# Dot Product

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a.d + b.e + c.f$$

Two vectors of same dimentions
OR

Project one of the vectors on the other and then take the product of this projected length and the length of the other vector. This value is the dot product.

$$\vec{v}.\vec{w} = l(\text{projection of } \vec{v}).\, l(\vec{w})$$
$$= l(\text{projection of } \vec{w}).\, l(\vec{v}) = \vec{w}.\vec{v}$$

Here sequence doesn't matter.

$\vec{v}.\vec{w} > 0 \Rightarrow$ if $\vec{v}$ & $\vec{w}$ are in same directions
$\vec{v}.\vec{w} = 0 \Rightarrow$ if $\vec{v} \perp \vec{w}$
$\vec{v}.\vec{w} < 0 \Rightarrow$ if $\vec{v}$ & $\vec{w}$ are in opposite directions

DUALITY

$\begin{bmatrix} x & y \end{bmatrix}$ Matrix $\Longleftrightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix}$ vector : Duality

Linear transformation $\Rightarrow$ Dual vector

Dotting two vectors together is a way to translate one of them into the world of transformations

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 x_2 + y_1 y_2$$

2D to 1D linear transformation is the same as taking a dot product with that vector

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Cross Product

$\vec{v} \times \vec{w}$ = Area of parallelogram

$\vec{v} \times \vec{w} > 0 \Rightarrow$ if $\vec{v}$ is on the right of $\vec{w}$

$\vec{v} \times \vec{w} < 0 \Rightarrow$ if $\vec{v}$ is on the left of $\vec{w}$

Here sequence matters

if $\vec{v} \times \vec{w} = A$ , $\vec{w} \times \vec{v} = -A$

if $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

then $\vec{v} \times \vec{w} = \det\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\right)$

Linear transformation

$\hat{i}$ & $\hat{j} \Rightarrow$ unit vector (Area of ||gm = 1)

$\vec{v}$ & $\vec{w} \Rightarrow$ Their transformations

(Area scaled equal to determinant)

If $\vec{v}$ is to left of $\vec{w}$, Area is flipped $\Rightarrow$ -ve value

More $\perp^r \Rightarrow \vec{v} \times \vec{w}$ is bigger

Similar direction $\Rightarrow \vec{v} \times \vec{w}$ is smaller

if $\vec{v} \to 3\vec{v}$   $\vec{v} \times \vec{w} \Rightarrow (3\vec{v}) \times \vec{w} = 3(\vec{v} \times \vec{w})$

In vectors (what actually cross product is)

$\vec{v} \times \vec{w} = \vec{p}$

where $l(\vec{p})$ = Area of the parallelogram and direction of $\vec{p}$ is $\perp^r$ to the plane of the parallelogram. sign is followed by RHR

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \hat{i} & v_1 & w_1 \\ \hat{j} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix}\right)$$

$$= \hat{i}(v_2 w_3 - v_2 w_2) + \hat{j}(v_3 w_1 - v_1 w_3) + \hat{k}(v_1 w_2 - v_2 w_1)$$

OR $$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 \cdot w_3 - w_2 \cdot v_3 \\ v_3 \cdot w_1 - w_3 \cdot v_1 \\ v_1 \cdot w_2 - w_1 \cdot v_2 \end{bmatrix}$$

3D to 1D linear transformation

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

variables $\vec{v}$ $\vec{w}$

$\vec{p}$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

- $\vec{p}$ is the vector such that applying the transformations is the same thing as taking a dot product with that vector
- $\vec{p}$ is the dual vector

Cramer's Rule

Applicable only when $\det(A) \neq 0$ i.e One Input, One Output

If $T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}$, for all $\vec{v}$ and $\vec{w}$,
T is 'Orthonormal' Dot products are preserved

i.e If a transformation leaves all the basis vectors perpendicular to each other and with unit length then that is called orthonormal.
- These are rotation matrices : rigid motion

If $A\vec{x} = \vec{v}$

$x \cancel{y} = \frac{\text{Area}}{\det(A)}$ $\qquad$ $y = \frac{\text{Area}}{\det(A)}$

If $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

then ~~x =~~

$$x = \frac{\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}}{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}, \quad y = \frac{\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}}{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}$$

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Coordinate system
- Any way to translate between vectors and set of numbers is called a 'coordinate system'.
- The two special vectors $\hat{i}$ and $\hat{j}$ are called the basis vectors

- We consider a standard grid for our coordinate system
- However, space has no grid
- So if the considered grid changes, basis vectors will also change.
- For all coordinate systems, origin coincides

- To convert our grid to a new grid, we consider the new grid as the transformation of ours. ~~The transformation converts~~

Eg. Our grid $\longrightarrow$ New grid

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

But

$$A\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

Our language $\longleftarrow$ New language

To get our grid back use inverse

New grid $\longrightarrow$ Our grid

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^{-1}\begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

New language $\longleftrightarrow$ Our language

$$\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$$

$$A\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

Inverse change of basis matrix

written in our language

written in new language

How to translate a matrix

1. start with any vector in new language
2. Translate it to our language using change of basis matrix This gives vector in our language
3. Apply transformation ~~to~~ in our language to what you get from the 1st 2 steps by multiplying the transformation matrix on the left.
   This tells where the vector lands but in our language
4. To this, apply the inverse change of basis matrix multiplying it on the left.

$$\overbrace{\begin{bmatrix}2&-1\\1&1\end{bmatrix}^{-1}\overbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}\overbrace{\begin{bmatrix}2&-1\\1&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}}^{\text{Same vector in our language}}}^{\text{Transformed vector in our language}}}^{\text{Transformed vector in New language}}$$

inverse change of basis matrix | transformation matrix in our language | change of basis matrix | vector in new language

represents mathematical empathy

$$A^{-1} \; M \; A \; \vec{v} \Rightarrow \text{Transformation in new language}$$

Transformation

The empathy

the shift in perspective

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If the effect of a linear transformation on a given vector is to scale it by some constant, it is called eigenvector.
The scaling factor is called eigenvalue.

Eigenvectors and Eigenvalue

- The vector which does not get knocked off its span during a transformation is called eigenvector. It just gets stretched or squished.
- The factor by which an eigenvector is stretched or squished is called the eigen value of that vector
- -ve eigenvalue implies that the vector is flipp
- In 3D, for a rotation, eigenvector is the axis of rotation.
- The eigenvalue will be 1 as rotation does not stretch or squish anything.

$$A\vec{v} = \lambda\vec{v}$$

Transformation matrix — $A$
Eigenvector — $\vec{v}$
Eigen value (const) — $\lambda$

Matrix-vector product $A\vec{v}$ gives the same result as scaling the eigenvector $v$ by some $\lambda$

scaling by $\lambda \Leftrightarrow$ Matrix multiplication by

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{identity}}$$

$$\therefore A\vec{v} = (\lambda I)\vec{v}$$
$$\therefore (A - \lambda I)\vec{v} = \vec{0}$$

For $A - \lambda I = 0$, $\det(A - \lambda I) = 0$

This gives the value of $\lambda$. ↳ characteristic polynomial

After getting $\lambda$, use
$$(A - \lambda I)\vec{v} = 0$$
and find $\vec{v}$

Not all 2D transformations have eigen values
Eg: $90°$ anticlockwise rotation
Transformation matrix = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\therefore \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = (-\lambda)(-\lambda) - (-1)(1)$$
$$= \underbrace{\lambda^2 + 1}_{\text{characteristic polynomial}} = 0$$
$$\lambda = \pm i$$

i.e eigen value is imaginary, hence no eigen vector is present.
complex eigen values generally correspond to some kind of rotation in the transformation.

For a shear, all vectors on $x$-axis are eigenvectors and have eigenvalue 1.
$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = \underbrace{(1-\lambda)(1-\lambda)}_{\text{characteristic polynomial}} = 0$$
$\therefore \lambda = 1 \Rightarrow$ only solution.

- A single eigenvalue can have more than a line full of eigenvectors.

If a matrix is a diagonal matrix, all the basis vectors are eigenvectors with eigen values equal to the elements of the diagonal matrix.
Multiplying with diagonal matrix n times → scaling to $n^{th}$ power
Use eigenvectors as basis using change of base
This helps to compute large powers of a matrix

This can be done only when there are minimum 2- eigen vectors

Trick for computing eigen values

For a 2x2 matrix,

1. $tr\left(\begin{bmatrix} a & b \\ b & d \end{bmatrix}\right) = \frac{a+d}{2} = \frac{\lambda_1 + \lambda_2}{2} = m$ (mean)

$\text{mean}(a, d) = \text{mean}(\lambda_1, \lambda_2)$

2. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \lambda_1 \lambda_2 = p$ (product)

3. $p = m^2 - d^2 = (m+d)(m-d)$

$d^2 = m^2 - p$

$\lambda = m \pm d$

i.e $\boxed{\lambda = m \pm \sqrt{m^2 - p}}$ m plus or minus square root of m square minus p

Eg :

$a_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$a_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$a_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$a^2 + b^2 + c^2 = 1$

$a a_x + b a_y + c a_z$

$a\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} c & a-bi \\ a+bi & -c \end{bmatrix}$ $m = 0$, $p = 1$

$\therefore \lambda = \pm 1.$

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Functions can be treated the same way as we do to vectors.
They can be added or scaled i.e
they can be linearly transformed from one function to another.
Eg: derivative is an example of transformation

$$\frac{d}{dx}\frac{1}{9}x^3 - 1 = \frac{1}{3}x^2 - 1.$$

These linear transformations are called linear operators.

Linear transformations
A transformation is linear if it satisfies two properties, called additivity and scaling
Additivity: $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$
Scaling: $L(c\vec{v}) = cL(\vec{v})$

Derivative is linear

For Polynomials,
Basis functions:
$b_0(x) = 1$ Eg $x^2+3x+5 = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
$b_1(x) = x$
$b_2(x) = x^2$
$b_3(x) = x^3$
$\vdots$

$$a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \\ 0 \\ \vdots \end{bmatrix}$$

~~derivatives of each~~ basis vectors

$$\frac{d}{dx} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

$$\frac{d}{dx}(1x^3 + 5x^2 + 4x + 5) = \underbrace{3x^2 + 10x + 4}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1\cdot 4 \\ 2\cdot 5 \\ 3\cdot 1 \\ 0 \\ \vdots \end{bmatrix}$$

| Linear Algebra | Functions |
|---|---|
| Linear Transformations | Linear operators |
| Dot products | inner products |
| Eigen vectors | Eigenfunctions |

All the vectorish things in maths like arrows or lists of numbers or functions are together called vector spaces.

The rules for vector addition and scaling are called 'Axioms'

They are to be applied to be able to proceed with linear algebra applications!

Axioms (Rules for vectors addition and scaling)
interface

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

3. There is a vector $0$ such that $0 + \vec{v} = \vec{v}$ for all $\vec{v}$.

4. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so that $\vec{v} + (-\vec{v}) = 0$

5. $a(b\vec{v}) = (ab)\vec{v}$

6. $1.\vec{v} = \vec{v}$

7. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$

8. $(a+b)\vec{v} = a\vec{v} + b\vec{v}$