IISER Mohali
MTH102: Analysis in One Variable
Homework No. 11
To be discussed during tutorial on April 15, 2016

- Please solve all the problems.
- **Tutorial problems** will be discussed during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorials.

Tutorial Problems:

(1) Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show that f is Riemann integrable on $[r, 1]$ for every $0 < r < 1$.
(b) Show that f is not Riemann integrable on $[0, 1]$.

(2) Let $f : [a, b] \to \mathbb{R}$ be a bounded function with finitely many points of discontinuities. Show that f is Riemann integrable.

(3) Let $P = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ be a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be the step function given by

$$f(x) = \begin{cases} c_k & \text{if } x \in [t_{k-1}, t_k) \text{ for } 1 \leq k \leq n-1 \\ c_n & \text{if } x \in [t_{n-1}, b]. \end{cases}$$

Show that f is Riemann integrable and compute $\int_a^b f(x)dx$.

(4) Let $f,\ g : [0, 1] \to \mathbb{R}$ be two continuous functions such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Prove that there exists a point $x \in [a, b]$ such that $f(x) = g(x)$.

(5) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$. Give an example showing that the result fails if f is not continuous.

(6) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t \leq 1 \\ 4 & \text{if } t > 1. \end{cases}$$

(a) Determine the function $F(x) = \int_0^x f(t)dt$.
(b) Find the points at which F is continuous.
(c) Find the points at which F is differentiable and find its derivative at these points.

(7) Let f be a bounded real valued function on $[a, b]$. Suppose that there exist a sequence $\{P_n\}$ of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \big(U(f, P_n) - L(f, P_n)\big) = 0.$$

(a) Show that f is integrable on $[a, b]$.
(b) Show that $\int_a^b f(x)dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.

Hint: Use the definition of Riemann integration and an equivalent criteria for Riemann integrability.