Homework 10

$*$ means difficult or important
$**$ Optional / too difficult.

0)$*$ Solve all problem/exercises mentioned in class.

1)$*$ If S_1 is orientable and S_2 is diffeomorphic to S_1 then S_2 is also orientable

2)$**$ Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $c \in \mathrm{Im}(f)$. Let $S = f^{-1}(c)$. Assume $\mathrm{grad}(f) \neq 0$ on S. We want to show that S is a regular surface. The idea is to solve $f(x,y,z) = c$ for ~~one~~ one of the variables; e.g. if $f = x^2+y^2+z^2$, $c = 1$ then we can solve $z = \pm\sqrt{1-x^2-y^2}$. Like the example shows the solution may not be unique but if we get a solution say $z = g(x,y)$ then S is the graph of g and hence a smooth surface.

Given $p \in S$ assume $f_z \neq 0$. We ~~c~~ claim we can solve $f(x,y,z) = c$ as $z = g(x,y)$ "near p". If $f_z = 0$ either $f_x \neq 0$ or $f_y \neq 0$. If $f_x \neq 0$ then "near p" we will have $x = h(y,z)$ etc. We deal with the case $f_z \neq 0$.

Consider $F(x,y,z) = (x, y, f(x,y,z))$,

$F: \mathbb{R}^3 \to \mathbb{R}^3$.

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* Check that $JF = \begin{pmatrix} 1 & 0 & f_x \\ 0 & 1 & f_y \\ 0 & 0 & f_z \end{pmatrix}$.

Hence $\det JF(p) \neq 0$. By Inverse function Theorem there is an open set $W \subseteq \mathbb{R}^3$ such that $W \ni p$ and $F: W \to F(W)$ is a diffeomorphism. Let $G: F(W) \to W$ be the inverse of F. Let $G(x,y,z) = (\varphi_1(x,y,z), \varphi_2(x,y,z), \varphi_3(x,y,z))$.

For any $(x,y,z) \in W \cap S$ we have

$$G \circ F(x,y,z) = (x,y,z) \quad \text{since } G \circ F = Id_W$$

$$\Rightarrow \quad G(x,y,f(x,y,z)) = (x,y,z)$$

$$\Rightarrow \quad G(x,y,c) = (x,y,z)$$

In particular, $z = \varphi_3(x,y,c)$.

Define $g(x,y) = \varphi_3(x,y,c)$.

** Determine the domain of g, say V.

Find a surface patch using this.

This shows that every point of S falls in a surface patch. Thus S is regular surface.

Check all conditions for allowable surface patches carefully.

3) Check that if $z = g(y)$ is a smooth curve in the yz-plane not touching or crossing the z-axis then revolving it about the z-axis we get an orientable surface.

4)* If S is a plane say $ax+by+cz=d$ then show that $\forall p \in S$, T_pS is the plane $ax+by+cz=0$.

5)* (A) Consider the map $f: S \to \mathbb{R}^2$ where $S: z = x^2+y^2$ and $f(x,y,z) = (x,y)$. Let $p = (1,1,2) \in S$. (i) Find T_pS

(ii) $\mathbb{R}^2$ can be thought of as the xy-plane in $\mathbb{R}^3$. Let $q = f(p)$. Determine the map $Df_p: T_pS \to T_q\mathbb{R}^2$. In otherwords find any basis of T_pS, say $(\vec{v}_1, \vec{v}_2)$ and evaluate $Df_p(\vec{v}_1)$, $Df_p(\vec{v}_2)$

(B)* Do the same for $S = S^2: x^2+y^2+z^2=1$, $p=(0,0,1)$, $f: S^2 \to \mathbb{R}^2$ $(x,y,z) \mapsto (y,z)$.

(C)* Do the same for $f: S_1 \to S_2$ where $S_1: z = \sqrt{x^2+y^2}$, $z>0$, $S_2: x^2+y^2 = 1$

$f(x,y,z) = \left(\frac{x}{z}, \frac{y}{z}, z\right)$, $p = (1,0,1)$ i.e.

(i) Find T_pS_1, $T_{f(p)}S_2$ and describe

$Df_p : T_pS_1 \to T_{f(p)}S_2$.

(ii) Show that Df is an isomorphism at all points of S_1. See next problem.

4. A map $f : S_1 \to S_2$ is called a local diffeomorphism if $\forall p \in S_1$ there is an open set $\mathcal{U} \subseteq S_1$, $p \in \mathcal{U}$ such that $f(\mathcal{U}) \subseteq S_2$ is open and $f : \mathcal{U} \to f(\mathcal{U})$ is a diffeomorphism

(i) If f is a local diffeomorphism then show that Df_p is an isomorphism $\forall p \in S_1$

(ii)** Prove the converse also: If Df_p is an isomorphism $\forall p \in S_1$ then f is a local diffeomorphism.

5. (A) Consider the eight surface patches on S^2 obtained by projecting points on the coordinate planes. Show that the transition of patches have positive Jacobian determinant.

(B)* Cover the cylinder $S : x^2+y^2 = 1$ by two patches ~~and~~ ~~check~~ ~~the~~ such that the transition of patches has positive Jacobian determinant.