IISER Mohali
MTH102: Analysis in One Variable
Homework No. 09
To be discussed during tutorial on March 29, 2016

- Please solve all the problems.
- Tutors will discuss **tutorial problems** during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorial sessions.

Tutorial Problems:

(1) Let $f(x) = |x-1|+|x-2|$ for all $x \in \mathbb{R}$. Determine the set of points at which f is not differentiable.

(2) Let
$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$
(a) Show that f is differentiable at all $x \in \mathbb{R}$. Determine the function $f' : \mathbb{R} \to \mathbb{R}$.
(b) Is the function f' continuous on $\mathbb{R}$?
(c) Is the function f' differentiable on $\mathbb{R}$?

(3) Let
$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
(a) Prove that f is continuous at $x = 0$.
(b) Prove that f is differentiable at $x = 0$ and find the derivative.
(c) What happens at points $x \neq 0$?

(4) Let f be a differentiable function on $\mathbb{R}$. Further, suppose that $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.
(a) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.
(b) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.
Hint: Use Mean Value Theorem.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Then prove that f is a constant function.
Hint: Show that f is differentiable with $f'(x) = 0$ for all $x \in \mathbb{R}$.

(6) Let f be a twice differentiable function on an open interval (a, b) such that $f''(x) = 0$ for all $x \in (a, b)$. Then prove that f has the form $f(x) = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$.
Hint: Use a consequence of the Mean Value Theorem.

(7) Let f and g be two differentiable functions on $\mathbb{R}$. Suppose that $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$ and $f(0) = g(0)$. Then prove that $f(x) \leq g(x)$ for all $x \geq 0$.
Hint: Use properties of increasing or decreasing functions.

Extra Problems:

(1) Let $f(x) = |\sin(x)|$ for all $x \in \mathbb{R}$. Give the exact set of points at which f is not differentiable.
Hint: Drawing a graph of $f(x)$ will be helpful.

(2) Let $f(x) = \sin(|x|)$ for all $x \in \mathbb{R}$. Give the exact set of points at which f is not differentiable.
Hint: You may assume that $\sin(x)$ is a differentiable function with $\cos(x)$ as its derivative.

(3) Let $f(x) = x^{1/3}$ for $x \in \mathbb{R}$. Use the definition of derivative to prove that $f'(x) = \frac{1}{3}x^{-2/3}$ for all $x \neq 0$. Is the function f differentiable at $x = 0$?

(4) Suppose that f is differentiable at a point $a \in \mathbb{R}$. Then prove that
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(5) Let f be a thrice differentiable function on an open interval (a, b) such that $f'''(x) = 0$ for all $x \in (a, b)$. What form does f have? Prove your claim.