MTH102: Analysis in One variable
Home Work No. 04
16 February 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- $\mathbb{N}$ denote the set of natural numbers.
- $\mathbb{Z}$ denote the ring of integers.
- $\mathbb{Q}$ denote the field of rational numbers.
- $\mathbb{R}$ denote the field of real numbers.

(1) Determine whether the following series converge.
  (a) $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 1$ is a natural number.
  (b) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$
  (c) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$
  (d) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$

(2) Does convergence of a series implies that it is absolutely convergent? Justify your answer.

(3) Let $f : (a, b) \to \mathbb{R}$ be a continuous function such that $f(r) = 0$ for each rational number $r \in (a, b)$. Then prove that $f(x) = 0$ for each $x \in (a, b)$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x}\sin(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is f continuous on $\mathbb{R}$? Justify your answer.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ \frac{1}{10^{10}} & \text{if } x < 0 \end{cases}$$

Show that f is not continuous at 0.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^3\cos(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use the $\epsilon - \delta$ definition to show that f is continuous at 0. Is f continuous at $x \neq 0$ and why?

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

Show that f is not continuous at any $x \in \mathbb{R}$.

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

Show that f is continuous at 0 and discontinuous at every other point of $\mathbb{R}$.

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is not continuous at 0.

(10) Let $A = [0, 1] \cup [2, 3]$ and $f : A \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0, 1] \\ x^3 & \text{if } x \in [2, 3] \end{cases}$$

Is f continuous on A? Justify your answer.

(11) Let $P_n(\mathbb{R})$ be the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree less than n. Show that $P_n(\mathbb{R})$ is a vector space over $\mathbb{R}$. What is the vector space dimension of $P_n(\mathbb{R})$ over $\mathbb{R}$?

(12) Is every continuous function defined on an open interval necessarily bounded? Justify your answer.