Q2 $P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$.

$P^{n+1} = P^n P$.

$= \begin{bmatrix} p_{11}^n & 1-p_{12}^n \\ p_{21}^n & 1-p_{21}^n \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$

$\begin{bmatrix} p_{11}^{n+1} & \_\_ \\ p_{21}^{n+1} & \_\_ \end{bmatrix} = \begin{bmatrix} p_{11}^n - \alpha p_{11}^n + \beta - \beta p_{11}^n & \_\_\_\_ \\ p_{21}^n - \alpha p_{21}^n + \beta - \beta p_{21}^n & \_\_\_\_ \end{bmatrix}$

$\Rightarrow p_{11}^{n+1} = (1-\alpha-\beta) p_{11}^n + \beta$

$\sim p_{21}^{n+1} = (1-\alpha-\beta) p_{21}^n + \beta$

$p_{11}^n = A(1-\alpha-\beta)^n + \frac{\beta}{\alpha+\beta}$; where A is a constant

But $p_{11}^0 = 1 = A_{11} \cdot 1 + \frac{\beta}{\alpha+\beta}$

$\Rightarrow A_{11} = 1 - \frac{\beta}{\alpha+\beta}$.

$\Rightarrow p_{11}^{(n)} = \begin{cases} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \text{for } \alpha+\beta > 0 \\ 1 & \text{for } \alpha+\beta = 0 \end{cases}$

2) If $\alpha+\beta=0 \Rightarrow \alpha=\beta=0$ since $\alpha, \beta \geq 0$.
But this is a trivial chain.

$p_{21}^0 = 0 = A_{21} + \frac{\beta}{\alpha+\beta} \Rightarrow A_{21} = -\frac{\beta}{\alpha+\beta}$

$$\Rightarrow P_{21}^{n} = \begin{cases} \frac{\beta}{\alpha+\beta}\left[1-(1-\alpha-\beta)^{n}\right] & \text{if } \alpha+\beta > 0 \\ 0 & \text{if } \alpha+\beta = 0 \end{cases}$$

Now to find $\lim_{n\to\infty} P^{n}$, we take P_{11}^{n} and P_{21}^{n} with $\lim_{n\to\infty}$, for notational simplicity let the limits be P^{∞}, P_{11}^{∞} and P_{21}^{∞}.

$0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. $\Rightarrow 0 \leq \alpha+\beta \leq 2$.

$0 \geq -(\alpha+\beta) \geq -2 \Rightarrow 1 \geq 1-\alpha-\beta \geq -1$.

Equality is achieved if

1) $\alpha = \beta = 0 \Rightarrow P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is trivial

2) $\alpha = \beta = 1 \Rightarrow P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which does not have a limit

If equality is not acheived,

$\lim_{n\to\infty} (1-\alpha-\beta)^{n} = 0$.

$$\Rightarrow P_{11}^{\infty} = \frac{\beta}{\alpha+\beta} = P_{21}^{\infty} \Rightarrow P^{\infty} = \begin{bmatrix} \beta/\alpha+\beta & \alpha/\alpha+\beta \\ \beta/\alpha+\beta & \alpha/\alpha+\beta \end{bmatrix}$$

Q4. $S = \{Y, N\}$.

$$P = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}.$$

$$P^2 = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} = \begin{bmatrix} (1-p)^2+p^2 & 2(p-p^2) \\ 2(p-p^2) & p^2+(1-p)^2 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

$$\Rightarrow x_2 = x_0P^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1-p^2+p^2 & 2p-2p^2 \\ 2(p-p^2) & p^2+(1-p)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(p-p^2) & p^2+(1-p)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(p-p^2) & p^2+(1-p^2) \end{bmatrix}.$$

Which can also be seen from the fact that we can be on N in 2 steps either if →

1. We "stay" both times ; $P = (1-p)^2$
2. We "jump" both times ; $P = p^2$.