MTH 101 - Symmetry
Assignment 11 & Notes

Recall : Let $B_V = \{v_1, \cdots, v_n\}$ be an ordered basis of $V|_{\mathbb{R}}$, $B_W = \{w_1, \cdots, w_k\}$ be an ordered basis of $W|_{\mathbb{R}}$ and let $T : V \to W$ be a linear transformation. If for $v_j \in B_V$,

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{kj}w_k,$$

then the **matrix of T relative to the ordered basis $[B_V : B_W]$**, is written as follows:

$$T_{[B_V:B_W]} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$$

Notice that the j^{th} *column of the matrix* $T_{[B_V,B_W]}$ *is a column representation of* $T(v_j)$ *with respect to the ordered basis* B_W of W and for $v = \sum_{i=1}^n c_i v_i \in V$ with $[v]_{B_V} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$,

$$[Tv]_{B_W} = T_{[B_V:B_W]}.[v]_{B_V},$$

where $T_{[B_V:B_W]}.[v]_{B_V}$ denotes the multiplication of the $k \times n$ matrix $T_{[B_V:B_W]}$, with the $k \times 1$ column matrix $[v]_{B_V}$.

- If $B'_V = \{v'_1, v'_2, \cdots, v'_n\}$ is another ordered basis of V, then the matrix of T relative to the ordered basis $[B'_V : B_W]$ would be such that the j^{th} column of $T_{[B'_V,B_W]}$ is equal to the column vector $[T(v'_j)]_{B_W}$. But by the above discussion,
$$[T(v'_j)]_{B_W} = T_{[B_V:B_W]}.[v'_j]_{B_V}.$$
Hence if $[c^*_{[B_V:B'_V]}$ is the matrix whose j^{th} column is given by $[v'_j]_{B_V}$, then
$$\mathbf{T_{[B'_V,B_W]}} = \mathbf{T_{[B_V:B_W]}}.\mathbf{c_{[B_V:B'_V]}}.$$
(Note that $c_{[B_V:B'_V]}$ is the change of basis matrix relative to $[B_V : B'_V]$ that was mentioned in Assignment 10.)

- If $B'_W = \{w'_1, w'_2, \cdots, w'_k\}$ is another ordered basis for W, then the j^{th} column of the matrix $c_{[B_W,B'_W]}$ is given by the column vector $[w'_j]_{B_W}$ and the j^{th} column of the matrix $c_{[B'_W,B_W]}$ is given by the column vector $[w_j]_{B'_W}$. Thus if
$$[Tv_j]_{B_W} = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{kj}w_k,$$
then
$$[Tv_j]_{B'_W} = a_{1j}[w_1]_{B'_W} + a_{2j}[w_2]_{B'_W} + \cdots + a_{kj}[w_k]_{B'_W}.$$
Thus if
$$[Tv_j]_{B_W} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{pmatrix}, \quad c_{[B'_W,B_W]} = \begin{pmatrix} [w_1]_{B'_W} & [w_2]_{B'_W} & \cdots & [w_k]_{B'_W} \end{pmatrix},$$
then
$$[Tv_j]_{B'_W} = \begin{pmatrix} [w_1]_{B'_W} & [w_2]_{B'_W} & \cdots & [w_k]_{B'_W} \end{pmatrix}.\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{pmatrix} = c_{[B'_W,B_W]}.[Tv_j]_{B_W}.$$
Hence ,
$$\mathbf{T_{[B_V,B'_W]}} = \mathbf{c_{[B'_W,B_W]}}.\mathbf{T_{[B_V,B_W]}}.$$

Recall: For a linear transformation $T : V \to W$,

i. The **null space of** T, denoted by N_T or $N(T)$ is given as follows:

$$N(T) = \{v \in V : Tv = 0\}.$$

(Check that N_T is a subspace of V.) The **nullity** of T is defined as the **dimensional of** N_T.

ii. The **range of** T is defined as follows:

$$\text{Range}(T) = \{w \in W : Tv = w, \text{ for some } v \in V\}.$$

(Check that Range(T) is a subspace of W.) The **rank** of T is defined as the **dimensional of Range(**T**)**.

Thm: (**Rank-Nullity Theorem**): For a linear transformation $T : V \to W$,

$$\text{dimension}_{\mathbb{R}} V = Rank\ T + Nullity\ of\ T.$$

1. If
$$\begin{aligned} v_1 &= (1, -1) & w_1 &= (1, 0) \\ v_2 &= (2, -1) & w_2 &= (0, 1) \\ v_3 &= (-3, 2) & w_3 &= (1, 1), \end{aligned}$$
is there a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(v_i) = w_i$ for $i = 1, 2, 3$?

2. Describe explicitly the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(1, 0) = (a, b)$ and $T(0, 1) = (c, d)$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by
$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$
   a. What are the conditions on a vector $(a, b, c) \in \mathbb{R}^3$ such that (a, b, c) is in the range of T ? What is the rank of T ?
   b. What are the conditions on a vector $(a, b, c) \in \mathbb{R}^3$ such that (a, b, c) is in the null space of T ? What is the nullity of T ?