**MTH 101 - Symmetry**
Assignment 9 & Notes

Recall: A subset $X$ of a vector space $V|_{\mathbb{R}}$ is said to be a **basis** of $V$ over $\mathbb{R}$ if

i. $\text{Span}_{\mathbb{R}}(X) = V$.

ii. $X$ is a linearly independent subset of $V|_{\mathbb{R}}$.

- A vector space $V$ over the reals $\mathbb{R}$, is said to be **finite-dimensional**, if it has a finite basis.
- Theorem: Any two bases of a finite-dimensional vector space $V$ over $\mathbb{R}$, have the same number of vectors. Proof: For a contradiction, suppose that

$$B_1 = \{v_1, \cdots, v_k\} \quad \text{and} \quad B_2 = \{w_1, \cdots, w_\ell\},$$

are two bases of $V|_{\mathbb{R}}$ with $k < \ell$.
Since $\text{Span}_{\mathbb{R}} B_1 = V$, and $B_2 \subset V$, every vector $w_i \in B_2$ can be written as a linear combination of elements in $B_1$. That is, there exists $a_{ij} \in \mathbb{R}$ such that for $i = 1, \cdots, \ell$,

$$w_1 = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1k}v_k, \tag{1}$$
$$w_2 = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2k}v_k, \tag{2}$$
$$\vdots \tag{3}$$
$$w_\ell = a_{\ell 1}v_1 + a_{\ell 2}v_2 + \cdots + a_{\ell k}v_k. \tag{4}$$

Now consider the equation

$$c_1w_1 + c_2w_2 + \cdots + c_\ell w_\ell = 0. \tag{5}$$

Since $B_2$ is a basis of $V|_{\mathbb{R}}$ (hence linearly independent set), by definition

$$c_1 = c_2 = \cdots = c_\ell = 0, \tag{6}$$

should be the only solution to (5). However substituting the values of $w_i$ for the above equations we get

$$c_1(\sum_{j=1}^{k} a_{1j}v_j) + c_2(\sum_{j=1}^{k} a_{2j}v_j) + \cdots + c_\ell(\sum_{j=1}^{k} a_{\ell j}v_j) = 0. \tag{7}$$

Regrouping the coefficients of the vectors $v_1, v_2, \cdots, v_k$, (7) can be rewritten as

$$(\sum_{j=1}^{\ell} c_i a_{i1})v_1 + (\sum_{1=1}^{\ell} c_i a_{i2})v_2 + \cdots + (\sum_{i=1}^{\ell} c_i a_{ik})v_k = 0. \tag{8}$$

But $B_1 = \{v_1, \cdots, v_k\}$ is a linearly independent set, hence the coefficients of the vectors in (8) must be 0. That is,

$$(\sum_{j=1}^{\ell} c_i a_{i1}) = (\sum_{1=1}^{\ell} c_i a_{i2}) = \cdots = (\sum_{i=1}^{\ell} c_i a_{ik}) = 0. \tag{9}$$

This gives us the system of linear equations

$$\begin{aligned} c_1a_{11} + c_2a_{21} + \cdots + c_\ell a_{\ell 1} &= 0 \\ c_1a_{12} + c_2a_{22} + \cdots + c_\ell a_{\ell 2} &= 0 \\ &\vdots \\ c_1a_{1k} + c_2a_{2k} + \cdots + c_\ell a_{\ell k} &= 0, \end{aligned} \tag{10}$$

which is equal to $AC = 0$, where $A = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{\ell 1} \\ a_{12} & a_{22} & \cdots & a_{\ell 2} \\ & \cdots & & \\ a_{1k} & a_{2k} & \cdots & a_{\ell 1} \end{pmatrix}$ and $C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{pmatrix}$.

Since $k < \ell$, notice that the row-reduced echelon form of the matrix $A$ is of the form $R_A = \begin{pmatrix} I_k : A' \end{pmatrix}$, where $I_k$ is the $k \times k$ identity matrix and $A' = (a'_{ij})_{1\le i\le k, k+1\le j\le \ell}$ is a $k \times (\ell - k)$ matrix which may or may not be the zero matrix. We know that

$$AC = 0, \text{ if and only if } R_A C = 0$$

and $R_A C = 0$ imples that

$$c_i + \sum_{j=k+1}^{\ell} a'_{ij} c_j = 0, \quad \text{for } 1 \le i \le k, \tag{11}$$

and for different values of $c_{k+1}, \cdots, c_\ell$ one can obtain different non-zero solutions for $C$. This is a contradiction to (6). Hence $k$ cannot be less than $\ell$. The same arguments show that $\ell$ cannot be less than $k$. Hence $k = \ell$.

- For a finite-dimensional vector space $V$ over $\mathbb{R}$, **dimension of $V$ over $\mathbb{R}$** is equal to the number of elements in a basis of $V|_{\mathbb{R}}$.

**Example** . Let $V = \{\begin{pmatrix} x & -x \\ y & z \end{pmatrix} : x, y, z \in \mathbb{R}\}$. Note that $V$ is a vector space over $\mathbb{R}$, since given $A = \begin{pmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{pmatrix}$, $B = \begin{pmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{pmatrix}$ in $V$, $rA + B = \begin{pmatrix} rx_1 + X_2 & -(rx_1 + x_2) \\ ry_1 + y_2 & rz_1 + z_2 \end{pmatrix}$ which is again an element of $V|_{\mathbb{R}}$. Also note that

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} = x\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $V|_{\mathbb{R}} = \text{Span}_{\mathbb{R}}\{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ and check that the set

$$B = \{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$$

is linearly independent. Hence $B$ is a basis of $V|_{\mathbb{R}}$ and dimension $V|_{\mathbb{R}} = 3$.

- For a finite-dimensional vector space $V$ over $\mathbb{R}$, an **ordered basis** is a finite sequence $B = \{v_1, \cdots, v_n\}$ of linearly independent vectors which span $V$ and given a vector $v = \sum_{i=1}^n a_i v_i$ in $V$, the real number $a_i$ is said to be the $i^{th}$ **coordinate of $v$ lative to the ordered basis $B$**.
For example consider the two ordered bases

$$B_1 = \{e_1 = (1, 0), e_2 = (0, 1)\}, \quad \text{and} \quad B_2 = \{v_1 = (0, 1), v_2 = (1, 0)\}$$

of $\mathbb{R}^2$. Notice that

$$v_1 = 0.e_1 + 1.e_2, \quad \text{and} \quad v_2 = 1.e_1 + 0.e_2. \tag{12}$$

Hence relative to the basis $B_1$, $1^{st}$ coordinate of $v_1$ is 0 and the $2^{nd}$ coordinate of $v_1$ is 1
and in the matrix notation (12) can be written as

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{13}$$

The coefficient matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is called the change of basis matrix relative to $[B_1, B_2]$. In particular, it is easy to see that the change of basis matrix relative to $[B_i, B_i]$ is $I_2$ for $i = 1, 2$.

- In general , when $B_1 = \{v_1, \cdots, v_k\}$ and $B_2 = \{w_1, \cdots, w_k\}$ are two ordered bases of a vector space $V|_{\mathbb{R}}$, there exists $a_{ij}, b_{rl} \in \mathbb{R}$ such that

$$w_i = \sum_{j=1}^{k} a_{ij} v_j, \text{ for } 1 \le i \le k, \tag{14}$$

$$v_r = \sum_{l=1}^{k} b_{rl} w_l, \text{ for } 1 \le r \le k. \tag{15}$$

In the matrix notation, (14) can be written as

$$\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = A \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}, \tag{16}$$

where $A = (a_{ij})$ is the change of basis matrix relative to $[B_1, B_2]$ (the coeffient matrix is the one obtained when the elements of the new basis, namely $B_2$ in this case, are written as a linear combination of the elements of the basis that we started of with, namely $B_1$ in this case ). Likewise, (15) can be written as

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = B \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}, \tag{17}$$

where $B = (b_{ij})$ is the change of basis matrix relative to $[B_2, B_1]$. Substituting (17) in (16), we see that

$$\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = AB \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}. \tag{18}$$

But the change of basis matrix relative to $[B_2, B_2]$ is the identity matrix $I_k$. Therefore, $AB = I_k$ which implies that the matrices $A$ and $B$ are invertible.

1. Show that the vectors
$$v_1 = (1, 1, 0), v_2 = (0, 0, 1), v_3 = (1, 0, 4)$$
form a basis of $\mathbb{R}^3$. Find the cocordinates of each of the standard basis vectors in the ordered basis $B = \{v_1, v_2, v_3\}$. If $S$ denotes the standard basis of $\mathbb{R}^3|_{\mathbb{R}}$, determine the change of basis matrix relative to $[B, S]$ and $[S, B]$.

2. Let $V$ be the vector space of all $2 \times 2$ matrices over $\mathbb{R}$. Prove that $V$ has dimension 4 by exhibiting a basis for $V$ which has 4 elements.

3. Let $V$ be the vector space of all $2 \times 2$ matrices $A = (a_{ij})$ over $\mathbb{R}$ such that $a_{11} + a_{22} = 0$. Give a basis for $V$ and determine its dimension over $\mathbb{R}$.