Gambler's Ruin

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1 The Game

Let there be finite Money M. A game is played with 2 players, A, B.

At each time step, t a coin is flipped, and if X loses, X -= 1, and Y+=1

$$P(A \text{ wins}) = p$$

Let X_n be amount of money that A has at time n. The Game stops when one of the players is ruined, which is if $X_n = 0$ or $X_N = M$. When a particular player hits 0, they are said to be ruined.

2 Eventual Ruin

Then, we are interested in calculating

$$P(A \text{ is eventually ruined}) = P(R_A) =?$$

We can note that A is eventually ruined if $X_n = 0$ for any n, which means,

$$R_A = \bigcup_{n \geq 0} X_n = 0$$

So now, we can formalise this problem as

$$p_M(k) = P(R_A | X_0 = k) =?$$

2.1 Examples

Let us consider some cases.

2.1.1 M = 1

$$p_1(1) = 1 \text{ and } p_1(0) = 0$$

This is trivial since the game cannot even be played.

2.1.2 M = 2

$$p_2(0) = 0,\ p_2(1) = q,\ p_2(2) = 1$$

Boundary cases are trivial, $p_2(1)$ is the probability that A loses the first round, which is q.

2.1.3 M = 3

$$p_3(0) = 0,\ p_3(1) = q\sum_i (pq)^i,\ p_3(2) = q^2 \sum_i (pq)^i,\ p_3(3) = 1$$

Boundary cases are trivial. $p_3(1)$ is the probability that A's trajectory is [WLWL ... (WL) i times ... WLL]. The (WL) i times has a probability of $(pq)^i$, and the last L has a probability q. A similar analysis can be done for $p_3(2)$

2.1.4 M = 4

This is already very difficult to enumerate

2.2 Actually Solving

2.2.1 First Step analysis

Lemma: For $k \in \{1, M-1\}$ -

$$P(R_A|X_0 = k) = pP(R_A|X_0 = k+1) + qP(R_A|X_0 = k-1)$$

Also trivially,

$$P(R_A|X_0 = 0) = 1 \text{ and } P(R_A|X_0 = M) = 0$$

since the game has ended before any rounds are played.

2.2.1.1 Proof of Lemma

$$P(R_A|X_0 = k) = P(R_A, X_1 = k+1|X_0 = k) + P(R_A, X_1 = k-1|X_0 = k)$$

$$\implies P(R_A|X_0 = k) = \frac{P(R_A, X_1 = k+1, X_0 = k)}{P(X_0 = k)} + \frac{P(R_A, X_1 = k-1, X_0 = k)}{P(X_0 = k)}$$

But $P(X_1 = k+1|X_0 = k) = p$

Trying to introduce p, and similarly q

$$\begin{aligned}\implies P(R_A|X_0 = k) =& P(R_A|X_1 = k+1, X_0 = k)P(X_1 = k+1|X_0 = k) \\ &+ P(R_A|X_1 = k-1, X_0 = k)P(X_1 = k-1|X_0 = k)\end{aligned}$$

Now all that is left to show is that

$$P(R_A|X_1 = k+1, X_0 = k) = P(R_A|X_0 = k+1)$$

and

$$P(R_A|X_1 = k-1, X_0 = k) = P(R_A|X_0 = k-1)$$

This is the Markovian assumption, that we defer to a later time.

2.3 Solution

From Lemma 1

$$(p+q)p_M(k) = pp_M(k+1) - qp_M(k-1)$$

$$\implies p_M(k+1) - p_M(k) = q/p(p_M(k) - p_M(k-1))$$

$$\implies p_M(k+1) - p_M(k) = (q/p)^k(p_M(1) - p_M(0))$$

$$\implies p_M(n) = p_M(0) + \sum_{k=0}^{n-1} p_M(k+1) - p_M(k)$$

$$\implies p_M(n) = 1 + (P_M(1) - P_M(0)) \sum_{k=0}^{n-1} (p/q)^k$$

$$\implies p_M(n) = 1 + (P_M(1) - 1)\frac{1-r^n}{1-r}, \text{ where } r = q/p$$

Using the fact that $p_M(M) = 0$,

$$p_M(1) - 1 = \frac{1-r}{1-r^M}$$

and then replacing into the previous equation,

$$p_M(n) = \begin{cases} \frac{(\frac{q}{p})^n - (\frac{q}{p})^M)}{1-(\frac{q}{p})^M} & q \neq p \\ 1 - \frac{n}{M} & q = p \end{cases}$$

Taking $\lim_{M\to\infty}$

$$\lim_{M\to\infty} p_M(n) = \begin{cases} (q/p)^n & q < p \\ 1 & \text{otherwise} \end{cases}$$

Exercise 1:

Solve the recursion for p = q and complete $\lim_{M\to\infty}$

Exercise 2:

Do the same for B, and show that $P(R_A \cup R_B) = 1$

2.3.1 Almost surely

An event E is said to occur almost surely iff $P(E) = 1$

2.4 Markovian assumption

$$\begin{aligned}
P\left(R_A | X_1 = k+1, X_0 = k\right) &= \frac{P\left(R_A, X_1 = k+1, X_0 = k\right)}{P\left(X_1 = k+1, X_0 = k\right)} \\
&= \frac{P\left(\bigcup_{n\geq 0}\{X_n = 0\}, X_1 = k+1, X_0 = k\right)}{P\left(X_1 = k+1, X_0 = k\right)} \\
&= \frac{P\left(\bigcup_{n\geq 2}\{X_n = 0\} \cap \{X_1 = k+1\} \cap \{X_0 = k\}\right)}{P\left(\{X_1 = k+1\} \cap \{X_0 = k\}\right)} \\
&= \frac{P\left(\bigcup_{n\geq 2}\{X_n = 0\} \cap \{X_1 - X_0 = 1\} \cap \{X_1 = k+1\}\right)}{P\left(\{X_1 - X_0 = 1\} \cap \{X_1 = k+1\}\right)} \\
&= \frac{P\left(\bigcup_{n\geq 2}\{X_n = 0\} \cap \{X_1 = k+1\}\right)}{P\left(\{X_1 = k+1\}\right)} \\
&= P(R_A | X_1 = x+1) = P(R_A | X_0 = x+1)
\end{aligned}$$

3 Expected duration of game

$$T = \inf\{n \geq 0 | X_n = 0 \text{ or } M\}$$

T is the first hitting time of the walk.

$$E[T|X_0 = k]$$

3.1 Expectation of a random Variable

Let X be a random variable, on $(\Omega, mathcalF, P)$

$$\begin{aligned} E[X] &= \int_\Omega X dP \\ &= \int_\Omega x f(x) dx \\ &= \sum_\Omega x_i P(x = x_i) \end{aligned}$$

3.1.1 Conditioned Expectation

$$\begin{aligned} E[X|A] &= 1/P(A) \int_A x dP \\ &= 1/P(A) \int_\Omega X \mathbf{I}_A dP \\ &= 1/P(A) E[X \mathbf{I}_A] \end{aligned}$$

where A is an event.

Similarly, $f_{X|Y}(x|y) = f_{X,Y}(x, y)/f_Y(y)$

Going back to the Game, we know that $E[T|X_0 = 0] = E[T|X_1 = M] = 0$

3.1.2 Cases

M = 2:

$E[T|X_0 = 1] = 1$

M = 3:

$E[T|X_0 = 1] = 1q + 3pq^2 + ... + 2p^2 + 4q^2p^2 + ...$

$$\begin{aligned} P(T = 2k|X_0) &= p^2(pq)^k - 1 \\ P(T = 2k + 1|X_0) &= q(pq)^k \end{aligned}$$

From these two, we can find the Expectation value.

Exercise 3: $E_2[T|X_0 = 2] = (2q^2 + p + qp^2)/(1 - pq)^2$

3.2 General Solution

3.2.1 Lemma 2

For $k = 0, \dots, M-1$

$$E[T|X_0 = k] = 1 + pE[T|X0 = k+1] + qE[T|X_0 = k-1]$$

$$(p+q)\mu(k) = 1 + p\mu(k+1) + q\mu(k-1)$$

Exercise 4: $mu_M(x) = \frac{1}{q-p}\left(k - M\frac{1-r^k}{1-r^M}\right)$ for $q \neq p$ $\mu_M(x) = k(M-k)$ for $q = p$

Taking $\lim_{M\to\infty}$

$$\mu_\infty(k) = \begin{cases} \infty & q \leq p \\ \frac{k}{q-p} & q > p \end{cases}$$

3.2.1.1 Proof of Lemma

$$\mathbf{I}_{X_0=k} = \mathbf{I}_{X_0=k,X_1=k+1} + \mathbf{I}_{X_0=k,X_1=k-1}$$

$$\begin{aligned} E[T|X_0 = k] &= E[TI_{X_0=k}]/P(x_0 = k) \\ &= 1/P(X_0 = k)(E[TI_{x0=k,x1=k+1}] + E[TI_{x_0=k,x_1=k-1}]) \end{aligned}$$

Multiplying and dividing $P(X_1 = k+1, X_0 = k)$ and $P(X_1 = k-1, X_0 = k)$ appropriately,

$$E[T|X_0 = k] = pE[T|X_1 = k+1, X_0 = k] + qE[T|X_1 = k-1, X_0 = k]$$

Now we shift time, but we must add 1 to the expectation because the walk actually happened for that step.

$$E[T|X_0 = k] = 1 + pE[T|X_0 = k+1] + qE[T|X_0 = k+1]$$

4 Markov Chains

A Markov Chain is a stochastic process $\{X_n\}_{n\in\mathcal{I}}$ that satisfies the "Markov Property".

It is called a chain if the index set is countable and X_n comes from a countable set S called the state space of the MC. Then, $\mathcal{I} = \mathbf{N} \cup \{0\}$

4.1 Homogenous MC

Defn: A stochastic process $\{X_n\}$ is called a Markov chain with initial distribution lambda and transition matrix P if

1. $P(X_0 = i) = \lambda_i$
2. $\begin{aligned} P(X_{m+1}) &= i_{m+1}|X_m = i_m, ...X_0 = i_0) \\ &= P(X_{m+1} = i_m + 1|X_m = i_m)\$ \\ &= p_{i_m->i_{m+1}} \end{aligned}$
3. Homogeneity - $p_{i\to j} = P(x_m = j|x_{m-1} = i)$ is independant of time.

Equivalently: Given the present X_n the future $\{X_m, m > n\}$ is independent of the past $\{X_m, m < n\}$

Equivalently: The $\{X_n\}$ is a $MC(\lambda, P)$ if $\forall n > 0$ and i_0,