MTH 101 - Symmetry
Assignment 4

1. Which of the following collections of 2×2 matrices with real entries forms a group under matrix multiplication ?
   (a) Those of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for which $ac \neq b^2$.
   (b) Those of the form $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ for which $a^2 \neq bc$.
   (c) Those of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for which $ac \neq 0$.
   (d) Those of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for which $ac \neq 0$ and $a, b, c \in \mathbb{Z}$.
   (e) Those of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which $ad \neq bc$ and $a, b, c, d \in \mathbb{Z}$.

2. Let G be the set of 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Check that G forms a group under matrix multiplication. (Use row-reduction to find the inverse of an element in G.) Let
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$
   What is the order of the elements A, B, AB and BA.

3. Let x and y be the elements of a group G. Prove that G contains elements w, z which satisfy $wx = y$ and $xz = y$ and show that these elemens are unique. (Hint: Use the closure property and existence of inverse of elements in G.)

4. Let D_4 be the set of all bijections that map a square to itself. Show that D_4 is a group and write the Cayley table for D_4.
   (a) List the cyclic subgroups of D_4.
   (b) Let r be an element of D_4 of order 4 and s be an D_4 of order 2 such that $s \notin < r >$. Determine the group $< r, s >$.

5. Let H be a subgroup generated by two elements a, b of a group G.
   (a) Prove that if $ab = ba$, then H is abelian.
   (b) Prove that if all three of a, b and ab are of order of 2, then H is abelian. In this case what is the order of H.

6. For a positive integer n, let $\mathbb{Z}_n$ be a cyclic group of order n.
   (a) Determine the orders of the elements in $\mathbb{Z}_5$ and $\mathbb{Z}_4$.
   (b) Determine the cyclic subgroups of $\mathbb{Z}_5$ and $\mathbb{Z}_4$.

7. $(*)$ Let G be the collection of all rational numbers x such that $0 \leq x < 1$. Show that the operation
$$x + y = \begin{cases} x + y & \text{if } 0 \leq x + y < 1 \\ x + y - 1 & \text{if } x + y \geq 1 \end{cases}$$
   makes G into an infinite abelian group all of whose elements have finite order.