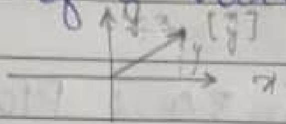
LINEAR ALGEBRA

Representation of a vector

$$\begin{bmatrix} x \\ y \end{bmatrix}$$



Vector addition

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

Scaling a vector
- Multiplying a vector with a scalar

$$2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Basis vectors : $\hat{i}$ and $\hat{j}$
- Unit vectors in x and y direction respectively

Linear combination :
- Linear combination of $\vec{v}$ and $\vec{w}$

$$a\vec{v} + b\vec{w}$$

- If one vector is kept fix, then tip of the other moving vector makes a straight line
- If both vectors are free to move, they can reach to any point in space

Span of the vectors
- The span of $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations $a\vec{v}+b\vec{w}$ where $a, b \in R$

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individual vectors : arrows
sets of vectors : points
vectors on a line : line → 1D span of linearly dependent vectors

LC of 2 vectors in 3D : Plane

LC of $\vec{v}$, $\vec{w}$ and $\vec{u}$: $a\vec{v} + b\vec{w} + c\vec{u}$

- If third vector is on the scale of first two, the resultant is a plane
- If third vector is taken at random in space, the resultant forms a cuboid.

Linearly dependent
- One vector is the linear combination of the others

$$2D: \vec{u} = a\vec{v} \quad OR \quad 3D: \vec{u} = a\vec{v} + b\vec{w}$$
$$OR \quad a\vec{v} + b\vec{u} = 0 \qquad a\vec{v} + b\vec{w} + c\vec{u} = 0$$

Linearly independent

$$2D: \vec{u} \neq a\vec{v} \quad OR \quad 3D: \vec{u} \neq a\vec{v} + b\vec{w}$$

- If each vector adds another dimension to the span then they are called linearly independent

Basis (technical definition)
The basis of a vector space is a set of linearly independent vectors that span the full space.

Linear Transformations:

Transformation - converts the input vector to output vector
- Acts like a function

Linear - Lines remain Lines, without getting curved
- The origin remains fixed in place

Linear transformations have lines kept parallel and equally spaced.
Also dots in 2D remain equidistant in 2D.

Eg: $\vec{v} = -1\hat{i} + 2\hat{j}$

Transformed $\vec{v} = -1(\text{Transformed } \hat{i}) + 2(\text{Transformed } \hat{j})$

Since $\vec{v}$ is the linear combination of $\hat{i}$ & $\hat{j}$, after transformation (linear), $\vec{v}$ is the linear combination of transformed $\hat{i}$ & $\hat{j}$

If, Transformed $\hat{i} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Transformed $\hat{j} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, then

2×2 matrix
$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$
where $\hat{i}$ lands — where $\hat{j}$ lands

Transformed $\vec{v} = -1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

If $\hat{i}$ lands on $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\hat{j}$ on $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

this can be written as $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$ i.e in "2×2" matrix

Linear transformations can be implemented using matrix multiplication.

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2x2 Matrix multiplication

$$\begin{array}{cc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{where } \hat{i} \text{ lands} \quad \text{where } \hat{j} \text{ lands} & \text{vector} \end{array} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}}_{\text{Transformed vector}}$$

90° rotation counterclockwise :-

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Leftarrow \text{New locations of } \hat{i} \text{ \& } \hat{j}$$

shear

$\hat{i}$ remains fixed and $\hat{j}$ moves to (1,1)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Linear transformations are a way to move around space such that the grid lines remain parallel and evenly spaced and the origin remains fixed
- These transformations can be described using only a handful of numbers, the coordinates of where each basis vector lands
- Matrices give a language to describe these transformations where columns represent those coordinates and matrix-vector multiplication just a way to compute what that transformation does to a given vector.

Composition matrix

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear } ②} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation } ①} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{composition}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}}$$

Geometric meaning : Product of two matrices means applying transformations one after the other from right to left

$$\overset{M_2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{M_1}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

① First Trace where $\hat{i}$ goes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} = e \begin{bmatrix} a \\ c \end{bmatrix} + g \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$$

First column of composite
(Final location of $\hat{i}$)

② Then trace where $\hat{j}$ goes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} = f \begin{bmatrix} a \\ c \end{bmatrix} + h \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} af + bh \\ cf + dh \end{bmatrix}$$

Second column of composite
(Final location of $\hat{j}$)

Matrices are non-comutative i.e $AB \neq BA$
Matrices are associative i.e $A(BC) = (AB)C$

All these transformations apply to 3D also, the third direction is denoted by $\hat{z}$.

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Linear transformation scales the area by a factor

Determinant
The scaling factor by which the transformation scales any area is called the determinant.

-ve value of determinant represents the inversion of the orientation of space (flipping)

In the begning, $\hat{j}$ is on left of $\hat{i}$. If after transformation, $\hat{j}$ goes on right of $\hat{i}$, it is said that orientation of space has been inverted

- In 3D, determinant tells the scaling factor of the volume
- If value of a 3D determinant is zero, it represents a point
- If right handed 3D vectors, after transformation become left handed 3D vectors, orientation has been flipped and the determinant is -ve.

$$\det(M_1 M_2) = \det(M_1)\det(M_2)$$

Calculating

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Linear system of equations
- contains variables and constants
- Variables are scaled using scalars and added to give a constant value. (Nothing else is allowed)

Eg:

$$\begin{aligned} 2x + 5y + 3z &= -3 \\ 4x + 0y + 8z &= 0 \\ 1x + 3y + 0z &= 2 \end{aligned} \rightarrow \begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

all constant coefficients variables constants

$$A \quad \vec{x} \quad = \quad \vec{v}$$

Orinal vector $\vec{x}$, after transformation A, lands on $\vec{v}$ ⇒ This is interpretation of the equation.

If $|A| \neq 0$: One and only one vector lands on $\vec{v}$.

Inverse matrix → Only possible if $\det(A) \neq 0$.
inverse of A is A^{-1}

A^{-1} is a unique transformation with the property that if you first apply A and then follow it ~~by~~ with the transformation A^{-1}, you end up back where you started

∴ According to matrix multiplication,

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity transformation
The transformation that does nothing.

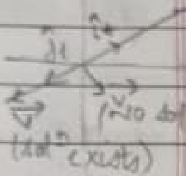
∴ we can find $\vec{x}$ by

$$A^{-1}A\vec{x} = A^{-1}\vec{v}$$

$$\therefore \vec{x} = A^{-1}\vec{v}$$

This means we are reversing the transformation from $\vec{v}$ to $\vec{x}$.

- If $\det(A) = 0$, i.e transformation squishes to a line, A^{-1} cannot be found.
- You cannot un-squish a line to a plane
- In 3D, if the transformation squishes the space to a plane or a line or even a point, then A^{-1} doesn't exist as all of this implies $\det(A) = 0$.



- In 2D, if $\vec{v}$ lies on the same line of the squished plane, solutions exist, otherwise they don't.

Rank of a matrix

Rank 1 → line

Best for 2D — Rank 2 → Plane

Best for 3D — Rank 3 → space

Rank 0 → Point

"Rank" means the number of dimensions in the output of a transformation.

For non-zero determinant,

In 2D : Rank 2 ⇒ Plane

In 3D : Rank 3 ⇒ space

Column space → Tells when a solution exists

- set of all possible outputs of $A\vec{v}$
- span of the columns of a matrix

∴ Rank is the number of dimensions in the column space

Full Rank

when rank is as high as it can be i.e Rank = No of columns, the matrix is full rank.

Null space or Kernal → Tells what the set of all possible solutions look like.
- It is the space of all vectors that become null, i.e they land on the zero vector
If in $A\vec{x} = \vec{v}$
$\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

the null space gives you all the possible solutions to the equation.

Non-square Matrices
A 3×2 matrix implies 2 basis vectors land on a plane cutting through the origin in 3D
A 2×3 matrix implies 3 basis vectors land on a plane in 2D
A 1×2 matrix implies 2 basis vectors land on a line.

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Dot Product

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a.d + b.e + c.f$$

Two vectors of same dimentions
OR

Project one of the vectors on the other and then take the product of this projected length and the length of the other vector. This value is the dot product.

$$\vec{v} \cdot \vec{w} = l(\text{projection of } \vec{v}) \cdot l(\vec{w})$$
$$= l(\text{projection of } \vec{w}) \cdot l(\vec{v}) = \vec{w} \cdot \vec{v}$$

Here sequence doesn't matter.

$\vec{v} \cdot \vec{w} > 0 \Rightarrow$ if $\vec{v}$ & $\vec{w}$ are in same directions
$\vec{v} \cdot \vec{w} = 0 \Rightarrow$ if $\vec{v} \perp^r \vec{w}$
$\vec{v} \cdot \vec{w} < 0 \Rightarrow$ if $\vec{v}$ & $\vec{w}$ are in opposite directions

$$\underset{\text{Matrix}}{\begin{bmatrix} x & y \end{bmatrix}} \Longleftrightarrow \underset{\text{vector}}{\begin{bmatrix} x \\ y \end{bmatrix}}$$

Dotting two vectors together is a way to translate one of them into the world of transformation

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 x_2 + y_1 y_2$$