Markov Chains

Dhruva Sambrani

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Contents

Definition

Equi 1: $X_{n n \geq 1}$ is a Markov Chain on a state space S (countable) with an initial distribution λ and transition matrix P if

1. $P(x_0 = i) = \lambda_i$
2. **Markov property**: $P(x_{m+1} = i_{m+1} | \text{PAST}) = P(x_{m+1} = i_{m+1} | x_m = i_m) = p_{i_m i_{m+1}}$

Equi 2: Given x_m the future $\{x_n : n > m\}$ and the past $\{x_n : n < m\}$ are independent.

Equi 3: $\{x_n\}$ is a MC(λ, P) if $P(x_0 = i_0, \dots x_m = i_m) = \lambda p_{i_0 i_1} p_{i_1 i_2} \dots$

Proof of equivalence

Showing 3 $\implies$ 1

Equi 3 $\implies$ Equi 1.1 is obvious.

$$P(x_m = i_m | \text{PAST}) = P(x_m = i_m, \text{PAST}) / P(\text{PAST})$$

From **Equi 3**,

$$P(x_m = i_m | \text{PAST}) = \frac{\lambda_{i_0} \prod_{k=1}^{m} p_{i_{k-1}, i_k}}{\lambda \prod_{k=1}^{m-1} p_{i_{k-1}, i_k}} = p_{i_{m-1}, i_m}$$

which is **Equi 1.2**.

Hence **Equi 3 $\implies$ Equi 1**

Showing 1 $\implies$ 3

$$P(x_m = i_m, \text{PAST}) = P(x_m = i_m | \text{PAST}) P(\text{PAST})$$

From **Equi 1.2**:

$$P(x_m = i_m | \text{PAST}) = p_{i_{m-1}, i_m}$$
$$\implies P(x_m = i_m, \text{PAST}) = p_{i_{m-1}, i_m} P(\text{PAST})$$

Now similarly pulling out each step from the past into the product, we get

$$P(x_0 = i_0, \dots x_m = i_m) = P(x_0 = i_0) \prod_{\substack{k=m \\ \Delta k = -1}}^{1} p_{i_{k-1}, i_k}$$

Finally, using **Equi 1.1**, we get **Equi 3**.

Transition Matrix

$$P = ((p_{ij}))_{i,j \in S}$$

where p_{ij} = probability that the chain jumps to state j if it is in state i.

Stochasticity

Row-wise sum is 1. $\sum_j p_{ij}$ is the sum of the probability that given we are at i, we jump to any possible j. Since we must be *somewhere* every step, this sum must be 1.

Chapman Kolmogorov equation / Semigroup Property

$P^(n + m) = P^n P^m \forall n, m >= 0$

$$\begin{aligned}
p_{ij}^{(n+m)} &= P(X_{n+m} = j | X_0 = i) \\
&= \sum_k P(X_{n+m} = j, X_m = k | X_0 = i) \\
&= \sum_k P(X_{n+m} = j | X_m = k, X_0 = i) P(X_m = k | X_0 = 1) \\
&= \sum_k P(X_{n+m} = j | X_m = k) P(X_m = k | X_0 = 1) \\
&= \sum_k p_{ik}^m p_{kj}^n \\
\implies P^{n+m} &= P^n P^m
\end{aligned}$$

Going back to the example,

$$P = (1 - \alpha, \alpha; \beta, 1 - \beta)$$

$$\begin{aligned}
p_{11}^{(n)} &= \sum_j p_{1j}^{n-1} p_{j1} \\
&= p_{11}^{n-1} p_{11} + p_{12}^{n-1} p_{21} \\
&= p_{11}^{n-1}(1 - \alpha) + \beta(1 - p_{11}^{n-1})
\end{aligned}$$

Exercise:

Similarly solve for other terms and find the values

Stationary distribution of an MC

Defn: A stationary distribution on the nodes of the MC is such that $(x_0, ...x_n)$ has the same distribution as $(x_m, ..., x_{m+n})$ for all m. That is, $X_m \sim X_l$ for any m and l.

$\mu_0(i) = P(x_0 = i) \forall i \in S$

$\mu_n(i) = P(x_n = i)$

$mu_1(i) = mu_0(i) * p_j i$

Which is $mu_i = mu_0 P^i$

A distribution π on S is called Stationary / invariant distribution of the chain MC(P) is $\pi = \pi P$ That is, π is a left eigenvector of P with eigenvalue 1.

Equi: S is finite, |S| = N, pi in |R^N_+ is called a stationary or invariant distn of the MS(P) if it satisfies 1. Balance Condition: piP = pi 2. pi mathbf 1 = 1

Exercise: Ehrenfest chain

Chain of length N.
$P(X_{n+1} = i+1 | X_n = i) = (N-i)/N; P(X_{n+1} = i-1 | X_n = i) = i/N$
Find π.

Example - 2 State MC

For $P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$, what is the stationary distribution? What are the entries of P^n?

Solved in Assignment 1.

Example - Bus Stop

Buses arrive at a bus stop st the inter-arrival times are iid. At time n, x_n is the time until the next bus arrives

$p_{i+1,i} = 1,\ p_{1,i} = q(i)$

$\pi P = \pi$

$$P = \begin{bmatrix} q(1) & q(2) & q(3) & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & & & \end{bmatrix}$$

Th balance equation also leads to

$$\pi(i) = \pi(i+1) + \pi(1)q(i)$$

Normalization leads to

$$\sum_{i \in S} \pi(i) = 1 => \pi(1) sum_i sum_{j \geq i} q(j) = 1$$

$$pi(1) = \frac{1}{sum_i sum_{j \geq i} q(j)}$$

For stationary distribution to exist, we need the double sum to be finite, else $\pi(1) = 0$ which implies that $pi(i) = 0$

$$sum_i sum_{j \geq i} q(j) = sum_j j q(j) = E[\text{time between arrival}]$$

Flow of a MC

Defn: For $A \subset S$, define $F(A, A^C) = \sum_{i \in A} \sum_{j in A^C} \pi(i) p_{ij}$

Thm: π satisfies the balance equation iff $F(A, A^C) = F(A^C, A) \forall A \subset S$

Proof: Suppose thm holds $for all A \subset S$. For $A = \{x\}$,

1. $F(AA^C) = \sum_{j \neq k} \pi(k) p_{kj}$
2. $F(A^C A) = \sum_{i \neq k} \pi(i) p_{ik}$

Then,

$$\sum_{j\neq k}\pi(k)p_{kj} = sum_{i\neq k}\pi(i)p_{ik}$$

$$sum_{j\neq k}\pi(k)p_{kj} = sum_{i\in S}\pi(i)p_{ik} - \pi(k)p_{kk}$$

$$\implies \pi(k) = sum_{jinS}p_{kj} = sum_{iinS}\pi(i)p_{ik}$$
$$\implies \pi(k) = sum_{iinS}\pi(i)p_{ik}$$

Conversely, $\pi(i) = \sum_{j\in S}\pi(j)p_{ji}$ $\sum_{j\in A}\pi(i)p_{ij} + sum_{jinA^C}\pi(i)p_{ij} = \sum_A pi(j)p_{ji} + sum_{A^C}\pi(j)p_{ji}$

Now, sum over $i \in A$ on both sides and conclude

> Exercise: Consider the Gambler's Ruin MC with $M = \infty$, reflecting boundary condition at 0. Take $A = \{0, 1, ..n-1\}$ Write $F(A, A^C) = F(A^C, A)$ and solve for π

From the balance equation

$pi(i) = ppi(i+1) + qpi(i-1)$

From the flow equation,

Let $A = \{0, 1, ..i-1\}$

$F(A, A^C) = p\pi(i-1) = q\pi(i) = F(A^C, A)$

This along with the normalization condition allows us to find π

Class structure of a MC

Def: $i, j \in S, i \longrightarrow j$ if there exists $n \geq 0 st p_{ij}^{(n)} > 0 \implies \exists$PATH from i to j. $\longrightarrow$ is transitive and reflexive.

Def: $\leftrightarrow$: $i \leftrightarrow j$, or "i communicates with j" iff $i \longrightarrow j$ and $j \longrightarrow i$. $\leftrightarrow$ is an equivalence relation.

Note, $S = \bigsqcup_i C_i$ where C_is are called communicating classes.

Closed communicating Classes

Def: $\sum_{j\in C} p_{ij} = 1 \forall i \in C$

If C is a closed communicating class then, if $i \in C$ and $i \longrightarrow j \implies j \in C$

Thm: If C is a closed communicating class of MC(P) then C is a closed communicating class of MC(P^n)

Proof:

If $i \in C$ and $i \to j$ in MC(P^n) $\exists$PATH$_{i\to j}$

Irreducable

If a chain has only one closed communicating class, it is called irreducable.

For any $i, j \in S \exists n > 0 st p_{ij}^{(n)} > 0$. If $\{i\}$ is a closed communicating class i is called an absorbing state.

Period of an state

Note that

$p_{ij}^{(nk)} \geq (p_{ii}^{(n)})^k$

from Chapman Kolmogorov theorem,

$p_{ii}^{(m)} > 0 => p_{ii}^{(n)} > 0$if $m|n$

Period of i is defined as $d(i) = gcd\{n : p_{ii}^{(n)} > 0\}$

i is called aperiodic if $d(i) = 1$

Period is a class property

If i and j are in same communicating class, $i \leftrightarrow j$, then $d(i) = d(j)$

$D_i = n : p_{ii}^n > 0; d_i = \gcd(D_i)$ $D_j = n : p_{jj}^n > 0; d_j = \gcd(D_j)$

Since $i \leftrightarrow j \implies \exists n_1, n_2, stp_{ij}^{(n_1)}, p_{ji}^{(n_2)} >= 0$

Note, d_i and d_j both divide $n_1 + n_2$

For any n in $D(i)$,

$p_{jj}^{(n_1+n_2+n)} \geq p_{ji}^{(n_2)} p_{ii}^{(n)} p_{ij}^{(n_1)} > 0$

$\implies d_j|n1 + n2 + n \implies d_j|n$

$\implies d_j|n \forall n \in D_i \implies d_j \leq d_i$

Similarly $d_i \leq d_j$.

Hence $d_i = d_j$.

Theorem

If $i \in S$ be aperiodic, then there exists n_0 st $p_{ii}^{(n)} > 0 \forall n \geq N$.

Proof: If $D_i = \{n \geq 0, p_{ii}^n > 0\}$ Take n_1, n_2 in D_1 st $n_2 - n_1 = 1$

for n in $\mathbf{N}$, $n = qn_1 + r$, $r \leq n - 1$ $n = (q - r)n_1 + rn_2$.

For large n, $q - r > 0$, and $(q - r)n_1$ and rn_2 are both positive and in D_1.

Exercise: An irreducable chain is aperiodic iff $\exists nstp_{ij}^{(n)} > 0 \forall i, j \in S$

Theorem

Let $\{X_n\}$ be irreducable of period $d > 1$. Then it can be decomposed to a disjoint union of sets $C_0, C_1, ...C_{d-1}$ such that

$\sum_{j \in C_{r+1}} p_{ij} = 1 \forall i \in C_r \forall r$

Pf:

Define a relation $i \leftrightarrow^d j \iff p_{ij}^(nd) > 0$ for some $n \in \mathbf{N}$ in an irreducable chain of period d.

This relation is transitive and reflexive.

Proof of Symmetric-ness -

$p_{ij}^{(\alpha d)} > 0$ for some α

Since the chain is irr, $j \to i$,

So exists $beta > 0 stp_{ji}^{(beta)} > 0$

$\implies p_{ii}^{(\alpha d+b)} \geq p_{ij}^{(\alpha d)} p_{ji}^{(beta)} > 0$

but period of i is d, which means $d|\alpha d + \beta \implies d|\beta$

Hence, this is an equivalence relation.

Lemma 2: S can be written as a disjoint union of the equivalence classes.

Pick $i_0 \in S$, and denote its equivalence class (under $\leftrightarrow^d$) as C_0. Then pick $i_1 \in Sstp_{i_0 i_1} > 0$. Denote its equivalence class as C_1. Similarly do until i_{d-1} and C_{d-1}.

Note that i_d MUST be in C_0, because there exists a path of length d.

Let i be in C_0, and $p_i j > 0$ for some $j \in S$, then j must be in C_1.

Suppose $j \notin C_1$, but in C_2.

Then consider PATH $(i_0 \to i \to j \to i_2 \to i_3 \to \cdots \to i_{d-1})$

This $i_0 - > i$ is of length ad, and $j - > i_2$ is of length bd. Then, the length of this new PATH is $d+1$, which contradicts the fact that i_d must be in C_0.

Similar argument can be made for all other pairs which are not $(r, r+1)$.