CSE 400: Fundamentals of Probability in Computing
Lecture 6: Discrete RVs, Expectation, and Problem Solving

Group-6 Scribe Refactoring

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1 Introduction to Random Variables

A **Random Variable (RV)** X on a sample space Ω is defined as a function $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each sample point $\omega \in \Omega$[cite: 183].

1.1 Discrete Random Variables

A random variable is considered **discrete** if it can take on at most a countable number of possible values[cite: 339, 354].

- **Support**: Countable (finite or countably infinite)[cite: 251].
- **Probability Mass Function (PMF)**: Defined as $P_X(x_k) = Pr(X = x_k)$ for $k = 1, 2, 3, \ldots$ [cite: 356, 357].
- **Constraint**: The sum of all probabilities in a PMF must equal 1: $\sum_{k=1}^{\infty} P_X(x_k) = 1$[cite: 368].

2 Independent Events

Two events A and B are **independent** if the occurrence of one does not provide information about the likelihood of the other[cite: 572, 582].

- **Mathematical Definition**: $Pr(A|B) = Pr(A)$ and $Pr(B|A) = Pr(B)$[cite: 583].
- **Joint Probability**: For independent events, $Pr(A, B) = Pr(A)Pr(B)$[cite: 583, 594].
- **Mutual Independence**: Three events A, B, and C are mutually independent if every pair is independent and $Pr(A, B, C) = Pr(A)Pr(B)Pr(C)$[cite: 595, 598].

3 Derivations and Bayes' Theorem

3.1 Bayes' Formula Derivation

Using the definition of conditional probability $Pr(AB_i) = Pr(B_i|A)Pr(A)$, we derive Bayes' Theorem[cite: 421, 424]:

$$Pr(B_i|A) = \frac{Pr(A|B_i)Pr(B_i)}{\sum_{j=1}^{n} Pr(A|B_j)Pr(B_j)} \tag{1}$$

Where $Pr(B_i)$ is the *a priori* probability and $Pr(B_i|A)$ is the *posteriori* probability[cite: 426, 427].

4 Types of Discrete Random Variables

4.1 Bernoulli Random Variable

Models a single trial with two outcomes: Success (1) or Failure (0)[cite: 670, 679].

- **PMF**: $P_X(1) = p$, $P_X(0) = 1 - p$[cite: 691].
- **Examples**: Tossing a coin, spam detection, or demographic classification[cite: 707, 709].

4.2 Binomial Random Variable

Denoted as $B(n, p)$, it represents the number of successes in n independent trials[cite: 732, 734].

- **PMF Derivation**: $p(i) = \binom{n}{i} p^i (1-p)^{n-i}$ for $i = 0, 1, \ldots, n$[cite: 747].
- **Usage**: Number of defective items in a sample or correct answers on a test[cite: 761, 762].

4.3 Geometric Random Variable

Represents the number of trials required to achieve the **first** success in a sequence of independent trials[cite: 779, 780].

- **PMF**: $P_X(X = n) = (1-p)^{n-1} p$[cite: 794].
- **Proof of PMF Summation**:

$$\sum_{n=1}^{\infty} Pr(X = n) = p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \left(\frac{1}{1-(1-p)} \right) = \frac{p}{p} = 1\text{[cite: 797]}. \tag{2}$$

4.4 Poisson Random Variable

Models the number of rare events occurring in a fixed interval[cite: 260].

- **PMF**: $p(i) = e^{-\lambda} \frac{\lambda^i}{i!}$ for $i = 0, 1, 2, \ldots$ [cite: 837, 847].
- **Constraint Proof**: $\sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$[cite: 848, 859].
- **Approximation**: Can approximate $B(n, p)$ when n is large and p is small[cite: 860].

5 Summary for Exam Review

- **RVs**: Understand mapping from Sample Space to Real Numbers[cite: 183].
- **Discrete vs. Continuous**: Discrete uses PMF (summation); Continuous uses PDF (integration)[cite: 162, 163, 173, 340, 342].
- **Independence**: Key requirement for Binomial and Geometric distributions[cite: 723, 770].
- **Bayes' Theorem**: Crucial for updating probabilities based on observed evidence[cite: 427].
- **Key Distributions**:
  - **Bernoulli**: 1 trial, 2 outcomes[cite: 668, 670].

- **Binomial**: n trials, counting successes[cite: 732].
- **Geometric**: Waiting for the first success[cite: 779].
- **Poisson**: Rare events over time/space[cite: 260, 830].