

# h, Recurrence Transience and invariant distributions

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$$h_i^{j(n)} = P_i(T_j = n) = f_{ij}^n$$

$$\text{To Prove: } p_{ij}^n = \sum f_{ij}^m p_{jj}^{(n-m)}$$

Proof:

$$\begin{aligned} p_{ij}^n &= P_i(X_n = j) = P(X_n = j | X_0 = i) \\ &= \sum_{m=1}^n P_i(X_n = j, T_j = m) \\ &= \sum_{m=1}^n P_i(X_n = j | T_j = m) P(T_j = m) \\ &= \sum_{m=1}^n f_{ij}^m P(T_j = m) \\ &= \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)} \} \text{ from SMP} \end{aligned}$$

Define  $N_i = \sum_{k=1}^{\infty} \mathbf{I}_{X_n=i}$  = no of visits to state  $i$

**Proposition:**

$$P_i(N_j = k) = \begin{cases} 1 - f_{ij} & k = 0 \\ f_{ij}(1 - f_{jj})f_{jj}^k - 1 & k > 0 \end{cases}$$

Where  $f_{ij} = P_i(T_j < \infty)$ .

For  $k = 0$ ,

$$\begin{aligned} P_i(N_j = 0) &= P_i(T_j = \infty) \\ &= 1 - P_i(T_j < \infty) \\ &= 1 - f_{ij} \end{aligned}$$

For  $k > 0$

**Proof by induction:**

$$\begin{aligned} P_i(N_j = k) &= P_i(T_j^{k+1} - T_j^k = \infty, T_j < \infty) \\ &= P_i(T_j^{k+1} - T_j = \infty | T_j^k < \infty) P_i(T_j^\infty < \infty) \\ &= P_j(T_j = \infty) P_i(T_j^k < \infty) \\ &= (1 - f_{jj}) P_i(T_j^k < \infty) \\ &= (1 - f_{jj}) P_i(N_j \geq k) \\ &= (1 - f_{jj}) (1 - \sum_{r=0}^{k-1} P_i(N_j = r)) \\ &= (1 - f_{jj}) (1 - [1 - f_{ij} + \sum_{r=0}^{k-1} f_{ij}(1 - f_{jj})f_{jj}^{r-1}]) \text{ from induction hypothesis} \\ &= (1 - f_{jj}) (f_{ij} - f_{ij}(f_{jj}^0 - f_{jj}^{k-1})) \end{aligned}$$

$$= (1 - f_{jj})(f_{ij}f_{jj}^{k-1})$$

$$P_i(N_i \geq k) \stackrel{?}{=}$$

## Recurrence and Transience of a Markov Chain

$i \in S$  is called recurrent if  $P(X_n = i \text{ for infinitely many } n) = 1$

it is called transient if  $P(X_n = i \text{ for infinitely many } n) = 0$

**Theorem:** TFAE

1.  $i \in S$  is recurrent
2.  $f_{ii} = 1$
3.  $P_i(N_i = \infty) = 1$
4.  $E_i[N_i] = \infty$
5.  $\sum_n p_{ii}^{(n)} = \infty$

**Proof:**

1.  $1 \iff 3$  by defn
2.  $3 \iff 4$ 
  - a.  $3 \implies 4$  is obvious
  - b.  $4 \implies 3$  requires the fact that  $N_i \text{ Geom}(f_{ii}) \implies P_i(N_i = \infty) = 1$
3.  $1 \iff 2$  -  $f_{ii} = P_i(T_i < \infty) = 1$ ; so if  $P(X_n = i) = 1$  for some  $n$ , it must hit  $i$  again in finite time.
4.  $4 \iff 5$  -  $E_i[\sum_{n=1}^{\infty} I_{X_n=i}] = \sum_n P_i(X_n = i) = \sum_n p_{ii}^n$

**Theorem:** TFAE

1.  $i \in S$  is transient
2.  $f_{ii} < 1$
3.  $P_i(N_i < \infty) = 1$
4.  $E_i[N_i] < \infty$
5.  $\sum_n p_{ii}^{(n)} < \infty$

In particular,  $i$  transient  $\implies \lim_{n \rightarrow \infty} p_{ii}^{(n)} = 0$

Proposition: If  $j$  is transient then  $p_{ij}^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$ .

Recall

$$\begin{aligned} p_{ij}^n &= \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)} \\ \sum_n p_{ij}^n &= \sum_n \sum_{m=1}^n f_{ij}^m p_{jj}^{(n-m)} \\ &= \sum_m f_{ij}^{(m)} \sum_{n=m}^{\infty} p_{jj}^{(n-m)} \\ &= (1 - E_j[N_j]) \sum_{m=1}^{\infty} f_{ij}^{(m)} \\ &= (1 - E_j[N_j]) P_i(T_j < \infty) \\ &< \infty \\ &\implies p_{ij}^n \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

## Transience is a class property

Let  $i \in S$  be transient.  $C$  be communicating class of  $i$  and take  $j \in C$ .

$$\exists n, m \text{ st } p_{ij}^{(n)}, p_{ji}^{(m)} > 0$$

$$\begin{aligned} \forall r \geq 0; p_{ii}^{(n+r+m)} &\geq p_{ij}^n p_{jj}^r p_{ji}^m \\ \sum_r^\infty p_{jj}^{(r)} &\leq \frac{1}{p_{ii}^n p_{ji}^m} \sum_{r=0}^\infty p_{ii}^{(n+r+m)} < \infty \text{ because } i \text{ is transient.} \\ \implies p_{jj}^{(r)} &\rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned}$$

Hence,  $j$  is also transient.

## Recurrent classes are closed

Let  $i \in C$ ;  $C$  not closed.

Then  $\exists j$  st  $i \rightarrow j$ ;  $j \nrightarrow i$ ;

Then  $\exists m$  st  $p_{ij}^{(m)} > 0$ ;

$$\begin{aligned} P_i(X_n = i, X_m = j) &= P_i(X_n = i | X_m = j) P_i(X_m = j) = 0 \\ P_i(X_n = i \text{ for infinitely many } n) &= P_i(A_n) \\ &= P_i(X_m = j \cap A_n) + P_i(X_m \neq j \cap A_n) \\ &= P_i(X_m \neq j \cap A_n) \\ &\leq P_i(X_m \neq j) = 1 - P_i(X_m = j) < 1 \end{aligned}$$

Therefore  $i$  is not recurrent, which is a contradiction.

## Positive Recurrence and Null Recurrence

$i \in S$ ,  $i$  recurrent, is positive recurrent if  $E_i[T_i] < \infty$

$i \in S$ ,  $i$  recurrent, is null recurrent if  $E_i[T_i] = \infty$

## Connection with the stationary distribution

Aim:

1. Irreducibility + Recurrence  $\implies$  balance equation is satisfied
2. Irreducibility + Positive Recurrence  $\implies$  normalization is satisfied

## Harmonic Functions

$h : S \rightarrow \mathbf{R}$  is a harmonic wrt to  $P$  where  $P$  is row stochastic if  $h(x) = \sum_{y \in S} p_{xy} h(y)$

If  $P$  is irreducible,  $h$  is a constant function.

Since  $S$  is finite,  $h$  attains maximum at some point say  $x_0$

$$h(x_0) \geq h(x) \forall x \in S$$

Let  $z \in S$ , st  $p_{x_0 z} > 0$  and suppose  $h(z) < h(x_0)$ .

$$h(x_0) = \sum_{x \in S} p_{x_0 x} h(x) = p_{x_0 z} h(z) + \sum_{x \neq z} p_{x_0 x} h(x) < h(x_0) \sum_{x \in S} p_{x_0 x} = h(x_0)$$

That means,  $h(z) \geq h(x_0) \implies h(z) = h(x_0)$

Now show that any  $z' \in S$ , Then exists  $PATH(x_0 \rightarrow z')$ , and  $h(k) = h(x_0)$  for each successive  $k$ .

If  $Ph = h$

$$(P - I)h = 0$$

$$h = cI$$

$$\implies \dim(\text{Ker}(P - I)) = 1$$

$$\begin{aligned}
&\implies \dim(\text{Ker}(P^T - I)) = 1 \\
&\implies (P^T - I)v^T = 0 \text{ has 1 dimensional solution} \\
&\implies vP = v \text{ has 1 dimensional solution.}
\end{aligned}$$

Hence any  $\pi P = \pi$  has to a constant multiple of  $v$ .

But due to normalization there is only one invariant probability distribution.

## Existence and Uniqueness of invariant measure

**Define:**

$$Y_i^a = E \left[ \sum_{n=0}^{T_a-1} I_{X_n=i} \right]$$

**Theorem:**

Let  $MC(P)$  be irreducible and recurrent, the following hold

1.  $Y_a^a = 1$
2.  $Y^a P = Y^a$ , where  $Y^a = (Y_i^a)$
3.  $0 < Y_i^a < \infty \forall i \in S$

**Proof:**

**1** is obvious.

$$\begin{aligned}
\mathbf{2} - Y_i^a &= E_a \left[ \sum_{n=0}^{T_a-1} I_{X_n=i} \right] \\
&= E_a \left[ \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} I_{X_n=i, T_a=m} \right] \\
&= E_a \left[ \sum_{n=1}^{\infty} I_{X_n=i, T_a \geq n} \right] - (1) \\
&= \sum_{n=1}^{\infty} \sum_j P_a(X_n = i, X_{n-1} = j, T_a \geq n) \\
&= \sum_{n=1}^{\infty} \sum_j P_a(X_n = i | X_{n-1} = j, T_a \geq n) P_a(X_{n-1} = j, T_a \geq n) \\
&= \sum_{n=1}^{\infty} \sum_j p_{ji} P_a(X_{n-1} = j, T_a \geq n) \\
&= \sum_{n=1}^{\infty} P(X_{n-1} = j, T_a \geq n) \\
&= E_a \left[ \sum_{n=1}^{\infty} I_{X_{n-1}=j, T_a \geq n} \right] \\
&= E_a \left[ \sum_{n=0}^{\infty} I_{X_n=j, T_a \geq (n+1)} \right] \\
&= Y_j^a
\end{aligned}$$

Hence, **2** holds

**3** - For  $i \in S \exists n_1, n_2 > 0$  st

$$p_{ia}^{(n_1)}, p_{ai}^{(n_2)} > 0$$

$$Y^a P^n = Y^a$$

$$\implies Y_i^a \geq p_{ai}^{(n_2)} Y_a^a > 0$$

**Theorem:** If  $\lambda$  is invariant measure of irreducible MC,  $\lambda_a = 1$  for some  $a$ . Then  $\lambda \geq Y^a$ . If chain is also recurrent,

$$\lambda = Y^a$$

$$\lambda P = \lambda$$

$$\begin{aligned}\lambda_j &= \sum p_{i_0j} \lambda_{i_0} = \sum_{i_0 \neq a} p_{i_0j} + p_{aj} \lambda_a \\ &= \sum_{i_0 \neq a} p_{i_0j} \sum_{i_1} p_{i_1 i_0} \lambda_{i_1} + p_{aj} \lambda_a \\ &= \sum_{i_0 \neq a} p_{i_0j} \sum_{i_1 \neq a} p_{i_1 i_0} \lambda_{i_1} + \sum_{i_0 \neq a} p_{i_0j} p_{a i_0} \lambda_a + p_{aj} \lambda_a \\ &= \sum_{i_0 \neq a} p_{i_0j} \sum_{i_1 \neq a} p_{i_1 i_0} \lambda_{i_1} + p_{aj} \lambda_a + \sum_{i_0 \neq a} p_{a i_0} p_{i_0j} \lambda_a + \sum_{i_1 \neq a} \sum_{i_0 \neq a} p_{a i_1} p_{i_1 i_0} p_{i_0j} \lambda_a + \dots\end{aligned}$$

The first term  $\geq 0$ , and hence,

$$\begin{aligned}\lambda_j &\geq \lambda_a [p_{aj} + \sum_{i_0 \neq a} p_{a i_0} p_{i_0j} \dots] \\ &= P_a(X_1 = j, T_a \geq 1) + P_a(X_2 = j, T_a \geq 2) + \dots \\ &= Y_j^a \text{ --- from (1)}\end{aligned}$$

If chain is recurrent,

$$\text{Define } \mu_j = \lambda_j - \lambda_a Y_j^a$$

$$0 = \mu_a = \sum_{i \in S} \mu_i p_{ia}^{(n)}$$

$$n \text{ st } p_{ja}^{(n)} > 0$$

$$\sum_{i \in S} \mu_i p_{ia}^{(n)} \geq \mu_i p_{ja}^{(n)}$$

$$\implies \mu_j = 0$$

This can be done for any  $k \in S$  since the chain is irreducible.  $\implies \mu = 0$

## Invariant Probability Distribution

**Theorem:** Consider an irreducible MC(S, P)

1. Some state  $i \in S$  is positive recurrent
2. All states are positive recurrent
3. The chain has an invariant probability distribution  $\lambda$ .

If the above holds,  $\lambda = 1/E_i[T_i] \forall i \in S$

**Proof:**

2  $\implies$  1 is obvious

1  $\implies$  3 -

$i \in S$  is recurrent, then MC is recurrent.  $\implies Y^i$  is an invariant measure.

$$\begin{aligned}\sum_{k \in S} Y_k^i &= \sum_k E_i[\sum_{n=0}^{T_a-1} I_{X_n=k}] = E_i[\sum_n^{T_a-1} \sum_k \mathbf{I}_{X_n=k}] \\ &= E_i[T_i]\end{aligned}$$

If positive recurrent,  $E_i[T_i] < \infty$ .

Hence  $\lambda_j = Y_j^i / E_i[T_i]$  is an invariant probability distribution.

3  $\implies$  2

Since  $\lambda$  is pdist,  $\exists j \text{ st } \lambda_j > 0$ .

$$\implies \lambda_k = \sum p_{ik}^{(n)} \lambda_i \forall n, j.$$

Pick an  $n$  such that  $p_{jk}^{(n)} > 0$ .

This  $\implies \lambda_k > 0 \forall k$ .

Fix  $a \in S$ ,

$$\pi_i = \lambda_i / \lambda_a$$

From before,

$$Y_i^a \geq \pi_i = \lambda_i / \lambda_a$$

Summing over all  $i \in S$  we get,

$$E_a[T_a] \geq 1/\lambda_a < \infty$$

*implies*  $a$  is positive recurrent.

But we can fix any  $a$ , since  $\lambda_i > 0$  for all  $i$ .

Hence, all  $a \in S$  are positive recurrent.

## Law of large numbers

Consider irreducible  $MC(\alpha, P)$ .

Define  $N_i(n) = \sum_{k=0}^{n-1} I_{X_k=i}$

1. If the chain is transient or Null recurrent,  $N_i(n)/n \rightarrow 0$  as  $n \rightarrow \infty$
2. If the chain is positive recurrent with invariant pdist  $\lambda$ .

$$N_i(n)/n \rightarrow 1/E_i[T_i]$$

$$T_i^{(r)} = \inf\{n > T_i^{r-1}; X_n = i\}$$

$$S_1 = T_i^{(1)}$$

$$S_2 = T_i^{(2)} - T_i^{(1)}$$

$\{S_j\}$  are i.i.d.