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Homework No1.

①a) $P(n) = 1^2+2^2+ \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in N$

$P(1) = 1 \quad , \quad \frac{1(1+1)(2\times1+1)}{6} = \frac{6}{6} = 1$ verified

$P(2) = 1^2+2^2 = 5 \quad , \quad \frac{2\times3\times5}{6} = 5$ verified.

They form the basis of our Induction

Let $P(K)$ be true

i.e $P(K) = 1^2+2^2+ \cdots +K^2 = \frac{K(K+1)(2K+1)}{6}$.

$P(K+1) = 1^2+2^2+ \cdots +K^2+(K+1)^2 = \frac{K(K+1)(2K+1)}{6} + (K+1)^2$

$= \frac{(K+1)\,(K(2K+1)+6K+6)}{6}$

$= \frac{(K+1)\,(2K^2+7K+6)}{6}$

$= \frac{(K+1)\,(2K^2+4K+3K+6)}{6}$

$= (K+1)\,\frac{2K(K+2)+3(K+2)}{6}$

$= \frac{(K+1)\,(2K+3)\,(K+2)}{6}$

thus $P(K+1)$ holds if $P(K)$ holds $\quad = \frac{(K+1)\,((K+1)+1)\,(2(K+1)+1)}{6}$

and therefore by PMI $P(n)$ holds $\forall n \in N$

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b) $P(n) : n^2 > n \ \forall n \in \mathbb{N} \quad n \geq 2$

$P(2) \Rightarrow \quad 2^2 = 4 > 2$ verified $\quad P(4) \Rightarrow 4^2 = 16 > 4$ verified.

$P(3) \Rightarrow \quad 3^2 = 9 > 3$ verified

thus, these forms our basis of Induction.

Let $P(K)$ be true $\qquad K \geq 2$

$$K^2 > K$$

$$K^2 + 1 > K + 1 \qquad \text{since } K > 0$$

$$2K > 0$$

Rough:

$K^2+1 > K+1$

K

i.e. $\quad K^2 + 1 + 2K > K^2 + 1 > K + 1$

$$(K+1)^2 > K+1$$

$P(K+1)$ holds if $P(K)$ holds, therefore By Principle of Mathematical Induction $P(n)$ holds $\forall n \geq 2$.

c) $P(n): \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \quad \forall n \in \mathbb{N}$.

$P(1): \frac{1}{2} \qquad 1 - \frac{1}{2} = \frac{1}{2}$ Verified

$P(2) = \frac{3}{4} \qquad 1 - \frac{1}{4} = \frac{3}{4}$ Verified

They form the basis of Induction.

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Let $P(K)$ be true.

$P(K): \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^K} = 1 - \frac{1}{2^K}$ holds.

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^K} + \frac{1}{2^{K+1}} = 1 - \frac{1}{2^K} + \frac{1}{2^{K+1}}$$

$$= 1 + \frac{(-2+1)}{2^{K+1}}$$

$$= 1 - \frac{1}{2^{K+1}}$$

therefore we have $P(K+1)$ true if $(P(K))$ is true therefore by PMI, $P(n)$ holds $\forall n \in \mathbb{N}$.

(2) $(2 + 5^{1/3})^{1/2}, \ (2 + 2^{1/2})^{1/2}, \ (5 - 3^{1/2})^{1/3}$

To show they are not rational numbers.

Assume they are rational.

i.e. $\quad p = (2 + 5^{1/3})^{1/2}$ is rational.

squaring again gives rational number.

$$p^2 = 2 + 5^{1/3}$$

$$p^2 - 2 = 5^{1/3} \qquad \text{Subtraction of rational is rational.}$$

$\Rightarrow 5^{1/3}$ is also rational.

$$5^{1/3} = \frac{p}{q} \qquad q \neq 0 \quad p \text{ and } q \text{ are coprime.}$$

$$5 = \frac{p^3}{q^3} \Rightarrow 5q^3 = p^3 \quad \cancel{p \text{ divides } q}$$

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we have

$$5q^3 = p^3 \quad — ①$$

p and q are co-prime. thus, 5 divides p

$p = 5K$ — ② p is multiple of 5.

putting ② in ①

$$5q^3 = 5^3 K^3$$

$$q^3 = 25K^3$$

Since we had p and q are coprime, ~~they have not factors in~~

K cannot divide q

so 5 divides q

$$q = 5K'$$

But p and q are coprime Contradiction! thus $5^{1/3}$ is irrational which equal to rational (p^2-2) Contradiction

Thus our assumption that they rational is wrong

Indeed they are irrational.

OR Zero rational theorem could also be used.

consider $(5-3^{1/2})^{1/3} = b$

$$(5-3^{1/2}) = b^3$$

$$5-b^3 = 3^{1/2}$$

$$(5-b^3)^2 = 3$$

if b is rational then, b^3 is rational $5-b^3$ is also rational.

Say $x = 5-b^3$ is rational.

i.e $x^2 = 3$

$$x^2 - 3 = 0$$

Only rational solution possible are ⇒ $\pm\frac{3}{1}, \pm 1$

none of these satisfies the equation.

thus $x = 5-b^3$ cannot be rational, so b cannot be Rational.

③ $|a| = |a-b+b|$ using triangle inequality $\forall a,b \in \mathbb{R}$.

we have

$$|a| \leq |a-b| + |b|$$

$$|a| - |b| \leq |a-b| \quad — ①$$

similarly

$$|b| = |b-a+a| \leq |a-b| + |a|$$

$$-(|a|-|b|) \leq |a-b| \quad — ②$$

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$\textcircled{1},\textcircled{2} \Rightarrow$

$-|a-b| \leq |a|-|b| \leq |a-b|$ — ③

lly, if

$-c \leq x \leq c \Rightarrow |x| \leq c$ — ④

~~where $c>0$~~ $\Rightarrow$ we have. ~~By def.~~

~~$c>0$~~

~~$-c<0$~~ $\quad -|c| \leq c \leq |c|$

~~$|-c| = c$~~ $\quad |c| \geq -c \geq |c|$

$|c| \leq -c \leq |c|$

$|c|$

$\textcircled{1},\textcircled{2} \Rightarrow \big||a|-|b|\big| \leq |a-b|$

$\textcircled{1},\textcircled{2} \quad |a|-|b| \leq |a-b|$

$-(|a|-|b|) \leq |a-b|$

By definition $\big||a|-|b|\big|$ is either $-(|a|-|b|)$, $(|a|-|b|)$

we can write $\big||a|-|b|\big| \leq |a-b|$

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④ To prove $|a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|$ ~~$\forall n\in$~~ for any n-real numbers

Consider, applying triangle inequality.

$|a_1+(a_2+\cdots+a_n)| \leq |a_1|+|a_2+\cdots+a_n|$

$\leq |a_1|+|a_2|+|a_3+\cdots+a_n|$

$\leq |a_1|+|a_2|+|a_3|+|a_4+\cdots+a_n|$

$\vdots$

$\leq |a_1|+|a_2|+|a_3|+\cdots+|a_n|$ proved

OR it can be proved using Principle of Mathematical Induction

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⑤ $|a-b| \leq c$ if and only if $b-c \leq a \leq b+c$

$\Rightarrow |a-b| \leq c$

$a-b \geq 0 \quad |a-b| = a-b \Rightarrow a-b \leq c$

$a \leq c+b$ — ①

$a-b<0 \quad |a-b| = -(a-b) \Rightarrow -(a-b) \leq c$

$a-b \geq -c$

$a \geq b-c$ — ②

$\textcircled{1},\textcircled{2} \Rightarrow b-c \leq a < c+b$

$\Leftarrow \quad b-c \leq a \leq b+c \Rightarrow b-c \leq a \Rightarrow b-a \leq c$ i.e. $-(a-b) \leq c$ — ③

$a \leq b+c \Rightarrow (a-b) \leq c$ — ④

By def. $|a-b|$ is either $a-b$ or $-(a-b)$ thus we conclude $\textcircled{3},\textcircled{4} \Rightarrow |a-b| \leq c$

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6. Let $a, b \in \mathbb{R}$.
To prove if $a \leq c \;\forall\; c > b$, then $a \leq b$

assume if $a \leq c \;\forall\; c > b$, then $a > b$

$\frac{a}{2} > \frac{b}{2}$ — ①

$\frac{a}{2} + \frac{b}{2} > \frac{b}{2} + \frac{b}{2}$

say $c = \frac{a+b}{2} > b$

But, adding $\frac{a}{2}$ in ① both side.

$\frac{a}{2} + \frac{a}{2} > \frac{b}{2} + \frac{a}{2}$

$a > c$ contradicts the hypothesis.

Our assumption was wrong.

⑦ To show denseness of Irrational in Set of Real numbers.

$x, y \in \mathbb{R}$ $\exists\; s \in \mathbb{I}$ st. $x < s < y$.

By denseness of rational number

$r \in \mathbb{Q}$ st. $x < r < y$ — ①

$y - r > 0$ and $\sqrt{2} > 0$ $\boxed{\sqrt{2} \text{ is irrational}}$

By Archimedian Property $\exists\; n \in \mathbb{N}$.
st. $n(y-r) > \sqrt{2}$
$y - r > \frac{\sqrt{2}}{n}$

$y > \frac{\sqrt{2}}{n} + r > r$ — ② $\boxed{\frac{\sqrt{2}}{n} \text{ is positive}}$

$\frac{\sqrt{2}}{n} + r$ is irrational

if rational $\frac{\sqrt{2}}{n} + r - r$ also rational

and $\frac{\sqrt{2}}{n} \times n$ also rational, but $\sqrt{2}$ is not rational. thus $\frac{\sqrt{2}}{n} + r$ is not rational.

①, ② ⇒ $x < \frac{\sqrt{2}}{n} + r < y$

irrational number b/w two Real number.

8.

(a) $A = \{ r \in \mathbb{Q} \mid r^2 < 4 \}$ Bounded.
$\text{Sup}(A) = 2$ $\inf(A) = -2$
Donot lie in A.

(b) $B = \left\{ 1 - \frac{1}{3^n} \;\middle|\; n \in \mathbb{N} \right\}$. Bounded.
$\inf(B) = \frac{2}{3}$ $\text{Sup}(B) = 1$
Do not lie in B.

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(c) $C = \{ n^{(-1)^n} \mid n \in \mathbb{N} \}$ Bounded Below By 0

$\inf(C) = 0$ NOT BONDED ABOVE

(d) $D = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$ Bound below By 0 & ABOVE by 1.

$\sup(D) = 1 \quad \inf(D) = 0$

(9) $A \subseteq \mathbb{R} \quad b \in \mathbb{R}$

$a < b + \epsilon \quad \forall a \in A$ and each $\epsilon > 0$.

Claim: then b is an upper Bound for A.

Suppose b is not an upper Bound for A.

then $\exists\ a' \in A$ st. $a' > b$

$a' - b > 0$

using Archimedian property.

$\exists n \in \mathbb{N}$. st.

$n(a' - b) > 1$

$a' - b > \frac{1}{n}$

$a' > b + \frac{1}{n} \qquad \epsilon = \frac{1}{n} > 0$

But it is a Contoradiction

with given conditions

thus our Supposition was wrong, the Claim is true.

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(10) A, B non-empty Sets of real numbers.

st. $x \leq y \quad \forall x \in A$ and $y \in B$.

Claim: $\sup A \leq \inf B$

Suppose: $\sup A > \inf B$

$\inf B$ cannot be Supremum of A.

thus. $\exists\ x_0 \in A$ st.

$x_0 > \inf B$

But now x_0 cannot be infimum of B.

$\exists\ y_0 \in B$ st.

$y_0 < x_0$ Contradiction!

we had $x \leq y \quad \forall x \in A$ and $y \in B$

Our Supposition was wrong, thus claim is true.

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(11) $P(n)$: $(a+b)^n = {}^nC_0\, a^0 b^n + {}^nC_1\, a^1 b^{n-1} + {}^nC_2\, a^2 b^{n-2} + \cdots + {}^nC_n\, a^n b^0$

$P(1)$: LHS $(a+b)$ RHS ${}^1C_0\, a^0 b^1 + {}^1C_1\, a^1 b^{1-1}$

$= b+a$ LHS=RHS verified

$P(2)$: $(a+b)^2$ RHS ${}^2C_0\, a^0 b^2 + {}^2C_1\, a^1 b^{2-1} + {}^2C_2\, a^2 b^{2-2}$

$= a^2+b^2+2ab$ $= b^2+2ab+a^2$

LHS=RHS verified.

$P(1)$, $P(2)$ forms the basis for Induction.

Suppose $P(K)$ is true.

$P(K)$: $(a+b)^K = {}^KC_0\, a^0 b^K + {}^KC_1\, a^1 b^{K-1} + \cdots + {}^KC_K\, a^K b^0$.

$(a+b)^K (a+b) = (a+b)^{K+1} = {}^KC_0\, a b^K + {}^KC_1\, a^2 b^{K-1} + \cdots + {}^KC_K\, a^{K+1} b^0$

${}^KC_0\, a^0 b^{K+1} + {}^KC_1\, a^1 b^K + \cdots + {}^KC_K\, a^K b$

Using the Identity ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, ${}^KC_0 = {}^{K+1}C_0$, ${}^KC_K = {}^{K+1}C_{K+1}$

$= {}^{K+1}C_0\, a^0 b^{K+1} + a^1 b^K ({}^KC_0 + {}^KC_1) + a^2 b^{K-1} ({}^KC_1 + {}^KC_2)$

$+ \cdots + ({}^KC_{K-1} + {}^KC_K)\, a^K b^1 + {}^{K+1}C_{K+1}\, a^{K+1} b^0$

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$= {}^{K+1}C_0\, a^0 b^{K+1} + {}^{K+1}C_1\, a^1 b^K + \cdots + {}^{K+1}C_{K+1}\, a^{K+1} b^0$

$= (a+b)^K$

Thus $P(K+1)$ holds if $P(K)$ holds.

Therefore by Principle of Mathematical Induction $P(n)$ holds or is proved.

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Homework No2.

(10) To prove $\lim \frac{2^n}{n^2} = +\infty$ $n \in N$.

$2^n = (1+1)^n = 1+n+\frac{n(n-1)}{2\times 1}+\frac{n(n-1)(n-2)}{3\times 2\times 1}+\cdots+{}^nC_n$

clearly.

$$2^n > n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{6}$$

Dividing by n^2 both side.

$$\frac{2^n}{n^2} > \frac{n^3+5n}{6n^2} = \frac{n^2+5}{6n}$$

Proof Let $M>0$ and $N=6M$

so $n>N$

$\Rightarrow n>6M$

$\frac{n}{6}>M$

$\frac{2^n}{n^2} > \frac{n^2+5}{6n} > \frac{n^2}{6n} > M$

$\frac{2^n}{n^2} > M$

thus $\lim \frac{2^n}{n^2} = +\infty$

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9. To prove
$$\lim a^n = \begin{cases} 0 & \text{if } 0\leq |a|<1 \\ 1 & \text{if } a=1 \\ +\infty & \text{if } a>1 \\ \text{Does not exist} & \text{if } a\leq -1 \end{cases}$$

- if $|a|<1$ $-1<a<1$

if $a=1$ $a^n=0$ $\forall n$ $\lim a^n=0$ obvious.

Suppose $a\neq 0$

Since $|a|<1$ $|a|=\frac{1}{1+b}$ $b>0$

$(|a|)^n$ consider

Binomial expansion: $(1+b)^n = 1+nb+\frac{n(n-1)}{2}b^2+\cdots+b^n$

$(1+b)^n \geq 1+nb > nb$

$$|a^n-0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}$$

Let $\epsilon>0$ and $N=\frac{1}{\epsilon b}$

so $n>N \Rightarrow n>\frac{1}{\epsilon b}$

$|a^n-0| < \frac{1}{nb} < \epsilon \Rightarrow \lim a^n = 0$

- if $a=1$ $a^n=1$ $\forall n\in N$ $\lim a_n = 1$ if $a=1$.
- $a>1$ $\frac{1}{a}<1$ By first case $\lim \left(\frac{1}{a}\right)^n = 0$

W.r.t $\lim s_n=+\infty$ if and only if $\lim s_n=0$, thus $\lim a^n=+\infty$, $a>1$

Premium

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- if $a \leq -1$
  for odd n $a^n < -1$
  for even n $a^n \geq 1$

for large n, $\lim a^n$ either goes to $+\infty$ or $-\infty$ which simultaneously cannot happen thus impossible.

if $\lim a_n = A$ $A \to$ finite Real number.
for $\epsilon > 0$ $n > N \Rightarrow |a_n - A| < 1$ —①
setting $\epsilon = 1$
for even n, if ① holds then $A > 0$
for odd n, if ① holds then $A < 0$

A cannot be simultaneously positive and negative a contradiction because.
Since limit of a sequence is unique.

Therefore $\lim a^n$ does not exist if $a \leq -1$

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8. given $S_n > 0$ $\forall n \in \mathbb{N}$.

To prove $\lim S_n = +\infty$ if and only if $\lim \frac{1}{S_n} = 0$

$\Rightarrow$ $\lim S_n = +\infty$

for every $M > 0$ $\exists N$ st $n > N \Rightarrow S_n > M$

$S_n > M$

$\frac{1}{S_n} < \frac{1}{M}$ $S_n > 0 \forall n \in \mathbb{N}$

$\left|\frac{1}{S_n}\right| < \frac{1}{M}$ setting $\frac{1}{M} = \epsilon$

$\left|\frac{1}{S_n} - 0\right| = \left|\frac{1}{S_n}\right| < \epsilon \Rightarrow \lim \frac{1}{S_n} = 0$

$\Leftarrow$ if $\lim \frac{1}{S_n} = 0$

for every $\epsilon > 0$ $\exists N$, $n > N \Rightarrow \left|\frac{1}{S_n} - 0\right| < \epsilon$

$\left|\frac{1}{S_n}\right| < \epsilon$

$S_n = |S_n| > \epsilon$

$S_n > \epsilon = M$ $M > 0$

$\Rightarrow \lim S_n = +\infty$

Premium

7. s_n and t_n sequences of real numbers

$\exists K \in \mathbb{N}$ s.t $s_n \leq t_n \quad \forall n > K$.

(a) if $\lim s_n = +\infty$, then $\lim t_n = +\infty$

for every $M > 0 \ \exists N_1$ st

$n > N_1 \Rightarrow$

$s_n > M$

if $N = \max\{N_1, K\}$.

then $n > N$

$\Rightarrow \quad t_n \geq s_n > M \Rightarrow \lim t_n = +\infty$

(b) if $\lim t_n = -\infty$ then $\lim s_n = -\infty$

for every $M < 0 \ \exists N_1$ st

$n > N_1 \Rightarrow t_n < M$

if $N = \max\{N_1, K\}$

then $n > N$

$\Rightarrow s_n \leq t_n < M \Rightarrow \lim s_n = -\infty$

6. $s_1 = 1 \quad s_{n+1} = \sqrt{s_n + 1} \quad \forall n \in \mathbb{N}$.

given s_n converges,

$\lim s_{n+1} = \lim \sqrt{s_n + 1}$

$\lim s_n = K \qquad K = \sqrt{K+1}$

$K^2 = K + 1$

$K^2 - K - 1 = 0$

$K = \frac{1 \pm \sqrt{5}}{2} \qquad K \neq \frac{1 - \sqrt{5}}{2} < 0$

since $s_n > 0 \ \forall n \in \mathbb{N}$

then $\lim s_n \geq 0$

$K = \frac{1 + \sqrt{5}}{2}$ is limit of s_n.

5 $\lim |s_n|$ exist but $\lim s_n$ does not exist

$s_n = (-1)^n$ Does not exist.

4. Given: $\lim s_n = s$.

Let $a \in \mathbb{R}$, such that $s_n \geq a$ for all but finitely many n. To Prove that $s \geq a$.

Now, let m be the max m s.t. $s_m \ngeq a$

and suppose $s < a$

$a - s > 0$

choosing ϵ as $\quad 0 < \epsilon < a - s$

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$\exists N,\ n>N \Rightarrow |S_n - s| < \epsilon$

$s-\epsilon < S_n < s+\epsilon < a-s+s$

$s-\epsilon < S_n < a$ for $n>N$

which Contradicts the maximality of

Our Supposition $s<a$ was wrong.

Thus, $s \geq a$ must hold by trichotomy.

Q2 and 3. for every $\epsilon>0$ $\quad r \in \mathbb{R}$. r is a finite Real number

$\exists N$

$n>N \Rightarrow |S_n - r| < \epsilon$

S_n is a sequence ~~$r \in \mathbb{R}$~~.

To show existence of S_n.

$r-\epsilon < S_n < r+\epsilon$

say $a = r-\epsilon \in \mathbb{R}$ say $b = r+\epsilon \in \mathbb{R}$

$a < S_n < b$

By denseness of rationals and irrationals between two Real numbers, we can have a sequence S_n been constructed as a sequence of rationals lying

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b/w a & b And also as a sequence of irrationals lying b/w a & b, that converges to $\frac{a+b}{2}$ i.e r.

Ques 1.

(e) $S_n = \sqrt{n^2+n} - n = \dfrac{n}{\sqrt{n^2+n}+n}$

$\lim S_n = \frac{1}{2}$

Let $\epsilon>0$, put $N = \frac{1}{2\epsilon}$, $n>N \Rightarrow n > \frac{1}{2\epsilon}$

$$\frac{1}{2n} < \epsilon \quad -①$$

$$\frac{1}{2n} > \frac{n}{2(n+1)\cdot n} = \frac{n}{2(\sqrt{n^2+n})^2} > \frac{n}{2(\sqrt{n^2+n}+n)^2}$$

$$\frac{n}{2(\sqrt{n^2+n}+n)(\sqrt{n^2+n}+n)} \times \frac{(\sqrt{n^2+n}-n)}{(\sqrt{n^2+n}-n)} = \frac{\sqrt{n^2+n}-n}{2(\sqrt{n^2+n}+n)}$$

$-②$

①,② $\Rightarrow \left| \dfrac{2n - (n+\sqrt{n^2+n})}{2(\sqrt{n^2+n}+n)} \right| < \epsilon$

$$\left| \frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2} \right| < \epsilon$$

$$\left| \sqrt{n^2+n} - n - \frac{1}{2} \right| < \epsilon$$

$$\Rightarrow \lim \left(\sqrt{n^2+n} - n\right) = \frac{1}{2}$$

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B+

(g) $S_n = (-1)^n n$. $n \in \mathbb{N}$.

even n. $\Rightarrow S_n = n \geq 2$ — ①

odd n $\Rightarrow S_n = -n \leq -1$ — ②

thus for large n $\lim S_n$ either goes to $+\infty$ and $-\infty$ which impossible simultaneously.

Consider. $\lim S_n = A$ $A \in \mathbb{R}$ a finite Real number

for every $\epsilon > 0$. $\exists N$,

$n > N \Rightarrow |S_n - A| < \epsilon = 1$ — ③

①,③ $\Rightarrow$ for even $n > N$ $A > 0$

②,③ $\Rightarrow$ for odd $n > N$ $A < 0$

But A cannot be simultaneously negative & positive which is absurd.

thus $\lim S_n$ does not exist

B+

(f) $S_n = \frac{1}{n} \sin n$

claim $\lim S_n \to 0$

let $\epsilon > 0$, $\exists N$ and let $N = \frac{1}{\epsilon}$ $n \in \mathbb{N}$.

$n > N \Rightarrow n > \frac{1}{\epsilon}$

$\frac{1}{n} < \epsilon$ —①

$-1 \leq \sin n \leq 1$

$0 \leq |\sin n| \leq 1$

$\therefore 0 \leq \frac{|\sin n|}{n} \leq \frac{1}{n}$ —②

①,② $\Rightarrow \left|\frac{\sin n}{n}\right| < \epsilon$

$\left|\frac{\sin n}{n} - 0\right| < \epsilon \Rightarrow \lim S_n \to 0$

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Home Work No. 03

(1) $S_n = \sin\left(\frac{n\pi}{3}\right) \quad \forall\, n \geq 1$

$S_n = \left\{ \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 1, -1, 0 \right\}$

$\lim\sup S_n = \frac{\sqrt{3}}{2} \qquad \lim\inf S_n = -1$

We have $\lim\sup S_n \neq \lim\inf S_n$

Thus $\lim S_n$ does not exist.

Monotone Subsequence: $S_{n_k} = \sin\left(2k\pi + \frac{\pi}{3}\right)$

Since it is bounded, so it is convergent as well

Recall: A Sequence S_n of real numbers is called a non-decreasing sequence if $S_n \leq S_{n+1} \ \forall\, n$ and (S_n) is called non-increasing sequence if $S_n \geq S_{n+1} \ \forall\, n$.

A Sequence that is non-decreasing or non-increasing will be called a monotone sequence.

B+

(2) $S_n = n\left(1 + (-1)^n\right) \quad \forall\, n \geq 1$

for $S_n = \begin{cases} 2n & n \to \text{even} \\ 0 & n \to \text{odd} \end{cases}$

Clearly $\lim\sup S_n = +\infty$ and $\lim\inf S_n = 0$

We have $\lim\sup S_n \neq \lim\inf S_n$

Thus $\lim S_n$ does not exist

(3) Given $\gamma > 0$ and a bounded Sequence (S_n):

NOT SURE

$\cancel{V_N = \sup\{S_n : n > N\} \to}$

Using property: $\sup a S = a \sup S$. {can be proved easily}

$\sup \gamma S_n = \gamma \sup S_n$

taking limit

[$\lim k S_n = k \lim S_n$]

$\lim\sup \gamma S_n = \lim \gamma \sup S_n$
$= \gamma \lim\sup S_n$

Premium

Premium

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Pg. ______

(4) Given: $a_n > 0 \ \forall n \in \mathbb{N}$

if it's bounded, then we are done,

Or if not, claim: it has a subsequence diverging to $+\infty$

Thus for each $n \in \mathbb{N}$, and $n = K$

$\exists \ a_n > K$ since a_n is bounded.

a_n is unbounded above.

for each $M > 0$, $\exists n$ st $a_n > M$

for $M = 1$, let a_{n_1} be s.t $a_{n_1} \geq 1$

Now for $M_2 = \max\{2, a_{n_1}\}$ $\exists a_{n_2} > M_2$

Continuing the process, for $M_{K+1} = \max\{K+1, a_{n_K}\}$

We have $a_{n_{K+1}} > K+1$

and $a_{n_{K+1}} > a_{n_K}$

Therefore a increasing subsequence such that $\lim a_{n_K} = +\infty$

Clearly

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Pg. ______

Exact formal proof required

(5) S non-empty, bounded subset of $\mathbb{R}$ st $\text{Sup}(S) \notin S$.

To prove $\exists$ an increasing sequence (x_n) of point in S converging to Sup S.

let $\text{Sup } S = x_0$

and clearly $\forall x \in S \ x < x_0$

now consider, $x_0 - \frac{1}{n} < x_0$ for each $n \in \mathbb{N}$.

$x_0 - \frac{1}{n}$ cannot be the Sup of S. Thus $\exists x_n \in S$

st. $x_0 - \frac{1}{n} < x_n < x_0$ $\left[\begin{matrix} x_n \in S \\ x < x_0 \ \forall x \in S \end{matrix}\right]$

$$x_0 - \frac{1}{n} < x_n < x_0 < x_0 + \frac{1}{n}$$

$$|x_n - x_0| < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

$\lim x_n = x_0$

Every Sequence has a monotone subsequence

let x_{n_K} be a monotonic subsequence of x_n. Since x_n converges to x_0, so subsequence, x_{n_K} also converges to same limit.

i.e. $\lim x_{n_K} = x_0$.

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B+

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B+

⑥ $S_n = (-1)^n$

if S_n is cauchy

then

for every $\epsilon > 0$ $\exists N$, st $n \geq m > N$

$\Rightarrow |S_n - S_m| < \epsilon$

Consider $\epsilon = 1$ and $n = m+1$

$|S_n - S_m| < 1$

$|S_{m+1} - S_m| < 1$

$|(-1)^{m+1} - (-1)^m| < 1$

m odd
m+1 even

$|1 - (-1)| < 1$

$|2| < 1$ false

m even
m+1 odd

$|-1 - 1| < 1$

$|2| < 1$ false.

thus S_n fails the cauchy definition.

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(7) let s_n and t_n be two Cauchy Sequence in $\mathbb{R}$.

> for every $\epsilon > 0$ $\exists N_1$ st. $n \geq m > N_1$ $\Rightarrow |s_n - s_m| < \epsilon/2$

for every $\epsilon > 0$ $\exists N_2$ st. $n \geq m > N_2$ $\Rightarrow |t_n - t_m| < \epsilon/2$

$N = \max\{N_1, N_2\}$ $\quad n \geq m > N \Rightarrow$

$$|s_n + t_n - (s_m + t_m)| = |s_n - s_m + t_n - t_m|$$
$$< |s_n - s_m| + |t_n - t_m|$$
$$< \epsilon/2 + \epsilon/2$$
$$< \epsilon$$

Thus, $s_n + t_n$ is also cauchy.

> $|s_n t_n - s_m t_m| = |s_n t_n - s_m t_n + s_n t_n - s_m t_m|$

$$\leq |s_n t_n - s_m t_n| + |s_m t_n - s_m t_m|$$
$$\leq |t_n| \, |s_n - s_m| + |s_m| \, |t_n - t_m|$$
$$\leq \frac{\epsilon}{2} |t_n| + |s_m| \frac{\epsilon}{2}$$

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Since s_n and t_n are cauchy Seq in $\mathbb{R}$, they are convergent and thus bounded.

say $|s_n| \leq M_1$ and $|t_n| < M_2$

$M = \max\{M_1, M_2\}$

$N = \max\{N_1, N_2\}$ $\quad n \geq m > N.$

$$|s_n t_n - s_m t_m| < M\epsilon$$

$s_n t_n$ is also cauchy.

Qu. 8. if $\sum_{n=1}^{\infty} |a_n|$ converge and (b_n) is a bounded Sequence,

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

for every $\epsilon > 0$, $\exists N$ st $n \geq m > N$

$$\Rightarrow \left|\sum_{k=m}^{n} |a_k|\right| < \epsilon/M.$$

b_n is bounded, thus $\exists M > 0$ st. $|b_n| < M$ $\forall n \in \mathbb{N}.$

$$M\left|\sum_{k=m}^{n} |a_k|\right| < \epsilon$$

$$\left|\sum_{k=m}^{n} |M a_k|\right| < \epsilon \quad -(2)$$

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Pg.

now look $|Ma_k|$ $K = m, m+1, \dots, n$

$|b_i| < M$ $i = m, m+1, \dots, n$
$\forall i$ thus as well.

for $i = K$:

$|b_k||a_k| < |Ma_k|$

$|b_k a_k| < |Ma_k|$

So, for $\sum_{K=m}^{n} |b_k a_k| < \sum_{K=m}^{n} |Ma_k|$ — ①

①, ② ⇒

$\left|\sum_{K=m}^{n} |b_k a_k|\right| < \epsilon$ $n > m > N$

$\Rightarrow \sum a_n b_n$ converges

Premium

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Pg.

9. $\sum_{n=1}^{\infty} |a_n|$ converges.

for every $\epsilon > 0$ $\exists N, n > m > N$

$\left|\sum_{K=m}^{n} |a_k|\right| < \epsilon$ — ①

Using triangle inequality.

$\sum_{K=m}^{n} |a_k| \geq \left|\sum_{K=m}^{n} a_k\right|$

$\Rightarrow \left|\sum_{K=m}^{n} |a_k|\right| \geq \left|\sum_{K=m}^{n} a_k\right|$ — ②

①② ⇒ $\left|\sum_{K=m}^{n} a_k\right| < \epsilon \Rightarrow \sum a_n$ converges

$\sum (a_n + b_n) = \lim (s_n + t_n) = A + B.$

(b) $KS_n = \sum_{i=1}^{n} K a_j$ $K \in \mathbb{R}$.

$\lim S_n = A$

$\lim KS_n = K \lim S_n = KA$

KS_n is n^{th} partial sum for $\sum_{n=1}^{\infty} K a_n$

thus, $\sum K a_n = \lim KS_n = KA$

Premium

$$\left|\sum_{k=m}^{n} |b_k a_k|\right| < \epsilon \quad n > m > N$$

$\Rightarrow \sum a_n b_n$ converges

10 $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$

i.e.

(i) $s_n = \sum_{j=1}^{n} a_j \qquad t_n = \sum_{j=1}^{n} b_j$

and given

$\lim s_n = A \qquad \lim t_n = B$

hence $\lim (s_n + t_n) = A + B$

clearly $s_n + t_n = \sum_{j=1}^{n} (a_j + b_j)$ is n^{th} partial sum for $\sum (a_n + b_n)$, so

$\sum (a_n + b_n) = \lim (s_n + t_n) = A + B$.

(ii) $K s_n = \sum_{j=1}^{n} K a_j \qquad K \in \mathbb{R}$.

$\lim s_n = A$

$\lim K s_n = K \lim s_n = KA$

$K s_n$ is n^{th} partial sum for $\sum_{n=1}^{\infty} K a_n$

then, $\sum K a_n = \lim K s_n = KA$

Premium

Premium

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(c) it fails even for series of two terms.

$$a_1 b_1 + a_2 b_2 \neq (a_1+a_2)(b_1+b_2)$$

(11) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ two Series Such that

$$a_n = b_n \ \forall \text{ but finitely many } n \in \mathbb{N}.$$

(a) $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

$s_n = \sum_{k=1}^{n} a_k$ if $\sum a_n$ converges.

then

B+

Homework no 4

(1)

(a) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p>1$ is a natural number.

$$\sum_{K=1}^{n} \frac{1}{K^p} \leq 1 + \int_1^n \frac{1}{x^p} dx = 1 + \frac{1}{x^{p-1}(1-p)}\Big|_1^n$$

$$= 1 + \frac{1}{n^{p-1}(1-p)} - \frac{1}{(1-p)}$$

$$= 1 + \frac{1}{(p-1)}\left[1 - \frac{1}{n^{(p-1)}}\right] \leq 1 + \frac{1}{p-1} = \frac{p}{p-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{p}{p-1} < +\infty$$

it converges $p>1$.

(b) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ Ratio test

- non-zero term.
- converges absolutely if $\lim \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$
- diverges if $\lim \inf \left|\frac{a_{n+1}}{a_n}\right| > 1$

$$\lim \sup \left|\frac{(n+1)^2}{3^{n+1}} \frac{3^n}{n^2}\right|$$

$$\lim \sup \left|\frac{1}{3}\left(1+\frac{1}{n}\right)^2\right| = \frac{1}{3} < 1$$

it series converges

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(c) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

$\frac{1}{2^n+n} < \frac{1}{2^n} \; \forall n \in \mathbb{N}$ and $\sum \frac{1}{2^n}$ converges as it is a geometric series. $a = \frac{1}{2} < 1$

so by comparison test

$\sum \frac{1}{2^n+n}$ also converges

(d) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$

$0 \leq \cos^2 n \leq 1$

$0 \leq \frac{\cos^2 n}{n^2} \leq \frac{1}{n^2} \quad \forall n \in \mathbb{N}.$

$\sum \frac{1}{n^2}$ converges

By comparison test

$\sum \frac{\cos^2 n}{n^2}$ also converges

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(2) No convergence of a series DONOT implies that it is absolutely convergent.

Counter example: $\sum \frac{(-1)^n}{n}$ it converges by alternating series test.

But
$$\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n} \text{ diverges.}$$

(3) Let $f:(a,b)\to\mathbb{R}$ continuous function such that

$f(r)=0$ for each rational number $r\in(a,b)$.

To prove $f(x)=0 \ \forall x\in(a,b)$

Let $c\in(a,b)$ and let $\epsilon>0$ be given.

Since f is continuous, $\exists\ \delta>0$ for which
$x\in(a,b)$ and $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\epsilon$

Since rational numbers are dense in any interval,

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$\exists$ rational numbers x, satisfying the above hypothesis, where $f(x)=0$ and so
$|f(c)|<\epsilon$ for any $\epsilon>0$
This forces $f(c)=0 \ \forall\ c\in(a,b)$.

(4) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by.

$$f(x)=\begin{cases}\frac{1}{x}\sin\left(\frac{1}{x^2}\right) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

~~lets look at x=0~~
~~for every ε>0~~ $x_n\to 0$ and $f(x_n)\neq f(0)=0$

$\frac{1}{x_n}\sin\left(\frac{1}{x_n^2}\right)=\frac{1}{x_n}$, $x_n\to 0$

$\sin\left(\frac{1}{x_n^2}\right)=1 \quad \frac{1}{x_n^2}=2n\pi+\frac{\pi}{2}$

$x_n^2=\frac{1}{2n\pi+\pi/2}$

$x_n=\frac{1}{\sqrt{2n\pi+\pi/2}}$

while.
$\lim f(x_n)=+\infty \qquad \lim x_n=0$

f ~~is~~ not continuous at $x=0$

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(5) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ \frac{1}{10^{10}} & \text{if } x < 0 \end{cases}$$

Consider a sequence $x_n = \frac{-1}{n}$ $\lim x_n = 0$ $x_n < 0$ $\forall n \in \mathbb{N}$.

But $\lim f(x_n) = \frac{1}{10^{10}} \neq f(0) = 0$

therefore function f is not continuous at $x = 0$.

(6) Let $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^3 \cos\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at $x_0 = 0$

given $\epsilon > 0$, $\exists\ \delta > 0$ and $\delta = \epsilon^{1/3}$

$x \in \mathbb{R}$ $|x - 0| < \delta$

$\Rightarrow |x| < \epsilon^{1/3}$

$\Rightarrow |x|^3 < \epsilon$

$\Rightarrow |x^3 - 0| < \epsilon$

$\Rightarrow \left|x^3 \cos\left(\frac{1}{x^2}\right)\right| \leq |x^3 - 0| < \epsilon$

$\Rightarrow \left|x^3 \cos\left(\frac{1}{x^2}\right) - f(0)\right| < \epsilon$

$\Rightarrow |f(x) - f(0)| < \epsilon$

thus $f(x)$ is continuous at $x = 0$

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Pg. B+

(7) every real number is the limit of a sequence of rational numbers.

and also the limit of a sequence of irrational numbers.

Consider a seq. of rational x_n converging to $x \in \mathbb{I}$ (irrat

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

$\lim x_n$ (rational) $= x$ (irrational)

$\lim f(x_n) = \lim 1 = 1 \neq f(x) = 0$

similarly x_n (irrational) converge to x (rational)

$\lim f(x_n) = \lim 0 = 0 \neq f(x) = 1$

so f is not continuous at any Real number.

Premium

Dt. ______ Pg. ______ B+

8. Let $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational number} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

Consider a sequence of rational number $\lim x_n \to 0$

$$\lim f(x_n) = \lim x_n = 0 = f(0) = 0$$

and now consider a sequence of irrational number $\lim s_n = 0$

$$\lim f(s_n) = \lim 0 = f(0) = 0 \quad \text{holds.}$$

Thus f is continuous at $x = 0$.

Consider $x \in \mathbb{Q}$ & $x \neq 0$ and a sequence of irrational converging to x. $\lim r_n = x$.

$$\lim f(r_n) = \lim 0 = 0 \neq f(x) = x$$, thus f is not continuous at $x \in \mathbb{Q}$ & $x \neq 0$

(9) Let $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

To show f is not continuous at 0.

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Let $x_n = \frac{1}{2n\pi + \pi/2}$ $\quad \lim x_n = 0$

$$\lim f(x_n) = \lim \sin(2n\pi + \pi/2) = \lim 1 = 1 \neq f(0) = 0$$

thus $f(x)$ is discontinuous at $x = 0$

(10) $A = [0,1] \cup [2,3]$ and $f: A \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0,1] \\ x^3 & \text{if } x \in [2,3] \end{cases}$$

Yes f is continuous on A.

take x_n be in any sequence in $[0,1]$ or $[2,3]$

so $\lim x_n \in [0,1]$ if $x_n \in [0,1]$ or $\lim x_n \in [2,3]$ if $x_n \in [2,3]$

then $\lim x_n = x_0 \in$ same interval as x_n.

Consider. $\lim f(x_n) = \lim x_n^2 = x_0^2 = f(x_0)$ $\quad x_n \in [0,1]$, $x_0 \in [0,1]$

$\lim f(x_n) = \lim x_n^3 = x_0^3 = f(x_0)$ $\quad x_n \in [2,3]$, $x_0 \in [2,3]$

thus, f is continuous in A.

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(11) $P_n(\mathbb{R})$ be the set of all polynomial function from $\mathbb{R} \to \mathbb{R}$. of degree less than n

To show: $P_n(\mathbb{R})$ is a vector space over $\mathbb{R}$.

> $P_n(\mathbb{R}) = \{$ Real polynomial of degree n or less $\}$

$$= \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R} \}$$

Consider

$$f = a_i x^i + a_{i-1} x^{i-1} + \cdots + a_{i_0} \qquad i \leq n$$

$$g = a_K x^K + a_{K-1} x^{K-1} + \cdots + a_{K_0} \qquad K \leq n$$

if $i > K$

$$f+g = a_i x^i + a_{i-1} x^{i-1} + \cdots + \underbrace{(a_i + a_K) x^K + \cdots + a_{i_0+K_0}}_{i=K}$$

gives a polynomial of degree i or less. In case $i = K$, and $a_i = -a_K$. we get a degree less.

and also. f in $P_n(\mathbb{R})$ when multiplied by a scalar $K \in \mathbb{R}$.

$$Kf = Ka_K x^K + Ka_{K-1} x^{K-1} + \cdots + Ka_0$$

gives a polynomial of degree K or less than n.

$$Kf \in P_n(\mathbb{R})$$

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> Clearly we have closure property
> also commutative $f+g = g+f$ as $a_i + a_K = a_K + a_i$ addition of real number.
> our Scalar field $\mathbb{R}$, $K \in \mathbb{R}$ $Kf \in P_n(\mathbb{R})$ give $f \in P_n(\mathbb{R})$

> $K \cdot (f+g) = K \cdot f + K \cdot g$ $\quad K \in \mathbb{R}.\ f, g \in P_n(\mathbb{R})$

$(K+l) f = Kf + lf$ $\quad f \in P_n(\mathbb{R})\ K, l \in \mathbb{R}$

> $K(lf) = (Kl) f$ $\quad K, l \in \mathbb{R},\ f \in P_n(\mathbb{R})$

> $1 \cdot f = f$

Thus $P_n(\mathbb{R})$ forms a Vector Space

(12.) Consider $f(x) = \frac{1}{x}$ on $(0,1)$

it is NOT BOUNDED.

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Homework No: 05.

① $f:[0,2]\to\mathbb{R}$ $\quad f(x)=x^3$

Let $\epsilon>0$ $\quad\exists\,\delta>0,\ \delta=\epsilon/4$ st $x,y\in[0,2]$

and $|x-y|<\delta$ $\qquad$ clearly

$\Rightarrow |x-y|<\epsilon/4$ $\qquad 0\leq x+y\leq 4$

$4|x-y|<\epsilon$

$|x+y||x-y|\leq 4|x-y|<\epsilon$

$|x^2-y^2|<\epsilon$

$|f(x)-f(y)|<\epsilon$

thus, f is uniformly continuous on $[0,2]$.

(2) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=x^3$

We need to check that

$\left[\exists\,\epsilon>0,\ \forall n\in\mathbb{N}\ \exists\,x_n,y_n\in\mathbb{R},\ |x_n-y_n|<\delta_n=\frac{1}{n}\ \Rightarrow |f(x_n)-f(y_n)|\geq\epsilon\right]$

i.e. $\delta=\frac{1}{n}>0$

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$\epsilon=1$ $\quad x_n=n\quad y_n=n+\frac{1}{2n}$ $\qquad x_n,y_n\in\mathbb{R}$.

$|x_n-y_n|=\frac{1}{2n}<\delta$

$\Rightarrow |f(x_n)-f(y_n)|=\left|\left(\frac{1}{n}\right)^3-\left(n+\frac{1}{2n}\right)^3\right|$

$=\frac{3}{2}n+\frac{3}{4n}+\frac{1}{8n^3}>1=\epsilon$

This means $f(x)=x^3$ is not uniformly continuous on $\mathbb{R}$.

~~OR~~

~~consider.~~ $\cancel{x_n=\frac{1}{n}}$ ~~a cauchy sequence in~~ $\cancel{\mathbb{R}}$.

$\cancel{f(x_n)=}$

③ $f:[0,3]\to\mathbb{R}$ $\quad f(x)=\frac{x}{x+2}$

Let $\epsilon>0,\ \exists\,\delta>0,\ \delta=\frac{\epsilon}{M}$ $\qquad M=1/2$

Clearly $\left|\frac{2}{(x+2)(y+2)}\right|\leq\frac{1}{2}$

now $x,y\in[0,3]$ $\quad |x-y|<\delta\Rightarrow|x-y|<2\epsilon$

$\left|\frac{2(x-y)}{(x+2)(y+2)}\right|<\frac{1}{2}|x-y|<\epsilon$

$\left|\frac{x}{x+2}-\frac{y}{y+2}\right|=|f(x)-f(y)|<\epsilon$

f is uniformly continuous on $[0,3]$.

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4.

(a) $\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{h \to 0} \frac{(0+h)}{|0+h|} = \lim_{h \to 0} \frac{h}{|h|} = 1$ ($h>0$)

$\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{h \to 0} \frac{(0-h)}{|0-h|} = \frac{-h}{|h|} = -1$

(b) $\lim_{x \to 1^+} \frac{1}{x-1} = \lim_{h \to 0} \frac{1}{(1+h)-1} = \frac{1}{h} = +\infty$

5. $\lim_{x \to a^+} f_1(x) = L_1$ and $\lim_{x \to a^+} f_2(x) = L_2$

and $f_1(x) \leq f_2(x) \ \forall x \in (a,b)$.

To show $L_1 \leq L_2$

Let x_n be a sequence in (a,b) which converge to a.

$\lim_{x \to a^+} f_1(x) = L_1 \quad \lim f_1(x_n) = L_1$

and since $\lim_{x \to a^+} f_2(x) = L_2 \quad \lim f_2(x_n) = L_2$

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also. $f_1(x_n) \leq f_2(x_n) \ \forall n$.

Thus $\lim f_1(x_n) \leq \lim f_2(x_n)$

$L_1 \leq L_2$ as desired.

If $f_1(x) < f_2(x) \ \forall x \in (a,b)$, then it need not be the case that $L_1 < L_2$.

6.

(a) $\sum_{n=0}^{\infty} \sqrt{n}\, x^n \qquad \beta = \lim\sup (\sqrt{n})^{1/n}$

$= \lim\sup n^{\frac{1}{2n}}$

$\Rightarrow \beta = 1$

$R = \frac{1}{\beta} = 1 \qquad |x| < 1$

$x = 1$ or -1 $\quad a_n = \sqrt{n}$ or $-\sqrt{n}$ $\quad \lim a_n = +\infty$ or $-\infty$

thus at $x = 1$ and -1 $\sum \sqrt{n}$ doesnot converge

thus interval of convergence $(-1, 1)$

(b) $\sum_{n=0}^{\infty} \frac{2^n}{n^2} x^n \qquad \beta = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{2^{n+1}}{(n+1)^2} \frac{n^2}{2^n} \right| = 2$

$R = \frac{1}{2} \qquad |x| < \frac{1}{2} < 1$

at $x = \frac{1}{2}$ $\sum \frac{1}{n^2}$ converges. at $x = -\frac{1}{2}$ $\sum \frac{(-1)^n}{n^2}$ converges by alternating series test

$x \in [-1/2, 1/2]$

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(c) $\sum_{n=0}^{\infty} n^2 x^n$ $\qquad \beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)^2}{n^2}\right| = 1$

$R = \frac{1}{\beta} = 1$

$|x| < 1$

at $x = 1$ and -1 $\sum n^2$ $\sum -n^2$ diverge

$\lim n^2 = +\infty \neq 0$

$x \in (-1, 1)$

(d) $\sum_{n=0}^{\infty} \left(\frac{x}{n}\right)^n$ $\qquad \beta = \lim\sup \left|\frac{1}{n^n}\right|^{1/n}$

$= \lim\sup \left|\frac{1}{n}\right|$

$\beta = 0$

$R = \frac{1}{\beta} = +\infty$

$|x| < +\infty$

$\Rightarrow$ $(-\infty, \infty)$ converges on $\mathbb{R}$.

(7) $\sum_{n=0}^{\infty} a_n x^n$ Radius of convergence R.

a_n are integers, all but finitely many a_n' are non-zero

then $\exists N$, s.t $n > N \Rightarrow b_n \geq 1$ and $b_n \neq 0$

$a_n \geq 1$ for infinitely many n

$b_n = \sup\{a_k : k \geq n\}$.

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$\beta = \lim\sup |a_n|^{1/n} = \lim\sup b_n^{1/n} \geq 1$

Case 1: $\beta = \infty$ $R = 0$

Case 2: $1 \leq \alpha < \infty$

$R = \frac{1}{\alpha} \leq 1$ Thus in either case $R \leq 1$.

Qu 8 $\quad x \in [-1, 1]$

$\sum \frac{1}{\sqrt{n}} x^n$

Qu 9. $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$

$R_1 \qquad R_2$

$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$.

To show $R \geq \min(R_1, R_2)$

Suppose: $R < \min(R_1, R_2)$

clearly $\Rightarrow$ $\exists x \in \mathbb{R}$ s.t. $R < |x| < \min(R_1, R_2)$

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(10) $R = \min\{R_1, R_2\}$

$\sum \frac{2^n}{n^2} x^n \quad R_1 = \frac{1}{2}$ $\qquad \sum \frac{2}{n^2} x^n \quad R_2 = 1$

$\sum \left(\frac{2^n}{n^2} + \frac{2}{n^2}\right) x^n \quad R = \frac{1}{2} = \min(R_1, R_2)$

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$R > \min\{R_1, R_2\}$

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Worksheet no: 06

① $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = \left(x - \frac{1}{n}\right)^2$.

$\lim f_n(x) = x^2 = f(x) \quad \forall x \in [0,1]$ → limit function.

The sequence f_n converges pointwise to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in S.$$

$$|f_n(x) - f(x)| = \left|\left(x - \frac{1}{n}\right)^2 - x^2\right| = \left|\frac{1}{n^2} - \frac{2x}{n}\right| \quad \left[\begin{array}{l} 0 \leq x \leq 1 \\ |x| \leq 1 \end{array}\right.$$

$$\leq \frac{2|x|}{n} + \frac{1}{n^2} \leq 2 \cdot \frac{1}{n} + \frac{1}{n^2} \leq \frac{3}{n}$$

Clearly for every $\epsilon = \frac{3}{n} > 0 \quad n \in \mathbb{N}$, $\forall$.

$\exists N, \quad n > N$

$\Rightarrow |f_n - f| < \epsilon \quad \forall x \in [0,1]$

Thus sequence f_n converges uniformly on $[0,1]$.

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② $f_n : [0,1] \to \mathbb{R}$.

$f_n(x) = x - x^n$

$$f(x) = \lim f_n(x) = \begin{cases} 0, & x = 0 \\ 0 & x = 1 \\ x & 0 < x < 1 \end{cases}$$

$\lim f_n(x) = f(x) = x \quad 0 < x < 1$

$f(0) = f(1) = 0$

$f(x)$ is not continuous at $x = 1$

Consider a sequence $x_n = 1 - \frac{1}{n} < 1 \; \forall n \in \mathbb{N}$

$\lim x_n = 1$.

$\lim f(x_n) = \lim x_n = 1 \neq f(1) = 0$

thus not continuous at $x = 1$.

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(3) $f_n : [0,\infty) \to \mathbb{R}$ given $f_n(x) = \dfrac{1}{1+x^n}$

$$\lim f_n(a) = \begin{cases} 1 & 0 \leq a < 1 \\ 1/2 & a = 1 \\ 0 & a > 1 \end{cases}$$

Since limit function is not continuous function, convergence is not uniform.

(4) $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \frac{5 + 3\sin^2(nx)}{\sqrt{n}} \quad x \in \mathbb{R}$$

$$\lim_{n\to\infty} f_n(x) = \lim \frac{5 + 3\sin^2(nx)}{\sqrt{n}}$$

$$= \lim \frac{5}{\sqrt{n}} + 3 \lim \frac{1}{\sqrt{n}} \sin^2(nx)$$

$$= 0 + 0$$

$$= 0 \qquad \forall x \in \mathbb{R}.$$

$f(x) = 0$ is the limit function it converges point wise.

$\epsilon > 0 \quad \exists N,\ n > N \Rightarrow |f_n(x) - f(x)| < \epsilon \quad n > N$

$$\left| \frac{5 + 3\sin^2(nx)}{\sqrt{n}} - 0 \right| < \epsilon$$

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$$\left| \frac{5 + 3\sin^2(nx)}{\sqrt{n}} \right| \leq \frac{8}{\sqrt{n}}$$

$\epsilon > \frac{8}{\sqrt{n}}$ can be choosen as such.

i.e $\epsilon^2 > \frac{8}{n} \quad n > \frac{8}{\epsilon^2} \quad N = \frac{8}{\epsilon^2}$

f is uniformly continuous.

(5) $S \subseteq \mathbb{R}$.

$f_n \to f$ uniformly $g_n \to g$ uniformly on S.

Claim $f_n + g_n \to f + g$ uniformly on S.

$$|f_n + g_n - (f + g)| = |f_n - f + g_n - g| \leq |f_n - f| + |g_n - g| \quad — (1)$$

for every $\epsilon > 0 \ \exists N_1$ st $n > N_1$
$\Rightarrow |f_n - f| < \epsilon/2 \ \forall x \in S$.

for every $\epsilon > 0 \ \exists N_2$ st $n > N_2$
$\Rightarrow |g_n - g| < \epsilon/2 \ \forall x \in S$.

$N = \max\{N_1, N_2\}$
$n > N$

(1) $\Rightarrow \quad \Rightarrow |f_n + g_n - (f + g)| < \epsilon/2 + \epsilon/2 = \epsilon \quad \forall x \in S$.

which mean $f_n + g_n$ converges to $f + g$ uniformly on S.

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⑥ $f_n, g_n : \mathbb{R} \to \mathbb{R}$

$f_n(x) = x \quad g_n(x) = \frac{1}{n}$.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions $f(x) = x \quad g(x) = 0$

$\lim f_n(x) = x = f(x) \quad \forall x \in \mathbb{R}$.

$\lim g_n(x) = 0 = g(x) \quad \forall x \in \mathbb{R}$.

Since g_n is independent of x, it uniformly converges to g.

Consider $|f_n - f| < \epsilon$

$|x - x| < \epsilon$

$0 < \epsilon$ for any $\epsilon > 0$, $n > N, \forall x \in \mathbb{R}$.

clearly.

f_n converges to f uniformly.

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Pg. ______

ⓑ $f_n g_n = \frac{x}{n} \qquad \lim f_n g_n = \lim \frac{x}{n} = 0$

$fg = 0$

discussing $|f_n g_n - fg| < \epsilon$

$\left|\frac{x}{n} - 0\right| < \epsilon$

$\left|\frac{x}{n}\right| < \epsilon$

clearly ϵ is not independent of x.
since x is not bounded on $\mathbb{R}$.

Suppose
given $\epsilon > 0$
$\exists N \in \mathbb{N}$.
$|f_n(x) g_n(x) - gf| < \epsilon \quad \forall n > N$
$\forall x \in \mathbb{R}$.
$|f_n(x) g_n(x)| < \epsilon$
$\left|\frac{x}{n}\right| < \epsilon \quad \forall n > N$
$\forall x \in \mathbb{R}$.

Contradiction!! $|x| < \epsilon n \quad \forall x \in \mathbb{R}, n > N$.

$f_n g_n$ do not converge uniformly on $\mathbb{R}$ to fg.

⑦ $S \subseteq \mathbb{R}$. $\quad f_n : S \to \mathbb{R}$, uniformly continuous function on S

$f_n \to f$ uniformly on S.

Claim: f is uniformly continuous on S.

f_n is uniformly continuous on S.

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for any $\epsilon > 0$ $\exists\, \delta > 0$
$x, y \in S$
$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon/3 \;\; \forall\, x, y \in S$

f_n converges to f uniformly.

$\epsilon > 0$ $\exists\, N$, $n > N \Rightarrow |f_n(x) - f(x)| < \epsilon/3 \;\; \forall\, x \in S$.

$$|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f(y) + f_n(y) - f_n(y)|$$

$$\leq \underbrace{|f_n(x) - f(x)|}_{\forall x \in S.} + \underbrace{|f_n(y) - f(y)|}_{\forall x \in S} + \underbrace{|f_n(x) - f_n(y)|}_{|x-y| < \delta}$$

so when $|x-y| < \delta$

$$\Rightarrow |f(x) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

In particular, if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$ $\epsilon > 0$
$x, y \in S$.

thus f is uniformly continuous on S.

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Pg. ______ B+

8. f_n a sequence of continuous functions on $[a, b]$
converging uniformly to a function f on $[a, b]$.
$\Rightarrow$ f is uniformly continuous on $[a, b]$
let x_n be sequence in $[a, b]$ converging to real number x_0.

To prove $\lim_{n \to \infty} f_n(x_n) = f(x_0)$

x_n a sequence in $[a, b]$ and $\lim x_n = x_0 \in [a, b]$

Since, $\lim f(x_n) = f(x_0)$

let $\epsilon > 0$

$$|f_n(x_n) - f(x_0)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x_0)|$$

$$\leq \underbrace{|f_n(x_n) - f(x_n)|}_{\substack{\forall x \in [a,b] \\ \text{also for } x = x_0}} + \underbrace{|f(x_n) - f(x_0)|}_{x = x_0}$$

$$< \epsilon/2 + \epsilon/2$$

$$< \epsilon$$

$\Rightarrow \lim f_n(x_n) = f(x_0)$

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Homework no: 7

(1) Let $a, b \in \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = |x-a| + |x-b|$

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Premium