

MTH102: Analysis in One variable
Home Work No. 06
Sent on 28 February 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- \mathbb{N} denote the set of natural numbers.
- \mathbb{Z} denote the ring of integers.
- \mathbb{Q} denote the field of rational numbers.
- \mathbb{R} denote the field of real numbers.

- (1) Consider the sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ given by $f_n(x) = (x - \frac{1}{n})^2$.
 - (a) Does the sequence (f_n) converge point wise on $[0, 1]$? If so, then determine the limit function.
 - (b) Does the sequence (f_n) converge uniformly on $[0, 1]$? If so, then prove your assertion.
- (2) Repeat the above exercise for the sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ given by $f_n(x) = x - x^n$.
- (3) Repeat the above exercise for the sequence of functions $f_n : [0, \infty) \rightarrow \mathbb{R}$ given by $f_n(x) = \frac{1}{1+x^n}$.
- (4) Repeat the above exercise for the sequence of functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ given by $f_n(x) = \frac{5+3\sin^2(nx)}{\sqrt{n}}$.
- (5) Let S be a subset of \mathbb{R} . Suppose (f_n) and (g_n) are two sequences of functions such that (f_n) converge to f and (g_n) converge to g uniformly on S . Then prove that the sequence $(f_n + g_n)$ converge to $f + g$ uniformly on S .
- (6) Consider two sequences of functions $f_n, g_n : \mathbb{R} \rightarrow \mathbb{R}$ given by $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be the functions given by $f(x) = x$ and $g(x) = 0$.
 - (a) Prove that (f_n) converge to f uniformly on \mathbb{R} and (g_n) converge to g uniformly on \mathbb{R} .
 - (b) Prove that the sequence $(f_n g_n)$ does not converge to fg uniformly on \mathbb{R} .
- (7) Let S be a subset of \mathbb{R} . Prove that if (f_n) is a sequence of uniformly continuous functions on S converging uniformly to f on S , then f is also uniformly continuous on S .
- (8) Let (f_n) be a sequence of continuous functions on $[a, b]$ converging uniformly to a function f on $[a, b]$. Let (x_n) be a sequence in $[a, b]$ converging to a real number x . Then prove that $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$.