Q2

i) $N_a(t) = \sum_{k=0}^{t-1} \mathbb{I}_{\{X_k = 1\}}$.

Since , the recurrent,

$$\lim_{t\to\infty} \frac{N_a(t)}{t} = \frac{1}{E_a[T_a]}$$

Let $U_a(t) = \sup\{r : T_a^r \le t\}$

$\Rightarrow N_a = U_a$

$$\Rightarrow \lim_{t\to\infty} U_a = \frac{1}{E_a[T_a]}$$

Let $f : S \longrightarrow \mathbb{R}^+$

$F_r = \sum_{T_a^{(r)} \le k \le T_a^{(r+1)}} f(X_k)$.

$T_0 = 0$.

$F = \sum_{T_a^{(U_t)} \le k \le t} f(X_k)$

$\Rightarrow \sum_{k=0}^{t} = F_0 + \sum_{r=1}^{U_t - 1} + F$

$|F_0| \le |f(X_0)| + \dots + |f(X_{T_a^{(1)}})|$

$|F_0| \le T_a^{(1)} \cdot C \quad \}$ f is bounded.

$\because$ Positive recurrent $T_a^{U_t+1} - T_a^{U_t} < \infty$

ie $t - T_a^{U_t} < \infty$, thus $|F|$ is bounded

by some arguement as above.

$$\frac{F}{t} \xrightarrow{t\to\infty} 0$$

$$\Rightarrow \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} f(x_k) = \lim_{t\to\infty} \sum_{r=1}^{U_t-1} F_r \times \frac{U_t}{U_t}$$

F_i are iid $\frac{\sum_{r=1}^{U_t-1} F_r}{U_t} \Rightarrow E[F_i] = E_a\left[\sum_{k=0}^{T_a-1} f(x_k)\right]$

$$\Rightarrow \lim_{t\to\infty} \frac{1}{t} \sum_{\text{Path}} f(x) = \frac{E_a\left[\sum_{k=0}^{T_a-1} f(x_k)\right]}{E_a[T_a]}$$

Setting $f(x) = \mathbb{I}_{\{x_n=i\}}$; $i \in S$.

$$\lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{t} \mathbb{I}_{\{x_n=i\}} = \frac{E_a\left[\sum_{k=0}^{T_a-1} \mathbb{I}_{\{X_k=i\}}\right]}{E_a[T_a]} \quad \forall i \in S$$

a is an arbitrary positive recurrent state in S.

ii) $\because$ The chain is irreducible $b \in S$ is also positive recurrent.

$$\Rightarrow \frac{E_a\left[\sum_{k=0}^{T_a-1} \mathbb{I}_{\{X_k=i\}}\right]}{E_a[T_a]}, \quad \frac{E_b\left[\sum_{k=0}^{T_b-1} \mathbb{I}_{\{X_k=i\}}\right]}{E_b[T_b]} \quad \forall i \in S.$$