MTH 101 - Symmetry
Assignment 6

Notes: For a positive integer n, we denote by S_n the set of all bijections from the set $I_n = \{1, 2, \cdots, n\}$ to itself. A k-cycle $(a_1, \cdots, a_k)$ in S_n denotes the map $f : I_n \to I_n$ such that $f(a_i) = a_{i+1}$ for $i1 \leq i \leq k-1$, $f(a_k) = a_1$ and $f(j) = j$ if $j \in I_n - \{a_1, \cdots, a_k\}$. For example (1325) in S_n, denotes the map $f : I_n \to I_n$ such that

$$f(1) = 3,\ f(3) = 2,\ f(2) = 5,\ f(5) = 1,\ f(j) = j, \text{for } j \in \{4, 6, 7, \cdots, n\}.$$

Two cycles $(a_1, a_2, \cdots, a_k)$ and $(b_1, \cdots, b_r)$ in S_n are said to be disjoint if the sets $\{a_1, \cdots, a_k\}$ and $\{b_1, \cdots, b_r\}$ are disjoint. For example (123) and (4567) are disjoint cycles but (1234) and (23)(56) are not disjoint since

$$2, 3 \in \{1, 2, 3, 4\} \cap \{2, 3, 5, 6\}.$$

Given two permutations $(a_1, \cdots, a_k)$ and $(b_1, \cdots, b_r)$, the element $(a_1, \cdots, a_k)(b_1, \cdots, b_r)$ in S_n denotes the composition of the two bijections. For example, if f_1, f_2 are the bijections corresponding to the elements (134) and (3456) then (134)(3456) corresponds to the map $f_1 \circ f_2$ and (3456)(134) corresponds to the map $f_2 \circ f_1$. Thus, as product of disjoint cycles

$$f_1 \circ f_2 = (134)(3456) = (13)(456), \text{ and } f_2 \circ f_1 = (3456)(134) = (14)(356).$$

By definition, order of an element a in a group G, is the least positive integer k such that $a^k = e_G$, where e_G is the identity element of the group. In particular given a bijection $f \in S_n$, order of f in S_n is the least positive integer k such that $f^k = Id_{S_n}$, where by Id_{S_n} we denote the identity map in S_n and $f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k-times}$

1. Using the binary operation in S_n, write the following as product of disjoint cycles. Also determine their orders and check if they are even or odd permutations.
   i. (123)(3562)(123)
   ii. (123)(3561)(132)
   iii. (5241)(9425)(12)
   iv. $(567489)^3$
   v. (153467)

2. Let $A_n = \{\sigma \in S_n : \sigma \text{ is an even permutation}\}$.
   i. Prove that A_n is a normal subgroup of S_n.
   ii. Show that every element of A_n can be written as a product of 3-cycles.(Hint: Use the fact that $(a_1a_3)(a_1a_2) = (a_1a_2a_3)$).
   iii. Let $\phi : S_n \to (Z_2, \oplus_2)$ be a map such that

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even permutation} \\ 1 & \text{otherwise} \end{cases}$$

   Prove that ϕ is a group homomorphism. Determine $Kernel\ \phi = K_\phi$ and the quotient group S_n/K_ϕ.

3. Let $\phi : G \to G'$ be a group homomorphism. Prove that, for an element $a \in G$, order of $(\phi(a))$ divides order of a. Hence prove that if p is a odd prime and $\phi : S_n \to (Z_p, \oplus_p)$ is a group homomorphism, then $\phi(x) = \bar{0}$ for all $x \in S_n$, where $\bar{0}$ is the identity element of $(Z_p, \oplus_p)$.

4. Check that for $a_1, a_2 \in I_n$, $(a_1a_2) = (1a_1)(1a_2)(1a_1)$. Hence from that the group S_n is generated by the set of elements $X = \{(12), (13), \cdots, (1n)\}$.