BEFORE YOU START WRITING, check the page numbers at the bottom to confirm that all the pages are present. You have **ONE** hour to complete this exam. You must explain your work clearly to get credit for your answer. Use the available space judiciously.

Name:__________ *Reg. No:*__________ Tutorial section: __________

Question:	1	2	3	4	5	Total
Points:	6	3	3	2	6	20
Score:						

1. People arrive at a queue according to the following scheme: During each minute of time either 0 or 1 person arrives. In a minute, the probability that 1 person arrives is p and that no person arrives is $q = 1 - p$. Answer the following questions (No explanation required):

(2 marks) (a) Let C be the number of customers arriving in the first 10 minutes.

- $P(C = 2) = \binom{10}{2} p^2 q^8$
- $E[C] = 10p$

(2 marks) (b) Let W be the time (in minutes) until the first person arrives.

- $P(W = 5) = (1 - p)^4 p$
- $E[W] = 1/p$

(2 marks) (c) Let T be the time (in minutes) until 4 people arrive.

- $P(T = 10) = \binom{9}{3} p^4 q^6$
- $E[T] = 4/p$

(3 marks) 2. Consider two independent random variables: $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$ for $\lambda, \mu > 0$. Determine the probability mass function of $Z = X + Y$.

Solution:

$$\begin{aligned}
P(Z = k) &= \sum_{r=0}^{k} P(X = r)P(Y = k - r) \\
&= \sum_{r=0}^{k} e^{-\lambda}\frac{\lambda^r}{r!}e^{-\mu}\frac{\mu^{k-r}}{(k-r)!} \\
&= \frac{e^{-(\lambda+\mu)}}{k!}\sum_{r=0}^{k}\frac{k!}{r!(k-r)!}\lambda^r\mu^{k-r} \\
&= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^k}{k!}
\end{aligned}$$

This implies $Z \sim Poi(\lambda + \mu)$.

3. Let $X \sim Unif([0, 1])$.

(2 marks) (a) Determine the probability density function of the random variable X^2.

(1 mark) (b) Compute $E[X^2]$.

Solution: (a) We first determine the distribution of X^2. For $0 \leq x \leq 1$,

$$P(X^2 \leq x) = P(X \leq \sqrt{x}) = \sqrt{x}$$

This implies, $f_{X^2}(x) = \frac{1}{2\sqrt{x}}$ for $x \in [0, 1]$ and 0 otherwise.

(b) $E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$. So,

$$E[X^2] = \int_0^1 x^2 dx = 1/3$$

(2 marks) 4. Let $Y \sim \mathcal{N}(3,9)$ and $\phi(x) = P(Z \leq x)$, where $Z \sim \mathcal{N}(0,1)$. Compute $P(Y > 3|Y > 1)$ in terms of $\phi(2/3)$.

Solution:

$$
\begin{aligned}
P(Y > 3|Y > 1) &= \frac{P(Y > 3, Y > 1)}{P(Y > 1)} \\
&= \frac{P(Y > 3)}{P(Y > 1)} \\
&= \frac{P\left(\frac{Y-3}{3} > 0\right)}{P\left(\frac{Y-3}{3} > -2/3\right)} \\
&= \frac{P(Z > 0)}{P(Z > -2/3)} \\
&= \frac{1}{2(1 - \phi(-2/3))} \\
&= \frac{1}{2\phi(2/3)}
\end{aligned}
$$

5. Define $C(X,Y) = E[(X - E[X])(Y - E[Y]]$. Show that:

(2 marks) (a) $C(X,Y) = E[XY] - E[X]E[Y]$.

(2 marks) (b) For all $a, b \in \mathbb{R}$, $a^2E[X^2] + 2abE[XY] + b^2E[Y^2] \geq 0$.

(2 marks) (c) $E[XY]^2 \leq E[X^2]E[Y^2]$.
(Hint: For $A, B, C \in \mathbb{R}, Ar^2 + 2Br + C \geq 0 \ \forall r \in \mathbb{R}$ implies $B^2 \leq AC$)

Solution:

(a) $E[(X - E[X])(Y - E[Y])] = E[XY] - E[XE[Y]] - E[YE[X]] + E[X]E[Y] = E[XY] - E[X]E[Y]$.

(b) Note that $E[(aX+bY)^2] \geq 0$. Expand and use linearity of Expectation to conclude.

(c) In part (b), take $b = 1$ and consider the quadratic in variable a. Then,

$$a^2E[X^2] + 2aE[XY] + E[Y^2] \geq 0$$

implies the result.