Solution to HW 4

1. (i) The function $\sqrt{1-t^2}$ is not smooth on $[0,1]$

i.e. we cannot find an open interval $(a,b) \supseteq [0,1]$ and a smooth function $f(t)$ on (a,b) such that $f|_{[0,1]} = \sqrt{1-t^2}$. Suppose we could.

Then $f'(t) = \frac{d}{dt}\left(\sqrt{1-t^2}\right) = \frac{-t}{\sqrt{1-t^2}} \quad \forall t \in (0,1)$.

$$\Rightarrow \quad \lim_{t \to 1} f'(t) = f'(1) = \lim_{t \to 1-} \frac{-t}{\sqrt{1-t^2}}$$ since f is smooth.

However, the right hand limit does not exist.

(ii) β is not smoothly ~~ex~~ extendable beyond 0.
The proof is similar to that of 1(i).

2. (i) Book's definition: $S \subseteq \mathbb{R}^3$ is called a surface if $\forall p \in S$ $\exists$ an open neighborhood V of p in S and a homeomorphism from V to an open subset of $\mathbb{R}^2$.

Clearly ~~our definition implies~~ if S is a surface by our definition then it is so as per the book's definition.

~~Suppose~~ Let S be a surface by the book's definition. ~~The~~ Let $p \in S$. Then there is an open set $V \subseteq S$, $p \in V$ and a homeomorphism $\varphi : V \longrightarrow \mathcal{U} \subseteq \mathbb{R}^2$ where $\mathcal{U} \subseteq \mathbb{R}^2$ is open. Let $q = \varphi(p) \in \mathcal{U}$. Since $q \in \mathcal{U}$ and $\mathcal{U}$ is open there is an open disc $B \subseteq \mathcal{U}$ with $q \in B$. Let $W = \varphi^{-1}(B)$. One just checks that $\varphi : \varphi^{-1}(B) \longrightarrow B$ is a homeomorphism

2.(ii) Suppose S is a surface. We will use the books definition. Let $V \subseteq S$ be open. Let $p \in V$. Then there is an open set $W \subseteq S$ and a homeomorphism $\varphi : V \to \mathcal{U}$ where $\mathcal{U} \subseteq \mathbb{R}^2$ is open.

Now, check V, W open in $S \Rightarrow$ so is $V \cap W$. Then check that $\varphi(V \cap W)$ is open in $\mathcal{U}$ and hence in $\mathbb{R}^2$. Finally $\varphi : V \cap W \to \varphi(V \cap W)$ is a homeomorphism.

3. Let $\psi : J \to I$ be the inverse of φ.
Then $\varphi \circ \psi : J \to J$ is the identity map
and $\psi \circ \varphi : I \to I$ is the identity map.

In particular $\psi \circ \varphi(t) = t \quad \forall t \in I$
Take derivative and apply chain rule.

$$\psi'(\varphi(t)) \cdot \varphi'(t) = 1$$

$$\Rightarrow \quad \varphi'(t) \neq 0.$$

4. a) $\alpha(t) = t(\cos t, \sin t)$

$$\Rightarrow \alpha'(t) = (\cos t, \sin t) + t(-\sin t, \cos t)$$

$$\Rightarrow \|\alpha'(t)\| = \sqrt{1+t^2} > 0 \quad \forall t$$

~~Since~~ Hence $\alpha'(t) \neq 0$.
Thus α is regular with speed $\|\alpha'(t)\| = \sqrt{1+t^2}$. Unit tangent vector $= \dfrac{1}{\|\alpha'(t)\|} \alpha'(t)$

$$= \left(\frac{\cos t - t\sin t}{\sqrt{1+t^2}}, \frac{\sin t + t\cos t}{\sqrt{1+t^2}} \right)$$

4. b) $\alpha(t) = (t - \sin t,\ 1 - \cos t)$

$\Rightarrow \alpha'(t) = (1 + \cos t,\ \sin t)$

$\Rightarrow \|\alpha'(t)\| = \sqrt{(1+\cos t)^2 + \sin^2 t}$

$= \sqrt{2 + 2\cos t} = \sqrt{2}\sqrt{1+\cos t}$

Clearly $\alpha'(\pi) = 0$. Hence α is not regular
and its speed $= \sqrt{2}\cdot\sqrt{1+\cos t} = 2\sqrt{2}|\cos t/2|$.
It's unit tangent vector, when defined is

$\frac{1}{\|\alpha'(t)\|}\alpha'(t) = \left(\frac{1+\cos t}{2\sqrt{2}|\cos t/2|},\ \frac{\sin t}{2\sqrt{2}|\cos t/2|}\right)$.

c) $\alpha(t) = e^{kt}(\cos t, \sin t)$

$\Rightarrow \alpha'(t) = ke^{kt}(\cos t, \sin t) + e^{kt}(-\sin t, \cos t)$

(k is constant)

$\Rightarrow \|\alpha'(t)\| = \sqrt{1+k^2}\, e^{kt}$

Thus $\alpha'(t) \neq 0 \ \forall t$.

Hence α is regular with speed $\sqrt{1+k^2}\,e^{kt}$
at time t and unit tangent vector $=$

$\frac{1}{\|\alpha'(t)\|}\alpha'(t) = \left(\frac{k\cos t - \sin t}{\sqrt{1+k^2}},\ \frac{k\sin t + \cos t}{\sqrt{1+k^2}}\right)$

e) $\alpha(t) = (t^2, t^2+1, t^2+2)$

$\Rightarrow \alpha'(t) = 2t(1,1,1)$

$\Rightarrow \|\alpha'(t)\| = 2\sqrt{3}\cdot t \quad , \forall t \in (0, \infty)$.

Thus α is regular, speed $2\sqrt{3}t$,
unit tangent vector $\frac{1}{\sqrt{3}}(1,1,1)$.

Note: α traces a straight line.

5. (c) $\alpha(t) = e^{kt}(\cos t, \sin t)$

$t_0 = 0$

$$s = \int_0^t \|\alpha'(t)\|\, dt = \int_0^t \sqrt{1+k^2}\, e^{kt}\, dt$$

$$= \frac{\sqrt{1+k^2}}{k}(e^{kt} - 1)$$

$$\Rightarrow \quad t = \frac{1}{k}\log\left(1 + \frac{ks}{\sqrt{1+k^2}}\right)$$

Hence, the arc length parametrization is

$$\beta(s) = \alpha(t) = \alpha\left(\frac{1}{k}\log\left(1+\frac{ks}{\sqrt{1+k^2}}\right)\right)$$

$$= \left(1+\frac{ks}{\sqrt{1+k^2}}\right)\left(\cos \frac{1}{k}\log\left(1+\frac{ks}{\sqrt{1+k^2}}\right), \sin \frac{1}{k}\log\left(1+\frac{ks}{\sqrt{1+k^2}}\right)\right)$$

(d) ✓

(e) $$s = \int_1^t \|\alpha'(u)\|\, du = \int_1^t 2\sqrt{3}\, u\, du$$

$$= \left[\sqrt{3}u^2\right]_1^t = \sqrt{3}t^2 - \sqrt{3}$$

$$\Rightarrow \quad t^2 = \frac{1}{\sqrt{3}}s + 1.$$

Hence, the arc length parametrization is $\beta(s) = \alpha(t) = (t^2, t^2+1, t^2+2)$

$$= \left(\frac{1}{\sqrt{3}}s+1, \frac{1}{\sqrt{3}}s+2, \frac{1}{\sqrt{3}}s+3\right)$$

(It is clear that β traces a straight line)

6. $\alpha''(t) = (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t)) = 0$

$\Rightarrow \alpha_i''(t) = 0, \ i = 1,2,3$ etc.