School of Engineering and Applied Science

Lecture Scribe: L6

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1) List of Topics Covered

1. **Random Variables (RVs)**
   - Motivation and Concept
   - Discrete vs. Continuous RVs
   - Probability Mass Function (PMF)
2. **Probability Distributions**
   - Bernoulli Random Variable
   - Binomial Random Variable
   - Geometric Random Variable
   - Poisson Random Variable
3. **Bayes' Theorem**
   - Definition and Formula
   - A Priori vs. Posteriori Probabilities
4. **Independent Events**
   - Definition for two and three events
5. **Expectation and Moments**
   - Expectation of RVs ($\mu = E[X]$)
   - Expectation of a Function of RV
   - Variance, Skewness, and Kurtosis
6. **Distribution Functions**
   - Cumulative Density Function (CDF)
   - Probability Density Function (PDF)

2) Explanation of Topics Covered

- **Random Variables (RVs)**
  A random variable X is a function that assigns a real number to each sample point in a sample space Ω. Discrete random variables take values in a finite or countably infinite range. Their distribution can be visualized as a bar diagram where the x-axis represents the values and the height of the bar represents the probability $Pr[X = a]$.

  [Image of discrete probability mass function graph]

- **Discrete vs. Continuous Variables**
  Discrete variables have "countable support" and use a Probability Mass Function (PMF) where probabilities are assigned to single values. Continuous variables have "uncountable support" and use a Probability Density Function (PDF) where probabilities are assigned to intervals, and any single value has zero probability.

- **Probability Mass Function (PMF)**
  The PMF, denoted $P_X(x_k) = P(X = x_k)$, gives the probability of each outcome for a discrete RV. A fundamental property is that the sum of all probabilities in the PMF must equal 1: $\sum_{k=1}^{\infty} P_X(x_k) = 1$.

- **Independent Events**
  Two events A and B are independent if the occurrence of one does not change the likelihood of the other, mathematically $Pr(A|B) = Pr(A)$ and $Pr(B|A) = Pr(B)$. This implies their joint probability is the product of their individual probabilities: $Pr(A, B) = Pr(A)Pr(B)$.

3) List of Definitions and Theorems

Definition 1: Random Variable A random variable X on a sample space Ω is a function $X : \Omega \to \mathbb{R}$ that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.

Definition 2: Bernoulli Random Variable A random variable X is a Bernoulli RV if it represents an experiment with only two outcomes: Success (1) or Failure (0).

- **Notation:** $X \in \{0, 1\}$.
- **PMF:**
  - $P_X(1) = p$ (probability of success)
  - $P_X(0) = 1 - p$ (probability of failure)

Definition 3: Binomial Random Variable Denoted $B(n, p)$, it represents the number of successes X in n independent trials, where each trial has a success probability p.

- **PMF Statement:** $p(i) = \binom{n}{i} p^i (1 - p)^{n-i}$ for $i = 0, 1, \ldots, n$.

Theorem: Bayes' Formula Given events $B_1, B_2, \dots, B_n$ that partition the sample space, the probability of B_i given event A is:

$$Pr(B_i|A) = \frac{Pr(A|B_i)Pr(B_i)}{\sum_{j=1}^{n} Pr(A|B_j)Pr(B_j)}$$

- **Proof/Recap:** Using the definition of conditional probability, $Pr(A \cap B_i) = Pr(B_i|A)Pr(A)$. Substituting the law of total probability for $Pr(A)$ into the denominator yields the formula.

4) Important Examples

Example 1: Tossing 3 Fair Coins Suppose an experiment consists of tossing 3 fair coins. Let Y denote the number of heads.

- **Step 1:** Identify the possible values for Y: $\{0, 1, 2, 3\}$.
- **Step 2:** Calculate individual probabilities:
  - $P(Y = 0) = P\{(t, t, t)\} = 1/8$
  - $P(Y = 1) = P\{(t, t, h), (t, h, t), (h, t, t)\} = 3/8$
  - $P(Y = 2) = P\{(t, h, h), (h, t, h), (h, h, t)\} = 3/8$
  - $P(Y = 3) = P\{(h, h, h)\} = 1/8$
- **Step 3:** Verify the sum: $\sum P(Y = i) = 1/8 + 3/8 + 3/8 + 1/8 = 1$.

Example 2: PMF with Constant C The PMF of a random variable X is $p(i) = C\lambda^i/i!$ for $i = 0, 1, 2, \dots$ and $\lambda > 0$. Find C.

- **Step 1:** Use the property $\sum p(i) = 1$.
- **Step 2:** $\sum_{i=0}^{\infty} C\frac{\lambda^i}{i!} = C\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = Ce^{\lambda}$.
- **Step 3:** Set $Ce^{\lambda} = 1$, which gives $C = e^{-\lambda}$.

Example 3: Urn with Replacement (Geometric RV) An urn contains N white and M black balls. Balls are drawn with replacement until a black one is obtained.

- **Step 1:** Define X as the number of draws needed. X follows a Geometric distribution with success probability $p = M/(M + N)$.
- **Step 2:** The probability that exactly n draws are needed is:

$$P(X = n) = (1 - p)^{n-1}p$$

$$P(X = n) = \left(\frac{N}{M + N}\right)^{n-1}\left(\frac{M}{M + N}\right) = \frac{MN^{n-1}}{(M + N)^n}$$

5) List of Important Formulas

Distribution / Concept	**Formula**	**Parameters / Notes**
Sum of PMF	$\sum P_X(x_k) = 1$	Discrete RVs
Bernoulli PMF	$P(X=1) = p; P(X=0) = 1-p$	$p \in (0,1)$
Binomial PMF	$p(i) = \binom{n}{i} p^i (1-p)^{n-i}$	n trials, p success
Geometric PMF	$P(X=n) = (1-p)^{n-1} p$	First success on trial n
Poisson PMF	$p(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	$\lambda > 0, i = 0, 1, 2, \ldots$
Bayes' Theorem	$Pr(B_i \mid A) = \frac{Pr(A \mid B_i) Pr(B_i)}{Pr(A)}$	Posteriori probability
Independence	$Pr(A, B) = Pr(A) Pr(B)$	Joint probability