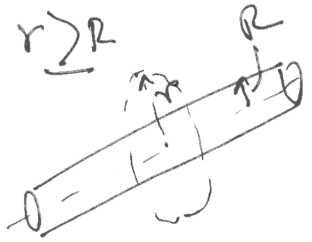
PHY102 : Quiz 1

1. A spherical charge distribution has a density ρ that is constant from $r = 0$ out to $r = R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R ? [2.5]

Assignment 7 (Solc).

1. $r \geq R$



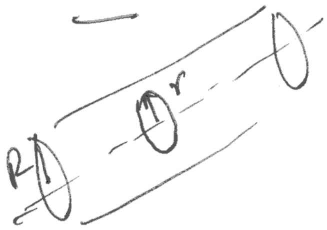
Consider Amperian loop around wire of radius r

$\vec{B}$ has same magnitude at all pts. on loop.

$$\oint \vec{B}.d\vec{s} = B.2\pi r \quad \& \text{ Since, } \oint \vec{B}.d\vec{s} = \mu_0 I_{enc.}$$

$$\therefore B.2\pi r = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}.$$

$r \leq R$



Current within a radius r, is,

$$I_r = J\pi r^2 \quad (J : \text{current density})$$

$$= \frac{I}{\pi R^2} . \pi r^2 = \frac{I r^2}{R^2}.$$

$$\therefore \oint \vec{B}.d\vec{s} = \mu_0 I_r \Rightarrow B.2\pi r = \mu_0 \frac{I r^2}{R^2}.$$

$$\therefore B = \frac{\mu_0 I r}{2\pi R^2}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta)\hat{z} = \hat{z} . \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\frac{\mu_0 I}{2\pi \cancel{r}}\right) = 0.$$

Since, current density, $\vec{J} = 0$ outside the wire.

$\therefore \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ is verified outside the wire.

2. Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of $\mathbf{E}$, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5\text{cm}$? [2.5]

Inside,
$$\vec{\nabla}\times\vec{B} = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\mu_0 I r}{2\pi R^2}\right)$$
$$= \hat{z}\frac{\mu_0 I}{\cancel{2}\pi R^2}\cdot\frac{1}{\cancel{r}}\cdot\cancel{2}\cancel{r} = \frac{\mu_0 I}{\pi R^2}\hat{z}$$
$$= \mu_0\left(\frac{I}{\pi R^2}\right)\hat{z} = \mu_0 J\hat{z} = \mu_0\vec{J}.$$

2. (a) Force on a particle largest in the rest frame of the particle.

Force in the rest frame, $F = qE = \dfrac{q\lambda}{2\pi\epsilon_0 r}$.

$\therefore$ Force in the frame moving with speed v,
$$F' = \frac{F}{\gamma} = \frac{q\lambda}{2\pi\epsilon_0\gamma r}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

(b) In the new frame, $\lambda' = \gamma\lambda$. (length contraction)
$$\therefore E' = \frac{\lambda'}{2\pi\epsilon_0 r'} = \frac{\lambda'}{2\pi\epsilon_0 r} \quad (\because r = r').$$
$$= \frac{\gamma\lambda}{2\pi\epsilon_0 r}$$

$\therefore$ Force due to E', $F_{E'} = qE' = \dfrac{q\gamma\lambda}{2\pi\epsilon_0 r}$

(repulsive)

2. Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of $\mathbf{E}$, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5$cm ? [2.5]

In the new frame, the magnetic field,

$$B' = \frac{\mu_0 I'}{2\pi r'} = \frac{\mu_0 I'}{2\pi r}$$

where, $I' = \lambda' v$ (since in this frame charges in the rod move with speed v).

$$\therefore B' = \frac{\mu_0 \lambda' v}{2\pi r} = \frac{\mu_0 \gamma \lambda v}{2\pi r}$$

$$\therefore F_{B'} = q v B' = q v \cdot \frac{\mu_0 \gamma \lambda v}{2\pi r} = \frac{\gamma q \lambda v^2}{2\pi \epsilon_0 r c^2}$$

(attractive)

using $\mu_0 \epsilon_0 = \frac{1}{c^2}$.

$\therefore$ Net force, $F_{E'} - F_{B'} = \frac{\gamma q \lambda}{2\pi\epsilon_0 r} - \frac{\gamma q \lambda v^2}{2\pi\epsilon_0 r c^2}$

$$= \frac{\gamma q \lambda}{2\pi\epsilon_0 r}\left(1 - \frac{v^2}{c^2}\right)$$

$$\left(\because \gamma^2 = \frac{1}{1 - v^2/c^2}\right)$$

$$= \frac{\cancel{\gamma} q \lambda}{2\pi\epsilon_0 r} \cdot \frac{1}{\cancel{\gamma^2}_{\gamma}} = \frac{q\lambda}{2\pi\epsilon_0 r \gamma}$$

$\therefore$ Net force is repulsive.

This is exactly what we calculated in (a)

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(c) In the rest frame of the charge, $\vec{B} = 0$.

& $\vec{E} = \vec{E}_\perp$

Now, $\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v}\times\vec{B}_\perp) = \gamma\vec{E}_\perp$

$\therefore |\vec{E}'_\perp| = \dfrac{\gamma\lambda}{2\pi\epsilon_0 r}$. (as found in (b)).

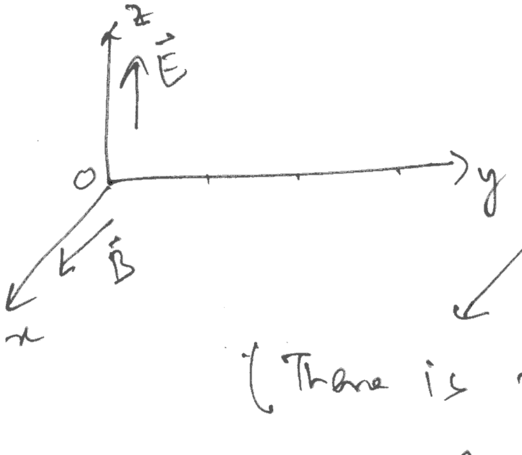
In the new frame,

$\vec{B}'_\perp = \gamma\left(\vec{B}_\perp - \dfrac{\vec{v}}{c^2}\times\vec{E}_\perp\right) = -\gamma\dfrac{\vec{v}}{c^2}\times\vec{E}_\perp$

$\therefore |\vec{B}'_\perp| = \dfrac{\gamma v}{c^2}\cdot\dfrac{\lambda}{2\pi\epsilon_0 r} = \dfrac{\gamma v \mu_0 \cancel{\epsilon_0}\lambda}{2\pi\cancel{\epsilon_0} r}$

$= \dfrac{\mu_0\gamma\lambda v}{2\pi r}$ (again as obtained in (b))

3.



$\vec{r}(t) = (0, y(t), z(t))$

$\vec{v}(t) = (0, \dot{y}, \dot{z})$

(There is no force in the x-direction).

$\therefore \vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = \hat{y}B\dot{z} - \hat{z}B\dot{y}$

2. Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of $\mathbf{E}$, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5$cm ? [2.5]

$$\therefore \vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) = q\left(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}\right)$$

$$\Rightarrow m\ddot{\vec{r}} = q(E - B\dot{y})\hat{z} + qB\dot{z}\hat{y}$$

$$\Rightarrow m\ddot{y}\hat{y} + m\ddot{z}\hat{z} = q(E - B\dot{y})\hat{z} + qB\dot{z}\hat{y}$$

$$\therefore m\ddot{y} = qB\dot{z} \quad \text{and} \quad m\ddot{z} = qE - qB\dot{y}$$

$$\Rightarrow \ddot{y} = \frac{qB}{m}\dot{z} \qquad \ddot{z} = \frac{qE}{m} - \frac{qB}{m}\dot{y}$$

Let, $\omega = \frac{qB}{m}$. $\Rightarrow$ cyclotron frequency.

$$\therefore \ddot{y} = \omega\dot{z} \; ; \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

General solution :

$$y(t) = C_1\cos\omega t + C_2\sin\omega t + (E/B)t + C_3.$$

$$z(t) = C_2\cos\omega t - C_1\sin\omega t + C_4.$$

$\dot{y}(0) = \dot{z}(0) = 0$ & $y(0) = z(0) = 0$.

~~Eliminate~~ Find the constants & show,

$$y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t) \; ; \quad z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$$

2

Let, $R = \frac{E}{\omega B}$. $\therefore (y - R\omega t)^2 + (z - R)^2 = R^2$.

$\Rightarrow$ circle of radius R whose center $(0, R\omega t, R)$ travels in y-direction at constant speed $v = \omega R = \frac{E}{B}$.

$\Rightarrow$ cycloid.

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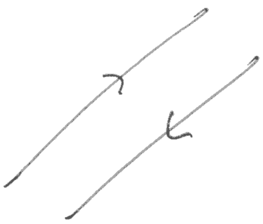
4.



Magnetic force / length,

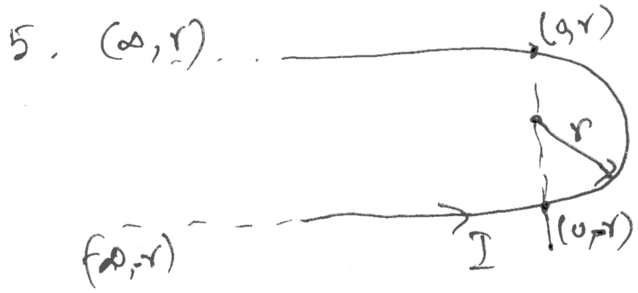
$$\frac{F}{l} = I_2 B = \frac{\mu_0 I_1 I_2}{2\pi r}$$

$$= \frac{\mu_0 I^2}{2\pi r} \quad (\because I_1 = I_2 = I)$$

$$= \left(\frac{\mu_0}{4\pi}\right) \cdot \frac{2I^2}{r} = \frac{10^{-7} \cdot 2 \cdot 4\cancel{00}\,^{80}}{8 \times 10^{-2}}$$

$$= 160 \times 10^{-5} \text{ N/m.}$$

5.



Break wire into 3 segments:
2 line segments (infinite)
1 semicircular segment

1 segment goes from $(-\infty, -r)$ to $(0, -r)$

$$B_1 = \frac{\mu_0 I}{4\pi r}(\cos\theta_1 - \cos\theta_2) = \frac{\mu_0 I}{4\pi r}\left(\cos 0^\circ - \cos\frac{\pi}{2}\right)$$

$$= \frac{\mu_0 I}{4\pi r}(1 - 0) = \frac{\mu_0 I}{4\pi r} = B_3$$ (contribution from other line segment)

Contribution from semicircular arc,

$$B_2 = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^{\pi} \frac{r\, d\theta}{r^2} = \frac{\mu_0 I}{4r}.$$

2. ~~Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of $\mathbf{E}$, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5$cm ?~~ [2.5]

Note that direction of all 3 magnetic fields are out of page. If that is the $+\hat{z}$ direction, then,

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 = \left(2 \cdot \frac{\mu_0 I}{4\pi r} + \frac{\mu_0 I}{4r}\right)\hat{z}$$

$$= \frac{\mu_0 I}{4\pi r}(2+\pi)\hat{z}.$$

6. Electron moving in circular orbit $\Rightarrow$ current

$$I = \frac{e}{\Delta t} = \frac{e}{(2\pi r/v)} \qquad v: \text{velocity}$$

$$= \frac{ev}{2\pi r} = \frac{(1.6\times10^{-19}C)(0.01\times3\times10^{8}\,m/s)}{2\pi\times(10^{-10}\,m)}$$

$$\simeq 7.6\times10^{-4}\,A \simeq 760\,\mu A.$$

$\therefore B = \frac{\mu_0 I}{2r}$ (Field at the centre of current carrying loop).

$$= \frac{\mu_0}{4\pi}\cdot\frac{2\pi I}{r} = 10^{-7}\cdot\frac{2\pi\cdot 760}{10^{-10}}$$

$$\simeq 4.8\,T.$$

PHY102 : Quiz ~~1~~

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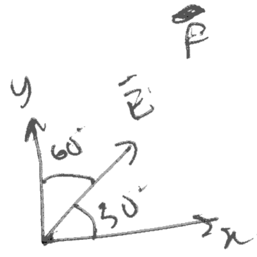
7. In frame F, $\vec{B} = 0$.

$\therefore \quad \vec{E}'_{\parallel} = \vec{E}_{\parallel} \quad , \quad \vec{E}'_{\perp} = \gamma \vec{E}_{\perp}$

$\vec{B}'_{\parallel} = 0 \quad , \quad \vec{B}'_{\perp} = -\gamma \frac{\vec{v}}{c^2} \times \vec{E}_{\perp}$

$\Rightarrow \vec{B}' = -\frac{\vec{v}}{c^2} \times \vec{E}'$



$\vec{E}$ is in x-y plane.

$\therefore |\vec{E}_{\perp}| = E_x = E\cos\theta \,, \; |\vec{E}_{\parallel}| = E_y = E\sin\theta$

Note : $\vec{E}_{\perp} = E\cos\theta\,\hat{x} \,, \quad \vec{E}_{\parallel} = E\sin\theta\,\hat{y}$

$\vec{v} = 0.6c\,\hat{y}$

Hence determine $\vec{E}'$ & $\vec{B}'$!