

Solution to HWS

①

2) If $\vec{T} = (\cos \varphi(s), \sin \varphi(s))$ then $\kappa_s = \frac{d\varphi}{ds}$ and

(i) $\vec{n}_s = (-\sin \varphi(s), \cos(\varphi(s)))$.

$$\Rightarrow \frac{d\vec{n}_s}{ds} = -\varphi'(s) (\cos \varphi(s), \sin \varphi(s)) = -\kappa_s \vec{T}.$$

(ii) Since φ is smooth, κ_s is smooth.

3) Signed Curvature of $\beta = \frac{\vec{r} \cdot (\beta'(t) \times \beta''(t))}{\|\beta'(t)\|^3}$

$$= \frac{\vec{r} \cdot (-\alpha'(a+b-t) \times \alpha''(a+b-t))}{\|\alpha'(a+b-t)\|^3} = -\text{Signed}$$

curvature of α at $(a+b-t)$

Note: $\beta(a) = \alpha(b)$, $\beta(b) = \alpha(a)$ β is the "same" curve traversed in the opposite direction.

Roughly speaking signed curvature depends not only on the "shape" of the curve but also on the direction.

4) $x = a \cos t$, $y = b \sin t$ is a parametrization of the ellipse.

Here, $\alpha(t) = (a \cos t, b \sin t)$

$$\Rightarrow \alpha'(t) = (-a \sin t, b \cos t),$$

$$\alpha''(t) = (-a \cos t, b \sin t)$$

$$\Rightarrow \alpha' \times \alpha'' = ab \vec{r}$$

$$\Rightarrow \kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$$

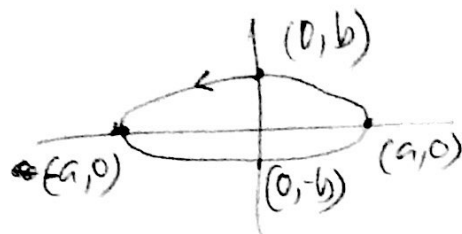
$$= \frac{ab}{\sqrt{(a^2 - b^2) \sin^2 t + b^2}}$$

Hence κ is maximum for $\sin t = 0$ and minimum for $\sin t = \pm 1$

In $[0, 2\pi]$, $\sin t = 0 \Leftrightarrow t = 0, \pi, 2\pi$ (2)

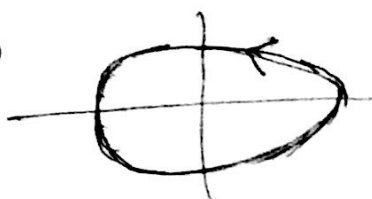
and $\sin t = \pm 1 \Leftrightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

It follows that curvature is maximum at the points $(\pm a, 0)$ and minimum at $(0, \pm b)$



5)

(i)



$$\kappa = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{3/2}}$$

$$r = 2 + \cos \theta \Rightarrow r' = -\sin \theta, r'' = -\cos \theta$$

$$\Rightarrow r^2 + r'^2 = (2 + \cos \theta)^2 + \sin^2 \theta = 4 + 3 \cos \theta$$

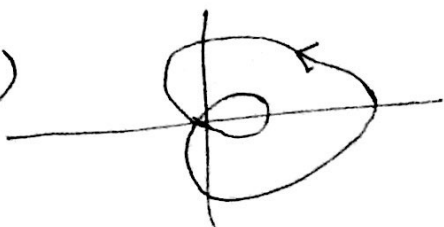
$$r^2 + 2r'^2 - rr'' = (2 + \cos \theta)^2 + 2\sin^2 \theta + \cos \theta (2 + \cos \theta)$$

$$= 4 + 4\cos \theta + \cos^2 \theta + 2\sin^2 \theta + 2\cos \theta + \cos^2 \theta$$

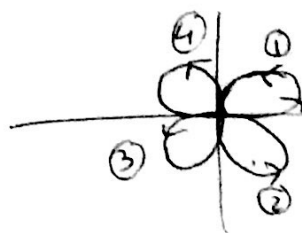
$$= 6 + 8 \cos \theta$$

$$\text{Hence } \kappa = \frac{|6 + 8 \cos \theta|}{(4 + 3 \cos \theta)^{3/2}}$$

(ii)



(iii)



The arrows show the direction as θ increases from 0 to 2π .

6) Apply $\kappa = \frac{|y'''|}{(1 + y'^2)^{3/2}}$

~~$\frac{d}{dx} \left(\frac{y''}{1 + y'^2} \right)$~~

~~$\frac{y'''}{1 + y'^2} - \frac{y' y''}{(1 + y'^2)^2}$~~

~~$A \left[\frac{y'''}{1 + y'^2} - \frac{y' y''}{(1 + y'^2)^2} \right]$~~ $\Rightarrow A \left[\frac{y'''}{1 + y'^2} - \frac{y' y''}{(1 + y'^2)^2} \right]$