Solutions for Binomial theorem

1)

(i) Using Binomial expansion,

$(1+x)^5 = {}^5C_0(1)^5(x)^0 + {}^5C_1(1)^4(x)^1 + {}^5C_2(1)^3(x)^2 + {}^5C_3(1)^2(x)^3 + {}^5C_4(1)^1(x)^4 + {}^5C_5(1)^0(x)^5$

$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

(ii) $(2x+y)^6 = {}^6C_0(2x)^6(y)^0 + {}^6C_1(2x)^5(y)^1 + {}^6C_2(2x)^4(y)^2 + {}^6C_3(2x)^3(y)^3 + {}^6C_4(2x)^2(y)^4 + {}^6C_5(2x)^1(y)^5 + {}^6C_6(2x)^0(y)^6$

$= 64x^6 + 6(32x^5)y + 15(16x^4)y^2 + 20(8x^3)y^3 + 15(4x^2)y^4 + 6(2x)y^5 + y^6$

$= 64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3 + 60x^2y^4 + 12xy^5 + y^6$

(iii) $(x-1/y)^5 = {}^5C_0(x)^5(-1/y)^0 + {}^5C_1(x)^4(-1/y)^1 + {}^5C_2(x)^3(-1/y)^2 + {}^5C_3(x)^2(-1/y)^3 + {}^5C_4(x)^1(-1/y)^4 + {}^5C_5(x)^0(-1/y)^5$

$= x^5 - 5x^4/y + 10x^3/y^2 - 10x^2/y^3 + 5x/y^4 - 1/y^5$

(iv) $(x^2 - 2/x)^4 = {}^4C_0(x^2)^4(-2/x)^0 + {}^4C_1(x^2)^3(-2/x)^1 + {}^4C_2(x^2)^2(-2/x)^2 + {}^4C_3(x^2)^1(-2/x)^3 + {}^4C_4(x^2)^0(-2/x)^4$

$= x^8 + 4(x^6)(-2/x) + 6(x^4)(4/x^2) + 4(x^2)(-8/x^3) + 16/x^4$

$= x^8 - 8(x^5) + 24x^2 - 32/x + 16/x^4$

2)

(i) Using Binomial theorem and writing $\sqrt{2} - 1 = \sqrt{2} + (-1)$

$(\sqrt{2} + 1)^5 - (\sqrt{2} - 1)^5 = ({}^5C_0(\sqrt{2})^5(1)^0 + {}^5C_1(\sqrt{2})^4(1)^1 + {}^5C_2(\sqrt{2})^3(1)^2 + {}^5C_3(\sqrt{2})^2(1)^3 + {}^5C_4(\sqrt{2})^1(1)^4 + {}^5C_5(\sqrt{2})^0(1)^5) - ({}^5C_0(\sqrt{2})^5(-1)^0 + {}^5C_1(\sqrt{2})^4(-1)^1 + {}^5C_2(\sqrt{2})^3(-1)^2 + {}^5C_3(\sqrt{2})^2(-1)^3 + {}^5C_4(\sqrt{2})^1(-1)^4 + {}^5C_5(\sqrt{2})^0(-1)^5)$

$= ((\sqrt{2})^5 + 5(\sqrt{2})^4 + 10(\sqrt{2})^3 + 10(\sqrt{2})^2 + 5(\sqrt{2})^1 + 1) - ((\sqrt{2})^5 - 5(\sqrt{2})^4 + 10(\sqrt{2})^3 - 10(\sqrt{2})^2 + 5(\sqrt{2})^1 - 1)$

$= 10(\sqrt{2})^4 + 20(\sqrt{2})^2 + 2$

$= 40 + 40 + 2$

$= 82$

(ii) $(2+\sqrt{3})^7 + (2-\sqrt{3})^7 = ({}^7C_0(2)^7(\sqrt{3})^0 + {}^7C_1(2)^6(\sqrt{3})^1 + {}^7C_2(2)^5(\sqrt{3})^2 + {}^7C_3(2)^4(\sqrt{3})^3 + {}^7C_4(2)^3(\sqrt{3})^4 + {}^7C_5(2)^2(\sqrt{3})^5 + {}^7C_6(2)^1(\sqrt{3})^6 + {}^7C_7(2)^0(\sqrt{3})^7) + ({}^7C_0(2)^7(-\sqrt{3})^0 + {}^7C_1(2)^6(-\sqrt{3})^1 + {}^7C_2(2)^5(-\sqrt{3})^2 + {}^7C_3(2)^4(-\sqrt{3})^3 + {}^7C_4(2)^3(-\sqrt{3})^4 + {}^7C_5(2)^2(-\sqrt{3})^5 + {}^7C_6(2)^1(-\sqrt{3})^6 + {}^7C_7(2)^0(-\sqrt{3})^7)$

$= (2)^7 + 7(2)^6(\sqrt{3}) + 21(2)^5(\sqrt{3})^2 + 35(2)^4(\sqrt{3})^3 + 35(2)^3(\sqrt{3})^4 + 21(2)^2(\sqrt{3})^5 + 7(2)^1(\sqrt{3})^6 + 1(2)^0(\sqrt{3})^7 + ((2)^7 + 7(2)^6(-\sqrt{3}) + 21(2)^5(-\sqrt{3})^2 + 35(2)^4(-\sqrt{3})^3 + 35(2)^3(-\sqrt{3})^4 + 21(2)^2(-\sqrt{3})^5 + 7(2)^1(-\sqrt{3})^6 + 1(2)^0(-\sqrt{3})^7)$

$= 2(2)^7 + 42(2)^5(\sqrt{3})^2 + 70(2)^3(\sqrt{3})^4 + 14(2)^1(\sqrt{3})^6$

$= 256 + 4032 + 5040 + 756$

$= 10084$

3) As n = 7 is odd so there will be n + 1 = 8 terms in binomial expansion so there will be 2 middle terms namely 4^{th} and 5^{th} terms. Now as (r+1)th term in expansion of $(a+b)^n$ is given by,

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

so 4^{th} term in expansion of $(3x-x^3/6)^7$ will be given by

$$T_4 = {}^7C_3\,(3x)^4\,(-x^3/6)^3$$

$$= -35(81x^4)(x^9/216)$$

$$= -105x^{13}/8$$

and 5^{th} term will be given by

$$T_5 = {}^7C_4(3x)^3\,(-x^3/6)^4$$

$= 35(27x^3)(x^{12}/6^4)$
$= 105x^{15}/144$

4) As (r+1)th term in expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

so (r+1)th term in expansion of $(x^2 + 2/x)^{15}$ is given by

$$T_{r+1} = {}^{15}C_r\, (x^2)^{15-r}\, (2/x)^r$$
$$= {}^{15}C_r\, (x^{30-2r})\, (1/x)^r (2^r)$$
$$= {}^{15}C_r\, (x^{30-3r})(2^r)$$

so for (r+1)th term to be independent of x, power of x should be 0 hence,

$$30\text{-}3r = 0 \text{ or } r = 10$$

Hence 11th term will be independent of x and 11th term will be

$$T_{11} = {}^{15}C_{10}\, (x^2)^5 (2/x)^{10}$$
$$= 3075072$$

5) As 1.01 = 1+0.01, so using binomial theorem

$$(1.01)^{1000000} = (1+0.01)^{1000000} = {}^{1000000}C_0(1)^{1000000}(0.01)^0 + {}^{1000000}C_1(1)^{1000000-1}(0.01)^1 + \text{-------------}$$
$$= 1 + 1000000(0.01) + \text{---------------------}$$
$$= 1 + 10000 + \text{positive terms}$$

From the above expansion it is clear that $1.01^{1000000}$ is greater than 10000

6) 17th and 18th term of expansion $(2+a)^{50}$ are given by

$$T_{17} = {}^{50}C_{16}\, (2)^{34}\, (a)^{16}$$
$$T_{18} = {}^{50}C_{17}\, (2)^{33}\, (a)^{17}$$

As 17th and 18th terms are equal so,

$$T_{17} = T_{18}$$
$${}^{50}C_{16}\, (2)^{34}\, (a)^{16} = {}^{50}C_{17}\, (2)^{33}\, (a)^{17}$$
$$2/34 = a/17$$
$$a = 1$$

Hence a = 1

7) In the expansion of $(x+2y)^9$, (r+1)th term will be given by

$$T_{r+1} = {}^9C_r(x)^{9-r}(2y)^r$$

so term with x^6y^3 will require r = 3

So it will be 4th term and given by

$$T_4 = {}^9C_3(x)^6(2y)^3$$
$$= 84(x)^6(8y^3)$$
$$= 672x^6y^3$$

Hence coefficient of x^6y^3 in expansion of $(x+2y)^9$ is 672.

Bonus:

8) 2nd, 3rd and 4th term in expansion of $(x+a)^n$ are240,720 and 1080 so

$$T_2 = 240,\ T_3 = 720 \text{ and } T_4 = 1080$$

Hence

$$T_2 = {}^nC_1\, (x)^{n-1}\, (a)^1 = 240 \qquad \text{-(i)}$$
$$T_3 = {}^nC_2\, (x)^{n-2}\, (a)^2 = 720 \qquad \text{-(ii)}$$
$$T_4 = {}^nC_3\, (x)^{n-3}\, (a)^3 = 1080 \qquad \text{-(iii)}$$

dividing (ii) by (i)

$${}^nC_2\, (x)^{n-2}\, (a)^2 / {}^nC_1\, (x)^{n-1}\, (a)^1 = 720/240$$

This gives,

$$(n-1)a/x = 6 \text{ so } a/x = 6/(n-1) \qquad \text{-(iv)}$$

dividing (iii) by (ii)

$$^{n}C_3 (x)^{n-3} (a)^3 / \, ^{n}C_2 (x)^{n-2} (a)^2 = 1080/720$$

This gives,

$$(n-2)a/x = 9/2 \text{ so } a/x = 9/2(n-2) \qquad \text{-(v)}$$

Using equations (iv) and (v)

$$6/(n-1) = 9/2(n-2)$$

Solving this gives $n = 5$

so $a/x = 3/2$

Now using (i)

$$^{5}C_1 (x)^4 (a)^1 = 240 \text{ so } 5x^4a = 240$$

As $a/x = 3/2$ so putting $a = 3x/2$ in above equation

$$15x^5/2 = 240 \text{ which gives } x = 2 \text{ and so } a = 3$$

Hence $n = 5$, $x = 2$ and $a = 3$.

9)

$$(1+x/2- 2/x)^4 = ((1+x/2) - 2/x)^4$$

{Grouping first two terms and considering it as one term, say $y = 1+x/2$}

$$(1+x/2- 2/x)^4 = (y-2/x)^4 = {}^4C_0(y)^4(-2/x)^0 + {}^4C_1(y)^3(-2/x)^1 + {}^4C_2(y)^2(-2/x)^2 + {}^4C_3(y)^1(-2/x)^3 + {}^4C_4(y)^0(-2/x)^4$$

$$= y^4 - 8y^3/x + 24y^2/x^2 - 32y/x^3 + 16/x^4$$

Now as $y = 1+x/2$

$$\text{so } y^4 = (1+x/2)^4 = {}^4C_0(1)^4(x/2)^0 + {}^4C_1(1)^3(x/2)^1 + {}^4C_2(1)^2(x/2)^2 + {}^4C_3(1)^1(x/2)^3 + {}^4C_4(1)^0(x/2)^4$$

$$= 1+2x+3x^2/2 + x^3/2 + x^4/16$$

$$\text{so } y^3 = (1+x/2)^3 = {}^3C_0(1)^3(x/2)^0 + {}^3C_1(1)^2(x/2)^1 + {}^3C_2(1)^1(x/2)^2 + {}^3C_3(1)^0(x/2)^3$$

$$= 1 + 3x/2 + 3x^2/4 + x^3/8$$

and $y^2 = 1+x+x^2/4$

Now,

$$(1+x/2-2/x)^4 = y^4 - 8y^3/x + 24y^2/x^2 - 32y/x^3 + 16/x^4$$

$$= (1+2x+3x^2/2 + x^3/2 + x^4/16) - 8(1/x + 3/2 + 3x/4 + x^2/8) + 24(1/x^2 + 1/x + 1/4) - 32(1/x^3 + 1/2x^2) + 16/x^4$$

$$= 16/x + 8/x^2 - 32/x^3 + 16/x^4 - 4x + x^2/2 + x^3/2 + x^4/16 - 5$$