MTH102: Analysis in One variable
Home Work No. 05
Sent on 25 Febraury 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- $\mathbb{N}$ denote the set of natural numbers.
- $\mathbb{Z}$ denote the ring of integers.
- $\mathbb{Q}$ denote the field of rational numbers.
- $\mathbb{R}$ denote the field of real numbers.

(1) Use the definition of uniform continuity to show that the function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = x^3$ is uniformly continuous.
(2) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is not uniformly continuous.
(3) Use the definition of uniform continuity to show that the function $f : [0, 3] \to \mathbb{R}$ defined by $f(x) = \frac{x}{x+2}$ is uniformly continuous.

(4) Determine the following limits:
  (a) $\lim_{x\to 0^+} \frac{x}{|x|}$ and $\lim_{x\to 0^-} \frac{x}{|x|}$.
  (b) $\lim_{x\to 1^+} \frac{1}{x-1}$.

(5) Let $\lim_{x\to a^+} f_1(x) = L_1$ and $\lim_{x\to a^+} f_2(x) = L_2$ and $f_1(x) \leq f_2(x)$ for all x in some interval (a, b). Then prove that $L_1 \leq L_2$.

(6) For each of the following power series, determine the radius of convergence and also the exact interval of convergence:
  (a) $\sum_{n=0}^{\infty} \sqrt{n}x^n$
  (b) $\sum_{n=0}^{\infty} \frac{2^n}{n^2}x^n$
  (c) $\sum_{n=0}^{\infty} n^2x^n$
  (d) $\sum_{n=0}^{\infty} (\frac{x}{n})^n$

(7) Let $\sum_{n=0}^{\infty} a_nx^n$ be a power series with radius of convergence R. Suppose that all the coefficients a_n are integers and all but finitely many a_n's are non-zero. Then prove that $R \leq 1$.
(8) Give an example of a power series whose exact interval of convergence is $(-1, 1]$.
(9) Let $\sum_{n=0}^{\infty} a_nx^n$ and $\sum_{n=0}^{\infty} b_nx^n$ be two power series with radius of convergence R_1 and R_2, respectively. Define their sum as

$$\sum_{n=0}^{\infty} a_nx^n + \sum_{n=0}^{\infty} b_nx^n = \sum_{n=0}^{\infty} (a_n + b_n)x^n.$$

Prove that if R is the radius of convergence of $\sum_{n=0}^{\infty}(a_n + b_n)x^n$, then $R \geq \min\{R_1, R_2\}$.
(10) In the preceding problem, give examples of power series where $R = \min\{R_1, R_2\}$, and where $R > \min\{R_1, R_2\}$.