

Solution to HW7

①

1) Try yourself.

2) Fix $m \neq 0$. Then $\lim_{y \rightarrow 0} f(x, y) = \frac{m}{1+m^2}$

Thus f is ~~not~~ not continuous at $(0,0)$

Given $v = (a, b)$

$$f'(0,0;v) = \lim_{t \rightarrow 0} \frac{f(tv) - f(0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{f(ta, tb)}{t} = \lim_{t \rightarrow 0} \frac{t^3 ab^2}{t(ta^2 + tb^4)}$$

$$= \lim_{t \rightarrow 0} \frac{ab^2}{a^2 + tb^4} = \begin{cases} \frac{b^2}{a} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

3) See below.

4) (i) $f(p+tv) = f(1+t, 1+2t, 1+3t)$

$$= (1+t)^2 + (1+2t)(1+3t) + (1+t)(1+3t)^2$$

$$= 3 + 2t + 5t + 7t + \text{higher order terms}$$

$$f(p) = 3$$

Hence, $f'(p;v) = \lim_{t \rightarrow 0} \frac{f(p+tv) - f(p)}{t} = 2+5+7=14$

(ii) $\text{grad } f = (f_x, f_y, f_z) = (y+z, x+z, x+y)$

$$\Rightarrow \text{grad } f(p) = (1, 2, 1)$$

$$\Rightarrow f'(p;v) = \text{grad } f(p) \cdot v = (1, 2, 1) \cdot (0, 1, 1) = 3$$

(iii) $\text{grad } f = (2xy+y^2, x^2+2xy) \Rightarrow \text{grad } f(p) = (8, 5)$

$$\Rightarrow f'(p;v) = (8, 5) \cdot (1, 1) = 13$$

5) See below.

②

6) Recall: If $f(x) = (f_1(x), \dots, f_m(x)) : u \in \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth then

$$Jf = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Thus when $n=1$, $Jf = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}$

Therefore, if $\alpha : (-\varepsilon, \varepsilon) \rightarrow u \in \mathbb{R}^n$ is a curve,

$\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t))$ then

$$J\alpha(0) = \begin{pmatrix} \alpha_1'(0) \\ \vdots \\ \alpha_n'(0) \end{pmatrix} = \alpha'(0) \text{ written as a column vector.}$$

The rest is chain rule.

7) (i) $f'(x) = e^x > 0$ Hence, we are done by the inverse function theorem. Here $\text{Im}(f) = (0, \infty)$ clearly we can directly prove that $\log : (0, \infty) \rightarrow \mathbb{R}$ is a smooth inverse of e^x .

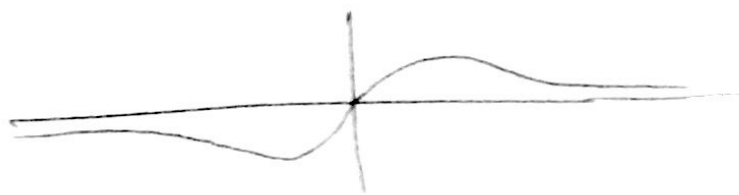
(ii) One can directly find f^{-1} and note that f, f^{-1} are smooth. Or just find $f'(x)$ and show that it is nonzero $\forall x \in \mathbb{R}$ if f were to be a diffeomorphism onto its image.

$$f'(x) = \frac{(x^2+1) - 2x \cdot x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Since $f'(\pm 1) = 0$ f cannot be a diffeomorphism

onto its image.

Ex: Check ~~if~~ if f is a homeomorphism onto its image. ③



(iii) $f'(0) = 0$ Hence f cannot be a diffeomorphism onto image. Note: $f^{-1}(x) = x^{1/3}$ is continuous. Hence, f is a homeomorphism $\mathbb{R} \rightarrow \mathbb{R}$.

$$(w) \quad Jf = \begin{pmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{pmatrix}$$

$$\Rightarrow \det Jf = x > 0$$

Hence, by the inverse function theorem we are done.

Note: This is just transformation of polar co-ordinates into Cartesian co-ordinates.

$$8. (i) \quad q_x = (1, 0, y), \quad q_y = (0, 1, x)$$

$$\Rightarrow q_x \times q_y = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = -y\vec{i} - x\vec{j} + \vec{k} \neq \vec{0} \quad \forall (x, y)$$

Hence, q is an allowable surface patch.

Well one needs to check that q is a homeomorphism onto its image $S = \{(x, y, z) : z = xy\}$. Clearly, q is bijective onto S and $q^{-1}(x, y, z) = (x, y)$ which is continuous.

$$(ii) \quad q_y = (0, 2y, 3y^2) \Rightarrow q_y(0, 0) = 0.$$

Hence, q cannot be an allowable surface patch.

Check if it is a surface patch.

(4)

(iii) Clearly φ is ~~not onto image~~ not even

$$1-1: \quad \varphi(0, y) = \varphi(-1, y).$$

Hence, φ is not a surface patch.

(iv) • φ is a surface patch for the cone

$z = \sqrt{x^2 + y^2}$ with vertex $= (0, 0, 0)$ removed.

• φ is smooth.

$$\bullet \quad \varphi_x = \left(1, 0, \frac{x}{\sqrt{x^2 + y^2}}\right), \quad \varphi_y = \left(0, 1, \frac{y}{\sqrt{x^2 + y^2}}\right)$$

Clearly $\varphi_x \times \varphi_y \neq \vec{0} \quad \forall (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Hence, φ is an allowable surface patch.

(v) • φ is a surface patch for the cylinder $x^2 + y^2 = 1$ with one line: $x=1, y=0$ removed.

• φ is smooth

$$\bullet \quad \varphi_x = (-\sin x, \cos x, 0)$$

$$\varphi_y = (0, 0, 1)$$

$$\Rightarrow \varphi_x \times \varphi_y = \sin x \vec{j} + \cos x \vec{i} \neq \vec{0} \quad \forall (x, y)$$

$\Rightarrow \varphi$ is an allowable surface patch.

③ and ⑤: The crucial thing to use is the Mean Value Theorem: If $f: [a, b] \rightarrow \mathbb{R}$ is C^1 then $\frac{f(b) - f(a)}{b - a} = f'(c)$ for some $c \in (a, b)$.

For (3) suppose $\{x_k\}$ is a sequence in \mathcal{U} converging to x where

$$x_k = (a_{1k}, a_{2k}, \dots, a_{nk}) \text{ and}$$

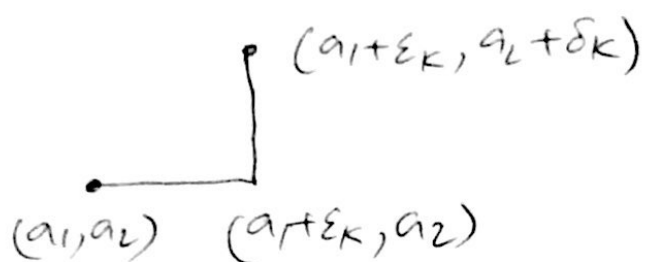
$$x = (a_1, a_2, \dots, a_n).$$

For simplicity assume $n=2$.

Write $a_{1k} = a_1 + \epsilon_k$

$$a_{2k} = a_2 + \delta_k$$

We have $\epsilon_k \rightarrow 0$, $\delta_k \rightarrow 0$ and we want to show that $f(a_1 + \epsilon_k, a_2 + \delta_k) \rightarrow f(a_1, a_2)$.



Now, $f(a_1 + \epsilon_k, a_2 + \delta_k) - f(a_1, a_2)$

$$= \left[f(a_1 + \epsilon_k, a_2 + \delta_k) - f(a_1 + \epsilon_k, a_2) \right] + \left[f(a_1 + \epsilon_k, a_2) - f(a_1, a_2) \right]$$

$$= f_y(a_1 + \epsilon_k, a_2 + \delta'_k) \cdot \delta_k + f_x(a_1 + \epsilon'_k, a_2) \cdot \epsilon_k$$

for some $0 < \delta'_k < \delta_k$ and $0 < \epsilon'_k < \epsilon_k$

Now, as f_x, f_y are continuous

$$f_y(a_1 + \epsilon_k, a_2 + \delta'_k) \rightarrow f_y(a_1, a_2) \text{ and}$$

$$f_x(a_1 + \epsilon'_k, a_2) \rightarrow f_x(a_1, a_2) \text{ as } \begin{matrix} \epsilon_k \rightarrow 0 \\ \delta_k \rightarrow 0 \end{matrix}$$

Hence, $\lim_{k \rightarrow \infty} \left[f(a_1 + \epsilon_k, a_2 + \delta_k) - f(a_1, a_2) \right] = 0$.

[Do (5) similarly.]