Sets and Functions

Maths Workshop 2020

Solutions

1

(a) $\{n \in \mathbb{Z} \mid n^2 \leq 16\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

Answer: 9

(b) Answer: 3

(c) $A \cap B = \{2, 5\}$

Answer: 2

(d) We have that $|A \cup B| = |A| + |B| - |A \cap B| = 16 + 25 - 8 = 33$

(e) $A - B = \{1, 3, 4, 6\}$

Answer: 4

2

Since $A \subset B$, every element of A also belongs to B, so $A - B = \phi$. Conversely, if $A - B = \phi$, there is no element in A that is also not in B, so $A \subset B$. Therefore, (a)$\Leftrightarrow$(b)

If $A \subset B$, then $A \cup B = B$ follows trivially. Conversely, let $A \cup B = B$. We know that in general, $X \subset X \cup Y$. Hence, $A \subset A \cup B = B$, so $A \subset B$. Therefore, (a)$\Leftrightarrow$(c).

Let $A \subset B$. If $x \in A$, then $x \in B$. Hence, $x \in A \cap B$, so $A \subset A \cap B$. Moreover, $A \cap B \subset A$, so together, we have $A \cap B = A$. Conversely, if $A \cap B = A$, then we have that $A = A \cap B \subset B$, so $A \subset B$. Therefore, (a)$\Leftrightarrow$(d).

Hence, (a),(b),(c),(d) are equivalent statements.

3

(a) When a is a multiple of b.

(b) $a\mathbb{Z} \cap b\mathbb{Z} = c\mathbb{Z}$, where $c = lcm(a, b)$

4

(a) For any $n \in \mathbb{Z}$, we have that 2 divides $n - n = 0$, so $n \sim n$.

If $n \sim m$, then 2 divides $n - m$, so 2 also divides $-(n - m) = m - n$, so $m \sim n$.

If $n \sim m$ and $m \sim k$, then 2 divides $n - m$ and $m - k$, and therefore divides the sum $n - m + m - k = n - k$, and so $n \sim k$.

The relation is reflective, symmetric, and transitive, and is therefore an equivalence relation.

(b) Let $n \sim 1$. Then 2 divides $n - 1$, so n is odd. The odd numbers, are therefore, related to 1.

5

(a) Let $f(x) = f(y)$, then automatically $x = y$, so f is one-one.

For each $x \in \mathbb{R}$, we have that $f(x) = x$. So f is onto.

(b) g is clearly not one-one (for instance, $g(1) = g(-1)$)

The negative reals do not have a pre-image under g. That is, if $y < 0$, there is no $x \in \mathbb{R}$ such that $g(x) = y$, so g is not onto.

6

(a) Let $f(x) = f(y)$. Then $4x + 3 = 4y + 3 \Rightarrow 4x = 4y \Rightarrow x = y$. f is therefore, one-one.

Let $y \in \mathbb{R}$. Working backwards, we see that if there exists x such that $4x + 3 = y$, then $x = \frac{y-3}{4}$. Clearly, $f(\frac{y-3}{4}) = y$, so f is onto.

Therefore, f is a bijection.

(b) We have already shown that $f(\frac{x-3}{4}) = x$. Define $g : \mathbb{R} \to \mathbb{R}, g(x) = \frac{x-3}{4}$. Then $f(g(x)) = x$, as we have already shown. Similarly, it can be shown that $g(f(x)) = x$. Hence g is the inverse of f.

7

(There can be more than one answer)

Let A be the set of odd integers.

Define $f : \mathbb{Z} \to A, f(n) = 2n + 1$ (note that $2n + 1$ is always odd). It is easy to check that f is a bijection. It's inverse is given by $g : A \to \mathbb{Z}, g(k) = \frac{k-1}{2}$.

(Interesting point: Since there is a bijection between the sets $\mathbb{Z}$ and A, it must be that they have the same 'cardinalities', even though it seems like $\mathbb{Z}$ is 'double' the size of A)