

# Hitting times and SMP

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## Hitting times

For A subset S, define hitting time of A as

$$T_A = \inf\{n \geq 0; X_n \in A\}$$

Convention  $\inf \emptyset = \infty$ .

$T_A$  takes values in  $\{0, 1, \dots\} \cup \{\infty\}$

1. Probability of hitting  $A = P(T_A < \infty | X_0 = i) = P_i(T_A < \infty)$
2. Expected hitting time  $E[T_A | X_0 = i] = E_i[T_A]$
3. Define vector  $h^A = (h_i^A | i \in S)$ , and  $\psi^A = (\psi_i^A | i \in S)$  by  $h_i^A = P_i(T_A < \infty)$ ,  $\psi_i^A = E[T_A]$

**Theorem:**  $h^A$  is the minimal solution of linear equations given by

$$h^A = \begin{cases} h_1^A = 1 & i \in A \\ h_i^A = \sum_{j \in S} p_{ij} h_j^A & i \notin A \end{cases}$$

*Proof:*

If  $i \in A$ ,  $h_i^A = 1$  is obvious

For  $i \notin A$  write  $T_A = 1 + T'_A$  where  $T'_A$  is the time to hit  $A$  after one step.

$$\begin{aligned} h_i^A &= P_i(T_A < \infty) \\ &= P_i(1 + T'_A < \infty) \\ &= \sum_j P_j(1 + T'_A < \infty | x_1 = j) p_{ij} \\ &= \sum_j P_j(T_A < \infty | x_0 = j) p_{ij} \\ &= \sum_j h_j^A p_{ij} \end{aligned}$$

Let  $\phi_A = (\phi_i^A, i \in S)$  be another solution.

Then,  $\phi_i^A = 1 = h_i^A$  for  $i \in A$

For  $i \notin A$

$$\begin{aligned} \phi_i^A &= \sum p_{ij} \phi_j^A \\ &= \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} \phi_j^A \\ &= \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} \sum_k p_{jk} \phi_k^A \\ &= \sum_{j \in A} p_{ij} + \sum_{j \notin A, k \in A} p_{ij} p_{jk} + \sum_{j \notin A, k \notin A} p_{ij} p_{jk} \phi_k^A \\ \phi_i^A &= P(T_a = 1) + P(T_a = 2) + P(T_a = 3) + \dots \geq P_i(T_A < \infty). \end{aligned}$$

Taking  $\lim_{n \rightarrow \infty}$ ,

$$\phi_i^A \geq P_i(T_A < \infty) = h_i^A$$

**Theorem:**  $\psi_A$  is the minimal solution of linear equations given by

$$\begin{cases} \psi_i^A = 0 & \text{if } i \in A \\ \psi_i^A = \sum_{j \in A^c} p_{ij} \psi_j^A & \text{if } i \notin A \end{cases}$$

*Proof:*

If  $i \in S$ , obvious.

$$\begin{aligned} E_i[T_A] &= E_i[1 + T_A] = \sum_{j \in S} E[1 + T'_A | X_1 = j] p_{ij} \\ &= 1 + E[T_A | X_0 = j] p_{ij} \\ &= 1 + \sum p_{ij} \psi_j^A \end{aligned}$$

Let  $k$  be another solution.

On  $i \in A$ , this is obvious

For  $i \notin A$ ,

$$\begin{aligned} k_i^A &= 1 + \sum_{j \notin A} p_{ij} \psi_j^A \\ &= 1 + \sum_{j \notin A} p_{ij} (1 + \sum_{l \notin A} \sum_{jl} k_l^A) \\ &= 1 + \sum_{j \notin A} p_{ij} + \sum_{j \in A^c, l \in A^c} p_{jl} k_l^A \\ &= \sum_n P_i(T_A \geq n) \end{aligned}$$

**Exercise:** Show  $\sum_n P_i(T_A \geq n) = E_i[T_A]$

## Stopping times

$\{X_n\}$  is a random process which is also a MC.

Given a filtration  $F_n$  on  $\Omega$ ,

$$\tau : \Omega \rightarrow W$$

is a stopping time if  $\{\tau = n\}$  is measurable wrt  $F_n$ . That is, if  $\{X_n\}$  is an MC,  $\tau$  is a stopping time wrt  $\{X_n\}$ , if  $\tau = n$  only depends on  $(x_0, x_1, x_2, \dots, x_n)$ .

$$\{\tau = n\} \in F_n$$

**Eg:**  $T_A$  is a ST.

**Eg:** Exit time of A.  $L_A = \sup\{n \geq 0; X_n \in A\}$ ; This is not a stopping time.

$F_T = \{A | A \cap \{\tau = n\} \in F_n\}$  ie  $A \cap \{\tau = n\}$  only depends on  $(X_0, \dots, X_n)$

**To Show:**

For any  $A \in \sigma(X_0, \dots, X_m)$

$$P(X_{m+1} = i_{m+1} \cap A | X_m = i) = p_{ii_{m+1}} p_{i_{m+1}i_{m+2}} \dots P(A | X_m = i)$$

Take  $A = (X_0 = i_0, X_1 = i_1 \dots X_m = i)$

Then,  $P(X_{m+1} = i_{m+1}, \dots \cap A | X_m = i) = \frac{\lambda_{i_0} p_{i_0 i_1} \dots}{P(X_m = i)} = p_{i_m i_{m+1}} p_{i_{m+1} i_{m+2}} \dots P(A | X_m = i)$

## Strong Markov Property

Let  $\{X_n\}$  be an  $MC(\lambda, P)$ .  $\tau$  is a Stopping Time with  $\{X_n\}$ . Suppose  $\tau < \infty$  then conditioned on  $\{\tau = n\}$ ,  $(X_{\tau+1}, X_{\tau+2}, \dots)$  is  $MC(\delta_n, P)$  and independent of PAST.

### Proof of SMP

Enough to show that for  $A$  in  $F_\tau$ ,

$$P(X_{\tau+1} = i_1, \dots, \cap A | X_\tau = i, \tau < \infty) = p_{ii_1} p_{i_1 i_2} \dots P(A | X_\tau = i, \tau < \infty)$$

$$p(x_{\tau+1} = i_1 \dots \cap A, x_\tau = i, \tau = n) = P(X_n + 1 = i_1 \dots \cap A, X_n = i, \tau = n)$$

$$= P(X_n + 1 = i_1 \dots | A, X_n = i, \tau = n) P(A, X_n = i, \tau = n)$$

$$= P(X_n + 1 = i_1 \dots | X_n = i) P(A, X_n = i, \tau = n) \Leftarrow \text{Because } A \text{ in } F_\tau$$

$$= p_{ii_1} p_{i_1 i_2} \dots P(A, X_n = i, \tau = n)$$

Hence,

$$P(X_{\tau+1} = i_1, \dots, \cap A | X_\tau = i, \tau < \infty) = \sum_n P(X_{\tau+1} = i_1, \dots, \cap A | X_\tau = i, \tau = n)$$

$$= p_{ii_1} p_{i_1 i_2} \dots \sum_n P(A, X_n = i, \tau = n)$$

Now, sum over  $n$  and divide by  $P(X_i = i, \tau < \infty)$ .