School of Engineering and Applied Science (SEAS), Ahmedabad University

CSE 400: Fundamentals of Probability in Computing

Tutorial-2: Solutions

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Overview

This lecture covers ten problems involving uniform distributions, normal distributions with Bayesian inference, minimization of expected absolute deviation, mixtures of exponential distributions, conditional distributions of Poisson variables, transformations of random variables, binomial approximation in stock modeling, exponential distribution properties, distribution of radial transformations, and joint distribution of order statistics.

Definitions and Notation

Uniform distribution on $(0, L)$:

$$f_X(x) = \frac{1}{L}, \quad 0 \leq x \leq L$$

Normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Exponential distribution (rate λ):

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

Poisson distribution:

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Binomial distribution:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Maximum of i.i.d. variables:

$$M_n = \max(X_1, X_2, \ldots, X_n)$$

Main Results

Q1. Uniform Point on a Line Segment

Let x be the distance from one end, $0 < x < L$.

$$R = \frac{\min(x, L-x)}{\max(x, L-x)}$$

Case 1: $x \leq \frac{L}{2}$

$$\frac{x}{L-x} < \frac{1}{4}$$
$$4x < L - x$$
$$5x < L$$
$$x < \frac{L}{5}$$

Case 2: $x > \frac{L}{2}$

$$\frac{L-x}{x} < \frac{1}{4}$$
$$4L - 4x < x$$
$$5x > 4L$$
$$x > \frac{4L}{5}$$

Valid region:

$$(0, L/5) \cup (4L/5, L)$$

Length:

$$\frac{2L}{5}$$

Probability:

$$\frac{2L/5}{L} = \frac{2}{5}$$

Q2. Normal Classification Error

Given:

$$X|W \sim N(4,4), \quad \sigma_1 = 2$$
$$X|B \sim N(6,9), \quad \sigma_2 = 3$$
$$P(B) = \alpha, \quad P(W) = 1 - \alpha$$

Require:

$$P(B|X=5) = \frac{1}{2}$$

Using Bayes' theorem:

$$\frac{\alpha f(5|B)}{\alpha f(5|B) + (1-\alpha) f(5|W)} = \frac{1}{2}$$

After simplification:

$$\alpha \approx 0.3827$$

Q3. Minimizing Expected Distance

(a) $X \sim$ **Uniform**$(0, A)$

$$E[|X-a|] = \int_0^a (a-x)\frac{1}{A}dx + \int_a^A (x-a)\frac{1}{A}dx$$

Derivative:

$$\frac{d}{da} = \frac{1}{2A}(4a - 2A) = 0$$

$$a = \frac{A}{2}$$

(b) $X \sim$ **Exponential**(λ)

$$\frac{d}{da}E[|X-a|] = 1 - 2e^{-\lambda a} = 0$$

$$e^{-\lambda a} = \frac{1}{2}$$

$$a = \frac{\ln 2}{\lambda}$$

Q4. Mixture of Exponentials

$$P(X > t) = p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}$$

$$P(X > t+s | X > t) = \frac{p_1 e^{-\lambda_1 (t+s)} + p_2 e^{-\lambda_2 (t+s)}}{p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}}$$

Q5. Conditional Poisson

$$P(X = k | X + Y = n) = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}$$

Q6. Transformation $Y = g(X)$, $X \sim U[-2, 2]$

PDF:

$$f_Y(y) = \frac{1}{4}, \quad -2 < y < -1$$

$$f_Y(y) = \frac{1}{4}, \quad 1 < y < 2$$

Point mass:

$$P(Y = 0) = \frac{1}{2}$$

Q7. Stock Model

$$S_{1000} = S_0 u^k d^{1000-k}$$

$$k \geq \frac{\ln(1.30) - 1000 \ln d}{\ln u - \ln d}$$

$$k \geq 470$$

$$k \sim \text{Binomial}(1000, 0.52)$$

$$P(k \geq 470) = \sum_{k=470}^{1000} \binom{1000}{k} (0.52)^k (0.48)^{1000-k}$$

Q8. Exponential ($\lambda = \frac{1}{2}$)

PDF:

$$f(x) = \frac{1}{2} e^{-x/2}$$

$$P(X > 2) = e^{-1}$$

$$P(X \geq 10 | X > 9) = e^{-1/2}$$

Q9. $Z = \sqrt{X^2 + Y^2}$

$$F_Z(z) = P(X^2 + Y^2 \leq z^2)$$

$$f_Z(z) = z \int_{-z}^{z} f_X(x) \frac{f_Y(\sqrt{z^2 - x^2}) + f_Y(-\sqrt{z^2 - x^2})}{\sqrt{z^2 - x^2}} \, dx$$

Q10. Joint Distribution of M_n and M_{n+1}

$$P(M_n \leq x) = F(x)^n$$

$$P(M_n \leq x, M_{n+1} \leq y) = \begin{cases} F(x)^n F(y), & x \leq y \\ F(y)^{n+1}, & x > y \end{cases}$$

Summary

- Uniform probability: $\frac{2}{5}$
- Normal Bayesian solution: $\alpha \approx 0.3827$
- Median minimizes expected absolute deviation
- Exponential minimizer: $\frac{\ln 2}{\lambda}$
- Poisson conditional $\rightarrow$ Binomial
- Transformation with point mass
- Binomial tail probability
- Exponential memoryless property
- Radial PDF via differentiation
- Joint CDF of maxima (piecewise)