Markov Chains

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Contents

Definition

Equi 1: $X_{n n \geq 1}$ is a Markov Chain on a state space S (countable) with an initial distribution λ and transition matrix P if

1. $P(x_0 = i) = \lambda_i$
2. **Markov property**: $P(x_{m+1} = i_{m+1} | \text{PAST}) = P(x_{m+1} = i_{m+1} | x_m = i_m) = p_{i_m i_{m+1}}$

Equi 2: Given x_m the future $\{x_n : n > m\}$ and the past $\{x_n : n < m\}$ are independent.

Equi 3: $\{x_n\}$ is a MC(λ, P) if $P(x_0 = i_0, \dots x_m = i_m) = \lambda p_{i_0 i_1} p_{i_1 i_2} \dots$

Proof of equivalence

Showing 3 $\implies$ 1

Equi 3 $\implies$ Equi 1.1 is obvious.

$$P(x_m = i_m | \text{PAST}) = P(x_m = i_m, \text{PAST}) / P(\text{PAST})$$

From **Equi 3**,

$$P(x_m = i_m | \text{PAST}) = \frac{\lambda_{i_0} \prod_{k=1}^{m} p_{i_{k-1}, i_k}}{\lambda \prod_{k=1}^{m-1} p_{i_{k-1}, i_k}} = p_{i_{m-1}, i_m}$$

which is **Equi 1.2**.

Hence **Equi 3 $\implies$ Equi 1**

Showing 1 $\implies$ 3

$$P(x_m = i_m, \text{PAST}) = P(x_m = i_m | \text{PAST}) P(\text{PAST})$$

From **Equi 1.2**:

$$P(x_m = i_m | \text{PAST}) = p_{i_{m-1}, i_m}$$
$$\implies P(x_m = i_m, \text{PAST}) = p_{i_{m-1}, i_m} P(\text{PAST})$$

Now similarly pulling out each step from the past into the product, we get

$$P(x_0 = i_0, \dots x_m = i_m) = P(x_0 = i_0) \prod_{\substack{k=m \\ \Delta k = -1}}^{1} p_{i_{k-1}, i_k}$$

Finally, using **Equi 1.1**, we get **Equi 3**.

Transition Matrix

$$P = ((p_{ij}))_{i,j \in S}$$

where p_{ij} = probability that the chain jumps to state j if it is in state i.

Properties of the Transition Matrix

Stochasticity

Row-wise sum is 1. $\sum_j p_{ij}$ is the sum of the probability that given we are at i, we jump to any possible j. Since we must be *somewhere* every step, this sum must be 1.

Stationarity

Defn: A stationary distribution on the nodes of the MC is such that $(x_0, ...x_n)$ has the same distribution as $(x_m, ..., x_{m+n})$ for all m. That is, $X_m \sim X_l$ for any m and l.

$\mu_0(i) = P(x_0 = i) \forall i \in S$

$\mu_n(i) = P(x_n = i)$

$mu_1(i) = mu_0(i) * p_j i$

Which is $mu_i = mu_0 P^i$

A distribution π on S is called Stationary / invariant distribution of the chain MC(P) is $\pi = \pi P$ That is, π is a left eigenvector of P with eigenvalue 1.

> Exercise: Ehrenfest chain
>
> Chain of length N.
> $P(X_{n+1} = i + 1 | X_n = i) = (N - i)/N; P(X_{n+1} = i - 1 | X_n = i) = i/N$
> Find π.

Example

For $P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$, what is the stationary distribution? What are the entries of P^n?

Chapman Kolmogorov equation / Semigroup Property

$P^(n + m) = P^n P^m \forall n, m >= 0$

$$\begin{aligned}
p_{ij}^{(n+m)} &= P(X_{n+m} = j | X_0 = i) \\
&= \sum_k P(X_{n+m} = j, X_m = k | X_0 = i) \\
&= \sum_k P(X_{n+m} = j | X_m = k, X_0 = i) P(X_m = k | X_0 = 1) \\
&= \sum_k P(X_{n+m} = j | X_m = k) P(X_m = k | X_0 = 1) \\
&= \sum_k p_{ik}^m p_{kj}^n \\
\implies P^{n+m} &= P^n P^m
\end{aligned}$$

Going back to the example,

$$P = (1 - \alpha, \alpha; \beta, 1 - \beta)$$

$$
\begin{aligned}
p_{11}^{(n)} &= \sum_j p_{1j}^{n-1} p_{j1} \\
&= p_{11}^{n-1} p_{11} + p_{12}^{n-1} p_{21} \\
&= p_{11}^{n-1}(1-\alpha) + \beta(1 - p_{11}^{n-1})
\end{aligned}
$$

Exercise:

Similarly solve for other terms and find the values