**IISER Mohali**
**MTH102: Analysis in One Variable**
**Homework No. 05**
**To be discussed during tutorial on February 19, 2016**

- Please solve all the problems.
- Tutors will discuss **tutorial problems** during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorial sessions.

**Tutorial Problems:**

(1) Determine whether the following series converge.
  (a) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$
  (b) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$
  (c) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$

(2) Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, where $A$ and $B$ are real numbers. Then prove the following:
  (a) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$.
  (b) $\sum_{n=1}^{\infty} ka_n = kA$ for all $k \in \mathbb{R}$.
  (c) Can we say that $\sum_{n=1}^{\infty} a_n b_n = AB$?

(3) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges and $(b_n)$ is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges. Hint: Use the Cauchy criteria.

(4) Let $f : (a, b) \to \mathbb{R}$ be a continuous function such that $f(r) = 0$ for each rational number $r \in (a, b)$. Then prove that $f(x) = 0$ for each $x \in (a, b)$.

(5) Let $f(x) = \frac{1}{x} \sin(\frac{1}{x^2})$ for $x \neq 0$ and $f(0) = 0$. Show that $f$ is not continuous at 0.

**Extra Problems:**

(1) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series such that $a_n = b_n$ for all but finitely many $n \in \mathbb{N}$. Then prove the following:
  (a) $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
  (b) $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\sum_{n=1}^{\infty} b_n$ diverges.

(2) Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x| + 2x$ is a continuous function.

(3) Give two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g \neq g \circ f$.