Matrices solution

Maths Workshop 2020

December 20, 2020

Solutions

1. (i) Order - 2×3 Transpose - $\begin{bmatrix} 1 & 7 \\ 2 & 8 \\ 3 & 9 \end{bmatrix}$

1. (ii) Order - 3×1 Transpose-$\begin{bmatrix} 5 & 2 & 1 \end{bmatrix}$

1. (iii) Order - 1×3 Transpose-$\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$

2. Given

$$\begin{aligned}
(A+B)^2 = A^2 + B^2 &\implies (A+B)(A+B) = A^2 + B^2 \\
&\implies A^2 + AB + BA + B^2 = A^2 + B^2 \\
&\implies AB + BA = 0 \\
&\implies \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\implies \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\implies \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\implies a = 1, b = 4
\end{aligned}$$

3. $P(x) = \begin{bmatrix} cos(x) & sin(x) \\ -sin(x) & cos(x) \end{bmatrix}$, then

$$\begin{aligned}
P(x)P(y) &= \begin{bmatrix} cos(x) & sin(x) \\ -sin(x) & cos(x) \end{bmatrix} \begin{bmatrix} cos(y) & sin(y) \\ -sin(y) & cos(y) \end{bmatrix} \\
&= \begin{bmatrix} cos(x)cos(y) - sin(x)sin(y) & cos(x)sin(y) + sin(x)cos(y) \\ -sin(x)cos(y) - sin(y)cos(x) & -sin(x)sin(y) + cos(x)cos(y) \end{bmatrix} \\
&= \begin{bmatrix} cos(x+y) & sin(x+y) \\ -sin(x+y) & cos(x+y) \end{bmatrix} \\
&= P(x+y)
\end{aligned}$$

Again,

$$
\begin{aligned}
P(y)P(x) &= \begin{bmatrix} cos(y) & sin(y) \\ -sin(y) & cos(y) \end{bmatrix} \begin{bmatrix} cos(x) & sin(x) \\ -sin(x) & cos(x) \end{bmatrix} \\
&= \begin{bmatrix} cos(x)cos(y) - sin(x)sin(y) & cos(x)sin(y) + sin(x)cos(y) \\ -sin(x)cos(y) - sin(y)cos(x) & -sin(x)sin(y) + cos(x)cos(y) \end{bmatrix} \\
&= \begin{bmatrix} cos(x+y) & sin(x+y) \\ -sin(x+y) & cos(x+y) \end{bmatrix} \\
&= P(x+y)
\end{aligned}
$$

Hence , $P(x)P(y) = P(x+y) = P(y)P(x)$

4. We find here :

$$
AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 7 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \\ 5 \end{bmatrix}
$$

Now $(AB)^t = \begin{bmatrix} 1 & 16 & 5 \end{bmatrix}$,$B^t = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$ and $A^t = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$

Thus,

$$
\begin{aligned}
B^t A^t &= \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 0 & 7 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 16 & 5 \end{bmatrix}
\end{aligned}
$$

Thus $(AB)^t = B^t A^t$

5.Given

$$
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\implies A^2 = AA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\implies A^3 = A^2 A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

From these computations we guess the general formula for A^n as
$A^n = \begin{bmatrix} 1 & 1 & 2n-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

We suppose that the formula is true for n=k.We will now prove it for n=k+1.

$$
A^{k+1} = A^k A = \begin{bmatrix} 1 & 1 & 2k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2(k+1)-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

$\therefore$ The formula holds for n=k+1 and thus by induction will hold for any natural number n.

6. Similarly as above we have

$$\begin{aligned} A &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \\ \implies A^2 = AA &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} \\ \implies A^3 = A^2 A &= \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} \end{aligned}$$

Thus we guess our answer for $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$. We suppose the formula to be true for n=k.We will now prove it for n=k+1.

$$A^{k+1} = A^k A = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{bmatrix}$$

$\therefore$ It holds for n=k+1.Thus it will hold for any natural number n that $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$