Assignment 9 (correction wrt revised notes)

①. $B = \{ v_1 = (1,1,0),\ v_2 = (0,0,1),\ v_3 = (1,0,4) \}$

$S = \{ e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1) \}$

the change of basis matrix relative to $[S\ B]$ is given by

$$\left(C_{[S,B]}\right)^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{pmatrix} = C_{[S,B]}$$

and

$$C_{[BS]} = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

(refer to the revised notes for the description of $C_{[BS]}$. As in literature, we shall refer to $C_{[BS]}$ as the change of basis matrix relative to $[BS]$.)

Assignment 9

1. To show the set

$$B = \{v_1 = (1,1,0),\ v_2 = (0,0,1),\ v_3 = (1,0,4)\}$$

form a basis of $\mathbb{R}^3|_{\mathbb{R}}$.

$\because \# X = \dim_{\mathbb{R}} \mathbb{R}^3 = 3$, it suffices to show X is linearly independent.

To check X is linearly ind, we show that

$$c_1(1,1,0) + c_2(0,0,1) + c_3(1,0,4) = (0,0,0)$$

implies $c_1 = c_2 = c_3 = 0$.

Consider $c_1 v_1 + c_2 v_2 + c_3 v_3 = \underline{0}$.

$$\Rightarrow \quad c_1 + c_3 = 0$$
$$c_1 = 0$$
$$c_2 + 4c_3 = 0.$$

$$\Rightarrow \quad c_3 = -c_1 = 0, \quad c_2 = -4c_3$$

This show $c_1 = c_2 = c_3 = 0$.

i.e X is lin. ind.

Let $(1,0,0) = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(1,0,0) = a_1(1,1,0) + a_2(0,0,1) + a_3(1,0,4)$

$\Rightarrow 1 = a_1 + a_3,\ 0 = a_1,\ a_2 + a_3 4 = 0$

$\Rightarrow a_3 = 1 \quad a_2 = -4.$

$\therefore (1,0,0) = 0 \cdot v_1 - 4v_2 + v_3.$

$$\Rightarrow \quad [(1,0,0)]_B = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}.$$

$$(0,1,0) = b_1 v_1 + b_2 v_2 + b_3 v_3$$
$$= b_1(1,1,0) + b_2(0,0,1) + b_3(1,0,4)$$

$$\Rightarrow \quad 0 = b_1 + b_3, \quad 1 = b_1, \quad 0 = b_2 + 4b_3$$

$$\Rightarrow \quad b_3 = -1 \qquad b_2 = -4(-1) = 4.$$

i.e $[(0,1,0)]_B = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}.$

$$(0,0,1) = c_1 v_1 + c_2 v_2 + c_3 v_3$$
$$= c_1(1,1,0) + c_2(0,0,1) + c_3(1,0,4)$$

$$\Rightarrow \quad 0 = c_1 + c_3, \quad 0 = c_1, \quad 1 = c_2 + 4c_3$$

$$\Rightarrow \quad c_3 = 0 \qquad c_2 = 1.$$

$$\Rightarrow [(0,0,1)]_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[v_1]_S = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad [v_2]_S = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad [v_3]_S = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

$\therefore$ Change of basis matrix relative to [S, B]

$$= \begin{pmatrix} 0 & -4 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Change of basis matrix relative to [B,S]

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

2. $V = M_2(\mathbb{R})\big|_{\mathbb{R}}$.

any general element of $M_2(\mathbb{R})$ is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

This shows, any element in V lies in the span of

$$X = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

we now show that X is linearly ind.

So we consider

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then equating the components we see that

$$c_1 = 0 = c_2 = c_3 = c_4.$$

$\Rightarrow$ X is lin. ind, Hence X is a basis of $V|_{\mathbb{R}}$.

3. $V = \{A \in M_2(\mathbb{R}) \mid a_{11} + a_{22} = 0\}$

$\Rightarrow A \in V$ if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{11} + a_{22} = 0$ or $a_{11} = -a_{22}$

This shows that any elt in V is of the form,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which implies that $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ spans V.

Now consider,

$$c_1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + c_2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c_3\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1 & c_2\\ c_3 & -c_1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

equating the components we see that

$c_1 = c_2 = c_3 = 0$.

$\Rightarrow$ This shows that B is a basis of $V|_{\mathbb{R}}$.