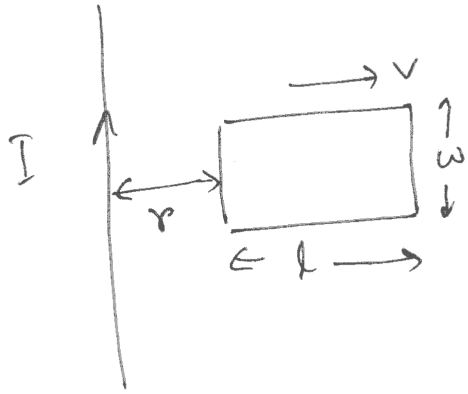
Assignment 7 (sols.)

1.



Method 1:

$$|\mathcal{E}| = vw(B_1 - B_2)$$

$$B_1 = \frac{\mu_0 I}{2\pi r}, \quad B_2 = \frac{\mu_0 I}{2\pi (r+l)}.$$

$r = 15\text{ cm} = 0.15\text{ m}; \quad l = 10\text{ cm} = 0.1\text{ m}.$

$v = 5\text{ m/sec}; \quad w = 8\text{ cm} = 0.08\text{ m}, \quad I = 100\text{ A}$

$$\therefore |\mathcal{E}| = 5 \times 0.08 \cdot \frac{\mu_0}{2\pi}\left(\frac{100}{0.15} - \frac{100}{0.25}\right)$$

$$= 0.4 \times 2 \cdot 10^{-7} \times 10^2 \; \frac{0.25 - 0.15}{0.25 \times 0.15}$$

$$= 2.13 \times 10^{-5} \text{ volts.}$$

Method 2:

$$\phi = \int \vec{B} \cdot d\vec{a} = \int_{r}^{r+l} \frac{\mu_0 I}{2\pi x} w \, dx$$

Note. $x \equiv x(t)$, $r \equiv r(t)$

$$= \frac{\mu_0 I w}{2\pi} \ln x \Big|_{r}^{r+l} = \frac{\mu_0 I w}{2\pi} \ln\left(\frac{r+l}{r}\right)$$

$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{\mu_0 I w}{2\pi} \cdot \frac{r}{r+l}\left(\frac{1}{r}\cdot\frac{dr}{dt} - \frac{1}{r^2}\cdot\frac{dr}{dt}(r+l)\right)$$

$$= -\frac{\mu_0 I w}{2\pi} \cdot \frac{r}{r+l}\left(\frac{v}{r} - \frac{v(r+l)}{r^2}\right).$$

$$= \frac{\mu_0 I w}{2\pi} \frac{lv}{r(r+l)} = vw\left(\frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I}{2\pi(r+l)}\right)$$

$$= vw(B_1 - B_2) \quad \text{(Same as above)}$$

Direction of $\mathcal{E}$ will be in a direction to drive current such that flux decreases. Since flux is downward & decreases, induced current is clockwise.

The current in the loop at a time t is $I(t) = \frac{\mathcal{E}(t)}{R}$ where R is a non-negligible resistance in the loop. This current creates a field ~~and~~ of its own and therefore a flux linked with it. Since $\mathcal{E}$ is changing with time, so does this flux. This creates an additional induced emf $\mathcal{E}'$ which we have ignored. We need to therefore find R so that, $\mathcal{E} \gg \mathcal{E}'$ & we can safely ignore $\mathcal{E}'$.

The field created by the current, $B' \sim \frac{\mu_0 I(t)}{2\pi \cdot L}$ where L is some length ~ 10 cm.

$$\therefore \phi' \sim B' \cdot \text{area of loop} = \frac{\mu_0 \times I(t) \times 0.08 \times \cancel{0.1}}{2\pi \times \cancel{0.1}}$$

$$= 2 \times 10^{-7} \times 0.08 \times I(t)$$

$$\therefore \phi' \sim 0.16 \times 10^{-5} \times I(t)$$

A typical time scale in the problem would be,

$$\tau' \sim \frac{l}{v} = \frac{0.1}{5} \text{ sec}$$

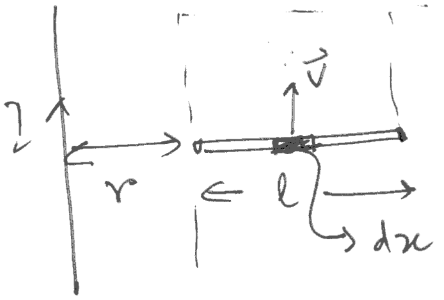
$$\therefore \mathcal{E}' \sim \frac{\phi'}{\tau'} = \frac{0.16 \times 10^{-5}}{0.1} \times 5 \times I(t) = 8 \times 10^{-5}\, I(t)$$

$$= 8 \times 10^{-5} \frac{\mathcal{E}(t)}{R}$$

$\therefore \quad R \sim \left(8 \times 10^{-5} \dfrac{\mathcal{E}}{\mathcal{E}'}\right) \Omega$

$\therefore$ Even for $R \gg 10^{-5}\,\Omega$ we will have $\mathcal{E} \gg \mathcal{E}'$.

2.



Let the conducting rod sweep an area in time t.

Area swept by an element dx of rod, a distance x from current carrying wire, is given as $\underbrace{vt}_{\text{vertical length}}\, dx = da$.

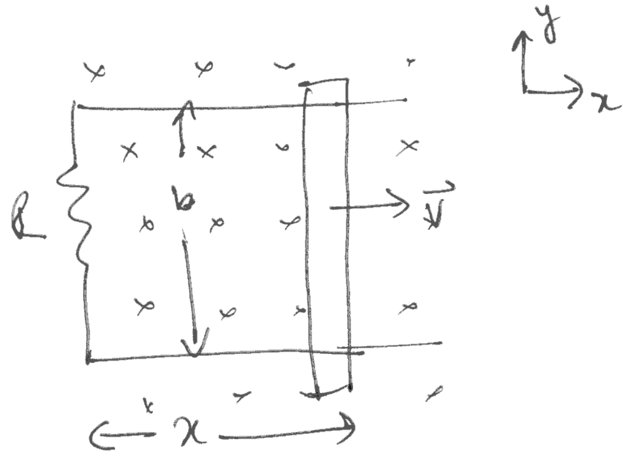
Magnetic field at a distance x from current carrying wire, $B = \dfrac{\mu_0 I}{2\pi x}$.

$$\therefore \quad \phi = \int B\, da = \int_r^{r+l} \frac{\mu_0 I}{2\pi x}\, vt\, dx$$

$$= \frac{\mu_0 I}{2\pi} vt \cdot \ln\left(\frac{r+l}{r}\right)$$

$$\therefore \quad |\mathcal{E}| = \frac{d\phi}{dt} = \frac{\mu_0 I}{2\pi} v \ln\left(\frac{r+l}{r}\right)$$

3.



$\times \Rightarrow$ denotes magnetic field direction

$\therefore \vec{B} = -B\hat{z}$ pointing into the page.

$\vec{v} = v\hat{x}$.

The rod moves ~~in~~ to the right in the $\hat{x}$ direction due to some external force.

Let at a time, t, the crossbar of mass m is at a distance x as shown.

Then it ~~is~~ forms closed loop and during its motion, area of loop increases with time.

With the magnetic field acting downwards, ~~mag~~ flux increases with time.

flux, $\phi = BA = Bbx$

$$\therefore \mathcal{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}(Bbx) = -Bb\frac{dx}{dt} = -Bbv.$$

This induced emf will cause current in the loop in the counterclockwise direction so that it opposes the increase in flux by giving rise to a magnetic field to oppose the external magnetic field $\vec{B}$.

Now, induced current, $I = \frac{|\mathcal{E}|}{R} = \frac{Bbv}{R}$

$\therefore$ Magnetic force experienced by bar,

$$\vec{F}_B = I\vec{l}\times\vec{B} = I\, b\hat{y}\times(-B\hat{z}).$$

$$= -IbB(\hat{y}\times\hat{z}) = -IbB\hat{x}$$

$$= -\frac{B^2b^2v}{R}\hat{x}$$

$\rightarrow$ opposite to $\vec{v}$.

$\therefore$ For the bar to keep moving at a constant speed, an external force needs to be present so that,

$$\vec{F}_{ext} = -\vec{F}_B = \frac{B^2b^2v}{R}\hat{x}.$$

Suppose at $t=0$, speed of rod $= v_0$ & the external agent stops pushing. Then,

$$F_B = -\frac{B^2b^2v}{R} = ma = m\frac{dv}{dt}$$

$$\Rightarrow \quad \frac{dv}{v} = -\frac{B^2b^2}{mR}dt = -\frac{1}{\tau}dt, \quad \tau = \frac{mR}{B^2b^2}$$

Integrating,

$$v(t) = v_0 e^{-t/\tau}.$$

$\therefore$ speed decreases exponentially in the absence of external force doing work.

$$x = \int_0^\infty v(t)\,dt = \int_0^\infty v_0 e^{-t/\tau}dt = v_0\tau.$$

$$= \frac{mR}{B^2b^2}v_0.$$

Power delivered by $\vec{F}_{ext}$ = power dissipated in resist

$$\therefore P = \vec{F}_{ext} \cdot \vec{v} = \frac{B^2 b^2 v}{R} \cdot v = \frac{(B b v)^2}{R} = \frac{\mathcal{E}^2}{R} = I^2 R.$$

4. (a) $$\phi = BA = (B_0 + bt)\pi a^2 = \pi(B_0 + bt)a^2$$

At $t = 0$, $\phi = \pi B_0 a^2$.

(b) $$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}(BA)$$

$$= -A\frac{dB}{dt} = -\pi a^2 \cdot \frac{d}{dt}(B_0 + bt)$$

$$= -\pi a^2 \cdot b.$$

(c) $$I = \frac{|\mathcal{E}|}{R} = \frac{\pi a^2 b}{R}.$$

(d) $$P = I^2 R = \left(\frac{\pi b a^2}{R}\right)^2 R = \frac{(\pi b a^2)^2}{R}$$

5. Faraday's law : $\int \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{a}$

(Integral form)

Inside solenoid, $B = \mu_0 n I$

$$\therefore \phi = \int \vec{B} \cdot da = B \cdot \pi r^2 = \mu_0 n I \pi r^2$$

$$= \mu_0 n (Ct) \pi r^2 \quad (\because I = Ct)$$

$\therefore$

$\therefore$ Faraday's law gives magnitude of tangential component of $\vec{E}$ as,

$$E_\theta \cdot 2\pi r = \frac{d}{dt}\left(\mu_0 n C t\, \pi r^2\right).$$

$$E_\theta \cdot 2\cancel{\pi} r = \mu_0 n C \cancel{\pi} r^2$$

$$\Rightarrow \quad E_\theta = \frac{\mu_0 n C r}{2}.$$

Outside solenoid, $\phi = (\mu_0 n C t)\, \pi R^2$

where R : radius of solenoid.

This happens because, field is zero outside.

$$\therefore E_\theta \cdot 2\pi r = \frac{d}{dt}\left(\mu_0 n C t\, \pi R^2\right)$$

$$E_\theta \cdot 2\cancel{\pi} r = \mu_0 n C \cancel{\pi} R^2$$

$$\Rightarrow \quad E_\theta = \frac{\mu_0 n C R^2}{2r}$$

To check that these results are correct, use,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \& \quad \vec{\nabla} \times \vec{E} = \hat{z}\, \frac{1}{r} \frac{\partial}{\partial r}\left(r E_\theta\right)$$

in cylindrical coordinates for this problem. This is the only term in curl which survives.