MTH102: Analysis in One variable
Home Work No. 05
Sent on 09 March 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- $\mathbb{N}$ denote the set of natural numbers.
- $\mathbb{Z}$ denote the ring of integers.
- $\mathbb{Q}$ denote the field of rational numbers.
- $\mathbb{R}$ denote the field of real numbers.

(1) Let $\alpha, \beta \in \mathbb{R}$ be two real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = |x - \alpha| + |x - \beta|.$$
Determine the set of points at which f is differentiable.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$
  (a) Show that f is differentiable at all $x \in \mathbb{R}$ and determine the function $f' : \mathbb{R} \to \mathbb{R}$.
  (b) Is the function f' continuous on $\mathbb{R}$?
  (c) Is the function f' differentiable on $\mathbb{R}$?

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
  (a) Prove that f is continuous at $x = 0$.
  (b) Prove that f is differentiable at $x = 0$ and find the derivative.
  (c) What happens at points $x \neq 0$?

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.
  (a) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.
  (b) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.
Hint: Use the Mean Value Theorem.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Then prove that f is a constant function.
Hint: Show that f is differentiable with $f'(x) = 0$ for all $x \in \mathbb{R}$.

(6) Prove that $\sin(x) \leq x$ for all real numbers $x \geq 0$.
Hint: Show that the function $x - \sin(x)$ is increasing.

(7) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Prove that the derivative of an even function is an odd function.

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $f(x)^3 + 2xf(x) - 3x^2 = 0$ for all x. Determine the function f'.

(9) Let f be a twice differentiable function on an open interval (a, b) such that $f''(x) = 0$ for all $x \in (a, b)$. Then prove that f has the form $f(x) = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$.