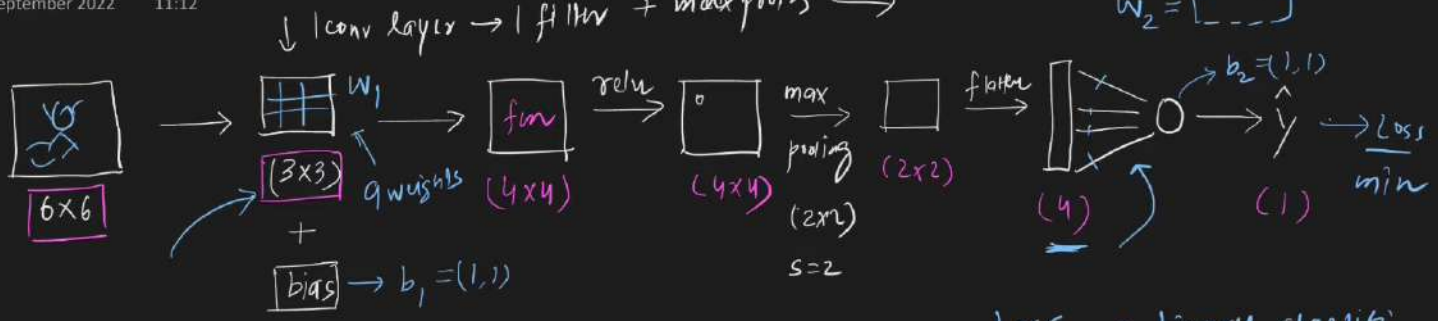
Backpropagation in CNN

10 September 2022 11:12



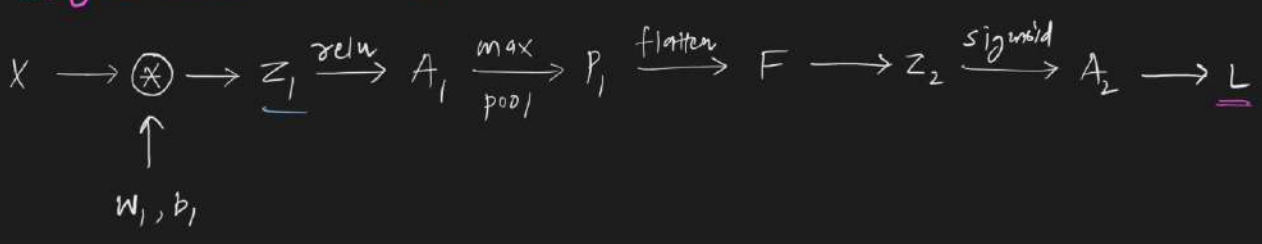
Trainable Parameters

$W_1 = (3,3)$ $W_2 = (1,4)$ $= 15$ trainable parameters

$b_1 = (1,1)$ $b_2 = (1,1)$

Loss → binary classifi

$$L = -y_i \log(\hat{y}_i) - (1-y_i) \log(1-\hat{y}_i)$$



Forward Prop

$Z_1 = conv(X, W_1) + b_1$

$A_1 = relu(Z_1)$

$P_1 = maxpool(A_1)$

$F = flatten(P_1)$

$Z_2 = W_2 F + b_2$

$A_2 = \sigma(Z_2)$

$W_2 = (1,4)$

$(1,4)\ (4,1)$

$(1,1)\ (1,1)$

$Z_2 = (1,1)$

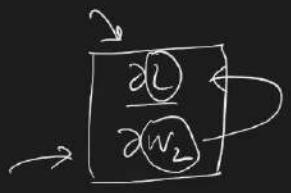
$A_2 = (1,1)$

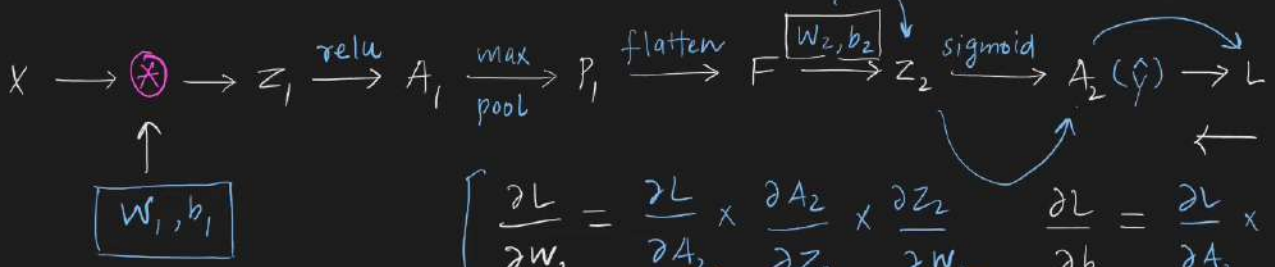
Gradient Descent

$$w_1 = w_1 - \eta \frac{\partial L}{\partial w_1} \qquad w_2 = w_2 - \eta \frac{\partial L}{\partial w_2}$$

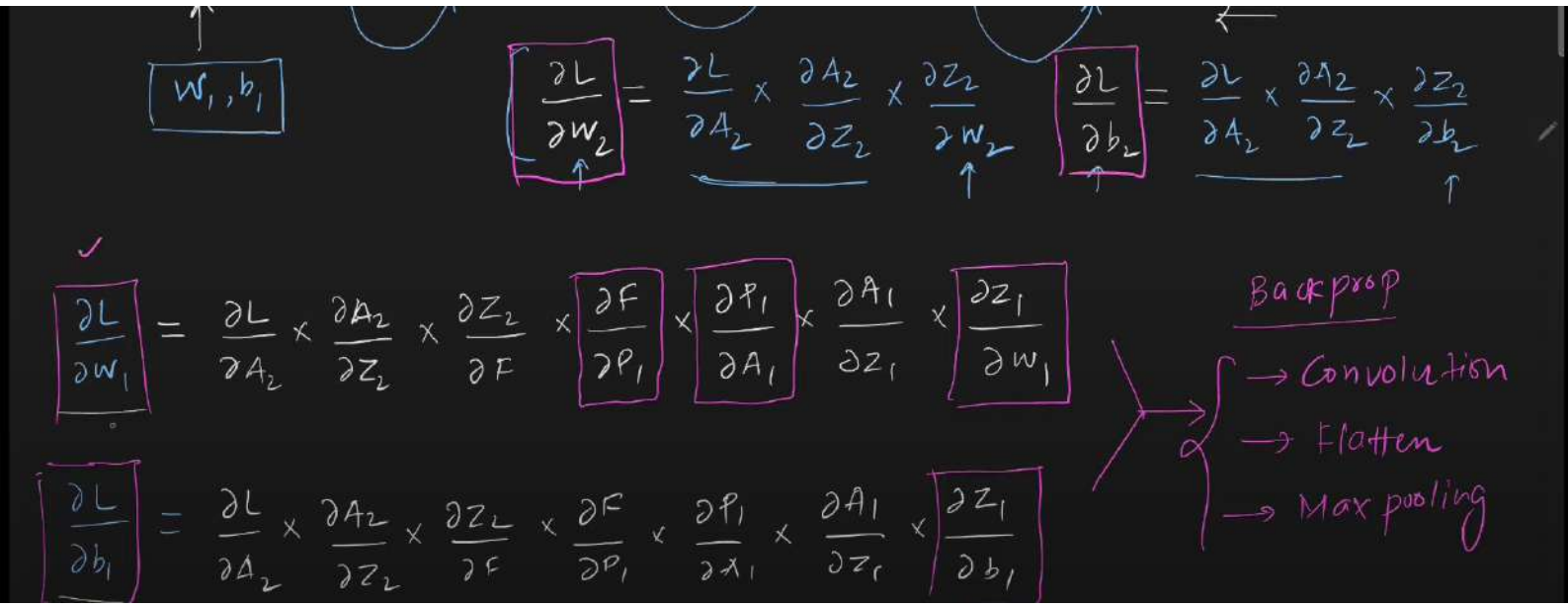
$$b_1 = b_1 - \eta \frac{\partial L}{\partial b_1} \qquad b_2 = b_2 - \eta \frac{\partial L}{\partial b_2}$$

Loss is minimized





$$\frac{\partial L}{\partial W_2} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial W_2} \qquad \frac{\partial L}{\partial b_2} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial b_2}$$

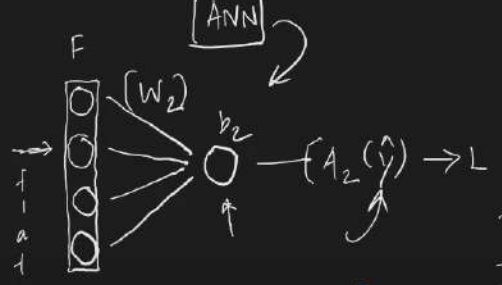

W_1, b_1
$\frac{\partial L}{\partial W_2} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial W_2}$
$\frac{\partial L}{\partial b_2} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial b_2}$
$\frac{\partial L}{\partial W_1} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial F} \times \frac{\partial F}{\partial P_1} \times \frac{\partial P_1}{\partial A_1} \times \frac{\partial A_1}{\partial Z_1} \times \frac{\partial Z_1}{\partial W_1}$
$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial F} \times \frac{\partial F}{\partial P_1} \times \frac{\partial P_1}{\partial A_1} \times \frac{\partial A_1}{\partial Z_1} \times \frac{\partial Z_1}{\partial b_1}$
Backprop
→ Convolution
→ Flatten
→ Max pooling

$$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial F} \times \frac{\partial F}{\partial P_1} \times \frac{\partial P_1}{\partial x_1} \times \frac{\partial A_1}{\partial Z_1} \times \frac{\partial Z_1}{\partial b_1}$$

→ Max pooling

$$\frac{\partial L}{\partial W_2} = \frac{\partial L}{\partial A_2} \frac{\partial A_2}{\partial Z_2} \frac{\partial Z_2}{\partial W_2}$$

$$\frac{\partial L}{\partial b_2} = \frac{\partial L}{\partial A_2} \frac{\partial A_2}{\partial Z_2} \frac{\partial Z_2}{\partial b_2}$$



Forward Prop eqⁿ

$$\begin{cases} Z_2 = W_2 F + b_2 \\ A_2 = \sigma(Z_2) \end{cases}$$

$$\frac{\partial L}{\partial a_2} = \frac{\partial}{\partial a_2}\left[-y_i \log(a_2) - (1-y_i)\log(1-a_2)\right]$$

$$= -\frac{y_i}{a_2} + \frac{(1-y_i)}{(1-a_2)} = \frac{-y_i(1-a_2) + a_2(1-y_i)}{a_2(1-a_2)}$$

$$\left\{ \begin{array}{l} Z_2 = W_2 F + b_2 \\ A_2 = \sigma(z_2) \end{array} \right\}$$

$$= \frac{-y_i}{a_2} + \frac{(1-y_i)}{(1-a_2)} = \frac{}{a_2(1-a_2)}$$

$$\frac{\partial L}{\partial a_2} = \frac{-y_i + y_i a_2 + a_2 - a_2 y_i}{a_2(1-a_2)} = \frac{(a_2 - y_i)}{a_2(1-a_2)}$$

$$\left[\frac{\partial A_2}{\partial z_2} = \sigma(z_2)[1-\sigma(z_2)] = a_2[1-a_2] \right.$$

$$\boxed{\frac{\partial Z_2}{\partial W_2} = F} \qquad \frac{\partial z_2}{\partial b_2} = 1$$

(1,4)

$$\frac{\partial L}{\partial W_2} = \frac{(a_2 - Y_i)}{a_2(1-a_2)} \times \cancel{a_2(1-a_2)} \times F = (a_2 - Y_i)\,F = (A_2 - Y)\,F^T$$

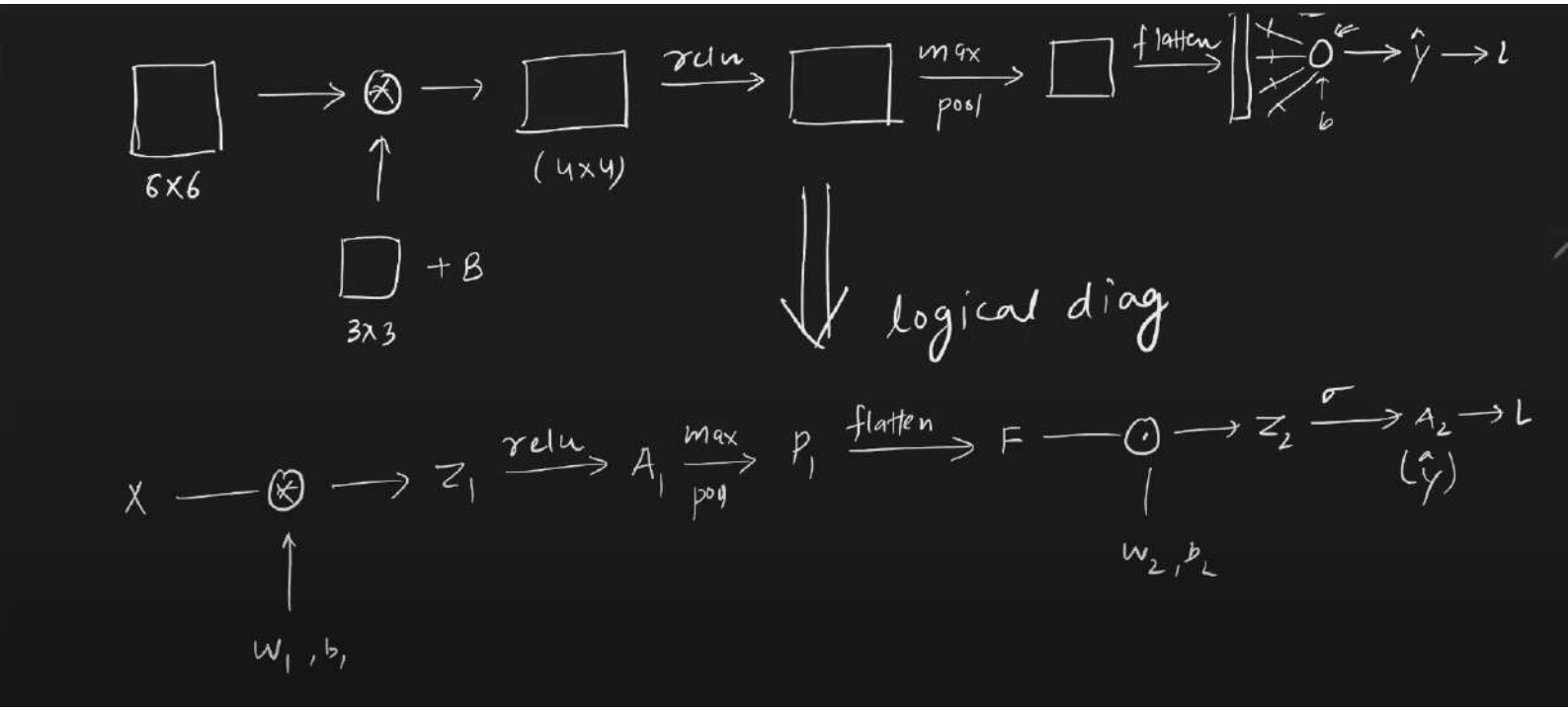
$A_2 - Y$: (1,1) (1,1) → (1,4)

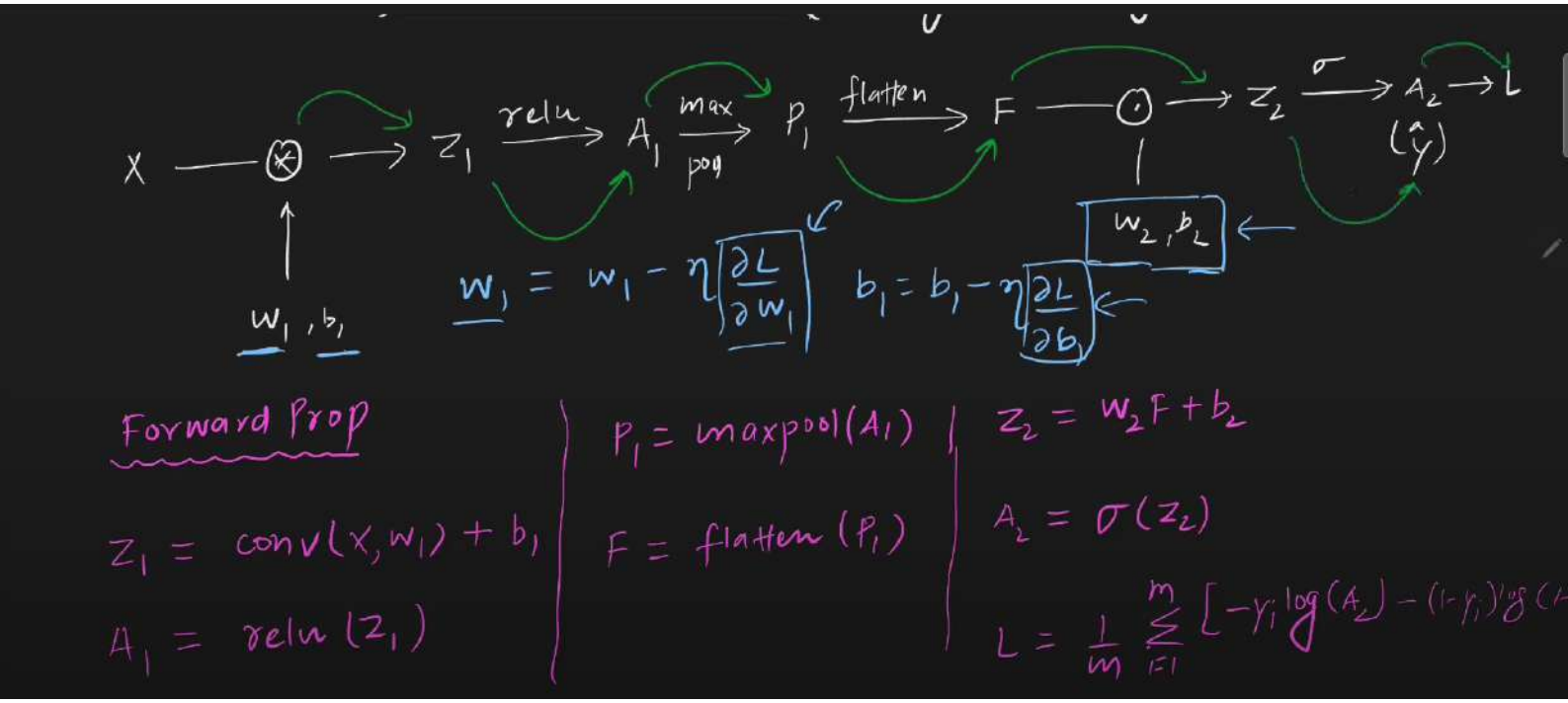
F^T : (4,1)

$$\frac{\partial L}{\partial b_2} = \frac{(a_2 - Y_i)}{\cancel{a_2(1-a_2)}} \cancel{a_2(1-a_2)} \times 1 = (A_2 - Y)$$

(1,4)

$$\boxed{\frac{\partial L}{\partial W_2} = (A_2 - Y)\,F^T} \qquad \boxed{\frac{\partial L}{\partial b_2} = (A_2 - Y)}$$


6X6
3X3
+B
(4x4)
relu
max
pool
flatten
b
ŷ
L
logical diag
X
Z1
relu
A1
max
pool
P1
flatten
F
Z2
σ
A2
(ŷ)
L
W1, b1
W2, b2



Forward Prop

$Z_1 = conv(X, W_1) + b_1$

$A_1 = relu(Z_1)$

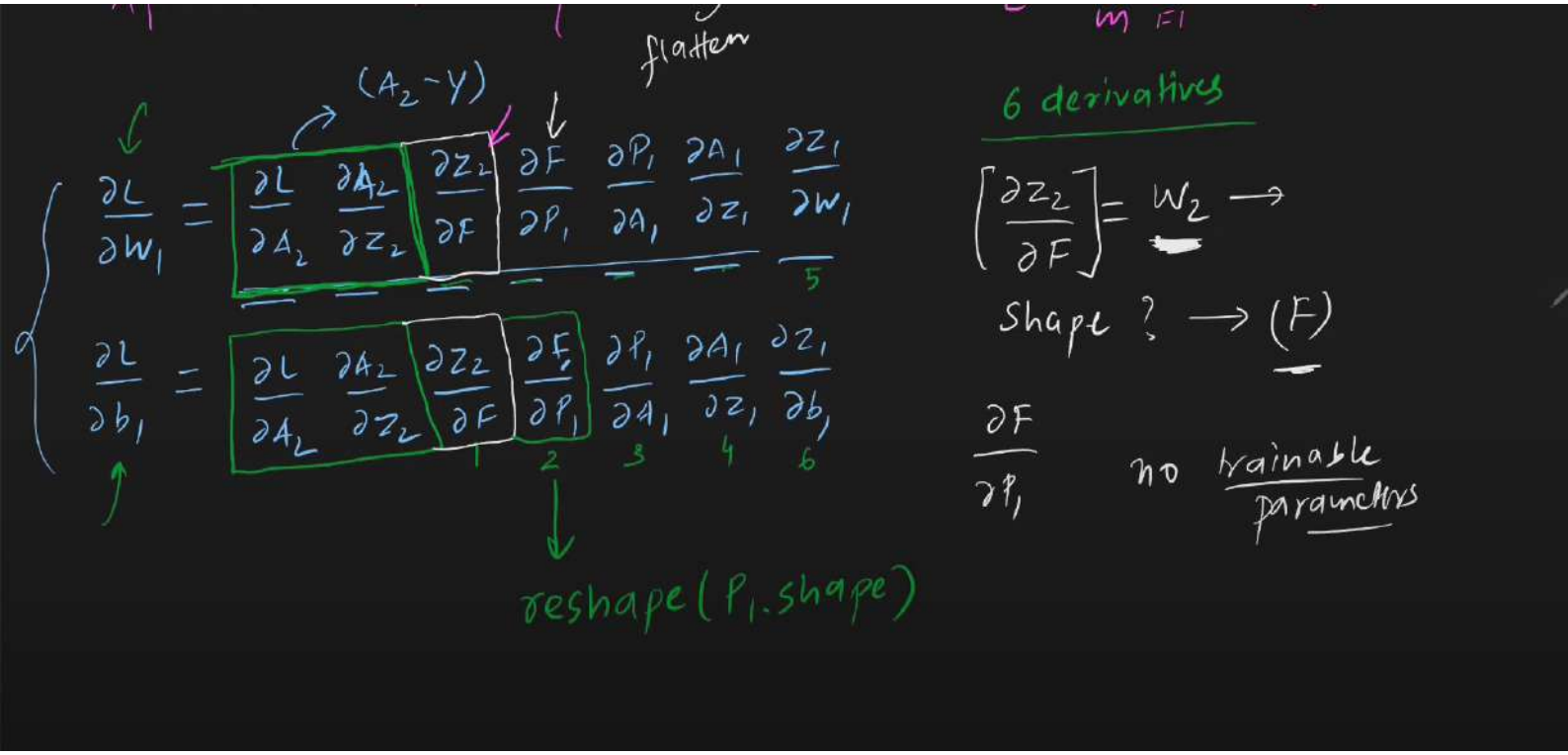
$P_1 = maxpool(A_1)$

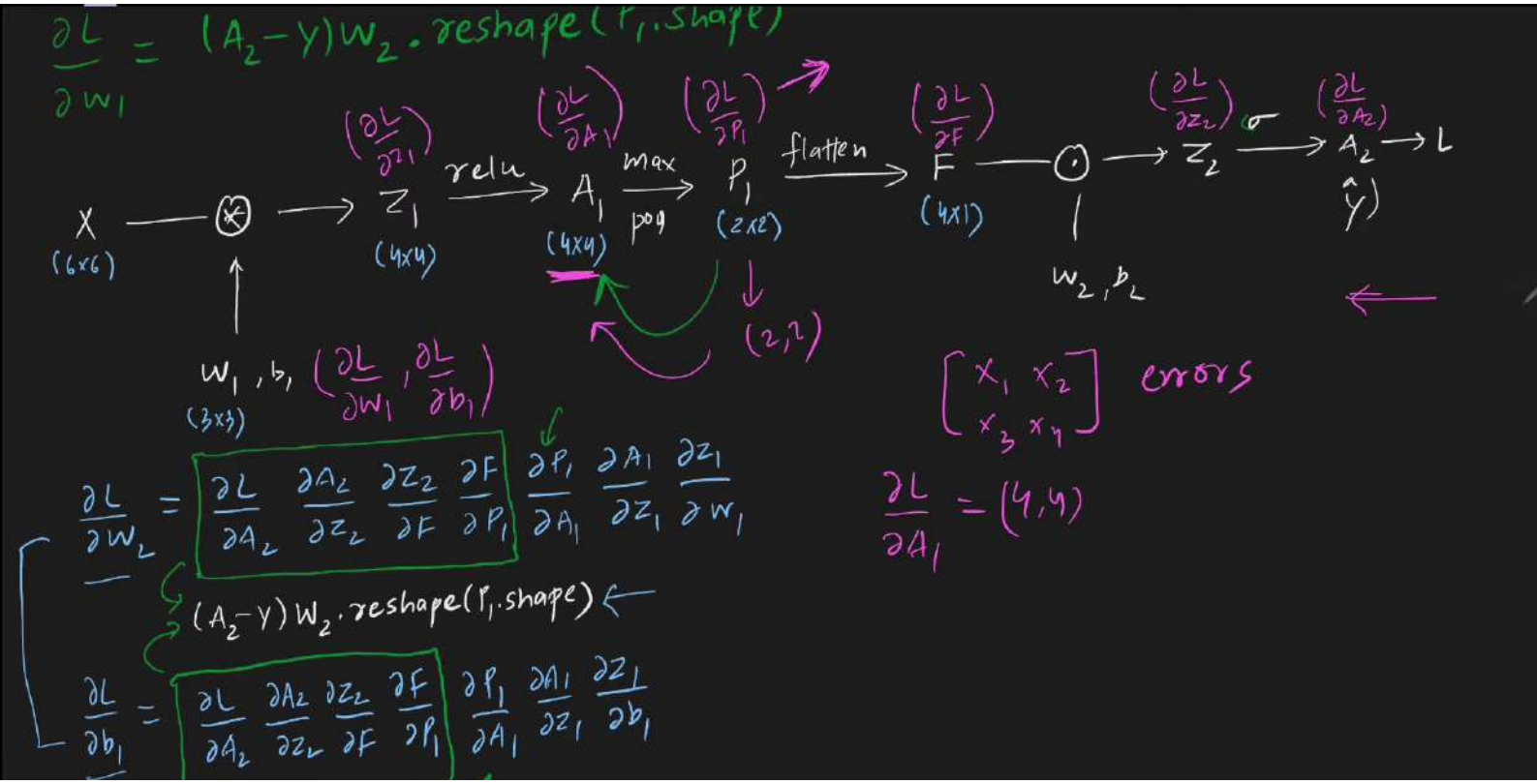
$F = flatten(P_1)$

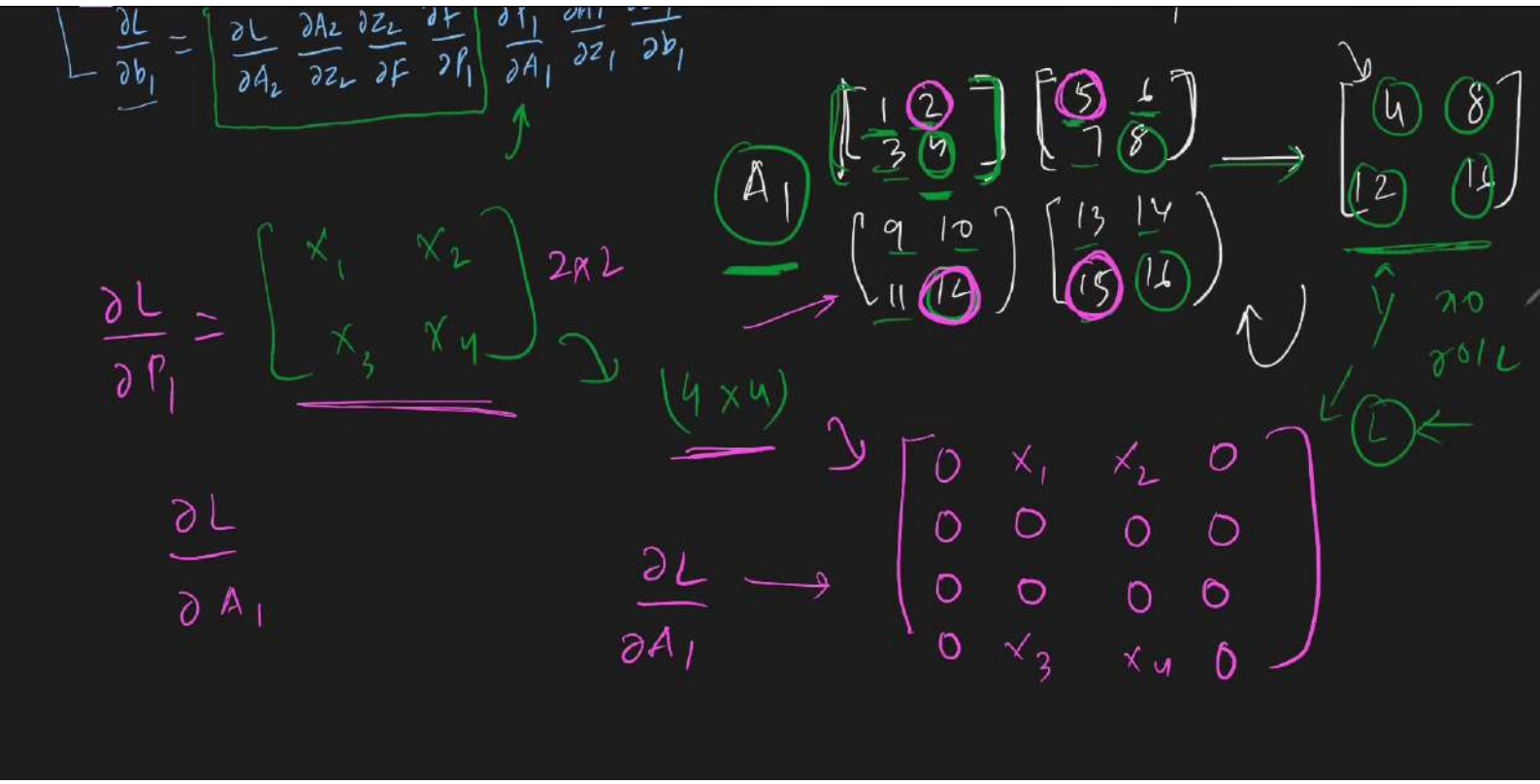
$Z_2 = W_2 F + b_2$

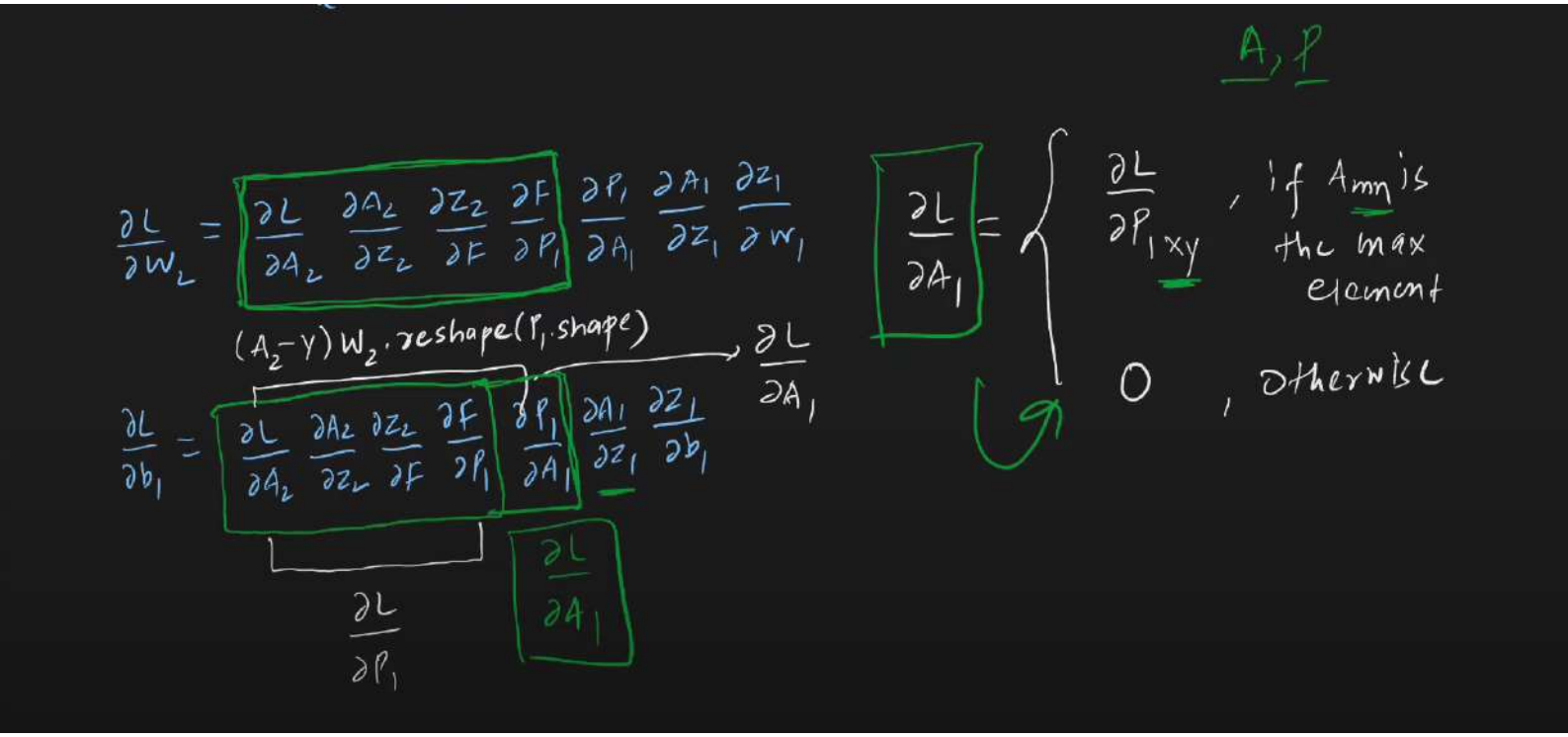
$A_2 = \sigma(Z_2)$

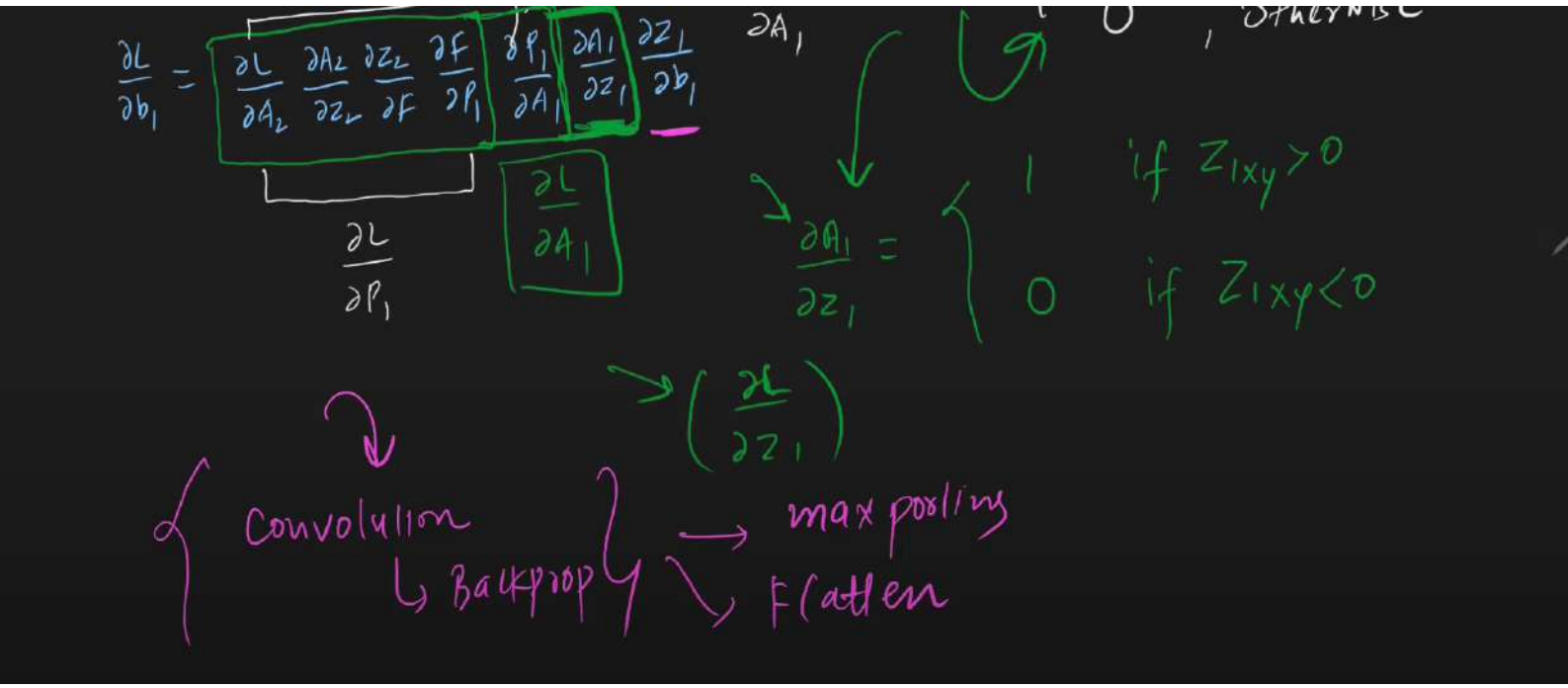
$L = \frac{1}{m} \sum_{i=1}^{m} [-y_i \log(A_2) - (1-y_i) \log(A$


flatten
$(A_2 - Y)$
$\frac{\partial L}{\partial W_1} = \frac{\partial L}{\partial A_2}\frac{\partial A_2}{\partial Z_2}\frac{\partial Z_2}{\partial F}\frac{\partial F}{\partial P_1}\frac{\partial P_1}{\partial A_1}\frac{\partial A_1}{\partial Z_1}\frac{\partial Z_1}{\partial W_1}$
$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2}\frac{\partial A_2}{\partial Z_2}\frac{\partial Z_2}{\partial F}\frac{\partial F}{\partial P_1}\frac{\partial P_1}{\partial A_1}\frac{\partial A_1}{\partial Z_1}\frac{\partial Z_1}{\partial b_1}$
1 2 3 4 5 6
reshape(P_1.shape)
6 derivatives
$\left[\frac{\partial Z_2}{\partial F}\right] = W_2 \rightarrow$
Shape ? $\rightarrow$ (F)
$\frac{\partial F}{\partial P_1}$ no trainable parameters


$\frac{\partial L}{\partial W_1} = (A_2 - Y) W_2 \cdot \text{reshape}(P_1.\text{shape})$
X
(6x6)
W_1, b_1
(3x3)
Z_1
(4x4)
relu
A_1
(4x4)
max pool
P_1
(2x2)
(2,2)
flatten
F
(4x1)
W_2, b_2
Z_2
A_2
$\hat{y}$
L
$\left(\frac{\partial L}{\partial W_1}, \frac{\partial L}{\partial b_1}\right)$
$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ errors
$\frac{\partial L}{\partial A_1} = (4,4)$
$\frac{\partial L}{\partial W_2} = \frac{\partial L}{\partial A_2}\frac{\partial A_2}{\partial Z_2}\frac{\partial Z_2}{\partial F}\frac{\partial F}{\partial P_1}\frac{\partial P_1}{\partial A_1}\frac{\partial A_1}{\partial Z_1}\frac{\partial Z_1}{\partial W_1}$
$(A_2 - Y) W_2 \cdot \text{reshape}(P_1.\text{shape})$
$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2}\frac{\partial A_2}{\partial Z_2}\frac{\partial Z_2}{\partial F}\frac{\partial F}{\partial P_1}\frac{\partial P_1}{\partial A_1}\frac{\partial A_1}{\partial Z_1}\frac{\partial Z_1}{\partial b_1}$


$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2} \frac{\partial A_2}{\partial Z_2} \frac{\partial Z_2}{\partial F} \frac{\partial F}{\partial P_1} \frac{\partial P_1}{\partial A_1} \frac{\partial A_1}{\partial Z_1} \frac{\partial Z_1}{\partial b_1}$
A_1
$\frac{\partial L}{\partial P_1} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ 2x2
(4x4)
$\frac{\partial L}{\partial A_1}$
$\frac{\partial L}{\partial A_1} \rightarrow \begin{bmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_3 & x_4 & 0 \end{bmatrix}$
$\hat{y}$
L

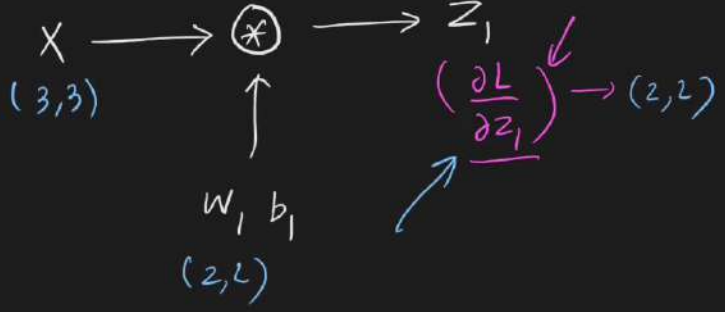

A, P
$\frac{\partial L}{\partial W_2} = \frac{\partial L}{\partial A_2}\frac{\partial A_2}{\partial Z_2}\frac{\partial Z_2}{\partial F}\frac{\partial F}{\partial P_1}\frac{\partial P_1}{\partial A_1}\frac{\partial A_1}{\partial Z_1}\frac{\partial Z_1}{\partial W_1}$
$\frac{\partial L}{\partial A_1} = \begin{cases} \frac{\partial L}{\partial P_{1xy}}, & \text{if } A_{mn} \text{ is the max element} \\ 0, & \text{otherwise} \end{cases}$
$(A_2 - Y)W_2 \cdot \text{reshape}(P_1.\text{shape}) \rightarrow \frac{\partial L}{\partial A_1}$
$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2}\frac{\partial A_2}{\partial Z_2}\frac{\partial Z_2}{\partial F}\frac{\partial F}{\partial P_1}\frac{\partial P_1}{\partial A_1}\frac{\partial A_1}{\partial Z_1}\frac{\partial Z_1}{\partial b_1}$
$\frac{\partial L}{\partial P_1}$
$\frac{\partial L}{\partial A_1}$


$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2}\frac{\partial A_2}{\partial Z_2}\frac{\partial Z_2}{\partial F}\frac{\partial F}{\partial P_1}\frac{\partial P_1}{\partial A_1}\frac{\partial A_1}{\partial Z_1}\frac{\partial Z_1}{\partial b_1}$
$\frac{\partial L}{\partial P_1}$
$\frac{\partial L}{\partial A_1}$
$\frac{\partial A_1}{\partial Z_1} = \begin{cases} 1 & \text{if } Z_{1xy} > 0 \\ 0 & \text{if } Z_{1xy} < 0 \end{cases}$
$\left(\frac{\partial L}{\partial Z_1}\right)$
Convolution
↳ Backprop
max pooling
Flatten

Backprop on Convolution

(2,2)

$$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial z_1} \times \boxed{\frac{\partial z_1}{\partial b_1}}$$



$$z_1 = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \quad W_1 = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

b_1

$(2,2)$

$$X = \begin{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} & x_{13} \\ \begin{bmatrix} x_{31} & x_{32} \end{bmatrix} & x_{33} \end{bmatrix} \circledast \quad W_1 = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

b_1

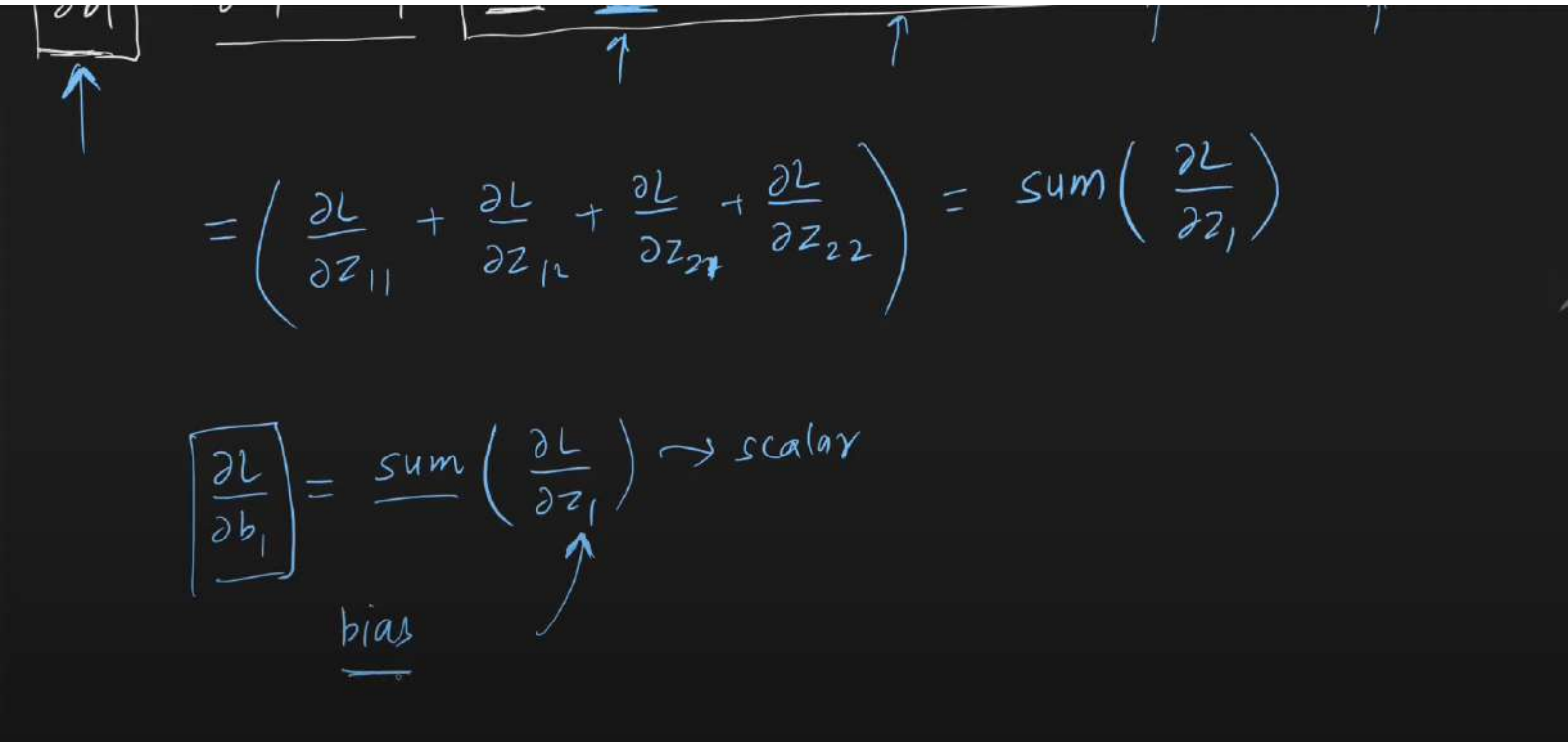
$$z_{11} = x_{11}w_{11} + x_{12}w_{12} + x_{21}w_{21} + x_{22}w_{22} + b_1$$

$$z_{12} = x_{12}w_{11} + x_{13}w_{12} + x_{22}w_{21} + x_{23}w_{22} + b_1$$

$$z_{21} = x_{21}w_{11} + x_{22}w_{12} + x_{31}w_{21} + x_{32}w_{22} + b_1$$

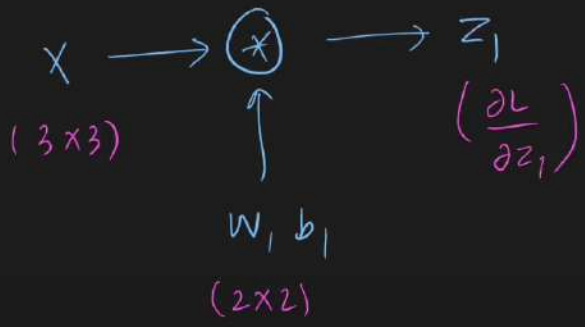
$$z_{22} = x_{22}w_{11} + x_{23}w_{12} + x_{32}w_{21} + x_{33}w_{22} + b_1$$

$$\boxed{\frac{\partial L}{\partial b_1}} = \frac{\partial L}{\partial z_1} \times \frac{\partial z_1}{\partial b_1} = \boxed{\frac{\partial L}{\partial z_{11}} \times \frac{\partial z_{11}}{\partial b_1} + \frac{\partial L}{\partial z_{12}} \times \frac{\partial z_{12}}{\partial b_1} + \frac{\partial L}{\partial z_{21}} \times \frac{\partial z_{21}}{\partial b_1} + \frac{\partial L}{\partial z_{22}} \times \frac{\partial z_{22}}{\partial b_1}}$$


$= \left(\frac{\partial L}{\partial z_{11}} + \frac{\partial L}{\partial z_{12}} + \frac{\partial L}{\partial z_{21}} + \frac{\partial L}{\partial z_{22}} \right) = \text{sum}\left(\frac{\partial L}{\partial z_1} \right)$
$\boxed{\frac{\partial L}{\partial b_1}} = \text{sum}\left(\frac{\partial L}{\partial z_1} \right) \rightarrow \text{scalar}$
bias

bias

$$\underline{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \quad \underline{W_1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} + b_1$$



$$z_{11} = x_{11}w_{11} + x_{12}w_{12} + x_{21}w_{21} + x_{22}w_{22} + b_1$$

$$z_{12} = x_{12}w_{11} + x_{13}w_{12} + x_{22}w_{21} + x_{23}w_{22} + b_1$$

$$z_{21} = x_{21}w_{11} + x_{22}w_{12} + x_{31}w_{21} + x_{32}w_{22} + b_1$$

$$z_{22} = x_{22}w_{11} + x_{23}w_{12} + x_{32}w_{21} + x_{33}w_{22} + b_1$$

$$\frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial w_1}$$

$$z_{11} = x_{11}w_{11} + x_{12}w_{12} + x_{21}w_{21} + x_{22}w_{22} + b_1$$

$$z_{12} = x_{12}w_{11} + x_{13}w_{12} + x_{22}w_{21} + x_{23}w_{22} + b_1$$

$$z_{21} = x_{21}w_{11} + x_{22}w_{12} + x_{31}w_{21} + x_{32}w_{22} + b_1$$

$$z_{22} = x_{22}w_{11} + x_{23}w_{12} + x_{32}w_{21} + x_{33}w_{22} + b_1$$

$$\frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial z_1} \times \frac{\partial z_1}{\partial w_1}$$

$$\frac{\partial L}{\partial w_{11}} = \frac{\partial L}{\partial z_{11}} \times \frac{\partial z_{11}}{\partial w_{11}} + \frac{\partial L}{\partial z_{12}} \times \frac{\partial z_{12}}{\partial w_{11}} + \frac{\partial L}{\partial z_{21}} \times \frac{\partial z_{21}}{\partial w_{11}} + \frac{\partial L}{\partial z_{22}} \times \frac{\partial z_{22}}{\partial w_{11}}$$

$$\frac{\partial L}{\partial w_{12}} = \frac{\partial L}{\partial z_{11}} \times \frac{\partial z_{11}}{\partial w_{12}} + \frac{\partial L}{\partial z_{12}} \times \frac{\partial z_{12}}{\partial w_{12}} + \frac{\partial L}{\partial z_{21}} \times \frac{\partial z_{21}}{\partial w_{12}} + \frac{\partial L}{\partial z_{22}} \times \frac{\partial z_{22}}{\partial w_{12}}$$

$$\frac{\partial L}{\partial w_{21}} = \frac{\partial L}{\partial z_{11}} \times \frac{\partial z_{11}}{\partial w_{21}} + \frac{\partial L}{\partial z_{12}} \times \frac{\partial z_{12}}{\partial w_{21}} + \frac{\partial L}{\partial z_{21}} \times \frac{\partial z_{21}}{\partial w_{21}} + \frac{\partial L}{\partial z_{22}} \times \frac{\partial z_{22}}{\partial w_{21}}$$

$$\frac{\partial L}{\partial w_{22}} = \frac{\partial L}{\partial z_{11}} \times \frac{\partial z_{11}}{\partial w_{22}} + \frac{\partial L}{\partial z_{12}} \times \frac{\partial z_{12}}{\partial w_{22}} + \frac{\partial L}{\partial z_{21}} \times \frac{\partial z_{21}}{\partial w_{22}} + \frac{\partial L}{\partial z_{22}} \times \frac{\partial z_{22}}{\partial w_{22}}$$

$$\frac{\partial L}{\partial w_{11}} = \frac{\partial L}{\partial z_{11}} x_{11} + \frac{\partial L}{\partial z_{12}} x_{12} + \frac{\partial L}{\partial z_{21}} x_{21} + \frac{\partial L}{\partial z_{22}} x_{22}$$

$$\frac{\partial L}{\partial w_{12}} = \frac{\partial L}{\partial z_{11}} x_{12} + \frac{\partial L}{\partial z_{12}} x_{13} + \frac{\partial L}{\partial z_{21}} x_{22} + \frac{\partial L}{\partial z_{22}} x_{23}$$

$$\frac{\partial L}{\partial w_{21}} = \frac{\partial L}{\partial z_{11}} x_{21} + \frac{\partial L}{\partial z_{12}} x_{22} + \frac{\partial L}{\partial z_{21}} x_{31} + \frac{\partial L}{\partial z_{22}} x_{32}$$

$$\frac{\partial L}{\partial w_{22}} = \frac{\partial L}{\partial z_{11}} x_{22} + \frac{\partial L}{\partial z_{12}} x_{23} + \frac{\partial L}{\partial z_{21}} x_{32} + \frac{\partial L}{\partial z_{22}} x_{33}$$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \qquad \frac{\partial L}{\partial z_1} = \begin{bmatrix} \frac{\partial L}{\partial z_{11}} & \frac{\partial L}{\partial z_{12}} \\ \frac{\partial L}{\partial z_{21}} & \frac{\partial L}{\partial z_{22}} \end{bmatrix}$$

$$\frac{\partial L}{\partial w_1} = \text{conv}\left(X, \frac{\partial L}{\partial z_1}\right)$$

$$\frac{\partial w_{22}}{\partial z_{11}} \quad \frac{\partial z_{12}}{\partial z_{21}} \quad \frac{\partial z_{22}}{}$$

$$\frac{\partial L}{\partial w_1} = \text{conv}\left(X, \frac{\partial L}{\partial z_1}\right)$$

$$\frac{\partial L}{\partial b_1} = \text{sum}\left(\frac{\partial L}{\partial z_1}\right)$$