ChatGPT 5.2

Tutorial_2_Solution.pdf
PDF

You are an expert lecture scribe generator

You have to generate a complete lecture scribe intended to serve as exam-oriented reference material.
This scribe must allow a student to fully reconstruct the entire lecture for a closed-notes,
reading-based exam using this document as the sole reference.

MANDATORY CONTEXT AND SCOPE CONSTRAINTS:

Use ONLY the provided lecture slides and permitted textbook material as context.
Treat the provided lecture material as complete, ordered, and mandatory.
Cover the lecture from the FIRST instructional slide to the FINAL instructional slide.
Do NOT stop early.
Do NOT assume logical stopping points.
Every section, subsection, definition, assumption, theorem, proposition, example,
derivation, proof, equation, table, figure, diagram, and stated result appearing
anywhere in the context MUST be included.
If new concepts appear later in the lecture, they MUST also be included.

MANDATORY CONTENT RULES:

Do NOT summarize, compress, or omit material.
Do NOT skip material even if sections appear conceptually independent.
ALL worked examples must be written fully with intermediate steps exactly as presented.
Proofs and derivations must be written step-by-step exactly as in the lecture.
If a slide contains only statements or definitions, reproduce them formally.
If a slide contains no mathematics, reproduce its content without interpretation.
Do NOT introduce new examples, intuition, motivation, commentary, or explanations.
Do NOT add alternative solution methods.
Do NOT add background knowledge.
Do NOT add transitional or conversational text.
Do NOT include anything not directly traceable to the provided context.

STRUCTURE REQUIREMENT:

Use EXACTLY the same structure as the example lecture scribe below.
Do NOT rename sections.
Do NOT reorder sections.
Do NOT add or remove sections.

Example Lecture Scribe (structure only, do not copy content):

Topic Title

Overview
Brief statement describing what the lecture covers.

Definitions and Notation
Formal definitions and symbols introduced in the lecture.

Main Results
Statements of theorems, propositions, or key results, section-wise.
Include figures, graphs, tables, or diagrams ONLY if they explicitly appear in the lecture.
If a figure is referenced in the lecture, include a formal textual description exactly as stated.

Derivations / Proofs
Step-by-step derivations or proofs exactly as presented.

Worked Examples
Example problems solved in class with all intermediate steps.

Summary
Concise recap strictly limited to formulas, results, and definitions already stated.

FINAL INSTRUCTION:

Follow the lecture order strictly from beginning to end.
The output must be complete rather than concise.
This document must be suitable as the sole exam-reference material.

Topic Title
CSE 400: Fundamentals of Probability in Computing
Tutorial-2: Solution

Overview
This lecture covers ten problems involving uniform distributions, normal distributions with Bayesian inference, minimization of expected absolute deviation, mixtures of exponential distributions, conditional distributions of Poisson variables, transformations of random variables, binomial approximation in stock modeling, exponential distribution properties, distribution of radial transformations, and joint distribution of order statistics.

Definitions and Notation

- Uniform distribution on (0, L):
$f_X(x) = 1/L,\ 0 \le x \le L$

- Normal distribution:
$f(x) = (1 / (\sigma\sqrt{2\pi}))\ e^{-(x-\mu)^2 / (2\sigma^2)}$

- Exponential distribution with rate λ:
$f(x) = \lambda e^{-\lambda x},\ x \ge 0$

- Poisson distribution with parameter λ:
$P(X = k) = e^{-\lambda} \lambda^k / k!$

• Binomial distribution:
P(X = k) = (n choose k) p^k (1 − p)^{n−k}

• Conditional probability:
P(A|B) = P(A ∩ B) / P(B)

• Maximum of i.i.d. variables:
Mn = max(X1, X2, ..., Xn)

Main Results

Q1. Uniform Point on a Line Segment

Let x be distance from one end, 0 < x < L.
R = min(x, L − x) / max(x, L − x)

Case 1: x ≤ L/2
x / (L − x) < 1/4
4x < L − x
5x < L
x < L/5

Case 2: x > L/2
(L − x) / x < 1/4
4L − 4x < x
5x > 4L
x > 4L/5

Valid region: (0, L/5) ∪ (4L/5, L)

Length = L/5 + L/5 = 2L/5

Probability = (2L/5)/L = 2/5

Q2. Normal Classification Error

Given:
X|W ~ N(4,4), σ1 = 2
X|B ~ N(6,9), σ2 = 3
P(B)=α, P(W)=1−α

Require:
P(B|X=5)=P(W|X=5)=1/2

Using Bayes:

α f(5|B) / [α f(5|B) + (1−α) f(5|W)] = 1/2

Compute densities:

f(5|B) = (1/(3√2π)) e^{−1/18}
f(5|W) = (1/(2√2π)) e^{−1/8}

Substitute:

α e^{−1/18}/(3√2π)
---------------------------------- = 1/2
α e^{−1/18}/(3√2π) + (1−α)e^{−1/8}/(2√2π)

Simplify:

3α / [3α + 2(1−α)e^{−5/72}] = 1/2

Solve:

$6\alpha = 3\alpha + 2(1-\alpha)(0.93)$
$3\alpha = 1.86 - 1.86\alpha$
$4.86\alpha = 1.86$
$\alpha \approx 0.3827$

Q3. Minimizing Expected Distance

(a) X ~ Uniform(0,A)

$E[|X-a|] = \int_0^a (a-x)(1/A)dx + \int_a^A (x-a)(1/A)dx$

Derivative:

$d/da\ [(2a^2 - 2aA + A^2)/(2A)] = (1/(2A))(4a - 2A)=0$

$4a = 2A$
$a = A/2$

(b) X ~ Exponential(λ)

$E[|X-a|] = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx$

Derivative:

d/da E[|X−a|] = 1 − 2e^{−λa} = 0

2e^{−λa}=1
e^{−λa}=1/2
−λa=ln(1/2)=−ln2
a = ln2/λ

Q4. Mixture of Exponentials

P(X>t)=p1 e^{−λ1 t} + p2 e^{−λ2 t}

P(X>t+s)=p1 e^{−λ1(t+s)} + p2 e^{−λ2(t+s)}

Conditional probability:

P(X>t+s|X>t) =
[p1 e^{−λ1(t+s)} + p2 e^{−λ2(t+s)}]

[p1 e^{−λ1 t} + p2 e^{−λ2 t}]

Q5. Conditional Poisson

P(X=k | X+Y=n) = P(X=k, Y=n−k)/P(X+Y=n)

Substitute:

(e^{−λ1} λ1^k/k!) (e^{−λ2} λ2^{n−k}/(n−k)!)

$e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^n / n!$

Simplify:

$\binom{n}{k} (\lambda_1/(\lambda_1+\lambda_2))^k (\lambda_2/(\lambda_1+\lambda_2))^{n-k}$

Q6. Transformation $Y=g(X)$, $X\sim\text{Uniform}[-2,2]$

$g(x)=$
x, $x\in[-2,-1]$
0, $x\in(-1,1)$
x, $x\in[1,2]$

CDF piecewise:

$y<-2$: 0

$-2\le y<-1$: $(y+2)/4$

$-1\le y<0$: $1/4$

At $y=0$: jump $1/2$
$F_Y(0)=3/4$

$0<y<1$: $3/4$

$1\le y\le 2$: $(y+2)/4$

$y>2$: 1

PDF:

$f_Y(y)=1/4$ for $-2<y<-1$
$f_Y(y)=1/4$ for $1<y<2$
0 otherwise

Point mass: $P(Y=0)=1/2$

Q7. Stock Model

$S_{1000}=S_0 u^k d^{1000-k}$

Require:

$u^k d^{1000-k} \ge 1.30$

Take logs:

$k \ln u + (1000-k) \ln d \ge \ln(1.30)$

$k(\ln u - \ln d) + 1000 \ln d \ge \ln(1.30)$

$k \ge [\ln(1.30) - 1000 \ln d] / (\ln u - \ln d)$

Substitute:

$\ln u \approx 0.01193$
$\ln d \approx -0.01005$
$\ln(1.30) \approx 0.26236$

$k \geq 469.4$
$k \geq 470$

$k \sim \text{Binomial}(1000, 0.52)$

$P(k \geq 470) = \Sigma$ from $k=470$ to 1000 $(1000 \text{ choose } k)(0.52)^k (0.48)^{1000-k}$

Q8. Exponential with $\lambda=1/2$

PDF: $(1/2)e^{-x/2}$

(a) $P(X>2)=e^{-1}\approx 0.3679$

(b)

$P(X\geq 10|X>9)=P(X\geq 10)/P(X>9)$

Numerator$=e^{-5}$
Denominator$=e^{-9/2}$

Result$=e^{-1/2}$

Q9. $Z = \sqrt{(X^2+Y^2)}$

CDF:

$FZ(z)=P(X^2+Y^2\leq z^2)$

$= \int\{-z\}^z \int\{-\sqrt{(z^2-x^2)}\}^{\{\sqrt{(z^2-x^2)}\}} fX(x)fY(y)\, dy\, dx$

Differentiate:

$d/dz \sqrt{(z^2-x^2)} = z/\sqrt{(z^2-x^2)}$

PDF:

$fZ(z)= z \int_{\{-z\}}^z fX(x)$
$[\ fY(\sqrt{(z^2-x^2)}) + fY(-\sqrt{(z^2-x^2)})\ ]$
---------------------------------- dx
$\sqrt{(z^2-x^2)}$

$z>0$

Q10. Joint Distribution of Mn and Mn+1

$Mn = \max(X1,\ldots,Xn)$
$Mn+1 = \max(Mn, Xn+1)$

CDF of Mn:

$P(Mn\leq x)=F(x)^n$

Joint CDF:

Case $x\leq y$:

$P(Mn\leq x, Mn+1\leq y)=F(x)^n F(y)$

Case $x>y$:

$P(M_n \le x, M_{n+1} \le y) = F(y)^{n+1}$

Final:

$P(M_n \le x, M_{n+1} \le y) =$

$F(x)^n F(y), x \le y$
$F(y)^{n+1}, x > y$

Summary

Uniform probability: 2/5
Normal Bayesian solution: $\alpha \approx 0.3827$
Median minimizes expected absolute deviation
Exponential minimizer: $\ln 2/\lambda$
Poisson conditional → Binomial
Transformation with point mass
Binomial tail probability
Exponential memoryless property
Radial PDF via differentiation
Joint CDF of maxima piecewise definition

Source: Tutorial_2_Solution

Sources

[redacted]