MTH102: Analysis in One variable
Home Work No. 03
02 February 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- $\mathbb{N}$ denote the set of natural numbers.
- $\mathbb{Z}$ denote the ring of integers.
- $\mathbb{Q}$ denote the field of rational numbers.
- $\mathbb{R}$ denote the field of real numbers.

(1) Let $s_n = \sin(\frac{n\pi}{3})$ for each $n \geq 1$.
Compute $\limsup s_n$ and $\liminf s_n$ and determine whether $\lim s_n$ exists.
Determine a monotone subsequence of (s_n).
Determine a convergent subsequence of (s_n).

(2) Let $s_n = n\big(1 + (-1)^n\big)$ for each $n \geq 1$. Compute $\limsup s_n$ and $\liminf s_n$. What can you say about $\lim s_n$.

(3) Given $r > 0$ and a bounded sequence (s_n). Show that $\limsup rs_n = r \limsup s_n$.

(4) Prove that a sequence of positive terms is either bounded or it has a subsequence diverging to $+\infty$.

(5) Let S be a non-empty bounded subset of $\mathbb{R}$ such that $\sup(S) \notin S$. Then prove that there is an increasing sequence (s_n) of points of S converging to $\sup(S)$.

(6) Using definition, explain why the sequence given by $s_n = (-1)^n$ is not a Cauchy sequence.

(7) Prove that the sum and the product of two Cauchy sequences is a Cauchy sequence.

(8) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges and (b_n) is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

(9) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(10) Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, where A and B are real numbers. Then prove the following:
(a) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$.
(b) $\sum_{n=1}^{\infty} ka_n = kA$ for all $k \in \mathbb{R}$.
(c) Can we say that $\sum_{n=1}^{\infty} a_n b_n = AB$?

(11) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series such that $a_n = b_n$ for all but finitely many $n \in \mathbb{N}$. Then prove the following:
(a) $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
(b) $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\sum_{n=1}^{\infty} b_n$ diverges.