APPENDIX

A

# Incompleteness of the Nineteen Rules

The nineteen Rules of Inference presented in Sections 3.1 and 3.2 of Chapter 3 are *incomplete*,[1] which is to say that there are truth functionally valid arguments whose validity cannot be proved using only those nineteen Rules. To discuss and establish this incompleteness, it is useful to introduce the notion of a characteristic that is 'hereditary with respect to a set of rules of inference'. We offer this definition: A characteristic $\Phi$ is *hereditary with respect to a set of rules of inference* if and only if whenever $\Phi$ belongs to one or more statements, it also belongs to every statement deduced from them by means of those Rules of Inference. For example, *truth* is a characteristic that is hereditary with respect to the nineteen Rules of Inference presented in the first two sections of Chapter 3. As we have already said, any conclusion must be true if it can be deduced from true premisses by means of our nineteen Rules of Inference. Indeed, we should not want to use any rules of inference with respect to which truth was *not* hereditary.

Now, to prove that a set of rules of inference is incomplete, we must find a characteristic $\Phi$ and a valid argument $\alpha$ such that

(1) $\Phi$ is hereditary with respect to the set of rules of inference; and
(2) $\Phi$ belongs to the premisses of $\alpha$ but not to the conclusion of $\alpha$.

The characteristic *truth* is hereditary with respect to any set of rules of inference in which we may be seriously interested and, therefore, satisfies condition (1) above. But where $\alpha$ is a valid argument, it follows immediately from our definition of validity that *truth* can never satisfy condition (2) above. Hence to prove the incompleteness of our nineteen Rules we must find a characteristic other than *truth* that is hereditary with respect to our nineteen Rules, and can belong to the premisses but not to the conclusion of some valid argument $\alpha$.

[1] The following proof of incompleteness was devised by my friend Professor Leo Simons of the University of Cincinnati. See Leo Simons, 'Logic Without Tautologies', *Notre Dame Journal of Formal Logic*, vol. 15 (1974), pp. 411–431, and 'More Logics Without Tautologies', ibid., vol. 19 (1978), pp. 543–557. For some discussion of the matter, see Robert L. Armstrong, 'A Question about Completeness', ibid., vol. 17 (1976), pp. 295–296, and Paul J. Campbell, 'An Answer to Armstrong's Question about Incompleteness in Copi', ibid., vol. 18 (1977), pp. 262–264.

To obtain such a characteristic, we introduce a three-element model in terms of which the symbols in our nineteen Rules can be interpreted. The three elements are the numbers **0**, **1**, and **2**, which play roles analogous to those of the truth values true (**T**) and false (**F**) introduced in Chapter 2. Every statement will have one of the three elements of the model assigned to it, and it will be said to assume, take on, or have one of the three *values* **0**, **1**, or **2**. Just as in Chapter 2 the statement variables $p, q, r, \ldots$, were allowed to range over the two truth values **T** and **F**, so here we allow the statement variables $p, q, r, \ldots$ to range over the three values **0**, **1**, and **2**.

The five symbols '$\sim$', '$\cdot$', '$\vee$', '$\supset$', and '$\equiv$' that occur in our nineteen Rules can be redefined for (or in terms of) our three-element model by the following three-valued tables:

| $p$ | $\sim p$ |
|---|---|
| 0 | 2 |
| 1 | 1 |
| 2 | 0 |

| $p$ | $q$ | $p \cdot q$ | $p \vee q$ | $p \supset q$ | $p \equiv q$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 2 | 2 | 0 | 2 | 2 |
| 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 2 | 1 | 1 | 1 |
| 2 | 0 | 2 | 0 | 0 | 2 |
| 2 | 1 | 2 | 1 | 0 | 1 |
| 2 | 2 | 2 | 2 | 0 | 0 |

Alternative (but equivalent) analytical definitions can be given as follows, where 'min$(x, y)$' denotes the minimum of the numbers $x$ and $y$, and 'max$(x, y)$' denotes the maximum of the numbers $x$ and $y$.

$$\sim p = 2 - p$$
$$p \cdot q = \max(p, q)$$
$$p \vee q = \min(p, q)$$
$$p \supset q = \min(2 - p, q)$$
$$p \equiv q = \max(\min(2 - p, q), \min(2 - q, p))$$

The desired characteristic $\Phi$ that is hereditary with respect to our nineteen Rules of Inference is the characteristic of having the value **0**. To prove that this characteristic is hereditary with respect to the nineteen Rules, it will suffice to show that it is hereditary with respect to each of the nineteen Rules. This can be shown for each rule by means of a three-valued table. For example, that having the value **0** is hereditary with respect to *Modus Ponens*, can be seen by examining the table above, which defines the value of '$p \supset q$' as a function of the values of '$p$' and of '$q$'. The two premisses '$p$' and '$p \supset q$' both have the value **0** only in the first row, and there the conclusion '$q$' has the value **0** also. Examining the same table shows that having the value **0** is hereditary also for Simplification, Conjunction, and Addition. Filling in additional columns for



'$\sim p$' and '$\sim q$' will show that having the value **0** is hereditary with respect to *Modus Tollens* and Disjunctive Syllogism also. That it is hereditary with respect to Hypothetical Syllogism can be shown by the following table:

| $p$ | $q$ | $r$ | $p \supset q$ | $q \supset r$ | $p \supset r$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 2 | 0 | 2 | 2 |
| 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 2 | 1 | 1 | 2 |
| 0 | 2 | 0 | 2 | 0 | 0 |
| 0 | 2 | 1 | 2 | 0 | 1 |
| 0 | 2 | 2 | 2 | 0 | 2 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 2 | 0 | 2 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 2 | 1 | 1 | 1 |
| 1 | 2 | 0 | 1 | 0 | 0 |
| 1 | 2 | 1 | 1 | 0 | 1 |
| 1 | 2 | 2 | 1 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 1 | 0 |
| 2 | 0 | 2 | 0 | 2 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 1 | 0 |
| 2 | 1 | 2 | 0 | 1 | 0 |
| 2 | 2 | 0 | 0 | 0 | 0 |
| 2 | 2 | 1 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 0 |

Only in the first, tenth, nineteenth, twenty-second, twenty-fifth, twenty-sixth, and twenty-seventh rows do the two premisses '$p \supset q$' and '$q \supset r$' both have the value **0**, and in each of them the conclusion '$p \supset r$' has the value **0** also. Even larger tables would be needed to show that having the value **0** is hereditary with respect to Constructive Dilemma and Destructive Dilemma. Larger tables can be constructed, but they are not absolutely necessary, because the alternative analytical definitions on page 334 can be used to show that having the value **0** is hereditary with respect to the Dilemmas, as on page 336.

When we construct three-valued tables to verify that having the value **0** is hereditary with respect to replacement of statements by their logical equivalents, we notice that although the biconditionals themselves need not have the value 0, the expressions flanking the equivalence sign necessarily have the same value. For example, in the table appropriate to the first of De Morgan's Theorems,

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \cdot q$ | $\sim(p \cdot q)$ | $\sim p \vee \sim q$ | $\sim(p \cdot q) \equiv (\sim p \vee \sim q)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 0 | 2 | 2 | 0 |
| 0 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 0 | 2 | 2 | 0 | 2 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 1 | 0 | 2 | 0 | 0 | 0 |
| 2 | 0 | 0 | 2 | 2 | 0 | 0 | 0 |
| 2 | 1 | 0 | 1 | 2 | 0 | 0 | 0 |
| 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 |

the equivalent expressions '$\sim(p \cdot q)$' and '$\sim p \vee \sim q$' have the same value in every row even though the statement of their equivalence fails to possess the value 0 in rows two, four, and five. It should be obvious, however, that having the value 0 is hereditary with respect to the replacement of all or part of any statement by any other statement that is equivalent to the part replaced.

Alternative proofs that having the value 0 is hereditary with respect to the nineteen Rules make use of our analytical definitions of the logical symbols. For example, that having the value 0 is hereditary with respect to Constructive Dilemma can be shown by the following argument. By assumption, '$(p \supset q) \cdot (r \supset s)$' and '$p \vee r$' both have the value 0. Hence both '$p \supset q$' and '$r \supset s$' have the value 0, so either $p = 2$ or $q = 0$ and either $r = 2$ or $s = 0$. Since '$p \vee r$' has the value 0, either $p = 0$ or $r = 0$. If $p = 0$, then $p \neq 2$, whence $q = 0$, and if $r = 0$, then $r \neq 2$ whence $s = 0$. Therefore, either $q = 0$ or $s = 0$ whence '$q \vee s$' has the value 0, which was to be shown.

Once it has been established that the characteristic of having the value 0 is hereditary with respect to the nineteen Rules, to prove the incompleteness of those Rules, one need only exhibit a valid argument whose premisses have the value 0, but whose conclusion does not have the value 0. Such an argument is

$$A \supset B$$
$$\therefore A \supset (A \cdot B)$$

whose validity is easily established by a truth table. Where '$A$' has the value 1 and '$B$' has the value 0, the premiss '$A \supset B$' has the value $1 \supset 0 = 0$ but the conclusion '$A \supset (A \cdot B)$' has the value $1 \supset (1 \cdot 0) = 1 \supset 1 = 1$. Therefore, the nineteen Rules of Inference are incomplete.