MTH 101 - Symmetry
Assignment 8

Corrections to Assignment 7:

1. Let G be a group. Then prove that the following are normal subgroups of G.
   ii. $[GG] =< aba^{-1}b^{-1} : a, b \in G >$

2. Let N be a normal subgroup of a group G. Then prove the following.
   i. If K is subgroup of N such that K a normal subgroup of G, then K is normal in G.

Notes: Let V be a vector space over the reals $\mathbb{R}$.

- Given a set of vectors $\{v_1, v_2, \cdots, v_k\}$ in V, a vector $v \in V$ such that
$$v = c_1v_1 + c_2v_2 + \cdots + c_kv_k,$$
with $c_1, \cdots, c_k \in \mathbb{R}$ is said to be a **linear combination** of the vectors $\{v_1, \cdots, v_k\}$.
For example, $(a, b) = a(1, 0) + b(0, 1) \in \mathbb{R}^2$ is a linear combination of the vectors $(1, 0), (0, 1)$.

- Given a set of vectors $X = \{v_1, \cdots, v_k\}$ in $V|_{\mathbb{R}}$,
$$\text{Span}_{\mathbb{R}}(X) = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k : c_1, c_2, \cdots, c_k \in \mathbb{R}\}.$$

- A set of vectors $\{v_1, v_2, \cdots, v_k\}$ in $V|_{\mathbb{R}}$ is said to be **linearly dependent** if there exists scalars $c_1, \cdots, c_k \in \mathbb{R}$ not all 0, such that
$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0.$$
For example $X = \{(1, 0, -1), (0, 1, 1), (1, 2, 1)\}$ is a linearly dependent set. Observe that
$$c_1(1, 0, -1) + c_2(0, 1, 1) + c_3(1, 2, 1) = (0, 0, 0)$$
whenever ,
$$c_1 + c_3 = 0, c_2 + 2c_3 = 0, -c_1 + c_2 + c_3 = 0. \tag{1}$$
Since the the system of equations (1) has a non-zero solution therefore X is linearly dependent.

- A set of vectors $X = \{v_1, v_2, \cdots, v_k\}$ in $V|_{\mathbb{R}}$ is said to be **linearly independent** if X is not linearly dependent. In other words, $\{v_1, v_2, \cdots, v_k\}$ is linearly independent if
$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0,$$
implies $c_1 = c_2 = \cdots = c_k = 0$. Thus the set $\{(1, 0, 1), (0, 1, 1), (1, 1, 1)\}$ is linearly independent in $\mathbb{R}^3$, since (0,0,0) is the only solution for the system of equations
$$c_1 + c_3 = 0, c_2 + c_3 = 0, c_1 + c_2 + c_3 = 0, \tag{2}$$
where the equations (2) that are obtained by considering the equation $c_1(1, 0, 1) + c_2(0, 1, 1) + c_3(1, 1, 1) = (0, 0, 0)$.

- A subset X of a vector space $V|_{\mathbb{R}}$ is said to be a **basis** of V over $\mathbb{R}$ if
   i. $\text{Span}_{\mathbb{R}}(X) = V$.
   ii. X is a linearly independent subset of $V|_{\mathbb{R}}$.

1. Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $(2, -1, 3, 2), (-1, 1, 1, -3)$, and $(1, 1, 9, -5)$.
2. Prove that the only sunspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.

3. Determine which of the following set of vectors in $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.
   i. $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 \geq 0\}$.
   ii. $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 = a_3\}$.
   iii. $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 = a_2\}$.
   iv. $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_2\}$.
   v. $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 a_2 = 0\}$.
   vi. $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 \in \mathbb{Q}\}$, where $\mathbb{Q}$ denotes the set of rational numbers.

4. Determine which of the following set of matrices in $V = M_2(\mathbb{R})$ are subspaces of V.
   i. $W = \{A \in M_2(\mathbb{R}) : A \text{ is invertible}\}$.
   ii. $W = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = b + c\}$.
   iii. $W = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + c = 0\}$.
   iv. $W = \{A \in M_2(\mathbb{R}) : AB = BA \text{ for a fixed } B \in M_2(\mathbb{R})\}$.
   v. $W = \{A \in M_2(\mathbb{R}) : A^2 = A\}$.

5. Determine which of the following subsets X of the vector spaces $V|_{\mathbb{R}}$ forms a basis of $V|_{\mathbb{R}}$.
   i. $V = \mathbb{R}^4,\ X = \{(1, 2, 0, 1), (2, 1, 0, -1), (1, 1, 0, 0), (1, 0, 1, -1)\}$
   ii. $V = \mathbb{R}^3,\ X = \{(1, 0, -1), (1, 2, 1), (0, -3, 2)\}$
   iii. $V = M_2(\mathbb{R}),\ X = \{\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -5 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}\}$
   iv. $V = M_2(\mathbb{R}), X = \{\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -5 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\}$.

6. Show that $X = \{(2, 3), (1, -1)\}$ and $X' = \{(1, 0), (0, 1)\}$ form two bases of $\mathbb{R}^2$. Express the vectors $(1, 0)$ and $(0, 1)$ as a linear combination of the vectors in X.

7. Find three vectors in $\mathbb{R}^3$ which are linearly dependent and are such that any two of them are linearly independent.