Assignment 10

1. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $T(x_1, x_2) = (x_1, 0)$

Let $B_1 = \{e_1 = (1,0),\ e_2 = (0,1)\}$

$B_2 = \{v_1 = (1,1),\ v_2 = (-1,2)\}$

$(1,0) = a_1(1,1) + a_2(-1,2) = (a_1 - a_2,\ a_1 + 2a_2)$

$\Rightarrow a_1 = -2a_2,\ 1 = -2a_2 - a_2 \Rightarrow a_2 = -\frac{1}{3}$

$\Rightarrow (1,0) = \frac{2}{3}v_1 - \frac{1}{3}v_2.$

$T(1,0) = (1,0) = \frac{2}{3}v_1 - \frac{1}{3}v_2.$

$T(0,1) = (0,0) = 0 \cdot v_1 + 0 \cdot v_2.$

- $[T]_{[B_1, B_2]} = \left([Te_1]_{B_2}\ [Te_2]_{B_2}\right)$
$= \begin{pmatrix} 2/3 & 0 \\ -1/3 & 0 \end{pmatrix}$

- $[T]_{[B_2, B_1]} = \left([Tv_1]_{B_1}\ [Tv_2]_{B_1}\right)$

$Tv_1 = T(1,1) = (1,0) = e_1$

$Tv_2 = T(-1,2) = (-1,0) = -e_1$

$\Rightarrow [T]_{[B_2, B_1]} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

- $[T]_{[B_2, B_2]} = \left([Tv_1]_{B_2}, [Tv_2]_{B_2} \right)$

$$Tv_1 = T(1,1) = (1,0) = \frac{2}{3} v_1 - \frac{1}{3} v_2$$

$$Tv_2 = T(-1,2) = (-1,0) = -\frac{2}{3} v_1 + \frac{1}{3} v_2$$

$$\Rightarrow [T]_{[B_2, B_2]} = \begin{pmatrix} 2/3 & -2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

- $B_3 = \{ \overset{w_1}{\overset{\shortparallel}{v_2}}, \overset{w_2}{\overset{\shortparallel}{v_1}} \}$

$$Tw_1 = Tv_2 = -\frac{2}{3} w_2 + \frac{1}{3} w_1$$

$$Tw_2 = Tv_1 = \frac{2}{3} v_1 - \frac{1}{3} v_2 = \frac{2}{3} w_2 - \frac{1}{3} w_1$$

$$[T]_{[B_3, B_3]} = \left([Tw_1]_{B_3}, [Tw_2]_{B_3} \right)$$

$$= \begin{pmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{pmatrix}$$

2. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is the linear map defined by

$$T(x_1, x_2) = (-x_2, x_1, x_1 + x_2)$$

$B_1 = \{ e_1 = (1,0),\ e_2 = (0,1) \}$

$B_2 = \{ \bar{e}_1 = (1,0,0),\ \bar{e}_2 = (0,1,0),\ \bar{e}_3 = (0,0,1) \}$

$B = \{ v_1 = (1,1,1),\ v_2 = (-1,2,0),\ v_3 = (1,0,1) \}$

$Te_1 = (0,1,1) \qquad Te_2 = (-1,0,1)$

- $[T]_{[B,B_2]} = \left([Te_1]_{B_2} \; [Te_2]_{B_2}\right)$

Note $Te_1 = (0,1,1) = 0.\bar{e}_1 + \bar{e}_2 + \bar{e}_3$

$Te_2 = (-1,0,1) = -\bar{e}_1 + 0.\bar{e}_2 + \bar{e}_3$

$$\Rightarrow [T]_{[B,B_2]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

- $[T]_{[B,B]} = \left([Te_1]_B \; [Te_2]_B\right)$

$Te_1 = (0,1,1) = a_1 v_1 + a_2 v_2 + a_3 v_3$

$= a_1(1,1,1) + a_2(-1,2,0) + a_3(1,0,1)$

$$\Rightarrow \left.\begin{array}{l} a_1 - a_2 + a_3 = 0 \\ 2a_2 + a_1 = 1 \\ a_1 + a_3 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} a_3 = 1 - a_1 \\ a_2 = \dfrac{1-a_1}{2} \end{array}$$

$a_1 - \dfrac{(1-a_1)}{2} + (1-a_1) = 0$

$\Rightarrow 2a_1 + 1 - a_1 = 0$

$\Rightarrow a_1 = -1.$

$\Rightarrow a_2 = \dfrac{1+1}{2} = 1$

$a_3 = 2$

$\Rightarrow Te_1 = -1.v_1 + v_2 + 2.v_3.$

$Te_2 = (-1,0,1) = b_1 v_1 + b_2 v_2 + b_3 v_3$

$= b_1(1,1,1) + b_2(-1,2,0) + b_3(1,0,1)$

$$\Rightarrow \left.\begin{array}{l} -1 = b_1 - b_2 + b_3 \\ 0 = b_1 + 2b_2 \\ 1 = b_1 + b_3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} b_2 = -b_1/2 \\ b_3 = 1 - b_1 \end{array}\right\} \begin{array}{l} -1 = \cancel{b_1} + \dfrac{b_1}{2} + 1 - \cancel{b_1} \\ \Rightarrow -2.2 = b_1 \end{array}$$

$\Rightarrow \quad Te_2 = -4v_1 + 2v_2 + 5v_3$

$$\Rightarrow \quad [T]_{[B_1, B]} = \left([Te_1]_B \; [Te_2]_B \right)$$

$$= \begin{pmatrix} -1 & -4 \\ 1 & 2 \\ 2 & 5 \end{pmatrix}$$

- Note $[T]_{[B_1, B]} = \left([Te_1]_B \; [Te_2]_B \right)$.

$[Te_1]_B = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

$\Rightarrow Te_1 = a_1 v_1 + a_2 v_2 + a_3 v_3.$

$[Te_1]_{B_2} = a_1 [v_1]_{B_2} + a_2 [v_2]_{B_2} + a_3 [v_3]_{B_2}$

$\therefore \left([v_1]_{B_2} \; [v_2]_{B_2} \; [v_3]_{B_2} \right) [T]_{[B_1, B]} = [T]_{[B_1, B_2]}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\therefore \quad \boxed{[T]_{[B_1, B_2]} = \left([v_1]_{B_2} \; [v_2]_{B_2} \; [v_3]_{B_2} \right) [T]_{[B_1, B]}}$$

where $\{v_1, v_2, v_3\}$ is an ordered basis of B.

3. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $Tv = Av$ where A is a 3×3 matrix and the vector v is written as a column vector wrt the standard ordered basis $\{e_1, e_2, e_3\}$.

Then Nullity of T = dimension of N_T

where
$$N_T = \{ [v]_{[e_1,e_2,e_3]} \in \mathbb{R}^3 : A[v]_{[e_1,e_2,e_3]} = 0 \}$$

i.e given such a question determine the spanning set for the space $AX = 0$, and determine $\dim N_T$.

and the rank $T = 3 = \dim \mathbb{R}^3 -$ nullity of T

by rank-nullity theorem.

4(i) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by
$T(x_1, x_2) = (x_1 - x_2, x_2)$.

If (x_1, x_2) is an eigenvector of T, then $\exists\, c \in \mathbb{R}$ st $T(x_1, x_2) = c(x_1, x_2)$
$$\Rightarrow (x_1 - x_2, x_2) = (cx_1, cx_2) \quad \text{—} (*)$$

equating the components of $(*)$ we get
$$x_1 - x_2 = cx_1 \qquad \Rightarrow \begin{cases} (1-c)x_1 = x_2 \\ (1-c)x_2 = 0 \end{cases}$$
$$x_2 = cx_2$$

Substituting the value of x_2 in the second eqn, we get

$$(1-c)^2 x_1 = 0.$$

if $x_1 \neq 0$, $(1-c)^2 = 0 \Rightarrow c = 1.$

$\therefore$ $c = 1$ is an eigenvalue of T

Note that if $(x_1 - x_2, x_2) = (x_1, x_2)$

then $x_1 - x_2 = x_1 \Rightarrow x_2 = 0$

$\therefore$ $(x_1, 0)$ is an eigenvector corresponding to eigenvalue $c = 1$,

and it is easy to see that

$T(1,0) = (1,0)$ $\therefore$ $(1,0)$ is infact an eigenvector of T.

(ii) $T(x_1, x_2) = (2x_1 + x_2, 2x_1 - x_2).$

If (x_1, x_2) is an eigenvector of T, then $\exists$ c st $T(x_1, x_2) = c(x_1, x_2)$

$$\Rightarrow (2x_1 + x_2, 2x_1 - x_2) = (cx_1, cx_2)$$

$$\Rightarrow \left.\begin{array}{l} (2-c)(x_1) + x_2 = 0 \\ 2x_1 - (1+c)x_2 = 0 \end{array}\right\} \Rightarrow x_1 = \frac{(1+c)}{2} x_2 \text{ and } (2-c)\frac{(1+c)}{2} x_2 + x_2 = 0$$

$\Rightarrow ((2-c)(1+c)+2)x_2 = 0.$

$\Rightarrow (2+2c-c-c^2+2)x_2 = 0.$

$\because x_2 = 0$ implies $x_1 = 0$ $\therefore x_2 \neq 0$ (as a 0 vector cannot be an eigenvector)

$\Rightarrow 2+2c-c-c^2+2 = 0$

$\Rightarrow c^2 - c - 4 = 0$

$$\Rightarrow c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2 \cdot 1}$$

(Using the formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ are the roots of the eq^n $ax^2+bx+c=0$)

$$= \frac{1 \pm \sqrt{1+16}}{2}$$

$$= \frac{1 \pm \sqrt{17}}{2} \in \mathbb{R}.$$

$\therefore T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $(2x_1+x_2, 2x_1-x_2)$ has two distinct eigenvalues, namely $\frac{1+\sqrt{17}}{2}$ and $\frac{1-\sqrt{17}}{2}$.

Using the relation $x_1 = \frac{(1+c)}{2}x_2$ and putting $x_2 = 1$ and the value of ~~the~~ c ~~eig~~ we see that

$$\left(\frac{1+\left(\frac{1+\sqrt{17}}{2}\right)}{2}, 1\right)$$ is ~~the~~ a eigenvector corresponding to eigenvalue $\frac{1+\sqrt{17}}{2}$

and $\left(\frac{1 + \left(\frac{1-\sqrt{17}}{2}\right)}{2}, 1 \right)$ is an eigenvector

corresponding to eigenvalue $\frac{1-\sqrt{17}}{2}$.