MTH202: Assignment 9

March 12, 2019

1. Suppose you choose a real number X from the interval $[2, 10]$ with a density function of the form
$$f_X(x) = Cx$$
where, C is a constant.
   (a) Find C.
   (b) $P(X > 5), P(X < 7)$.
   (c) Find $E[X]$.
   (d) Find $E[X^2]$.

Solution: (a) Since f_X is a probability density function, $\int_{-\infty}^{\infty} f_X(x)dx = 1$. Now,

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)dx &= \int_2^{10} Cxdx \\ &= \frac{C}{2}(10^2 - 2^2) \\ &= 48C\end{aligned}$$

This means (since $\int_{-\infty}^{\infty} f_X(x)dx = 1$), $C = 1/48$.

(b) Now we know the density function of X. So,

$$\begin{aligned}P(X > 5) = \int_5^{10} f_X(x)dx &= \frac{1}{48}\int_5^{10} xdx \\ &= 75/96\end{aligned}$$

Similarly,

$$\begin{aligned}P(X < 7) = \int_2^{7} f_X(x)dx &= \frac{1}{48}\int_2^{7} xdx \\ &= 45/96\end{aligned}$$

(c) Note that $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$. So,

$$\begin{aligned} E[X] = \frac{1}{48}\int_2^{10} x^2 dx &= \frac{1}{48 \times 3}(10^3 - 2^3) \\ &= 992/144 \end{aligned}$$

(c) Note that $E[X^k] = \int_{-\infty}^{\infty} x^k f_X(x)dx$ for any $k \geq 1$. So,

$$\begin{aligned} E[X^2] = \frac{1}{48}\int_2^{10} x^3 dx &= \frac{1}{48 \times 4}(10^4 - 2^4) \\ &= 9984/192 \end{aligned}$$

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2. Consider a random variable $X \sim Unif([0, 10])$. Compute the following:
   (a) $P(X < 3)$.
   (b) $P(X > 3)$.
   (c) $P(3 < X < 8)$.
   (d) $E[4X^2 - 2X]$.
   (e) $E[e^X]$.

Solution: Since $X \sim Unif([0, 10])$, the probability density function of X is given by:

$$f_X(x) = \begin{cases} 1/10 & \text{for } 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(c) Therefore,

$$\begin{aligned} P(3 < X < 8) &= \int_3^8 f_X(x)dx \\ &= \frac{1}{10}\int_3^8 dx \\ &= 5/10 \\ &= 1/2 \end{aligned}$$

(d) Using linearity of Expectation:

$E[4X^2 - 2X] = 4E[X^2] - 2E[X]$

$$\begin{aligned} &= 4\int_0^{10} x^2 f_X(x)dx - 2\int_0^{10} x f_X(x)dx \\ &= \frac{4}{10}\int_0^{10} x^2 dx - \frac{2}{10}\int_0^{10} x dx \\ &= \frac{4}{10} \times 10^3 - \frac{2}{10} \times 10^2 \\ &= 380 \end{aligned}$$

Please note that $E[X^2] \neq (E[x])^2$

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3. Let $\phi(z)$ denote $P(Z \leq z)$ for a standard normal random variable Z. Let $X \sim \mathcal{N}(2, 4)$ and $Y = 3 - 2X$.

   - Find $P(X > 1)$.
   - $P(-2 < Y < 1)$.
   - $P(X > 2|Y < 1)$

   Solution: First of all, note that $E[Y] = 3 - 2E[X] = 3 - 2 \times 2 = -1$ and $Var(Y) = 2^2 Var(X) = 16$. Therefore, $Y \sim \mathcal{N}(-1, 16)$. Also, recall

   - If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim Z \sim \mathcal{N}(0, 1)$.
   - $\phi(0) = 1/2$ and $\phi(-x) = 1 - \phi(x)$

   (a) Since $X \sim \mathcal{N}(2, 4)$, we have:

$$\begin{aligned} P(X > 1) &= P\left(\frac{X-2}{2} > \frac{1-2}{2}\right) \\ &= P(Z > -1/2) \\ &= 1 - P(Z \leq -1/2) \\ &= 1 - \phi(-1/2) \\ &= \phi(1/2) \end{aligned}$$

   (b) Since $Y \sim \mathcal{N}(-1, 16)$, we have:

$$\begin{aligned} P(-2 < Y < 1) &= P\left(\frac{-2+1}{4} < \frac{Y+1}{4} > \frac{1+1}{4}\right) \\ &= P(-1/4 < Z < 1/2) \\ &= \phi(1/2) - \phi(-1/4) \\ &= \phi(1/2) + \phi(1/4) - 1 \end{aligned}$$

You can also solve this by converting Y in terms of X first.

(b)Note that $Y < 1$ is same as $3 - 2X < 1$, i.e. $X > 1$

$$\begin{aligned} P(X > 2|Y < 1) &= P\left(X > 2, Y < 1\right)/P(Y < 1) \\ &= P(X > 2, X > 1)/P(X > 1) \\ &= P(X > 2)/P(X > 1) \end{aligned}$$

We have computed $P(X > 1)$ in part (a) and similarly,

$$\begin{aligned} P(X > 2) &= P\left(\frac{X-2}{2} > \frac{2-2}{2}\right) \\ &= P(Z > 0) \\ &= 1 - P(Z \leq 0) \\ &= 1 - \phi(0) \\ &= 1/2 \end{aligned}$$

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4. Let $U \sim Unif([0,1])$ and $X = -\ln(1-U)$. Show that $X \sim Exp(1)$.

Solution: We want to find the probability density function of X. For that, we first determine the probability distribution function of X. For $0 \leq x \leq 1$:

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(-\ln(1-U) \leq x) \\ &= P(U \leq 1 - e^{-x}) \\ &= 1 - e^{-x} \end{aligned}$$

Now the density function is given by $f_X(x) = \frac{d}{dx}F_X(x) = e^{-x}$, which is the density function of $Exp(1)$ random variable. Hence, $X \sim Exp(1)$.