

Vectors Solutions

A.1) $\vec{a} = 3\hat{i} + 12\hat{j} + 4\hat{k}$

\therefore Unit vector in the direction of \vec{a}

$$= \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i} + 12\hat{j} + 4\hat{k}}{\sqrt{3^2 + 12^2 + 4^2}} = \frac{1}{13}(3\hat{i} + 12\hat{j} + 4\hat{k})$$

A.2) $\vec{a} = 3\hat{i} + 4\hat{j}$

$\vec{b} = 4\hat{i} + 4\hat{j} + 2\hat{k}$

$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta \therefore \vec{a} \cdot \vec{b} = (3\hat{i} + 4\hat{j}) \cdot (4\hat{i} + 4\hat{j} + 2\hat{k}) = 12 + 16 = 28$

$|\vec{a}| = \sqrt{3^2 + 4^2} = 5$

$|\vec{b}| = \sqrt{4^2 + 4^2 + 2^2} = 6$

$\therefore 28 = 30 \cos\theta$

$\implies \cos\theta = \frac{14}{15}$

$\implies \theta = \cos^{-1}\left(\frac{14}{15}\right)$

A.3) Projection of \vec{A} on $\vec{B} = \frac{(\vec{A} \cdot \vec{B})}{|\vec{B}|}$

$\vec{A} = 3\hat{i} + 2\hat{j}$

$\vec{B} = 4\hat{i} - \hat{j}$

$$\begin{aligned} \text{Projection} &= \frac{(3\hat{i} + 2\hat{j}) \cdot (4\hat{i} - \hat{j})}{\sqrt{4^2 + 1^2}} \\ &= \frac{12 - 2}{\sqrt{17}} = \frac{10}{\sqrt{17}} \end{aligned}$$

A.4) $|\vec{A}| = 3, |\vec{B}| = 8$

$|\vec{A} \times \vec{B}| = 12$

$\therefore |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin\theta$

$\implies 12 = 24 \sin\theta$

$\implies \sin\theta = \frac{1}{2}$

A.5) $\vec{A} = 3\hat{i} + 4\hat{j} - 4\hat{k}$. Take any vector \vec{B} in the x-y plane. Then, \vec{B} is given by

$$\vec{B} = a\hat{i} + b\hat{j}$$

Since \vec{B} is perpendicular to \vec{A} ,

$$\vec{A} \cdot \vec{B} = 0$$

$$3a + 4b = 0$$

$$a = -\frac{4b}{3}$$

We can take any value of a and b which satisfies this equation. So, choose b=3 and a=-4. Then, $\vec{B} = -4\hat{i} + 3\hat{j}$ is one such vector.

A.6) If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, show that $\vec{a} \perp \vec{b}$.

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$$

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

$$\therefore \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

$$\implies |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\implies 4|\vec{a}||\vec{b}|\cos\theta = 0$$

$$\implies \cos\theta = 0$$

$$\therefore \theta = \frac{\pi}{2}$$

$$\therefore \vec{a} \perp \vec{b}$$

A.7) i) Prove

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{aligned} LHS &= (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) \\ &= \begin{vmatrix} \hat{i}(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) & \hat{j}(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) & \hat{k}(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = RHS \end{aligned}$$

ii) $(2\hat{i} + 3\hat{j}) \cdot [(2\hat{i} + 3\hat{j}) \times (4\hat{j} + 7\hat{k})]$

This is of form

$$\begin{aligned} &= [\vec{a} \ \vec{b} \ \vec{c}] \\ &= \begin{vmatrix} 2 & 3 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 7 \end{vmatrix} \\ &= 0 \end{aligned}$$

iii) Prove that $[\vec{a} \ \vec{b} \ \vec{c}]$ is cyclic, i.e $[\vec{a} \ \vec{b} \ \vec{c}] = [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{c} \ \vec{a} \ \vec{b}]$.

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$(\vec{b} \times \vec{c}) \cdot \vec{a} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(\vec{c} \times \vec{a}) \cdot \vec{b} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Upon calculating the determinant they all turn out equal.

A.8) i) Prove $[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = 2[\vec{a} \vec{b} \vec{c}]$.

$$\begin{aligned} LHS &= [\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] \\ &= (\vec{a} + \vec{b}) \cdot \{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\} \\ &= (\vec{a} + \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\} \\ &= (\vec{a} + \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\} \\ &= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c}) + (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{a}) + (\vec{a} + \vec{b}) \cdot (\vec{c} \times \vec{a}) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c}) + 0 + \vec{b} \cdot (\vec{c} \times \vec{a}) \\ &= [\vec{a} \vec{b} \vec{c}] + [\vec{b} \vec{c} \vec{a}] \\ &= [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{b} \vec{c}] \quad \left\{ \because [\vec{a} \vec{b} \vec{c}] = [\vec{a} \vec{b} \vec{c}] \right\} \\ &= 2[\vec{a} \vec{b} \vec{c}] = RHS \end{aligned}$$

ii) Prove $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = [\vec{a} \vec{b} \vec{c}]^2$.

$$\begin{aligned} LHS &= (\vec{a} \times \vec{b}) \cdot \{(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})\} \\ &= (\vec{a} \times \vec{b}) \cdot \{[\vec{b} \vec{c} \vec{a}]\vec{c} - [\vec{b} \vec{c} \vec{a}]\vec{a}\} \end{aligned}$$

$[\vec{b} \vec{c} \vec{c}] = 0$ because $[\vec{b} \vec{c} \vec{c}]$ has two rows equal and determinant of matrix having two rows equal is zero.

$$\begin{aligned} LHS &= (\vec{a} \times \vec{b}) \cdot [\vec{b} \vec{c} \vec{a}]\vec{c} \\ &= \{(\vec{a} \times \vec{b}) \cdot \vec{c}\}[\vec{b} \vec{c} \vec{a}] \\ &= [\vec{a} \vec{b} \vec{c}][\vec{b} \vec{c} \vec{a}] \\ &= [\vec{a} \vec{b} \vec{c}]^2 = RHS \end{aligned}$$