CREDIT DERIVATIVES

A Project Report Submitted

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CERTIFICATE

This is to certify that the work contained in this project report entitled "Credit Derivatives" submitted by Dhruvesh Bhure (Roll No.: 200123018) & Dev Shah (Roll No.: 200123074) to the Department of Mathematics, Indian Institute of Technology Guwahati towards partial requirement of Bachelor of Technology in Mathematics and Computing has been carried out by him/her under my supervision.

It is also certified that this report is a survey work based on the references in the bibliography.

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ABSTRACT

The main aim of the project is to understand credit derivatives, different types of credit derivatives and quantify risk associated with any credits. Modelling credit risk associated with different types of credits helps in valuing credit derivatives. The main objective is to model pricing for credit derivatives.

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Chapter 1

Credit Risk Modelling : Structural Models

To value credit derivatives it is necessary to be able to model credit risk. 2 used approaches to model credit risk are -

1. **Structural models or firm value models** - company defaults on its debt if the value of the assets of the company falls below a certain default point.

   - Fisher black and myron scholes - how equity owners hold a call option on the firm.
   - Black scholes merton model - robert merton extended the framework and analyzed risk debt behavior with the model.
   - Robert Geske extended the BSM model to include multiple debts.

2. **Reduced form models** - do not look inside the firm, they model directly the likelihood of default or downgrade

   - Jarrow-Turnbull
   - Duffie-singleton

1.1 Complexities in credit risk modelling

Default risk - It is a result of an inability to pay for corporate debtors. It is a very rare event. It is not an universal concept.

- Causes of defaulting
  - Microeconomic factor : poor management
  - Macroeconomic factors : high interest rates and recession

write about concept of senior creditors are paid before other creditors.

1.2 Overview of current model

Credit risk models used by the insurance and corporate finance literature concentrate on default rates, credit ratings, and credit risk premiums. Traditional models assume default risk can be diversified away in large portfolios similar to portfolio theory that employs the CAPM where only market/systematic risk matters.

1.3 The black-scholes-merton model

Corporate liabilities can be viewed as a covered call.
covered call : own the asset but short a call option.
Debt holder : the investor/creditor who lends the money based on an agreement to be paid back more than what is lend today. E(t) : market value of the issued equity.
D(t,T) : market value of the debt at time t Maturing at T.
K : face value of the zero coupon bond maturing at time T.
T : maturity of the debt

At time T, the market value of the issued equity of the company is the amount remaining after the debts have been paid out of the firm's asset; i.e.

$$E(T) = \max\{A(T) - K, 0\} \tag{1.1}$$

This payoff is similar to that of a call option on value of the firm's asset struck at the face value of the debt.

Hence the value of the debt on the maturity date is given by -

$$D(T,T) = min(A(T), K) \tag{1.2}$$

$$= A(T) - max(A(T) - K, 0) \tag{1.3}$$

$$= K - max(K - A(T), 0) \tag{1.4}$$

Equation 1.3 decomposes the risky debt into asset and a short call. This interpretation shows that the equity owners owns the call option of the company. If the company performs well, then equity holders should call the company.

Equation 1.4 decomposes the risky debt into a risk-free debt and a short put.The issuer(equity owners) can put the company back to the debt owner when the performance is bad. Hence the default risk = put option.

paste the graphs here and explain them a bit

$$A(t) = E(t) + D(t,T) \tag{1.5}$$

Compare

A(t) with S(t)

E(t) with V(t)

Since any corporate debt is contingent (dependent) claim on the form's future asset value at the time the debt matures, this is what we must model in order to capture the default. BSM assumed the dynamics of the asset value follow a lognormal stochastic process of the form

$$\frac{dA(t)}{dt} = rdt + \sigma dW(t) \tag{1.6}$$

where :

$$\begin{aligned} r &= \text{instantaneous risk-free rate(assumed constant)} \\ \sigma &= \text{precentage volatility} \\ W(t) &= \text{Wiener process under risk neutral measure} \end{aligned}$$

Since same assumption as in equity markets for the evolution of stock prices.

- $A(t) \geq 0$
- $dA(t) \propto A(t)$

It is possible to use the option pricing equation developed by BSM to price risky corporate liabilities.

At t $= T$:

$$A(T) \geq K \implies \text{No default}$$

$$A(T) < K \implies \text{Default}$$

The probability of defaulting at maturity is

$$p = \int_{-\infty}^{K} \phi(A(T))\, dA(T) = 1 - N(d_2) \tag{1.7}$$

where :

$$\phi(.) = \text{log normal density function}$$
$$N(.) = \text{cummulative normal probability}$$
$$d_2 = \frac{\ln \frac{A(t)}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$N(d_2)$ is the **suvival probability**. USing BSM to find the currentvalue of the equity.We have,

$$E(t) = A(t)N(d_1) - e^{-r(T-t)}KN(d_2) \tag{1.8}$$

where

$$d_1 = d_2 + \sigma\sqrt{T-t}$$

The current value of the debt is a covered call value :

$$D(t,T) = A(t) - E(t) \tag{1.9}$$
$$= A(t) - [A(t)N(d_1) - e^{-r(T-t)}KN(d_2)] \tag{1.10}$$
$$= A(t)[1 - N(d_1)] + e^{-r(T-t)}KN(d_2) \tag{1.11}$$

NOTE : first term in 1.11 is the ***recovery value*** whereas the second term is the ***present value of probability weighted face value of the debt***.

Yield of the debt(y) is calculated by solving

$$D(t,T) = Ke^{-y(T-t} \tag{1.12}$$
$$y = \frac{\ln K - \ln D(t,T)}{T-t} \tag{1.13}$$

1.4 Implications of BSM model

increase in debt-to-asset $\implies$ increase in risky yield

write code to plot the y-r Vs T graph (maturity dependency of the credit spread)

The general downward trend of these spread curves at the long end due to the fact that on average the asset value grows at the riskless rate, and so given enough time, will always grow to cover the fixed debt.

1.5 Geske compound option model

Now let us look at the case where company has a series of debts (zero coupon). Defaults are a series of contingent events. Later defaults are contingent upon prior no-default.
company may default when :

- $A(T_1) < K_1$ (i.e fails to pay its first debt)
- $A(T_1) < K_1 + Market\_value(K_2)$

1.6 Barrier structural model

This model also extends BSM model to multiple periods. This model views defaults as knockout options (down and out barrier)

Chapter 2

Credit Risk Modelling : Reduced form Models

Mainly represented by **Jarrow-Turnbull** and **Duffie-Singleton**
Properties of these models:

- arbitrage free
- employ risk neutral measure to price securities
- default is exogenous
- The computations of debt values of different maturities are independent

2.1 The poisson process

It is the theoretical framework for reduced form models.
Let the value be N_t at time t and λ be the intensity parameter.

$$P(N_{t+dt} - N_t = 1) = \lambda dt$$

$$P(N_{t+dt} - N_t = 0) = 1 - \lambda dt$$

As dt is small, there is negligible probability of two jumps occurring in the same time interval.

When the poisson process jump from 0 to 1 is viewed as default in reduced form models. Time taken till first default event will give **default time distribution**.

$$P(T > t) = e^{-\lambda(T-t)}$$

The survival probability before time t :

$$Q(t, T) = p(T > t) = e^{-\lambda(T-t)}$$

2.2 The Jarrow-Turnbull model

Assumption : The recovery payment will be paid at maturity in case of defaults. The coupon bond value is given as :

$$\begin{aligned} B(t) &= p(t, T)R(T) \int_t^T -dQ(t, u)du + \sum_{j=1}^{n} P(t, T_j)c_j e^{-\lambda(T_j - t)} \\ &= P(t, T)R(T)(1 - e^{-\lambda(T-t)}) + \sum_{j=1}^{n} P(t, T_j)c_j e^{-\lambda(T_j - t)} \end{aligned}$$

where :

$$\begin{aligned} P(t, T) &= \text{risk-free discount factor} \\ c_j &= \text{j-th coupon} \\ Q(t, T) &= \text{survival probability up to time t} \\ R &= \text{recovery ratio} \end{aligned}$$

if R = 0 , zero coupon bond with face value = \$1 then :

$$B(t) = P(t, T)e^{-\lambda(T-t)}$$

this equation is comparable with one-period binomial model with bond's forward yield spread y

$$D(t,T) = P(t,T)e^{-y(T-t)}$$

Therefore intensity parameter λ is the bond's forward yield spread in this case. D(t,T) is known as the **risky discount factor**, which is the present value of \$1 if there is no default.

If λ is a function of time and the recovery is paid at the time of default then

$$\begin{aligned} B(t) &= \int_t^T p(t,u)R(u)(-dQ(u)) + \sum_{j=1}^{n} P(t,T_j)c_j Q(t,T_j) \\ &= \int_t^T P(t,u)R(u)\lambda(u)e^{-\int_t^u \lambda(w)dw} + \sum_{j=1}^{n} P(t,T_j)c_j e^{-\int_t^{T_j} \lambda(w)dw} \end{aligned}$$

To actually implement this equation, we take λ to be constant in adjacent time points. If we further assume that default can occur only at coupon times then

$$B(t) = \sum_{j=1}^{n} P(t,T_j)R(T_j)\lambda(T_j)e^{-\sum_{k=1}^{j}\lambda(T_k)} + \sum_{j=1}^{n} P(t,T_j)c_j e^{-\sum_{k=1}^{n}\lambda(T_k)}$$

2.2.1 The calibration of Jarrow-Turnbull model

Since a debt contract pays interest under survival and pays recovery upon default, the expected payment is naturally the weighted average of the two payoffs.

From now on let the survival probability from now to any future time t be $Q(0,t)$. Therefore $Q(0,s) - Q(0,t)$ where $s > t$ is the default probability between two future time points t and s.

- **Geske Model** : asset value is recovered.

- **Duffie-Singleton Model** : fraction of the market debt value is recovered.

- **Jarrow-Turnbull Model** : arbitrary recovery value is assumed.

Forward default probability is the difference of two survivals weighted by the previous survival as shown below:

$$p(j-1,j) = \frac{Q(0,j-1) - Q(0,j)}{Q(0,j-1)}$$

Default probability increases with the increase in the recovery value. Draw the plot to show above thing. If the default probability is kept constant then the bond should be priced above par. take constant default probability and show that the bond price increases with increase in the recovery value

2.2.2 Transition Matrix

Now we can even incorporate the rating changes called migration risk.This extended model is called Jarrow,Lando and Turnbull. Migration risk is different from default risk in that a downgrade in credit ratings only widens the credit spread of the debt issuer and does not cause default. No default means no recovery to worry about.

2.2.3 conclusion

- It generates closed form solution for the bond price.

Drawbacks compared to reality:

- recovery actually occurs upon(or soon after) default.
- recovery amount can fluctuate randomly over time.
- recovery rate being exogenously specified percentage of the default-free recovery, may actually exceed the price of the bond at the moment of default.

2.3 The Duffie-Singleton model (fractional recovery model)

- This model allows the payment of recovery to occur at any time.
- the amount of recovery is restricted to be the proportional of the bond price at default time.

$$R(t) = \delta D(t,T)$$

where :

$$\begin{aligned} R(t) &= \text{recovery ratio} \\ \delta &= \text{fixed ratio} \\ D(t,T) &= \text{debt value if default did not occur} \end{aligned}$$

The debt value at time t is given by

$$D(t,T) = \frac{1}{1+r\Delta t}\{p\delta E[D(t+\Delta t,T)] + (1-p)E[D(t+\Delta t,T)]\}$$

By recursive substitution, current value of the bond as its terminal payoff if no default occurs :

$$D(t,T) = \left[\frac{1-p\Delta t(1-\delta)}{1+r\Delta t}\right]^n X(T)$$

The instantaneous default probability being $p\Delta t$ is consistent with Poisson distribution,

$$-\frac{dQ}{Q} = p\Delta t$$

$$D(t,T) = \frac{exp(-p(1-\delta)T)}{exp(rT)}X(T) = exp(-(r+s)T)X(T) \tag{2.1}$$

when r and s are not constant, 2.1 can be written as follows,

$$D(t,T) = E_t \left[exp \left(\int_t^T [r(u) + s(u)]du \right) \right] X(T)$$

where $s(u) = p_u(1-\delta)$

Intuitive interpretation : $p(1-\delta)$ serves as spread over the risk-free discount rate.

- p is small $\implies$ credit spread is small
- R(t) is high (i.e 1-δ is small) $\implies$ credit spread is small

2.3.1 Disadvantages

- If contract has no payoff at maturity such as CDS then it implies zero value today which is not true.

2.4 General observations on reduced form models

They are easy to calibrate as default probabilities and recovery are exogenously specified. They suffer from the constraint that default is always a surprise. This is rare, defaults is usually the end of a series of downgrades and spread widenings.The Jarrow-Turnbull and Duffie-Singleton models assume that defaults occur unexpectedly and follow the Poisson process. This assumption greatly reduces the complexity since the Poisson process has very nice mathematical properties. In order to further simplify the model, Jarrow Turnbull and Duffie-Singleton respectively make other assumptions so that there exist closed-form solutions to the basic underlying asset.

2.5 other models

2.5.1 Spread-based models

This model uses bond spread to detect default.

Assumption : Asset swap spreads follow a lognormal process of the following kind :

$$\frac{ds(t)}{s(t)} = \sigma dW(t)$$

$$s(t) = s(0)exp\left(-\frac{\sigma^2 t}{2} + \sigma W(t)\right)$$

2.5.2 Hazard models

2.6 Summary