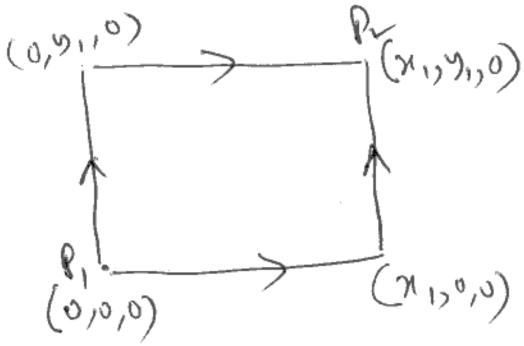
Assignment 2.

1. 

Path I:

$$\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = \int_{(0,0,0)}^{(x_1,0,0)} \vec{E}\cdot d\vec{s} + \int_{(x_1,0,0)}^{(x_1,y_1,0)} \vec{E}\cdot d\vec{s}$$

$$\vec{E}\cdot d\vec{s} = 6xy\,dx + (3x^2-3y^2)\,dy.$$

$y=0$ for $(0,0,0)\to(x_1,0,0)$ & $dy=0$ $\Big\}$ $\therefore$ 1st integral vanishes.

$x=x_1$ & $dx=0$ for $(x_1,0,0)\to(x_1,y_1,0)$.

$$\therefore \int_{(x_1,0,0)}^{(x_1,y_1,0)} \vec{E}\cdot d\vec{s} = \int_{(x_1,0,0)}^{(x_1,y_1,0)} (3x_1^2-3y^2)\,dy$$

$$= 3x_1^2y_1 - y_1^3.$$

$$\therefore \int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = 3x_1^2y_1 - y_1^3 \quad \text{for path I}$$

Show that for path II you get the same result

$$\therefore \phi = \int \vec{E}\cdot d\vec{s} + \text{const} = 3x^2y - y^3 + \text{const}.$$

Now, $\vec{E} = -\vec{\nabla}\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$

$$\therefore E_x = \frac{\partial\phi}{\partial x} = 6xy,\quad E_y = \frac{\partial\phi}{\partial y} = 3x^2-3y^2,\quad E_z = \frac{\partial\phi}{\partial z} = 0$$

2. Keeping in mind that we are using SI for our course, ~~we~~ it is better to include a factor of $\frac{1}{4\pi\epsilon_0}$ for the potential as,

$$\phi = \begin{cases} \frac{\rho_0}{4\pi\epsilon_0}(x^2+y^2+z^2) & x^2+y^2+z^2 < a^2 \\ \frac{\rho_0}{4\pi\epsilon_0}\left(-a^2+\frac{2a^3}{(x^2+y^2+z^2)^{1/2}}\right) & x^2+y^2+z^2 > a^2 \end{cases}$$

ρ_0 has dimension of volume charge density.

Now, $\vec{E} = -\vec{\nabla}\phi = -\left(\hat{x}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial y}+\hat{z}\frac{\partial\phi}{\partial z}\right)$

$$\frac{\partial\phi}{\partial x} = \frac{\rho_0}{4\pi\epsilon_0}\cdot 2x \quad \text{for } x^2+y^2+z^2<a^2$$

$$= \frac{\rho_0}{4\pi\epsilon_0}\cdot 2a^3\left(-\frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2+y^2+z^2)^{3/2}}\right) \quad \text{for } x^2+y^2+z^2>a^2$$

$$= -\frac{\rho_0}{4\pi\epsilon_0}\cdot\frac{2a^3x}{(x^2+y^2+z^2)^{3/2}}$$

$\therefore$ and similarly for $\frac{\partial\phi}{\partial y}$, $\frac{\partial\phi}{\partial z}$.

$$\therefore \vec{E} = -\vec{\nabla}\phi = -\frac{\rho_0}{4\pi\epsilon_0}(2x\hat{x}+2y\hat{y}+2z\hat{z})$$

$$= -\frac{2\rho_0}{4\pi\epsilon_0}(x\hat{x}+y\hat{y}+z\hat{z}) \quad \text{for } x^2+y^2+z^2<a^2.$$

Similarly, $\vec{E} = +\frac{2\rho_0 a^3}{4\pi\epsilon_0}\cdot\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$ for $x^2+y^2+z^2>a^2$.

To find the charge distribution, we will calculate the Laplacian ∇^2.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

For $x^2+y^2+z^2<a^2$, $\frac{\partial^2\phi}{\partial x^2} = \frac{\rho_0}{4\pi\epsilon_0}\cdot 2$.

Similarly for $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial z^2}$.

$$\therefore \nabla^2\phi = \frac{\rho_0}{4\pi\epsilon_0}(2+2+2) = \frac{6\rho_0}{4\pi\epsilon_0} \text{ for } x^2+y^2+z^2<a^2$$

But, by Poisson's Eqn, $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$.

where ρ is the charge distribution.

$$\therefore \frac{\cancel{6}^{3}\rho_0}{{}_{2}\cancel{4}\pi\cancel{\epsilon_0}} = -\frac{\rho}{\cancel{\epsilon_0}} \Rightarrow \rho = -\frac{3\rho_0}{2\pi} \text{ for } x^2+y^2+z^2<a^2$$

For $x^2+y^2+z^2>a^2$, $\frac{\partial^2\phi}{\partial x^2} = -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\cdot\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right]$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\left[-\frac{3}{\cancel{2}}\cdot\frac{x\cdot \cancel{2}x}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\left[\frac{-3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[\frac{-3x^2}{x^2+y^2+z^2}+1\right].$$

Similarly for $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial z^2}$.

$$\therefore \nabla^2\phi = -\frac{2\rho_0 a^3}{4\pi\epsilon_0(x^2+y^2+z^2)^{3/2}}\left[-\frac{3x^2}{x^2+y^2+z^2}+1-\frac{3y^2}{x^2+y^2+z^2}+1 - \frac{3z^2}{\dots} + 1\right]$$

$$\therefore \nabla^2\phi = \frac{-2\rho_0 a^3}{4\pi\epsilon_0 (x^2+y^2+z^2)^{3/2}}\left[\frac{-3\cancel{(x^2+y^2+z^2)}}{\cancel{(x^2+y^2+z^2)}} + 3\right] = 0.$$

for $x^2+y^2+z^2 > a^2$.

$\therefore \rho = 0$ for $x^2+y^2+z^2 > a^2$.

- Note that just like probs. Assignment 1, there is a discontinuity in the electric field at the surface of the sphere since,

$$|\Delta\vec{E}| = |\vec{E}|\Big|_{\substack{x^2+y^2+z^2=a^2 \\ \text{outside}}} - |\vec{E}|\Big|_{\substack{x^2+y^2+z^2=a^2 \\ \text{inside}}} \neq 0.$$

$\longrightarrow$ implies existence of surface charge !!

This is best seen by going to spherical polar coordinates with $r^2 = x^2+y^2+z^2$.

$$\therefore \vec{E} = \frac{-2\rho_0}{4\pi\epsilon_0} r\hat{r} \quad \text{for } r < a.$$

$$= \frac{2\rho_0 a^3}{4\pi\epsilon_0}\cdot\frac{\hat{r}}{r^2} \quad \text{for } r > a.$$

$$|\Delta\vec{E}| = \frac{\rho_0}{4\pi\epsilon_0}\left[2a - (-2a)\right] = \frac{\cancel{4}a\rho_0}{\cancel{4}\pi\epsilon_0} = \frac{\rho_0 a}{\pi\epsilon_0}.$$

Since, $|\Delta\vec{E}| = \dfrac{\sigma}{\epsilon_0}$ with σ: surface charge density.

$$\therefore \frac{\sigma}{\epsilon_0} = \frac{\rho_0 a}{\pi\epsilon_0} \Rightarrow \sigma = \frac{\rho_0 a}{\pi}.$$

3. Let R be the radius of the basketball.

Then, $\dfrac{Q}{4\pi\epsilon_0 R} = -1000 \Rightarrow Q = -4\pi\epsilon_0 R \cdot 1000$.

$$\text{Charge}/m^2 = \frac{Q}{4\pi R^2} = -\frac{\cancel{4\pi}\epsilon_0 \cancel{R} \cdot 1000}{\cancel{4\pi} R^{\cancel{2}}}$$

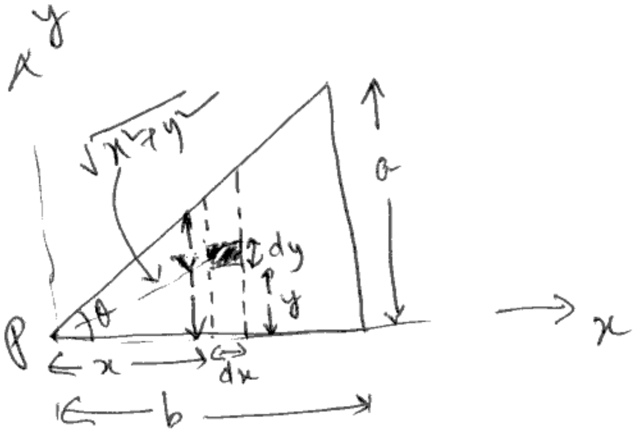
$$= -\frac{1000\,\epsilon_0}{R}.$$

$\therefore$ # of extra electrons/m^2 $= \dfrac{\text{Charge}/m^2}{\text{electronic charge}}$

$$= \frac{1000\,\epsilon_0/R}{e}.$$

Assuming $R \simeq 0.15\,m$.

$$\# = \frac{8.85\times10^{-12}\times1000}{1.6\times10^{-19}\times0.15} \simeq 3.7\times10^{11}/m^2.$$

4. 

To do this, we need to first divide the triangle into strips as shown & find the contribution of a strip at pt. P.

The potential at point P due to an infinitesimal area $dxdy$ (shaded) is,

$$d\phi' = \frac{\sigma\,dx\,dy}{4\pi\epsilon_0\sqrt{x^2+y^2}}.$$

$\therefore$ Potential due to the strip,

$$d\phi = \int_{y=0}^{y=Y} d\phi' = \frac{1}{4\pi\epsilon_0}\int_{y=0}^{y=Y} \frac{\sigma\,dx\,dy}{\sqrt{x^2+y^2}}$$

Now, $\frac{Y}{x} = \frac{a}{b}$ (similar triangles). $\Rightarrow Y = \frac{ax}{b}$.

$$\therefore d\phi = \frac{1}{4\pi\epsilon_0} \int_{y=0}^{y=ax/b} \frac{\sigma\, dx\, dy}{\sqrt{x^2+y^2}} = \frac{\sigma\, dx}{4\pi\epsilon_0} \int_0^{ax/b} \frac{dy}{\sqrt{x^2+y^2}}$$

$$= \frac{\sigma\, dx}{4\pi\epsilon_0} \left[\ln\left(y + \sqrt{x^2+y^2}\right) \right]_0^{ax/b}$$ (You can do the integration)

$$= \frac{\sigma\, dx}{4\pi\epsilon_0} \left[\ln\left(\frac{ax}{b} + \sqrt{x^2 + \frac{a^2x^2}{b^2}} \right) - \ln x \right]$$

$$= \frac{\sigma\, dx}{4\pi\epsilon_0} \ln\left(\frac{a}{b} + \sqrt{1 + \frac{a^2}{b^2}} \right)$$

$\tan\theta = \frac{a}{b}$ $\quad \therefore$ $d\phi = \frac{\sigma\, dx}{4\pi\epsilon_0} \ln(\tan\theta + \sec\theta)$.

$$= \frac{\sigma\, dx}{4\pi\epsilon_0} \ln\left(\frac{1+\sin\theta}{\cos\theta} \right).$$

$\therefore$ Contribution at P due to whole triangle,

$$\phi = \int_{x=0}^{x=b} d\phi = \frac{\sigma}{4\pi\epsilon_0} \ln\left(\frac{1+\sin\theta}{\cos\theta} \right) \int_0^b dx.$$

$$= \frac{\sigma b}{4\pi\epsilon_0} \ln\left(\frac{1+\sin\theta}{\cos\theta} \right).$$

5. Total energy = Energy of shell 1 + Energy of shell 2 + Energy of one shell due to potential of the other.

Energy of shell 1 = Energy of shell 2 = $\frac{Q^2}{8\pi\epsilon_0 R}$.

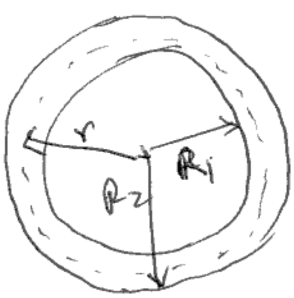
Energy required to build one shell given that the other shell is present $= Q\phi$

where ϕ = potential due to a shell = $\frac{Q}{4\pi\epsilon_0 R}$

$\therefore$ Total energy = $\frac{Q^2}{8\pi\epsilon_0 R} + \frac{Q^2}{8\pi\epsilon_0 R} + Q\phi$.

$$= \frac{Q^2}{4\pi\epsilon_0 R} + Q.\left(\frac{Q}{4\pi\epsilon_0 R}\right) = \frac{Q^2}{2\pi\epsilon_0 R}$$

6. (a) Already shown (Assignment 1),



for $0 \leq r \leq R_1$, $E(r) = 0$ (electric field inside shell)

This is a thick shell not a thin one

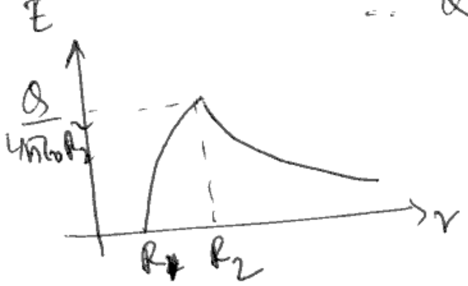
The electric field inside the shell,

$$E = \frac{Q_r}{4\pi\epsilon_0 r^2} \quad \text{for } R_1 \leq r \leq R_2.$$

To find Q_r note that $Q = \rho \cdot \frac{4}{3}\pi(R_2^3 - R_1^3)$

where ρ = charge density.

$$\therefore Q_r = \rho \cdot \frac{4}{3}\pi(r^3 - R_1^3) = \frac{Q \cancel{\frac{4}{3}\pi}(r^3 - R_1^3)}{\cancel{\frac{4}{3}\pi}(R_2^3 - R_1^3)}$$



$$= Q \cdot \frac{r^3 - R_1^3}{R_2^3 - R_1^3}$$

$$\therefore E = \frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{R_2^3 - R_1^3} \cdot \left(\frac{r^3 - R_1^3}{r^2}\right) \quad R_1 \leq r \leq R_2.$$

For $R_2 \leq r \leq \infty$, $E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$.

(b) Potential at $r=0$ is,

$$\phi(0) = -\int_\infty^0 E\,dr = -\left[\int_\infty^{R_2} + \int_{R_2}^{R_1} + \int_{R_1}^{0}\right] E\,dr$$

$$\therefore\ \phi(0) = -\int_{\infty}^{R_2} \frac{Q}{4\pi\epsilon_0 r^2}\,dr - \int_{R_1}^{R_2} \frac{Q}{4\pi\epsilon_0 (R_2^3 - R_1^3)} \cdot \frac{r^3 - R_1^3}{r^2}\,dr$$

$$= \frac{9}{14}\,\frac{Q}{4\pi\epsilon_0 R} \quad \left(\text{choosing } R_2 = 2R_1 = 2R\right)$$

i.e. $R = R_1$.

Please do the integration!