Hitting times and SMP

Dhruva Sambrani

31 August

Hitting times

For A subset S, define hitting time of A as

$T_A = \inf\{n \geq 0; X_n \in A\}$

Convention $\inf \phi = \infty$.

T_A takes values in $\{0, 1, \ldots\} \cup \{\infty\}$

1. Probability of hitting $A = P(T_A < \infty | X_0 = i) = P_i(T_A < \infty)$
2. Expected hitting time $E[T_A | X_0 = i] = E_i[T_A]$
3. Define vector $h^A = (h_i^A | i \in S)$, and $\psi^A = (\psi_i^A | i \in S)$ by $h_i^A = P_i(T_A < \infty)$, $\psi_i^A = E[T_A]$

Theorem: h^A is the minimal solution of linear equations given by

$$h^A = \begin{cases} h_1^A = 1 & i \in A \\ h_i^A = \sum_{j \in S} p_{ij} h_j^A & i \notin A \end{cases}$$

Proof:

If $i \in A$, $h_i^A = 1$ is obvious

For $i \notin A$ write $T_A = 1 + T'_A$ where T'_A is the time to hit A after one step.

$h_i^A = P_i(T_A < \infty)$

$= P_i(1 + T'_A < \infty)$

$= \sum_j P_j(1 + T'_A < \infty | x_1 = j) p_{ij}$

$= \sum_j P_j(T_A < \infty | x_0 = j) p_{ij}$

$= \sum_j h_j^A p_{ij}$

Let $\phi_A = (\phi_i^A, i \in S)$ be another solution.

Then, $\phi_i^A = 1 = h_i^A$ for $i \in A$

For $i \notin A$

$\phi_i^A = \sum p_{ij} \phi_j^A$

$= \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} \phi_j^A$

$= \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} \sum_k p_{jk} \phi_k^A$

$= \sum_{j \in A} p_{ij} + \sum_{j \notin A, k in A} p_{ij} p_{jk} + \sum_{j \notin A, k \notin A} p_{ij} p_{jk} \phi_k^A$

$\phi_i^A = P(T_a = 1) + P(T_a = 2) + P(T_a = 3) + \cdots \geq P_i(T_A <= n)$.

Taking $\lim_{n\to\infty}$,

$$\phi_i^A \geq P_i(T_A < \infty) = h_i^A$$

Theorem: ψ_A is the minimal solution of linear equations given by

$$\begin{cases} \psi_i^A = 0 & if i \in A \\ \psi_i^A = \sum_{j\in A^C} p_{ij}\psi_j^A & if i \notin A \end{cases}$$

Proof:

If $i \in S$, obvious.

$E_i[T_A] = E_i[1 + T_A] = \sum_{j\in S} E[1 + T'_A|X_1 = j]p_{ij}$

$= 1 + E[T_A|X_0 = j]p_{ij}$

$= 1 + \sum p_{ij}\psi_j^A$

Let k be another solution.

On $i \in A$, this is obvious

For $i \notin A$,

$k_i^A = 1 + \sum_{j\notin A} p_{ij}\psi_j^A$

$= 1 + \sum_{j\notin A} p_{ij}(1 + \sum_{l\notin A}\sum_{jl} k_l^A)$

$= 1 + \sum_{j\notin A} p_{ij} + \sum_{j\in A^C, l\in A^C} p_{jl}k_l^A$

$= \sum_n P_i(T_A \geq n)$

Exercise: Show $\sum_n P_i(T_A \geq n) = E_i[T_A]$

Stopping times

$\{X_n\}$ is a random process which is also a MC.

Given a filteration F_n on Ω,

$$\tau : \Omega \to W$$

is a stopping time if $\{\tau = n\}$ is measurable wrt F_n. That is, if $\{X_n\}$ is an MC, τ is a stopping time wrt $\{X_n\}$, if $\tau = n$ only depends on $(x_0, x_1, x_2, \ldots, x_n)$.

$$\{\tau = n\} \in F_n$$

Eg: T_A is a ST.
Eg: Exit time of A. $L_A = \sup\{n \geq 0; X_n \in A\}$; This is not a stopping time.

$F_T = \{A|A \cap \{T = n\} \in F_n\}$ ie $A \cap \{T = n\}$ only depends on $(X_0, \ldots X_n)$

To Show:

For any $A \in \sigma(X_0, \ldots, X_m)$

$$P(X_{m+1} = i_{m+1} \cap A|X_m = i) = p_{ii_{m+1}}p_{i_{m+1}i_{m+2}} \ldots P(A|X_m = i)$$

Take $A = (X_0 = i_0, X_1 = i_1 \dots X_m = i)$

Then, $P(X_{m+1} = i_{m+1}, \dots \cap A | X_m = i) = \frac{\lambda_{i_0} p_{i_0 i_1} \dots}{P(X_m = i)} = p_{i_m i_{m+1}} p_{i_{m+1} i_{m+2}} \dots P(A | X_m = i)$

Strong Markov Property

Let $\{X_n\}$ be an $MC(\lambda, P)$. τ is a Stopping Time with $\{X_n\}$. Suppose $\tau < \infty$ then conditioned on $\{\tau = n\}, (X_{\tau+1}, X_{\tau+2}, \dots)$ is $MC(\delta_n, P)$ and independent of PAST.

Proof of SMP

Enough to show that for A in F_τ,

$P(X_{\tau+1} = i_1, \dots, \cap A | X_\tau = i, \tau < infty) = p_{ii_1} p_{i_1 i_2} \dots P(A | X_\tau = i, \tau < \infty)$

$p(x_{\tau+1} = i_1 \dots \cap A, x_\tau = i, \tau = n) = P(X_n + 1 = i_1 \dots \cap A, X_n = i, \tau = n)$

$= P(X_n + 1 = i_1 \dots | A, X_n = i, \tau = n) P(A, X_n = i, \tau = n)$

$= P(X_n + 1 = i_1 \dots | X_n = i) P(A, X_n = i, \tau = n) \Leftarrow Because A in F_\tau$

$= p_{ii_1} p_{i_1 i_2} \dots P(A, X_n = i, \tau = A)$

Hence,

$P(X_{\tau+1} = i_1, \dots, \cap A | X_\tau = i, \tau < \infty) = \sum_n P(X_{\tau+1} = i_1, \dots, \cap A | X_\tau = i, \tau = n)$

$= p_{ii_1} p_{i_1 i_2} \dots \sum_n P(A, X_n = i, \tau = A)$

Now, sum over n and divide by $P(X_i = i, \tau < \infty)$.