School of Engineering and Applied Science

Lecture Scribe: Lecture 14 (Tutorial 2)

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1) List of Topics Covered

1. **Continuous Uniform Distribution**
   - Analysis of points chosen on line segments and roads.
2. **Bayes' Theorem and Normal Distribution**
   - Calculating posterior probabilities and minimizing error in region classification.
3. **Exponential Distribution**
   - Modeling repair times, fire station placement, and battery lifetimes.
4. **Law of Total Probability and Conditional Probability**
   - Determining probabilities for sequential events and multi-type components.
5. **Poisson Distribution and Binomial Structure**
   - Finding conditional probabilities of independent counts.
6. **Transformation of Random Variables**
   - Deriving Cumulative Distribution Functions (CDF) and Probability Density Functions (PDF) for piecewise and non-linear functions.
7. **Independent Trials and Binomial Distribution**
   - Approximating stock price movements over multiple periods.
8. **Joint Distributions and Order Statistics**
   - Analyzing the relationship between successive maximums of i.i.d. variables.

2) Explanation of Topics Covered

- **Uniform and Exponential Spatial Distributions**
  The tutorial explores how to model events occurring "at random" over a defined space. For a road of length A, a fire station is placed to minimize the expected distance $\mathbb{E}[|X - a|]$. In a uniform setting, the optimal location is the midpoint, whereas, for an exponentially distributed fire distance, the location is determined by the rate λ.

- **Bayesian Classification and Normal Density**
  Classification problems are addressed by comparing posterior probabilities. In the case of an image with black and white regions, we use the Normal density formula to determine the likelihood of a specific reading given the region. The goal is to find an image fraction α where the probability of error is identical for both region conclusions.

- **Reliability and Memoryless Properties**
  The tutorial examines lifetimes of components using the exponential distribution. It specifically looks at conditional survival: the probability a battery operates for an additional s hours given it has already lasted t hours.

- **Transformation and Joint CDFs**
  When a random variable Y is a function of X ($Y = g(X)$), the distribution of Y is found piecewise. This includes identifying "point masses" where a range of X values maps to a single Y value. For joint distributions, such as M_n and M_{n+1}, the tutorial demonstrates how to define the CDF based on whether the evaluation point x is less than or greater than y.

3) List of Definitions and Theorems

1) Bayes' Theorem (Posterior Probability)
Used to update the probability of a hypothesis (region type) given observed evidence (a reading $X = 5$):

$$P(B|X = 5) = \frac{\alpha f(5|B)}{\alpha f(5|B) + (1 - \alpha)f(5|W)}$$

where α is the prior probability $P(B)$.

2) Normal Density Formula
The probability density function for a normally distributed random variable with mean μ and variance σ^2:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

3) Law of Total Probability
Used to find the overall probability of an event by considering all mutually exclusive scenarios:

$$P(X > t) = p_1 P(X > t|T = 1) + p_2 P(X > t|T = 2)$$

4) Leibniz's Rule

Applied when differentiating an integral whose limits depend on the variable of differentiation (z):

$$\frac{d}{dz}\int_{-z}^{z} g(x,z)dx$$

This is crucial for finding the PDF of a transformed variable $Z = \sqrt{X^2 + Y^2}$.

4) Important Examples

Example 1: Finding Probability on a Line Segment

- **Context:** A point is chosen at random on a line segment of length L. Find the probability that the ratio of the shorter to the longer segment is less than $1/4$.
- **Process:**
  1. Let the point be at distance x from one end, where $0 < x < L$.
  2. The ratio $R = \frac{\min(x,L-x)}{\max(x,L-x)}$.
  3. **Case 1 ($x \leq L/2$):** $\frac{x}{L-x} < \frac{1}{4} \Rightarrow 4x < L - x \Rightarrow x < L/5$.
  4. **Case 2 ($x > L/2$):** $\frac{L-x}{x} < \frac{1}{4} \Rightarrow 4L - 4x < x \Rightarrow x > 4L/5$.
  5. **Valid region:** $(0, L/5) \cup (4L/5, L)$. Total length $= 2L/5$.
  6. **Probability:** $P = \frac{2L/5}{L} = \frac{2}{5}$.

Example 2: Conditional Poisson Distribution

- **Context:** Given $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ are independent, find $P(X = k|X + Y = n)$.
- **Process:**
  1. **Formula:** $P(X = k|X + Y = n) = \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)}$.
  2. **Substitute PMFs:**
  $$\frac{\left(\frac{e^{-\lambda_1}\lambda_1^k}{k!}\right)\left(\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\right)}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$
  3. **Simplify:** This gathers into a binomial structure:
  $$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

Example 3: Joint CDF of Successive Maximums

- **Context:** Find the joint distribution of $M_n = \max(X_1, \ldots, X_n)$ and $M_{n+1} = \max(M_n, X_{n+1})$.
- **Process:**
  1. **Case 1 ($x \leq y$):** Both $M_n \leq x$ and $X_{n+1} \leq y$ must occur.
  $$P(M_n \leq x, M_{n+1} \leq y) = F(x)^n F(y)$$
  2. **Case 2 ($x > y$):** If $M_{n+1} \leq y$, then M_n is naturally $\leq y$, which satisfies $M_n \leq x$.
  $$P(M_n \leq x, M_{n+1} \leq y) = P(M_{n+1} \leq y) = F(y)^{n+1}$$
  3. **Final Joint CDF:**
  $$P(M_n \leq x, M_{n+1} \leq y) = \begin{cases} F(x)^n F(y), & x \leq y \\ F(y)^{n+1}, & x > y \end{cases}$$

5) List of Important Formulas

Formula / Concept	**Result**
Uniform PDF	$f_X(x) = \frac{1}{A}$ for $0 \leq x \leq A$
Exponential PDF	$f_X(x) = \lambda e^{-\lambda x}$
Poisson PMF	$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$
Binomial PMF	$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$
Exponential Survival	$P(X > t) = e^{-\lambda t}$
Conditional Prob.	$P(A\|B) = \frac{P(A \cap B)}{P(B)}$
Transformed PDF	$f_Y(y) = f_X(x) \left\| \frac{dx}{dy} \right\|$
Optimal Station (U)	$a = \frac{A}{2}$ (Midpoint)
Optimal Station (Exp)	$a = \frac{\ln 2}{\lambda}$