MTH202: Solutions (Selected exercises)

February 1, 2019

1. How many different letter arrangements can be formed from the letters $PEPPER$?

   Solution: Suppose we have distinct P's and E's. Number of ways to arrange $P_1, E_1, P_2, P_3, E_2, R$ is given by 6!. Number of ways to permute P's and E's among themselves is 3!2!. So the answer is $\frac{6!}{3!2!}$.

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2. In how many ways can 100 identical books be distributed into 10 different bags so that no bag is empty?

   Solution: Suppose ith bag gets x_i books. Then we want to compute number of positive integer values solutions to the equation $x_1 + \ldots + x_{10} = 100$. Hence, the answer is $\binom{99}{9}$.

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3. Let A, B be events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Show that $\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$.

   Solution: $P(A \cap B) \leq P(A) = 1/2$ (since $A \cap B \subset A$). Also, $P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1$ implies $P(A \cap B) \geq 1/2 + 2/3 - 1 = 1/6$.

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4. For events A, B, prove that:

$$|P(A) - P(B)| \leq P(A \Delta B)$$

   Solution:

$$\begin{aligned}
P(A) - P(B) &\leq P(A \cup B) - P(B) \\
&= P(A \setminus B) \\
&\leq P(A \setminus B) + P(B \setminus A) \\
&= P((A \setminus B) \cup (B \setminus A)) \quad \text{since } (A \setminus B) \cap (B \setminus A) = \phi \\
&= P(A \Delta B)
\end{aligned}$$

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5. E, F, G be events in Ω such that $E \cap F = \phi$ and $P(G) \neq 0$. Show that $P(E \cup F|G) = P(E|G) + P(F|G)$.

Solution:

$$\begin{aligned} P(E \cup F|G)P(G) &= P((E \cup F) \cap G) \\ &= P((E \cap G) \cup (F \cap G)) \\ &= P(E \cap G) + P(F \cap G) \\ &= P(E|G)P(G) + P(F|G)P(G) \end{aligned}$$

Cancel $P(G)$ from both sides.

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6. A laboratory blood test is 95 percent efficient in detecting a certain disease when it is, in fact, present. However the test also yields a false positive for 1 percent of healthy persons tested. If 0.5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

Solution: Let D be the event that the person tested has the disease and E be the event that the test is positive. Then,

$$\begin{aligned} P(D|E) &= \frac{P(D \cap E)}{P(E)} \\ &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} \\ &= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} \\ &= \frac{95}{294} \end{aligned}$$

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7. Suppose that we have 3 cards that are identical in form except that both sides of the first card are coloured red, both sides of second card are coloured black and one side of third card is red while the other side is black. One card is randomly selected and put down on table. If the visible side of the chosen card is red, what is the probability that other side is black?

Solution: Let RR, BB, and RB denote, respectively, the events that the chosen card is all red, all black, or the redblack card. Also, let R be the event that the upturned side of the chosen card is red. Then the desired probability is obtained by

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\begin{aligned}
P(RB|R) &= \frac{P(RB \cap R)}{P(R)} \\
&= \frac{P(R|RB)P(RB)}{P(R|RR)P(RR) + P(R|RB)P(RB) + P(R|BB)P(BB)} \\
&= \frac{\frac{1}{2} \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3}} \\
&= \frac{1}{3}
\end{aligned}
$$

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8. Suppose A and B are independent events. Show that A^c and B^c are also independent.

Solution: Enough to show that A, B^c are independent.

$$
\begin{aligned}
P(A \cap B^c) &= P(A) - P(A \cap B) \\
&= P(A) - P(A)P(B) \\
&= P(A)P(B^c)
\end{aligned}
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9. A fair coin is tossed three times. Consider the following events:

- $E = \{\text{Toss 1 and toss 2 produce different outcomes}\}$
- $F = \{\text{Toss 2 and toss 3 produce different outcomes}\}$
- $G = \{\text{Toss 3 and toss 1 produce different outcomes}\}$

Show that $P(E) = P(E|F) = P(E|G)$ but $P(E) \neq P(E|F \cap G)$.

Solution: Total 8 equally likely possibilities. We have $|\Omega| = 8$ and $|E| = |F| = |G| = 4$. So, $P(E) = P(F) = P(G) = 1/2$.
Similarly, count the number of outcomes in intersection events, i.e. $|E \cap F| = |F \cap G| = |G \cap E| = 2$. We get: $P(E \cap F) = P(F \cap G) = P(G \cap E) = 1/4$. Now,

$$
\begin{aligned}
P(E|F) &= \frac{P(E \cap F)}{P(F)} = 1/2 = P(E) \\
P(E|G) &= \frac{P(E \cap G)}{P(G)} = 1/2 = P(E)
\end{aligned}
$$

Observe that $E \cap F \cap G$ is empty. This implies:

$$
P(E|F \cap G) = \frac{P(E \cap F \cap G)}{P(F \cap G)} = 0 \neq P(E).
$$

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10. X be the number shown when a single fair die is rolled. Compute $E[X]$.

Solution: $1 \times 1/6 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 + 5 \times 1/6 + 6 \times 1/6 = 7/2$.

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11. Let X be a discrete random variable with p.m.f.

$$P(k) = \begin{cases} 0.2 & \text{for } k = 0 \\ 0.2 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Define $Y = X(X-1)(X-2)$. Find the p.m.f. of Y.

Solution: Note that $Y \in \{0, 6\}$. We need to compute $P(Y = 0)$ and $P(Y = 6)$.

$$P(Y = 0) = P(X = 0) + P(X = 1) + P(X = 2) = 0.7$$

and,

$$P(Y = 6) = P(X = 3) = 0.3$$

Let P_Y denote the p.m.f. of Y. Then,

$$P_Y(k) = \begin{cases} 0.7 & \text{for } k = 0 \\ 0.3 & \text{for } k = 6 \\ 0 & \text{otherwise} \end{cases}$$

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12. Let X be a discrete random variable with p.m.f.

$$P(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Compute the following:

- Find $E[X]$.
- Find $Var(X)$.
- If $Y = (X-2)^2$, find $E[Y]$.

Solution:

- $E[X] = 0(0.1) + 1(0.4) + 2(0.3) + 3(0.2) = 1.6$
- Recall that $Var(X) = E[X^2] - (E[X])^2$. We have:

  $$E[X^2] = 0^2(0.1) + 1^2(0.4) + 2^2(0.3) + 3^2(0.2) = 3.4$$

  So, $Var(X) = 3.4 - (1.6)^2 = 0.84$.
- $E[Y] = (0-2)^2(0.1) + (1-2)^2(0.4) + (2-2)^2(0.3) + (3-2)^2(0.2) = 1$.

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13. A standard (6-sided) fair die is rolled two times (each roll being independent). Let X_i denote the value obtained on the i-th roll. Is this a random variable? What about $S = X_1 - X_2$, the difference of the values rolled? Determine the p.m.f of S.

    Solution: Since both rolls are independent, we have:

    $$\begin{aligned}
    P(S=-5) &= P(\{X_1 = 1\} \cap \{X_2 = 6\}) = 1/36 \\
    P(S=-4) &= P((\{X_1 = 1\} \cap \{X_2 = 5\}) \cup (\{X_1 = 2\} \cap \{X_2 = 6\})) = 2/36 \\
    P(S=-3) &= P(\cup_{k=1}^{3}(\{X_1 = k\} \cap \{X_2 = 3+k\})) = 3/36 \\
    P(S=-2) &= P(\cup_{k=1}^{4}(\{X_1 = k\} \cap \{X_2 = 2+k\})) = 4/36 \\
    P(S=-1) &= P(\cup_{k=1}^{5}(\{X_1 = k\} \cap \{X_2 = 1+k\})) = 5/36 \\
    P(S=0) &= P(\cup_{k=1}^{6}(\{X_1 = k\} \cap \{X_2 = k\})) = 6/36
    \end{aligned}$$

    By symmetry $P(S = k) = P(S = -k)$.

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14. In an experiment, we can observe whether the solution is acidic (event A) and whether the solution is coloured (event B). Assume that $P(B) > 0$. We now carry out the experiment repeatedly (each trial being independent) until B is observed. What is the probability that A is observed at the same time as B?

    Solution: Probability that we do not observe B till $n-1$-th experiment and in the n-th trial we observe B and A is given by $(1-P(B))^{n-1}P(A \cap B)$ (note that we are using independent of trials here). So the required probability is given by:

    $$\sum_{n=1}^{\infty}(1-P(B))^{n-1}P(A \cap B) = \frac{P(A \cap B)}{1-(1-P(B))} = P(A|B)$$

    This presents an alternate interpretation of conditional probability.

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15. A communication system consists of n components (called a n-C system), each of which function (independently) with probability p. The overall

system can operate effectively if at least one-half of its components function. For what values of p is a 5-C system more likely to operate efectively than a 3-C system?

Solution: X_n = number of functioning components in a n-C system. Then $X_n \sim Bin(n,p)$. We want to find p such that $P(X_5 \geq 3) > P(X_3 \geq 2)$. That is,

$$10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$$

This gives $p > 1/2$.

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16. A fair die is rolled repeatedly (each roll being independent) until a 6 appears. The number of rolls is recorded. What is the probability that we never see a 6?

    Solution: $P(6 \text{ appears on i-th roll }) = 1/6$. Let E_n be the event that we do not see a 6 in n rolls of the dice. Then, $E_{n+1} \subset E_n$ is a decreasing chain of events. The event E, that we do not see a 6 at all is given by $\cap_n E_n$. We get that $P(E) = \lim_{n\to\infty} P(E_n) = \lim_{n\to\infty}(5/6)^n = 0$.