Assignment 6

1 (i) $\underset{f}{(123)}\underset{g}{(3562)}\underset{f}{(123)} = (1)(2563)$

$f \cdot g \cdot f(1) = f \cdot g(2) = f(3) = 1$

$\because$ (2563) is a 4-cycle its order is 4.

$(2563) = (23)(26)(25)$ (product of 3 2-cycles $\therefore$ (2563) is odd permutation)

(iv) $(567489)^3$

Note $(567489)^2 = (578)(649)$ — order 3.

2nd place

3rd place or skip 2 inbetween

and $(567489)^3 = (54)(68)(79)$

— odd permutation since product of odd nos of 2 cycles.

$o((54)(68)(79)) = 2$

$\because$ order of 2 cycle is 2 $\therefore$ order of $(54) = 2 = $ order $(68) = $ or $(79) = $ order $(54)(68)(79)$ as they are disjoint and $\therefore$ commute.

(V) (153467) — 6-cycle

$\therefore$ order $(153467) = 6$

$(153467) = (17)(16)(14)(13)(15)$ — odd permutation

2. $A_n = \{\sigma \in S_n : \sigma \text{ is an even permutation}\}$.

(i) Claim: A_n is a normal subset of S_n.

i.e $\forall\ \sigma \in A_n$ and $\rho \in S_n$ $\rho\sigma\rho^{-1} \in A_n$.

Let $\rho \in S_n$ be such that ρ can be written as product of k 2cycles,

i.e $\rho = (a_1 a_2)(a_3 a_4) \cdots (a_{2k-1}, a_{2k})$

$\Rightarrow \rho^{-1} = ((a_1 a_2) \quad (a_{2k-1}, a_{2k}))^{-1}$

$= (a_{2k-1}, a_{2k})^{-1} (\quad) \cdots (a_3, a_4)^{-1}(a_1, a_2)^{-1}$

$= (a_{2k-1}, a_{2k}) \cdots (a_3 a_4)(a_1 a_2)$

$\left(\because (a, b)^{-1} = (a, b) \right.$.

and $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$ $\left.\right)$
group.

$\Rightarrow \rho^{-1}$ is also product of k-2cycles.

$\therefore$ if $\sigma \in A_n$ is product of $2r$ 2cycles,

then $\rho\sigma\rho^{-1}$ is the product of $k+2r+k$ 2cycles

$\Rightarrow \underbrace{2k+2r}_{\text{even}}$ 2cycles

Hence $\rho\sigma\rho^{-1} \in A_n$ $\forall$ $\sigma \in A_n, \rho \in S_n$.

(ii) Every element of A_n can be written as product of 3cycles.

let $\sigma \in A_n$, then we know that σ can be written as product of $2r$ 2-cycles for some $r \in \mathbb{N}$.

Suppose

$$\sigma = (a_1 a_2)(a_3 a_4)(a_5 a_6)(a_7 a_8) \cdots$$

then grouping the consequtie 2-cycles together

σ Can be written as

$$\sigma = \big((a_1 a_2)(a_3 a_4)\big)\big((a_5 a_6)(a_7 a_8)\big) \cdots$$

If given $(a_i \, a_{i+1})(a_{i+2} \, a_{i+3})$

$$\{a_i, a_{i+1}\} \cap \{a_{i+2}, a_{i+3}\} \neq \phi$$

say $\{a_i, a_{i+1}\} \cap \{a_{i+2}, a_{i+3}\} = \{a\}$

with $a_{i+1} = a = a_{i+2}$

then $(a_i \; a)(a \; a_{i+3}) = (a \; a_{i+3} \; a_i)$

If $\{a_i, a_{i+1}\} \cap \{a_{i+2}, a_{i+3}\} = \phi$

then we write using the fact that $o(a\,b) = 2$ we write

$$(a_i, a_{i+1})(a_{i+2}, a_{i+3}) = \underbrace{(a_i \; a_{i+1})(a_{i+1}, a_{i+2})}\underbrace{(a_{i+1} \, a_{i+2})(a_{i+2}, a_{i+3})}$$

$$= (a_{i+1} \; a_{i+2}, a_i)(a_{i+2} \; a_{i+3} \; a_{i+1})$$

Hence by regrouping the elts we see that the elts in A_n Can be written as product of 3-cycles.

(iii). Let $\phi : S_n \longrightarrow (\mathbb{Z}_2, \oplus_2)$ be given by

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even permutation} \\ 1 & \text{if } \sigma \text{ is odd permutation} \end{cases}$$

Then for $\sigma, \rho \in S_n$

$0 = \phi(\sigma\rho)$ — even if σ, ρ are even or σ, ρ are odd

$1 = \phi(\sigma\rho)$ — odd if one is odd and another even.

if σ, ρ are both even

then $\phi(\sigma\rho) = 0$

on the other hand

$\phi(\sigma) = 0, \ \phi(\rho) = 0$

$\Rightarrow \phi(\sigma) \oplus_2 \phi(\rho) = 0$.

if σ, ρ are both odd

then $\phi(\sigma) = 1, \ \phi(\rho) = 1$

and $\phi(\sigma) \oplus_2 \phi(\rho) = 0$.

$\Rightarrow \phi(\sigma\rho) = 0 = \phi(\sigma) \oplus_2 \phi(\rho) \quad \forall \sigma, \rho \in S_n$

if σ — even and ρ — odd.

then $\phi(\sigma\rho) = 1$ and $\phi(\sigma) \oplus_2 \phi(\rho) = 0 \oplus_2 1 = 1$.

$\Rightarrow \phi(\sigma\rho) = \phi(\sigma) \oplus_2 \phi(\rho)$. $\therefore$ ϕ is a group homo.

$$\ker\phi = \{\sigma \in S_n \mid \phi(\sigma) = 0\}$$
$$= \{\sigma \in S_n \mid \sigma \text{ is even}\} = A_n.$$

$\therefore \ker\phi = A_n.$

and $S_n/A_n = \{A_n, (12)A_n\}$

$\because$ for every $\sigma \in A_n$, $\sigma A_n = A_n$,

and if $\rho \in S_n$ is a odd permutation

then $(12)\rho$ is a even permutation

hence $(12)\rho \in A_n$.

But $\rho = (12)(12)\rho$ (as $(12)(12) = e$)

$\therefore \rho A_n = (12)\underbrace{(12)\rho}A_n \quad \forall \rho \in S_n - A_n$

$\Rightarrow \rho A_n = (12)A_n \quad \forall \rho \in S_n - A_n$

This shows that

$$S_n/A_n = \{A_n, (12)A_n\}.$$

3. Let $\phi : G \longrightarrow G'$ be a group homomorphism. (6)

For $a \in G$, $\phi(a) \in G'$.

If $o(a) = n$, then $\phi(a^n) = \phi(e_G) = e_{G'}$, where $e_G, e_{G'}$ are respy the identity elements in G and G'.

$$\Rightarrow \quad \phi(a)^n = e_{G'}$$

If $o(\phi(a)) = k$, then using division algorithm we ~~know~~ know, $\exists\, q_1, r_1 \in \mathbb{Z}$, $0 \le r_1 < k$ st

$$n = kq_1 + r_1$$

$$\Rightarrow \phi(a)^n = (\phi(a))^{kq_1 + r_1} = (\phi(a)^k)^{q_1} \cdot \phi(a)^{r_1} = e$$

$$\Rightarrow \phi(a)^{r_1} = e_{G'} \quad (\because o(\phi(a)) = k) \text{ — } (*)$$

If $r_1 \neq 0$, $(*)$ would contradict the fact that $k =$ smallest positive integer st $\phi(a)^k = e_{G'}$.

$\therefore r_1 = 0$.

$$\Rightarrow n = kq_1 \Rightarrow k = o(\phi(a)) \text{ divides } n = o(a)$$

$\longrightarrow$

For the second part, recall that every element of S_n can be written as the product of 2-cycles. This implies $X = \{(a,b) \mid a, b \in \{1, \dots, n\}\}$ generates

S_n. Hence if $\phi : S_n \longrightarrow (\mathbb{Z}_p, \oplus_p)$ is a group homomorphism, then ϕ is determined ~~by~~ exclusively by the ~~elements~~ values of $\phi((a,b))$ for any 2-cycle $(ab) \in S_n$.

Note order of a 2-cycle $(a,b) = 2$ and

~~of $\phi((a,b))$ ... $\phi((ab)) \neq \bar{0}$ in $(\mathbb{Z}_p, \oplus_p)$~~

order of any non-zero element of $(\mathbb{Z}_p, \oplus_p)$, where p is a odd prime is p.

$\therefore$ if $\phi(ab) \neq \bar{0}$ for any 2-cycle (ab), then $o(\phi(ab))$ would be p.

By the first part this implies that

$p \mid 2$ which is a contradiction.

Hence $\phi((ab)) = \bar{0} \quad \forall \quad (ab) \in S_n$.

$\Rightarrow \phi(f) = \bar{0} \quad \forall \quad f \in S_n$ as f can be written product of 2-cycles σ_i,

$f = \sigma_1 \sigma_2 \cdots \sigma_r$

$\phi(\sigma_1 \sigma_2 \cdots \sigma_r) = \phi(\sigma_1) \oplus \phi(\sigma_2) \oplus \cdots \phi(\sigma_r)$

$= \bar{0} \oplus_p \bar{0} \cdots \oplus_p \bar{0} = \bar{0}$.

4. For $a_1, a_2 \in I_n = \{1, 2, \dots, n\}$, it can be easily checked that (8)

$$(a_1\, a_2) = (1\, a_1)(1\, a_2)(1\, a_1).$$

(Notice that we want $\begin{pmatrix} 1 & a_1 & a_2 \\ 1 & a_2 & a_1 \end{pmatrix}$ but at a time we are only allowed to swap the position 1 with exactly one other than 1.

So the way to do it will be

Step 1: $\begin{pmatrix} 1 & a_1 & a_2 \\ a_1 & 1 & a_2 \end{pmatrix}$ (exchanges 1 and a_1

Step 2: $\begin{pmatrix} 1 & a_1 & a_2 \\ a_2 & a_1 & 1 \end{pmatrix}$ (exchanges position 1 and a_2 since a_1 was in position 1 by step 1, by step 2, $a_1 \to a_2$ so it is now in its desired position. and a_2 is in position 1 and 1 is in position a_1.)

Step 3: $\begin{pmatrix} 1 & a_1 & a_2 \\ a_1 & 1 & a_2 \end{pmatrix}$ exchanges position 1 and a_1. Since a_2 was in position 1 (by step 2), using $(1\, a_1)$, a_2 comes to position a_1 and as 1 was in position a_1, by $(1, a_1)$, it goes to 1, which means that the sean

$(1\ a_1)\cdot(1\ a_2)(1\ a_1)$ gives

$$\begin{pmatrix} 1 & a_1 & a_2 \\ 1 & a_2 & a_1 \end{pmatrix}.$$

So $(a_1\ a_2) = (1\ a_1)(1\ a_2)(1\ a_1)$.

Since every element in S_n is the product of disjoint cycles and every cycle can be written as product of 2 cycles which in turn (by first part) can be written as product of elements of the form $(1\,a)$; $a \in I_n$. It follows that every element of S_n can be written as product of elements from

$$X = \{(1\,a) : a \in \{2, \dots, n\}\}.$$

This completes the proof.