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MIDSEM 2 - NLD.

Q1. $f(x) = 3x \mod 1$

This can be looked at as a shift map in the ternary representation. That is,

x_{n+1} can be found by right shifting x_n in the ternary basis, and then removing the leading digit.

Hence for p-orbits, we need a sequence of p digits which are repeated.

$p = 1$.

$0, 0.\bar{1}, 0.\bar{2}$ are fixed points. $= 3$ orbits.

$p = 2$.

$0.\overline{00}, 0.\overline{01}, \ldots, 0.\overline{22}$ are fixed points of f^2. $= 9$.

But, of these, 3 are trivial $\Rightarrow$ 6. period 2 points

Every 2-orbit has 2 points, so $= 3$ orbits

$p = 3$.

Similarly, there are 27 fixed points of f^3

But 3 are trivial $\Rightarrow$ 24 points

Each 3 orbit has 3 points $\Rightarrow$ 8 orbits

$p = 4$

Similarly, there are 81 fixed points, of which 3 period 1 and 6 period-2 points are trivial $\Rightarrow$ 72 points

$\Rightarrow \frac{72}{4} = 18$ orbits.

b) At each point,

$f'(x) = 3$.

$\Rightarrow \lambda = \frac{1}{n} \sum_{i=1}^{n} \ln(f'(x_i)) = \ln 3$.

Q2 $x_{n+1} = r x_n (1 - x_n)$.

Say x_1, x_2 are 2 cycle points.

For superstability, $f'(x_1) f'(x_2) = 0$.

$\Rightarrow$ x_1 or x_2 is such that $f'(x_i) = 0$. WLOG, let x_1 be the

$\Rightarrow r \cdot (1 - 2x_1) = 0 \Rightarrow x_1 = 1/2$.

Now, $x_2 = f(x_1) = \frac{r}{2}(1 - 1/2) = \frac{r}{4}$.

$\Rightarrow f(x_2) = \frac{r^2}{4}\left(1 - \frac{r}{4}\right) = 1/2$.

$\Rightarrow R_2$ is solution of above equation.

$r^2(4 - r) = 8$.

$\Rightarrow (r-2)(r^2 - 2r - 4) = 0$

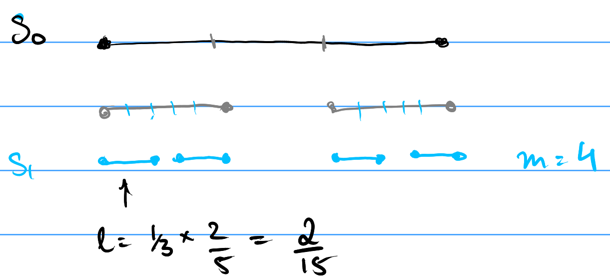
But $r = 2 \Rightarrow R_1$

Hence, $R_2 = \dfrac{2 \pm \sqrt{4 + 16}}{2} = 1 \pm \sqrt{5}$

But $R_2 > 0 \Rightarrow R_2 = 1 + \sqrt{5}$

Q3. Similarity dimension is given by how many pieces of equal size we need to cover the set.

For S_n,



Hence, we need $m = 4^n$ if we scale one piece by $\left(\frac{2}{15}\right)^n = r$

$\Rightarrow d_s = \dfrac{\ln(m)}{\ln(1/r)} = \dfrac{\ln 4}{\ln(15) - \ln(2)} = 0.688\ldots$

b) The measure of $S_n \geq S_{n+1}$, since S_{n+1} is completely covered by S_n.

measure of $S_n = 4^n \times \left(\frac{2}{5} \times \frac{1}{3}\right)^n = \left(\frac{8}{15}\right)^n$.

$S_\infty \leqslant S_n \Rightarrow S_\infty \leq \left(\frac{8}{15}\right)^n \quad \forall\, n.$

Taking $n \to \infty$,

$S_\infty \leq 0$, but $S_n > 0$ by definition.

$\Rightarrow S_\infty = 0$.

Q4 $\dot{x} = (r + x^2 - x^4)\, x$

fixed points of the system are when $\dot{x} = 0$.

$\Rightarrow x = 0$ or $(r + x^2 - x^4) = 0$.

Letting $y = x^2$,

$r + y - y^2 = 0$

$\Rightarrow y = \dfrac{1 \pm \sqrt{1 + 4r}}{2}$

$\boxed{x = 0 \text{ is always a fixed point.}}$

$x^2 = \dfrac{+1 + \sqrt{1+4r}}{2}$ is a solution when

$1 + 4r > 0$

$\Rightarrow r > -1/4$

Therefore solution

$\boxed{x = \pm\sqrt{\dfrac{+1 + \sqrt{1+4r}}{2}} \text{ is fixed when } r > -1/4}$

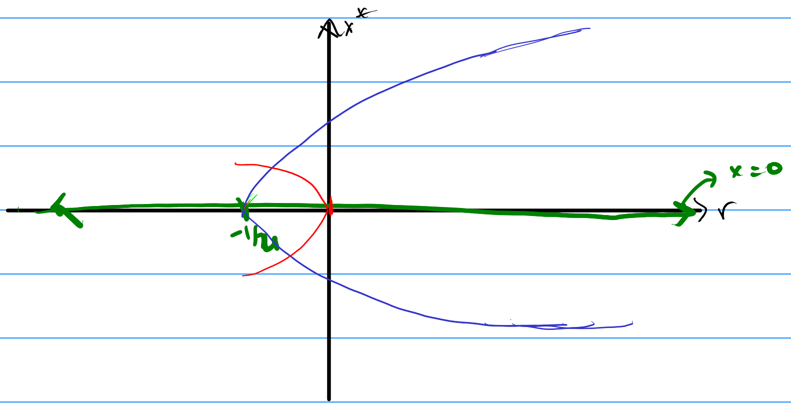
$x^2 = \dfrac{+1 - \sqrt{1+4r}}{2}$ is a solution when

$1-4r > 0$ And $\sqrt{1-4r} < 1$.

$r > -1/4$ And $1-4r < 1$

$\Rightarrow r > -1/4$ And $r < 0$

$$\Rightarrow \boxed{x = \pm\sqrt{\frac{1-\sqrt{1-4r}}{2}} \quad \text{is a solution for } -1/4 < r < 0}$$



Q5. For phase locking, $\dot{\theta} = \dot{\Theta} \Rightarrow \theta - \Theta$ is a constant.

$$\omega + A f(\theta - \Theta) = \Omega.$$

Let $\theta - \Theta \in [-\pi/2, \pi/2] + 2n\pi$

$\Rightarrow \omega + A(\theta - \Theta) = \Omega$

$\Rightarrow \theta - \Theta = \frac{\Omega - \omega}{A} = \mu \Rightarrow \mu \in [-\pi/2, \pi/2] + 2n\pi$.

For $\theta - \Theta \notin [-\pi/2, \pi/2] + 2n\pi$.

$\pi - \theta + \Theta = \frac{\Omega - \omega}{A} = \mu \Rightarrow \pi - \mu \in [-\pi, -\frac{\pi}{2}] + 2n\pi \cup [\frac{\pi}{2}, \pi] + 2n\pi$.

$\Rightarrow \mu \in [\frac{\pi}{2}, \pi] + (2n-1)\pi \cup [-\pi, -\pi/2] + (2n-1)\pi$.