = r))$

$= (1 - f_{jj})(1 - [1 - f_{ij} + \sum_{r=0}^{k-1} f_{ij}(1 - f_{jj}) f_{jj}^{r-1}])$ from induction hypothesis

$= (1 - f_{jj})(f_{ij} - f_{ij}(f_{jj}^0 - f_{jj}^{k-1}))$

$= (1 - f_{jj})(f_{ij} f_{jj}^{k-1})$

$P_i(N_i \geq k) \stackrel{?}{=}$

Recurrence and Transience of a Markov Chain

$i \in S$ is called recurrent if $P(X_n = i \text{ for infinitely many } n) = 1$

it is called transient if $P(X_n = i \text{ for infinitely many } n) = 0$

Theorem: TFAE

1. $i \in S$ is recurrent
2. $f_{ii} = 1$
3. $P_i(N_i = \infty) = 1$
4. $E_i[N_i] = \infty$
5. $\sum_n p_{ii}^{(n)} = \infty$

Proof:

1. $1 \iff 3$ by defn
2. $3 \iff 4$
   a. $3 \implies 4$ is obvious
   b. $4 \implies 3$ requires the fact that N_i Geom$(f_{ii}) \implies P_i(N_i = \infty) = 1$
3. $1 \iff 2$ - $f_{ii} = P_i(T_i < \infty) = 1$; so if $P(X_n = i) = 1$ for some n, it must hit i again in finite time.
4. $4 \iff 5$ - $E_i[\sum_{n=1}^{\infty} I_{X_n = i}] = \sum_n P_i(X_n = i) = \sum_n p_{ii}^n$

Theorem: TFAE

1. $i \in S$ is transient
2. $f_{ii} < 1$
3. $P_i(N_i < \infty) = 1$
4. $E_i[N_i] < \infty$
5. $\sum_n p_{ii}^(n) < \infty$

In particular, i transient $\implies \lim_{n\to\infty} p_{ii}^{(n)} = 0$

Proposition: If j is transient then $p_{ij}^{(n)} \to 0$ as $n \to \infty$.

Recall

$p_{ij}^n = \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)}$

$\sum_n p_{ij}^n = \sum_n \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)}$

$= \sum_m f_i j^{(m)} \sum_{n=m}^{\infty} p_{jj}^{(n-m)}$

$= (1 - E_j[N_j]) sum_{m=1}^{\infty} f_{ij}^{(m)}$

$= (1 - E_j[N_j]) P_i(T_j < \infty)$

$< \infty$

$\implies p_{ij}^n \to 0$ as $n \to \infty$

Transience is a class property

Let $i \in S$ be transient. C be communicating class of i and take $j \in C$.

$\exists n, m \ st \ p_{ij}^{(n)}, p_{ji}^{(m)} > 0$

$\forall r \geq 0; p_{ii}^{(n+r+m)} \geq p_{ij}^{n} p_{jj}^{r} p_{ji}^{m}$

$\sum_{r}^{\infty} p_{jj}^{(} r) \leq \frac{1}{p_{ii}^{n} p_{ji}^{m}} \sum_{r=0}^{\infty} p_{ii}^{(n+r+m)} < \infty$ because i is transient.

$\implies p_{jj}^{(r)} \to 0$ as $r \to \infty$.

Hence, j is also transient.

Recurrent classes are closed

Let $i \in C$; C not closed.

Then $\exists j$ st $i \to j; j \nrightarrow i$;

Then $\exists m$ st $p_{ij}^{(m)} > 0$;

$P_i(X_n = i, X_m = j) = P_i(X_n = i | X_m = j) P_i(X_m = j) = 0$

$P_i(X_n = i$ for infinitely many $n) = P_i(A_n)$

$= P_i(X_m = j \cap A_n) + P_i(X_m \neq j \cap A_n)$

$= P_i(X_m \neq j \cap A_n)$

$\leq P_i(X_m \neq j) = 1 - P_i(X_m = j) < 1$

Therefore i is not recurrent, which is a contradiction.

Positive Recurrence and Null Recurrence

$i \in S$, i recurrent, is positive recurrent if $E_i[T_i] < \infty$

$i \in S$, i recurrent, is null recurrent if $E_i[T_i] = \infty$

Connection with the stationary distribution

Aim:

1. Irreducability + Recurrence $\implies$ balance equation is satisfied
2. Irreducability + Positive Recurrence $\implies$ normalization is satisfied

Harmonic Functions

$h : S \to \mathbf{R}$ is a harmonic wrt to P where P is row stochastic if $h(x) = \sum_{y \in S} p_{xy} h(y)$

If P is irreducible, h is a constant function.

Since S is finite, h attains maximum at some point say x_0

$h(x_0) \geq h(x) \forall x \in S$

Let $z \in S$, st $p_{x_0 z} > 0$ and suppose $h(z) < h(x_0)$.

$h(x_0) = \sum_{x \in S} p_{x_0 x} h(x) = p_{x_0 z} h(z) + \sum_{x \neq z} p_{x_0 x} h(x) < h(x_0) \sum_{x \in S} p_{x_0 x} = h(x_0)$

That means, $h(z) \geq h(x_0) \implies h(z) = h(x_0)$

Now show that any $z' \in S$, Then exists $PATH(x_0 \to z')$, and $h(k) = h(x_0)$ for each successive k.

If $Ph = h$

$(P - I)h = 0$

$h = cI$

$\implies \dim(\mathrm{Ker}(P - I)) = 1$

$\implies \dim(\mathrm{Ker}(P^T - I)) = 1$

$\implies (P^T - I)v^T = 0$ has 1 dimensional solution

$\implies vP = v$ has 1 dimensional solution.

Hence any $\pi P = \pi$ has to a constant multiple of v.

But due to normalization there is only one invariant probability distribution.

Existance and Uniqueness of invariant measure

Define:

$$Y_i^a = E\left[\sum_{n=0}^{T_a - 1} I_{X_n = i}\right]$$

Theorem:

Let $MC(P)$ be irreducible and recurrent, the following hold

1. $Y_a^a = 1$
2. $Y^a P = Y^a$, where $Y^a = (Y_i^a)$
3. $0 < Y_i^a < \infty \forall i \in S$

Proof:

1 is obvious.

2 - $Y_i^a = E_a[\sum_{n=0}^{T_a-1} I_{X_n=i}]$

$= E_a[\sum_{m=1}^{\infty} \sum_{n=0}^{m-1} I_{X_n=i, T_a=m}]$

$= E_a[\sum_{n=1}^{\infty} I_{X_n=i, T_a \geq n}]$ — (1)

$= \sum_{n=1}^{\infty} \sum_j P_a(X_n = i, X_{n-1} = j, T_a \geq n)$

$= \sum_{n=1}^{\infty} \sum_j P_a(X_n = i | X_{n-1} = j, T_a \geq n) P_a(X_m = j, T_a \geq n)$

$= \sum_{n=1}^{\infty} \sum_j p_{ji} P_a(X_{n-1} = j, T_a \geq n)$

$\sum_{n=1}^{\infty} P(X_{n-1} = j, T_a \geq n)$

$= E_a[\sum_{n=1}^{\infty} I_{X_{n-1}=j, T_a \geq n}]$

$= E_a[\sum_{n=0}^{\infty} I_{X_n=j, T_a \geq (n+1)}]$

$= Y_j^a$

Hence, 2 holds

3 - For $i \in S \exists n_1, n_2 > 0$ st

$p_{ia}^{(n_1)}, p_{ai}^{(n_2)} > 0$

$Y^a P^n = Y^a$

$\implies Y_i^a \geq p_{ai}^{(n_2)} Y_a^a > 0$

Theorem: If λ is invariant measure of irreducible MC, $\lambda_a = 1$ for some a. Then $\lambda \geq Y^a$. If chain is also recurrent,

$$\lambda = Y^a$$

$$\lambda P = \lambda$$

$\lambda_j = \sum p_{i_0 j}\lambda_{i_0} = \sum_{i_0 \neq a} p_{i_0 j} + p_{aj}\lambda_a$

$= \sum_{i_0} \neq a p_{i_0 j} \sum_{i_1} p_{i_1 i_0}\lambda_{i_1} + p_{aj}\lambda_a$

$= \sum_{i_0} \neq a p_{i_0 j} \sum_{i_1 \neq a} p_{i_1 i_0}\lambda_{i_1} + sum_{i_0 \neq a} p_{i_0 j} p_{a i_0}\lambda_a + p_{aj}\lambda_a$

$= sum_{i_0} \neq a \cdots \sum_{i_{n-1} \neq a} + p_{aj}\lambda_a + \sum_{i_0 \neq a} p_{a i_0} p_{i_0 j}\lambda_a + \sum_{i_1 \neq a}\sum_{i_0 \neq a} p_{a i_1} p_{i_1 i_0} p_{i_0 j}\lambda_a + \dots$

The first term ≥ 0, and hence,

$\lambda_j >= \lambda_a[p_{aj} + \sum_{i_0 \neq a} p_{a i_0} p_{i_0 j} \dots]$

$= P_a(X_1 = j, T_a >= 1) + P_a(X_2 = j, T_a >= 2) + \dots$

$= Y_j^a$ — from (1)

If chain is recurrent,

Define $\mu_j = \lambda_j - \lambda_a Y_j^a$

$0 = \mu_a = \sum_{i \in S} \mu_i p_{ia}^{(n)}$

$n \ st \ p_{ja}^{(n)} > 0$

$\sum_{i \in S} \mu_i p_{ia}^{(n)} \geq \mu_i p_{ja}^{(n)}$

$\implies \mu_j = 0$

This can be done for any $k \in S$ since the chain is irreducible. $\implies \mu = 0$

Invariant Probability Distribution

Theorem: Consider an irreducible MC(S, P)

1. Some state $i \in S$ is positive recurrent
2. All states are positive recurrent
3. The chain has an invariant probability distribution lambda.

If the above holds, $\lambda = 1/E_i[T_i] \forall i \in S$

Proof:

$2 \implies 1$ is obvious

$1 \implies 3$ -

$i \in S$ is recurrent, then MC is recurrent. $\implies Y^i$ is an invariant measure.

$\sum_{k \in S} Y_k^i = \sum_k E_i[\sum_{n=0}^{T_a - 1} I_{X_n = k}] = E_i[\sum_n^{T_a - 1}\sum_k \mathbf{I}_{X_n = k}]$

$= E_i[T_i]$

If positive recurrent, $E_i[T_i] < \infty$.

Hence $\lambda_j = Y_j^i / E_i[T_i]$ is an invariant probability distribution.

$3 \implies 2$

Since λ is pdist, $\exists j st \lambda_j > 0$.

$\implies \lambda_k = \sum p_{ik}^{(n)}\lambda_i \forall n, j$.

Pick an n such that $p_{jk}^{(n)} > 0$.

This $\implies \lambda_k > 0 \forall k$.

Fix $a \in S$,

$\pi_i = \lambda_i / \lambda_a$

From before,

$Y_i^a \geq \pi_i = \lambda_i / \lambda_a$

Summing over all $i \in S$ we get,

$E_a[T_a] \geq 1/\lambda_a < \infty$

$impliesa$ is positive recurrent.

But we can fix any a, since $\lambda_i > 0$ for all i.

Hence, all $a \in S$ are positive recurrent.

Law of large numbers

Consider irreducible $MC(\alpha, P)$.

Define $N_i(n) = \sum_{k=0}^{n-1} I_{X_k = i}$

1. If the chain is transient or Null recurrent, $N_i(n)/n \to 0$ as $n \to \infty$
2. If the chain is positive recurrent with invariant pdist λ.

$N_i(n)/n \to 1/E_i[T_i]$

$T_i^{(r)} = \inf\{n > T_i^{r-1}; X_n = i\}$

$S_1 = T_i^(1)$

$S_2 = T_i^(2) - T_i^(1)$

$\{S_j\}$ are i.i.d.