Solutions to be posted by next weekend 27/01/2018

1. A free electron has the wave function $\Psi(x,t) = sin(kx - \omega t)$.
   (a) Determine the electrons de Broglie wavelength, momentum, kinetic energy and speed when $k = 50$ nm^{-1}.
   (b) Determine the electrons de Broglie wavelength, momentum, total energy, kinetic energy and speed when $k = 50$ pm^{-1}.
2. Is the function $\psi(x) = Ax[1 - (x/a)]$ an acceptable wave function for the particle in an infinite box of length a? Calculate the normalization constant A.
3. If an electron in a certain excited energy level in a 1-D box of length 2×10^{-10} m makes a transition to the ground state emitting a photon of wavelength 8.79 nm, find the quantum number of the excited state.
4. In a region of space, a particle with mass m and with zero energy has a time-independent wave function $\psi(x) = Axe^{-x^2/L^2}$ where A and L are constants. Determine the potential energy of the particle.
5. A particle is in the n th energy state $\psi_n(x)$ of an infinite square well potential of width a.
   (a) Determine the probability $P_n(1/b)$ that the particle is confined to the first $1/b$ fraction of the width of the well.
   (b) Comment on what happens to $P_n(1/b)$ as $n \to \infty$. Compare it with the classical probability of the particle in the same region.
6. A proton is confined in an infinite square well of width 10 fm. (The nuclear potential that binds protons and neutrons in the nucleus of an atom is often approximated by an infinite square well potential.)
   (a) Calculate the energy and wavelength of the photon emitted when the proton undergoes a transition from the first excited state (n = 2) to the ground state (n = 1).
   (b) In what region of the electromagnetic spectrum does this wavelength belong?
7. This problem will familiarize you with the integrals needed for solving particle in a box problems. Show that:
   (a) $\int_0^a \sin^2 \frac{n\pi x}{a} dx = \frac{a}{2}$
   (b) $\int_0^a x \sin^2 \frac{n\pi x}{a} dx = \frac{a^2}{4}$
   (c) $\int_0^a x^2 \sin^2 \frac{n\pi x}{a} dx = \left(\frac{a}{2\pi n}\right)^3 \left(\frac{4\pi^3 n^3}{3} - 2n\pi\right)$
   All of these integrals can be evaluated from:
   $$I(\beta) = \int_0^a e^{\beta x} \sin^2 \frac{n\pi x}{a} dx$$
   Show that the above integrals are given by $I(0)$, $I'(0)$ and $I''(0)$ respectively. The prime denotes differentiation by β. Evaluate $I(\beta)$ and check the integrals.
   (d) Evaluate $\int_0^a e^{\pm 2\pi n x/a} dx$ for $n \neq 0$.
8. Consider the wave function:
   $$\Psi(x,t) = Ae^{-\lambda|x|e^{-i\omega t}}$$
   where A, λ and ω are positive real constants.
   (a) Normalize Ψ.
   (b) Determine the average values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x\rangle + \sigma)$ and $(\langle x\rangle - \sigma)$, to understand the sense in which σ represents the *spread* in x. What is the probability of finding the particle outside this range?

9. You have seen in class that *stationary states* are *standing wave* solutions of the time-dependent Schrödinger equation in the case of a time independent potential $V(x)$. Suppose I have two solutions of the time independent Schrödinger equation, $\psi_1(x)$ and $\psi_2(x)$ corresponding to energies ϵ_1 and ϵ_2 respectively. If we take a linear combination of the two states, will the corresponding probability density be that of a stationary state? If not then what is the time dependence with which the state moves?

10. Let $P_{ab}(t)$ be the probability of finding a particle in the range $(a < x < b)$, at time t,

(a) Show that

$$\frac{d}{dt}P_{ab} = J(a,t) - J(b,t)$$

where $J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$

(b) What are the units of $J(x,t)$? This quantity is known as the *probability current.*

(c) What does this quantity tell you?