

Markov Chains

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25 August, 2022

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Definition

Equi 1: $X_{n \geq 1}$ is a Markov Chain on a state space S (countable) with an initial distribution λ and transition matrix P if

1. $P(x_0 = i) = \lambda_i$
2. **Markov property:** $P(x_{m+1} = i_{m+1} | \text{PAST}) = P(x_{m+1} = i_{m+1} | x_m = i_m) = p_{i_m i_{m+1}}$

Equi 2: Given x_m the future $\{x_n : n > m\}$ and the past $\{x_n : n < m\}$ are independent.

Equi 3: $\{x_n\}$ is a MC(λ, P) if $P(x_0 = i_0, \dots, x_m = i_m) = \lambda p_{i_0 i_1} p_{i_1 i_2} \dots$

Proof of equivalence

Showing 3 \implies 1

Equi 3 \implies Equi 1.1 is obvious.

$$P(x_m = i_m | \text{PAST}) = P(x_m = i_m, \text{PAST}) / P(\text{PAST})$$

From **Equi 3**,

$$P(x_m = i_m | \text{PAST}) = \frac{\lambda_{i_0} \prod_{k=1}^m p_{i_{k-1}, i_k}}{\lambda \prod_{k=1}^{m-1} p_{i_{k-1}, i_k}} = p_{i_{m-1}, i_m}$$

which is **Equi 1.2**.

Hence **Equi 3 \implies Equi 1**

Showing 1 \implies 3

$$P(x_m = i_m, \text{PAST}) = P(x_m = i_m | \text{PAST}) P(\text{PAST})$$

From **Equi 1.2**:

$$\begin{aligned} P(x_m = i_m | \text{PAST}) &= p_{i_{m-1}, i_m} \\ \implies P(x_m = i_m, \text{PAST}) &= p_{i_{m-1}, i_m} P(\text{PAST}) \end{aligned}$$

Now similarly pulling out each step from the past into the product, we get

$$P(x_0 = i_0, \dots, x_m = i_m) = P(x_0 = i_0) \prod_{\substack{k=m \\ \Delta k=-1}}^1 p_{i_{k-1}, i_k}$$

Finally, using **Equi 1.1**, we get **Equi 3**.

Transition Matrix

$$P = ((p_{ij}))_{i,j \in S}$$

where p_{ij} = probability that the chain jumps to state j if it is in state i .

Stochasticity

Row-wise sum is 1. $\sum_j p_{ij}$ is the sum of the probability that given we are at i , we jump to any possible j . Since we must be *somewhere* every step, this sum must be 1.

Chapman Kolmogorov equation / Semigroup Property

$$P^{(n+m)} = P^n P^m \forall n, m \geq 0$$

$$\begin{aligned} p_{ij}^{(n+m)} &= P(X_{n+m} = j | X_0 = i) \\ &= \sum_k P(X_{n+m} = j, X_m = k | X_0 = i) \\ &= \sum_k P(X_{n+m} = j | X_m = k, X_0 = i) P(X_m = k | X_0 = i) \\ &= \sum_k P(X_{n+m} = j | X_m = k) P(X_m = k | X_0 = i) \\ &= \sum_k p_{ik}^m p_{kj}^n \\ \implies P^{n+m} &= P^n P^m \end{aligned}$$

Going back to the example,

$$P = (1 - \alpha, \alpha; \beta, 1 - \beta)$$

$$\begin{aligned} p_{11}^{(n)} &= \sum_j p_{1j}^{n-1} p_{j1} \\ &= p_{11}^{n-1} p_{11} + p_{12}^{n-1} p_{21} \\ &= p_{11}^{n-1} (1 - \alpha) + \beta (1 - p_{11}^{n-1}) \end{aligned}$$

Exercise:

Similarly solve for other terms and find the values

Stationary distribution of an MC

Defn: A stationary distribution on the nodes of the MC is such that (x_0, \dots, x_n) has the same distribution as (x_m, \dots, x_{m+n}) for all m . That is, $X_m \sim X_l$ for any m and l .

$$\mu_0(i) = P(x_0 = i) \forall i \in S$$

$$\mu_n(i) = P(x_n = i)$$

$$\mu_{n+1}(i) = \mu_n(i) * p_j i$$

$$\text{Which is } \mu_i = \mu_0 P^i$$

A distribution π on S is called Stationary / invariant distribution of the chain $MC(P)$ is $\pi = \pi P$ That is, π is a left eigenvector of P with eigenvalue 1.

Equi: S is finite, $|S| = N$, π_i in \mathbb{R}^N_+ is called a stationary or invariant distn of the $MC(P)$ if it satisfies 1.

Balance Condition: $\pi_i P_{ij} = \pi_j P_{ji}$

Exercise: Ehrenfest chain

Chain of length N .

$$P(X_{n+1} = i + 1 | X_n = i) = (N - i)/N; P(X_{n+1} = i - 1 | X_n = i) = i/N$$

Find π .

Example - 2 State MC

For $P = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix}$, what is the stationary distribution? What are the entries of P^n ?

Solved in Assignment 1.

Example - Bus Stop

Buses arrive at a bus stop st the inter-arrival times are iid. At time n , x_n is the time until the next bus arrives

$$p_{i+1,i} = 1, p_{1,i} = q(i)$$

$$\pi P = \pi$$

$$P = \begin{bmatrix} q(1) & q(2) & q(3) & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & & & \end{bmatrix}$$

The balance equation also leads to

$$\pi(i) = \pi(i + 1) + \pi(1)q(i)$$

Normalization leads to

$$\sum_{i \in S} \pi(i) = 1 \Rightarrow \pi(1) \sum_i \sum_{j \geq i} q(j) = 1$$

$$p_i(1) = \frac{1}{\sum_i \sum_{j \geq i} q(j)}$$

For stationary distribution to exist, we need the double sum to be finite, else $\pi(1) = 0$ which implies that $p_i(i) = 0$

$$\sum_i \sum_{j \geq i} q(j) = \sum_j j q(j) = E[\text{time between arrival}]$$

Flow of a MC

Defn: For $A \subset S$, define $F(A, A^C) = \sum_{i \in A} \sum_{j \in A^C} \pi(i) p_{ij}$

Thm: π satisfies the balance equation iff $F(A, A^C) = F(A^C, A) \forall A \subset S$

Proof: Suppose thm holds *forall* $A \subset S$. For $A = \{x\}$,

1. $F(AA^C) = \sum_{j \neq k} \pi(k) p_{kj}$
2. $F(A^C A) = \sum_{i \neq k} \pi(i) p_{ik}$

Then,

$$\sum_{j \neq k} \pi(k) p_{kj} = \sum_{i \neq k} \pi(i) p_{ik}$$

$$\sum_{j \neq k} \pi(k) p_{kj} = \sum_{i \in S} \pi(i) p_{ik} - \pi(k) p_{kk}$$

$$\implies \pi(k) = \sum_{j \in S} p_{kj} = \sum_{i \in S} \pi(i) p_{ik}$$

$$\implies \pi(k) = \sum_{i \in S} \pi(i) p_{ik}$$

Conversely, $\pi(i) = \sum_{j \in S} \pi(j) p_{ji} = \sum_{j \in A} \pi(i) p_{ij} + \sum_{j \in A^C} \pi(i) p_{ij} = \sum_A \pi(j) p_{ji} + \sum_{A^C} \pi(j) p_{ji}$

Now, sum over $i \in A$ on both sides and conclude

Exercise: Consider the Gambler's Ruin MC with $M = \infty$, reflecting boundary condition at 0.
Take $A = \{0, 1, \dots, n-1\}$ Write $F(A, A^C) = F(A^C, A)$ and solve for π

From the balance equation

$$p_i(i) = p p_i(i+1) + q p_i(i-1)$$

From the flow equation,

$$\text{Let } A = \{0, 1, \dots, i-1\}$$

$$F(A, A^C) = p \pi(i-1) = q \pi(i) = F(A^C, A)$$

This along with the normalization condition allows us to find π

Class structure of a MC

Def: $i, j \in S, i \longrightarrow j$ if there exists $n \geq 0$ st $p_{ij}^{(n)} > 0 \implies \exists \text{PATH from } i \text{ to } j$. \longrightarrow is transitive and reflexive.

Def: $\leftrightarrow: i \leftrightarrow j$, or "i communicates with j" iff $i \longrightarrow j$ and $j \longrightarrow i$. \leftrightarrow is an equivalence relation.

Note, $S = \bigsqcup_i C_i$ where C_i s are called communicating classes.

Closed communicating Classes

Def: $\sum_{j \in C} p_{ij} = 1 \forall i \in C$

If C is a closed communicating class then, if $i \in C$ and $i \longrightarrow j \implies j \in C$

Thm: If C is a closed communicating class of $\text{MC}(P)$ then C is a closed communicating class of $\text{MC}(P^n)$

Proof:

If $i \in C$ and $i \rightarrow j$ in $\text{MC}(P^n) \exists \text{PATH}_{i \rightarrow j}$

Irreducible

If a chain has only one closed communicating class, it is called irreducible.

For any $i, j \in S \exists n > 0$ st $p_{ij}^{(n)} > 0$. If $\{i\}$ is a closed communicating class i is called an absorbing state.

Period of an state

Note that

$$p_{ij}^{(nk)} \geq (p_{ii}^{(n)})^k$$

from Chapman Kolmogorov theorem,

$$p_{ii}^{(m)} > 0 \Rightarrow p_{ii}^{(n)} > 0 \text{ if } m|n$$

Period of i is defined as $d(i) = \gcd\{n : p_{ii}^{(n)} > 0\}$

i is called aperiodic if $d(i) = 1$

Period is a class property

If i and j are in same communicating class, $i \leftrightarrow j$, then $d(i) = d(j)$

$$D_i = n : p_{ii}^n > 0; d_i = \gcd(D_i) \quad D_j = n : p_{jj}^n > 0; d_j = \gcd(D_j)$$

Since $i \leftrightarrow j \Rightarrow \exists n_1, n_2, stp_{ij}^{(n_1)}, p_{ji}^{(n_2)} > 0$

Note, d_i and d_j both divide $n_1 + n_2$

For any n in $D(i)$,

$$p_{jj}^{(n_1+n_2+n)} \geq p_{ji}^{(n_2)} p_{ii}^{(n)} p_{ij}^{(n_1)} > 0$$

$$\Rightarrow d_j | n_1 + n_2 + n \Rightarrow d_j | n$$

$$\Rightarrow d_j | n \forall n \in D_i \Rightarrow d_j \leq d_i$$

Similarly $d_i \leq d_j$.

Hence $d_i = d_j$.

Theorem

If $i \in S$ be aperiodic, then there exists n_0 st $p_{ii}^{(n)} > 0 \forall n \geq N$.

Proof: If $D_i = \{n \geq 0, p_{ii}^n > 0\}$ Take n_1, n_2 in D_1 st $n_2 - n_1 = 1$

for n in \mathbf{N} , $n = qn_1 + r$, $r \leq n_1 - 1$ $n = (q - r)n_1 + rn_2$.

For large n , $q - r > 0$, and $(q - r)n_1$ and rn_2 are both positive and in D_1 .

Exercise: An irreducible chain is aperiodic iff $\exists n st p_{ij}^{(n)} > 0 \forall i, j \in S$

Theorem

Let $\{X_n\}$ be irreducible of period $d > 1$. Then it can be decomposed to a disjoint union of sets C_0, C_1, \dots, C_{d-1} such that

$$\sum_{j \in C_{r+1}} p_{ij} = 1 \forall i \in C_r \forall r$$

Pf:

Define a relation $i \leftrightarrow^d j \iff p_{ij}^{(nd)} > 0$ for some $n \in \mathbf{N}$ in an irreducible chain of period d .

This relation is transitive and reflexive.

Proof of Symmetric-ness -

$$p_{ij}^{(\alpha d)} > 0 \text{ for some } \alpha$$

Since the chain is irr, $j \rightarrow i$,

So exists $\beta > 0$ $stp_{ji}^{(\beta)} > 0$

$$\implies p_{ii}^{(\alpha d + \beta)} \geq p_{ij}^{(\alpha d)} p_{ji}^{(\beta)} > 0$$

but period of i is d , which means $d | \alpha d + \beta \implies d | \beta$

Hence, this is an equivalence relation.

Lemma 2: S can be written as a disjoint union of the equivalence classes.

Pick $i_0 \in S$, and denote its equivalence class (under \leftrightarrow^d) as C_0 . Then pick $i_1 \in S$ $stp_{i_0 i_1} > 0$. Denote its equivalence class as C_1 . Similarly do until i_{d-1} and C_{d-1} .

Note that i_d MUST be in C_0 , because there exists a path of length d .

Let i be in C_0 , and $stp_{ij} > 0$ for some $j \in S$, then j must be in C_1 .

Suppose $j \notin C_1$, but in C_2 .

Then consider PATH $(i_0 \rightarrow i \rightarrow j \rightarrow i_2 \rightarrow i_3 \rightarrow \dots \rightarrow i_{d-1})$

This $i_0 \rightarrow i$ is of length αd , and $j \rightarrow i_2$ is of length βd . Then, the length of this new PATH is $d + 1$, which contradicts the fact that i_d must be in C_0 .

Similar argument can be made for all other pairs which are not $(r, r + 1)$.