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MTH102: Analysis in One Variable
Homework No. 02
To be discussed during tutorial on January 22, 2016

- Please solve all the problems.
- **Tutorial Problems** will be discussed during tutorial sessions.
- If time permits, tutors may also discuss **Extra Problems** during tutorial sessions.

Tutorial Problems:

(1) Let S and T be non-empty bounded subsets of $\mathbb{R}$. Let $S+T = \{s+t \mid s \in S \text{ and } t \in T\}$.
  (a) Prove that if $S \subseteq T$, then $inf(T) \leq inf(S) \leq sup(S) \leq sup(T)$.
  (b) Prove that $sup(S \cup T) = \max\{sup(S), sup(T)\}$.
  (c) Prove that $inf(S \cup T) = \min\{inf(S), inf(T)\}$.
  (d) Prove that $sup(S+T) = sup(S) + sup(T)$.
  (e) Prove that $inf(S+T) = inf(S) + inf(T)$.
(2) Let $\mathbb{I}$ be the set of all irrational numbers. Prove that if $a < b$ are two real numbers, then there exists $x \in \mathbb{I}$ such that $a < x < b$.
(3) Prove that if $0 < a$ is a real number, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.
(4) Determine the limits of the following sequences and prove your claims:
  (a) $\lim \frac{3n+1}{5n-2}$.
  (b) $\lim \frac{(-1)^n}{n}$.
(5) Give an example of a sequence of rational numbers converging to an irrational number.
(6) Let (s_n) be a sequence of non-negative real numbers and let $\lim s_n = s$. Prove that $\lim \sqrt{s}_n = \sqrt{s}$.
(7) Let (s_n) be a sequence such that $\lim s_n = s$. Let $a \in \mathbb{R}$ and $s_n \geq a$ for all but finitely many n. Prove that $s \geq a$.

Extra Problems:

(1) Let S and T be non-empty subsets of $\mathbb{R}$, not necessarily bounded. Prove that if $S \subseteq T$, then $inf(T) \leq inf(S) \leq sup(S) \leq sup(T)$.
(2) Determine the limits of the following sequences and prove your claims:
  (a) $\lim \frac{n}{n^2+1}$.
  (b) $\lim(\sqrt{n^2+n} - n)$.
  (c) $\lim \frac{1}{n}\sin(n)$.
(3) Prove that the following sequences do not converge.
  (a) $(-1)^n n$.
  (b) $\sin(\frac{n\pi}{3})$.
  (c) $\cos(\frac{n\pi}{3})$.
(4) Give an example of a sequence of irrational numbers converging to a rational number.
(5) Give an example of a sequence (s_n) such that $\lim |s_n|$ exists but $\lim s_n$ does not exist.