IISER Mohali
MTH102: Analysis in One Variable
Homework No. 06
To be discussed during tutorial on 04 March, 2016

- Please solve all the problems.
- Tutors will discuss **tutorial problems** during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorial sessions.

Tutorial Problems:

(1) Use the $\epsilon - \delta$ definition of uniform continuity to show that the function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.
(2) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is not uniformly continuous.
Remark: In fact, the only polynomials that are uniformly continuous on $\mathbb{R}$ are those of degree at most 1.
(3) For each of the following power series, determine the radius of convergence and also the exact interval of convergence:
(a) $\sum_{n=0}^{\infty} \sqrt{n} x^n$
(b) $\sum_{n=0}^{\infty} \frac{2^n}{n^2} x^n$
(4) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Suppose that all the coefficients a_n are integers and all but finitely many a_n's are non-zero. Then prove that $R \leq 1$.
Hint: Use the definition of radius of convergence.
(5) Give an example of a power series whose exact interval of convergence is $(-1, 1]$.
Hint: Take a power series which becomes harmonic series at 1 or -1.
(6) Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series with radius of convergence R_1 and R_2, respectively. Define their sum as

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

Prove that if R is the radius of convergence of $\sum_{n=0}^{\infty} (a_n + b_n) x^n$, then $R \geq \min\{R_1, R_2\}$.
(7) In problem 5 above, give examples of power series where $R = \min\{R_1, R_2\}$.

Extra Problems:

(1) Use the $\epsilon - \delta$ definition of uniform continuity to show that the function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = \frac{x}{x+2}$ is uniformly continuous on $[0, 2]$.
(2) Let $f : S \to \mathbb{R}$ be a uniformly continuous function on a subset S of $\mathbb{R}$. Show that if (x_n) is a Cauchy sequence in S, then $\big(f(x_n)\big)$ is also a Cauchy sequence in $\mathbb{R}$.
(3) For each of the following power series, determine the radius of convergence and also the exact interval of convergence:
(a) $\sum_{n=0}^{\infty} n^2 x^n$
(b) $\sum_{n=0}^{\infty} (\frac{x}{n})^n$
(4) In problem 5 above, give examples of power series where $R > \min\{R_1, R_2\}$.