Poisson Processes

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Poisson Processes

Definition 1

Let $N(t)$ be a counting process. If

1. $N(t)$ has Independent increments
2. $P(N(t+s) - N(s) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \forall S \geq 0$ some $\lambda > 0$

then it is a Poisson Process.

Definition 2

Let there be a counting process $N(t)$ such that

1. $N(0) = 0$
2. Independent increments
3. $P(N(t) = 1) = \lambda t + o(t)$
4. $P(N(t) >= 2) = o(t)$

Theorem: The definitions are equivalent.

Proof:

(1) $\implies$ (2) is obvious from taking the taylor expansion.

Define $p_n(t) = P(N(t) = n)$

For $n = 0$,

$$\begin{aligned} p_0(t+h) = P(N(t+h) = 0) &= P(N(t) = 0, N(t+h) - N(t) = 0) \\ &= P(N(t)) + P(N(t+h) - N(t) = 0) \leftarrow \text{ from independance} \\ &= p_0(t)(1 - (P(N(t) = 1) + P(N(t) >= 2)) \\ &= p_0(t)(1 - (\lambda h + o(h))) \end{aligned}$$

$$\implies \frac{p_0(t+h) - p_0(t)}{h} = -\lambda p_0 + o(h)/h$$

$$\implies \frac{dp_0}{dt} = -\lambda p_0$$

$p_0 = e^{-\lambda t}$

For $n \neq 0$,

$$\begin{aligned}p_n(t+h) =& P(N(t+h)=n, N(t)=n) \\ &+ P(N(t+h)=n, N(t)=n-1) \\ &+ P(N(t+h)=n, N(t)=n-2) + \dots \\ =& p_n(t)(1-\lambda h) + p_{n-1}(t)\lambda h + 0\end{aligned}$$

$$\implies \frac{p_n(t+h)-p_n(t)}{h} = -\lambda(p_n(t) - p_{n-1}(t))$$

$$\frac{dp_n}{dt} = -\lambda(p_n - p_{n-1})$$

Now using the poisson distribution as an ansatz and induction, we can prove

$$\begin{aligned}\frac{dp_1}{dt} &= -\lambda(p_1 - p_0) \\ &= -\lambda p_1 + \lambda e^{-\lambda t}\end{aligned}$$

$$\implies p_1'(t) + \lambda p_1(t) = \lambda e^{-\lambda t}$$

One can solve using Linear differential equation using I.F.

Inter-arrival times

1. $\{X_i\}$ are i.i.d with distribution $\text{Exp}(\lambda)$
2. $P(X_1 < s | N(t) = 1) = s/t I_{s<t} + I_{s>t}$

$$\begin{aligned}P(X_1 > y) &= P(N(y) = 0) \\ &= e^{-\lambda t}\end{aligned}$$

$$\bar{F}(x) = 1 - F(x) = 1 - P(X_1 > x)$$

It is easy to show that $\bar{F}(x+h) = \bar{F}(x)\bar{F}(h)$, which is the memorylessness property