MTH202: Solutions (Selected exercises)

March 9, 2019

1. Consider the following games and answer the questions.

   (a) Toss a fair coin, repeatedly and independently each time. If it shows HEADS you win a point otherwise you lose a point. What is your expected score after n tosses?

   (b) Toss a fair coin, repeatedly and independently each time. If it shows HEADS you win one point otherwise you win zero points. How many times should you toss the coin to make your expected score larger than 10?

   (c) Toss a coin, repeatedly and independently each time. Play the game described in (a). If after n tosses your expected score is $n/2$, what is the probability of getting HEADS with this coin?

   (d) Toss a coin, repeatedly and independently each time. Play the game described in (a). Is it possible to have an expected score of $2n$ after n tosses?

   Solution: X_i = no. of points won in i^{th} toss. Let X_n denote the score at n^{th} toss.

   (a) $E[X_n] = E[\sum_i X_i] = \sum_i E[X_i] = 0$.

   (b) $E[X_n] = E[\sum_i X_i] = \sum_i E[X_i] = n/2$. So, for $E[X_n] \geq 10$, we need $n \geq 20$.

   (c) $E[X_n] = E[\sum_i X_i] = \sum_i E[X_i] = n(2p-1)$ and $E[X_n = n/2$. This implies, $p = 3/4$.

   (d) No. Because that would mean $n(2p-1) = 2n$, i.e., $p > 1$.

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2. Suppose $X \sim Poi(1)$ denote the number of earthquakes in a region in one year. What is the probability that there was at least two earthquakes this year?

   Solution: $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} = 1 - 3e^{-2} \approx 0.6$

3. Let X and Y be distinct real-valued random variables.

   (a) Show that: for all real numbers a and b check that $a^2E[X^2] + 2abE[XY] + b^2E[Y^2)]$ is a non-negative number.

   (b) What is $E[X - E[X]]$?

   (c) Show that $E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$.

   Solution:

   (a) Note that $E[(aX + bY)^2] \geq 0$. Expand and use linearity of Expectation to conclude.

   (b) $E[X - E[X]] = E[X] - E[E[X]] = E[X] - E[X] = 0$.

   (c) $E[(X - E[X])(Y - E[Y])] = E[XY] - E[XE[Y]] - E[YE[X]] + E[X]E[Y] = E[XY] - E[X]E[Y]$.

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4. Roll a die twice. Let X be the number of times 4 comes up. Compute $E[X]$.

   Solution: $X \sim Bin(2, 1/6)$. Then, $E[X] = 1/3$.

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5. Consider n random variables $Y_i \sim Geom(p_i)$ for $1 \leq i \leq n$. Compute $E[Z]$, where $Z = \sum_{i=1}^{n} Y_i$.

   Solution: $E[Z] = \sum_{i=1}^{n} E[Y_i] = \sum_{i=1}^{n} \frac{1}{p_i}$

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6. Consider a fair die. Answer the following questions:

   (a) What is the probability of getting the outcome 1 after exactly 10 rolls?

   (b) What is the probability of getting the outcome 1 after at least 10 rolls?

   (c) Find the expected value of the times one must throw the die until outcome 1 occurs?

   (d) Find the expected value of the times one must throw the die until outcome 1 occurs 4 times?

   Solution:

   (a) Let $X \sim Geom(1/6)$, Then the desired probability is given by $P(X = 10)$.

   (b) Let $X \sim Geom(1/6)$, Then the desired probability is given by $P(X \geq 10)$.

(c) Let $X \sim Geom(1/6)$, Then $E[X] = 6$.

(d) This is an example of negative binomial distribution with parameters $r = 4$ and $p = 1/6$. So, the expected number of throws equals $r/p = 24$.

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7. A club contains 50 members; 20 are men and 30 are women. A committee of 10 members is chosen at random. Compute the probability mass function of the number of women on the committee. What is the expected number of women on the committee?

   Solution: Let X number of women on the committee. Then, for $0 \leq k \leq 10$,

   $$P(X = k) = \frac{\binom{30}{k}\binom{20}{50-k}}{\binom{50}{10}}$$

   This is an example of Hypergeometric distribution with parameters $n = 10, N = 50, m = 30$. So,

   $$E[X] = \frac{nm}{N} = \frac{10 \times 30}{50} = 6$$

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8. Show that the sum of two independent Bernoulli random variables with parameter p is a Binomial random variable with parameters $2, p$.

   Solution: $X, Y \sim Ber(p)$ and $Z = X + Y$. Then, Z takes values in $\{0, 1, 2\}$.

   $$\begin{aligned} P(Z = k) &= \sum_{r=0}^{k} P(X = r)P(Y = k - r) \\ &= P(X = 0)P(Y = k) + P(X = 1)P(Y = k - 1) \end{aligned}$$

   This gives,

   $$P(Z = 0) = (1 - p)^2, P(Z = 1) = 2(1 - p)p, P(Z = 2) = p^2$$

   which is same as saying $Z \sim Bin(2, p)$.

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9. Let $X \sim Bin(n - 1, p)$ and $Y \sim Ber(p)$. Show that $X + Y \sim Bin(n, p)$.

Solution: Let $Z = X + Y$ and $Y \in \{0, 1\}$. So,

$$\begin{aligned}
P(Z = k) &= \sum_{r=0}^{k} P(Y = r)P(X = k - r) \\
&= P(Y = 0)\binom{n-1}{k}p^k(1-p)^{n-1-k} + P(Y = 1)\binom{n-1}{k-1}p^{k-1}(1-p)^{n-1-(k-1)} \\
&= \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] p^k(1-p)^{n-k+1} \\
&= \binom{n}{k}p^k(1-p)^{n-k}
\end{aligned}$$

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10. Let $X \sim Geom(p)$ and $Y \sim Geom(q)$ be two independent Geometric random variables. Find the probability mass function of $Z = X + Y$.

    Solution: Note that $Z \in \{2, 3, \ldots\}$.

$$\begin{aligned}
P(Z = k) &= \sum_{r=1}^{k-1} P(X = r)P(Y = k - r) \\
&= \sum_{r=1}^{k-1} (1-p)^{r-1}p(1-q)^{k-r-1}q \\
&= pq(1-q)^{k-2}\sum_{r=1}^{k-1}\left(\frac{1-p}{1-q}\right)^{r-1} \\
&= \frac{pq}{q-p}[(1-p)^{k-2} - (1-q)^{k-2}]
\end{aligned}$$

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11. In a physics experiment involving estimating the value of g, several independent trials are performed. The value obtained is deemed acceptable with error ≤ 0.2. Suppose that in each trial the probability that your estimated value is in $[9.6, 10]$ is p. You stop when you get five estimates within the permissible error. What is the probability that you achieve this in 8 trials?

    Solution: p = Probability of "success". X be the number of trials required to achieve 5 successes. Then, X is Negative Binomial with parameters $5, p$ and

$$P(X = 8) = \binom{7}{4}p^5(1-p)^3$$

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12. People arrive at a queue according to the following scheme: During each minute of time either 0 or 1 person arrives. The probability that 1 person arrives is p and that no person arrives is $q = 1 - p$. Answer the following questions:

(a) Let C_r be the number of customers arriving in the first r minutes. What is $E[C_r]$?

(b) Let W be the time (in minutes) until the first person arrives. What is $E[W]$?

(c) Let W_r be the time (in minutes) until the r people arrive. What is $E[W_r]$?

Solution: Note that $C_r \sim Bin(r, p), W \sim Geom(p)$ and $W_r \sim NegBin(r, p)$. Therefore, $E[C_r] = rp, E[W] = 1/p$ and $E[W_r] = r/p$.

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13. A die is rolled twice (independently each time) with outcomes X, Y. Let $Z = \min(X, Y)$. Find the cumulative distribution function of Z in terms of cumulative distribution functions of X and Y.

Solution: Note that $\{Z > k\}$ if and only if $\{X > k, Y > k\}$. So,

$$P(Z > k) = P(X > k, Y > k) = P(X > k)P(Y > k)$$

So,

$$P(Z \leq k) = 1 - P(X > k)P(Y > k)$$

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14. A fair die is rolled until the first time T that a six turns up. Compute the following:

(a) $P(T > 3)$.

(b) $P(T > 6|T > 3)$.

Solution: Note that $T \sim Geom(1/6)$, i.e. $P(T = k) = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1}$. So,

(a) $P(T > 3) = \sum_{k=4}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} = \left(\frac{5}{6}\right)^3$.

(b) Note that: $P(T > 6) = P(T > 6|T > 3)P(T > 3) + P(T > 6|T \leq 3)P(T \leq 3)$, but $P(T > 6|T \leq 3) = 0$. Thus,

$$P(T > 6|T \leq 3) = P(T > 6)/P(T > 3) = \left(\frac{5}{6}\right)^3$$

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15. Consider two discrete random variables X and $Z \sim Poi(\lambda)$ such that $P(X = i|Z = k) = \binom{k}{i}p^i(1-p)^{k-i}$. What is $P(X = 0)$?

Solution: Note that: $P(X=i)=\sum\limits_{k=0}^{\infty} P(X=i|Z=k)P(Z=k)$. So, we have:

$$
\begin{aligned}
P(X=0) &= \sum_{k=0}^{\infty} P(X=i|Z=k)P(Z=k) \\
&= \sum_{k=0}^{\infty} \binom{k}{0}(1-p)^k e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \sum_{k=0}^{\infty} e^{-\lambda}\frac{[(1-p)\lambda]^k}{k!} \\
&= e^{-\lambda}e^{(1-p)\lambda} \\
&= e^{-p\lambda}
\end{aligned}
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16. Let X be a real-valued random variable and $F_X(a)=P(X\le a)$ denotes its cumulative distribution function. Then, prove the following:

   (a) F_X is a non-decreasing function.

   (b) $\lim_{a\to\infty} F_X(a)=1$.
   (Hint: consider a sequence of real numbers $b_n\to\infty$ and events $\{X\le b_n\}$).

   (c) $\lim_{a\to-\infty} F_X(a)=0$.

   (d) F_X is right continuous.
   (Hint: consider a sequence of real numbers $b_n\to b$ and events $\{X\le b_n\}$).

   (e) $P(s<X\le t)=F_X(t)-F_X(s)$.
   (Hint: Write the event $\{X\le t\}$ as a disjoint union of two other events.)

Solution:

   (a) $a\le b$ implies $\{X\le a\}\subseteq\{X\le b\}$. This implies, $P(X\le a)\le P(X\le b)$, i.e., $F_X(a)\le F_X(b)$.

   (b) Consider a sequence of real numbers $b_n\to\infty$, then the events $\{X\le b_n\}$ form an increasing sequence. Therefore,

   $$\lim_{n\to\infty} F_X(b_n)=\lim_{n\to\infty} P(X\le b_n)=P(\cup_n\{X\le b_n\})=P(X\le\infty)=1$$

   (c) Similar to (b).

(d) Consider a sequence of real numbers $b_n \to b$ and use an argument similar to (b) to get $\lim_n P(X \le b_n) = P(X \le b)$.

(e) Note that $P(X \le t) = P(X \le s) + P(s < X \le t)$. So,

$$P(s < X \le t) = P(X \le t) - P(X \le s) = F_X(t) - F_X(s)$$

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17. For what values of constant C do these functions define a probability density on $\mathbb{R}$.

(a)

$$f_X(x) = \begin{cases} C(x^3 - x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) $f_X(x) = Ce^{-x^2} \quad \text{for} \quad x \in \mathbb{R}$.

(c)

$$f_X(x) = \begin{cases} Ce^{-x/10} & \text{for } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution: Use the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$ to get:

(a) $C = -1/4$.
(b) $C = 1/\sqrt{\pi}$.
(c) $C = 1/10$.

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18. Consider the following probability density functions and compute the Expectation.

(a)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$f_X(x) = \begin{cases} 1/x & \text{for } 1 \le x \le e \\ 0 & \text{otherwise} \end{cases}$$

Solution: Note that $E[X] = \int_{-\infty}^{\infty} xf(x)dx$. So,

(a) $E[X] = 1/\lambda$.

(b) $E[X] = 1/2$.

(c) $E[X] = 3/4$.

(d) $E[X] = 1$