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Q2

i) Irreducible if there exists a single closed communicating class.

Let $i, j \in S$

If $i < j$

Note there exists a path from $i \to j$

$(i, i+1, i+2, \dots j)$

Where all hopping probabilities are $= p > 0$.

$\Rightarrow \quad i \longrightarrow j$

Similarly, $j < i$.

$i, i-1, i-2, \dots j$

is such a path. with all hopping probability $= 1-p > 0$

$\Rightarrow \quad i \to j$

Hence, all $i \to j$. $\Rightarrow$ any c.c.class must include all nodes, and hence there is only 1 ccclass, which is S itself.

Similarly, every subset has flow outward except for S itself. Hence, we have a single communicating class.

ii) Period of state is a class property.

Note $P_{ii}^{(1)} = 0 \; \forall i$, $P_{ii}^{(2)} = 2p(1-p) \quad \forall i \neq \{0, 1, N-1, N\}$.

$P_{11}^{(2)} = 1-p + p(1-p)$, $P_{N-1N-1}^{(2)} = p + p(1-p)$

$P_{00}^{(2)} = 1-p$, $P_{NN} = p$

$\Rightarrow \quad P_{ii}^{(2)} \neq 0 \; \forall i$

$\Rightarrow \quad P_{ii}^{2n} \neq 0 \; ; \quad P_{ii}^{2n+1} = 0 \qquad \forall n \in \mathbb{N}$

$\Rightarrow \quad d_i = \gcd(\{2n \mid n \in \mathbb{N}\}) = 2. \quad \forall i$

iii) let $A_i = \{0, 1, 2 \dots i\} \quad \forall i \in \{1 \dots N-1\}$.

$F(A_i, A_i^c) = p\pi_i = (1-p)\pi_{i+1} = F(A_i^c, A_i)$.

$\Rightarrow \dfrac{\pi_i}{\pi_{i+1}} = \dfrac{1-p}{p}$.

$\dfrac{\pi_0}{\pi_1} = (1-p)$.

$\sum \pi_i = 1$.

$\pi_i = \dfrac{\pi_i}{\pi_{i-1}} \cdot \dfrac{\pi_{i-1}}{\pi_{i-2}} \cdot \dots \dfrac{\pi_1}{\pi_0} \cdot \pi_0 \qquad \forall i \geq 1$ — 1

$= \left(\dfrac{p}{1-p}\right)^{i-1} \dfrac{1}{1-p} \pi_0 \qquad \forall i \geq 1$.

$\sum_i \pi_i = \dfrac{\pi_0}{1-p} \sum_{i=1}^{N} \left(\dfrac{p}{1-p}\right)^{i-1} + \pi_0$.

$= \dfrac{\pi_0}{1-p} \left[\sum_{i=0}^{N-1} \left(\dfrac{p}{1-p}\right)^{i}\right] + \pi_0$.

let $r = \dfrac{p}{1-p}$.

$\Rightarrow \sum \pi_i = \dfrac{\pi_0}{1-p} \cdot \dfrac{1-r^n}{1-r} + \pi_0 = 1$.

$\Rightarrow \pi_0 = \dfrac{1}{\dfrac{1}{1-p}\dfrac{1-r^n}{1-r} + 1}$; $\pi_{i \neq 0}$ can be found from 1

Q5 Let i and l be nodes such that they are max

$$P^n_{kj} \leq P^n_{lj} \quad \forall k$$

$$P_{ik} P^n_{kj} \leq P^n_{kj} \leq P^n_{lj} \quad \forall k, i$$

$$P(X_{n+1} = j \mid X_0 = i) = \sum_k P(X_{n+1} = j \mid X_1 = k) P(X_1 = k \mid X_0 = i)$$

$$= \sum_k P(X_n = j \mid X_0 = k) P_{ik}$$

$$\leq P(X_n = j \mid X_0 = l) \sum_k P_{ik}$$

$$= P(X_n = j \mid X_0 = l) \, 1.$$

$$\Rightarrow P^{(n+1)}_{ij} \leq P^{(n)}_{lj}$$

Now assume i, l st they are min.

$$P(X_{n+1} = j \mid X_0 = i) = \sum_k P(X_n = j \mid X_0 = k) P_{ik}.$$

$$\geq P(X_n = j \mid X_0 = l) \sum_k P_{ik}.$$

$$\Rightarrow P^{(n+1)}_{ij} \geq P^{(n)}_{lj}.$$