Dheeraj Sambrani
MS18163.

Q2

i) Irreducible if there exists a single closed communicating class.

Let $i, j \in S$

If $i < j$

Note there exists a path from $i \to j$

$(i, i+1, i+2, \dots j)$

Where all hopping probabilities are $= p > 0$.

$\Rightarrow P(X_n = j \mid X_0 = i) > 0 \quad \forall \; j > i. \Rightarrow i \to j \quad \forall \; j > i$

Similarly, $j < i$.

$i, i-1, i-2, \dots j$

is such a path. with all hopping probability $= 1-p > 0$

$\Rightarrow P(X_n = j \mid X_0 = i) > 0 \quad \forall \; j < i. \Rightarrow i \to j \quad \forall \; j < i$

Hence $i \to j \quad \forall \; i, j \in S. \Rightarrow$ any c.c.class must include all nodes, and hence there is only 1 cc class, which is S itself. } Definition of C.C class.

Similarly, every subset has flow outward except for S itself. Hence, we have a single communicating class.

ii) Period of state is a class property.

Note $P_{ii}^{(1)} = 0 \; \forall i$, $P_{ii}^{(2)} = 2p(1-p) \quad \forall \; i \neq \{0, 1, N-1, N\}$.

$P_{11}^{(2)} = 1-p + p(1-p)$, $P_{N-1 N-1}^{(2)} = p + p(1-p)$

$P_{00}^{(2)} = 1-p$, $P_{NN} = p$

$\Rightarrow P_{ii}^{(2)} \neq 0 \; \forall i$

$\Rightarrow P_{ii}^{2n} \neq 0$; $P_{ii}^{2n+1} = 0$ $\quad \forall n \in \mathbb{N}$ from K.C. Theorem

$\Rightarrow \quad d_i = \gcd(\{2n \mid n \in \mathbb{N}\}) = 2. \qquad \forall i$

iii) let $A_i = \{0, 1, 2 \ldots i\} \quad \forall i \in \{1 \ldots N-1\}$.

$F(A_i, A_i^c) = p\,\pi_i = (1-p)\,\pi_{i+1} = F(A_i^c, A_i)$.

$\Rightarrow \dfrac{\pi_i}{\pi_{i+1}} = \dfrac{1-p}{p}$.

$\dfrac{\pi_0}{\pi_1} = (1-p)$.

$\sum \pi_i = 1$.

$$\underline{\pi_i} = \frac{\pi_i}{\pi_{i-1}} \cdot \frac{\pi_{i-1}}{\pi_{i-2}} \cdot \ldots \frac{\pi_1}{\pi_0} \cdot \pi_0 \qquad \forall i \geq 1 - (1)$$

$$= \left(\frac{p}{1-p}\right)^{i-1} \frac{1}{1-p}\,\pi_0 \qquad \forall i \geq 1.$$

$$\sum_i \pi_i = \frac{\pi_0}{1-p} \sum_{i=1}^{N} \left(\frac{p}{1-p}\right)^{i-1} + \pi_0.$$

$$= \frac{\pi_0}{1-p}\left[\sum_{i=0}^{N-1} \left(\frac{p}{1-p}\right)^{i}\right] + \pi_0.$$

let $r = \dfrac{p}{1-p}$.

$$\Rightarrow \sum \pi_i = \frac{\pi_0}{1-p} \cdot \frac{1-r^n}{1-r} + \pi_0 = 1.$$

$\Rightarrow \pi_0 = \dfrac{1}{\dfrac{1}{1-p}\dfrac{1-r^n}{1-r} + 1}$; $\pi_{i \neq 0}$ can be found from (1)

Q5 Let i and l be nodes such that they are max

$$P^{n}_{kj} \leqslant P^{n}_{lj} \; ; \; P^{(n+1)}_{kj} \leqslant P^{(n+1)}_{ij} \quad \forall k \in S$$

$$P(X_{n+1}=j \mid X_0=i) = \sum_{k\in S} P(X_{n+1}=j \mid X_1=k)\,P(X_1=k \mid X_0=i) \Big\} \text{C.K.T.}$$

$$= \sum_{k\in S} P(X_n=j \mid X_0=k)\,P_{ik} \Big\} \text{ Markov property.}$$

$$\leqslant P(X_n=j \mid X_0=l) \sum_k P_{ik}$$

$$= P(X_n=j \mid X_0=l) \; 1.$$

$$\Rightarrow \quad P^{(n+1)}_{ij} \leq P^{(n)}_{lj}$$

Now assume i, l st they are min.

$$P^{(n+1)}_{ij} \leq P^{(n+1)}_{kj} \; ; \; P^{(n)}_{lj} \leq P^{(n)}_{kj} \quad \forall k \in S$$

$$P(X_{n+1}=j \mid X_0=i) = \sum_{k\in S} P(X_{n+1}=j \mid X_1=k)\,P_{ik} \Big\} \text{ K.C.T}$$

$$= \sum_{k\in S} P(X_n=j \mid X_0=k) \; P_{ik} \Big\} \text{ Markov property.}$$

$$\geqslant P(X_n=j \mid X_0=l) \sum_{k\in S} P_{ik}.$$

$$\Rightarrow \quad P^{(n+1)}_{ij} \geqslant P^{(n)}_{lj}.$$