Lecture 6: Discrete RVs, Expectation and Problem Solving

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1 Outline

The lecture covers the following key topics:

- Previous Lecture Recap: Random Variables (RVs) and Independent Events.
- Definition and Examples of Discrete RVs.
- Expectation of RVs: $\mu = E[X] = \sum x_i p_x(x_i)$.
- Cumulative Density Function (CDF) and Probability Density Function (PDF).
- Moments: Variance, Skewness, and Kurtosis.
- **Types of Discrete RVs:** Bernoulli, Binomial, Geometric, and Poisson.

2 Random Variables (RV) Concept

2.1 Definition

A random variable X on a sample space Ω is a function $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each sample point $\omega \in \Omega$.

2.2 Discrete Random Variables

An RV is **discrete** if it takes values in a range that is finite or countably infinite.

- **Probability Mass Function (PMF):** For a discrete RV X, the function $P_X(x_k) = P(X = x_k)$ is the PMF.
- **Summation Property:** The sum of all probabilities in a PMF must equal 1: $\sum_{k=1}^{\infty} P_X(x_k) = 1$.

3 Examples and Problem Solving

3.1 Tossing 3 Fair Coins

Let Y be the number of heads appearing.

- $P(Y = 0) = P\{(t, t, t)\} = 1/8$.
- $P(Y = 1) = P\{(t, t, h), (t, h, t), (h, t, t)\} = 3/8$.
- $P(Y = 2) = P\{(t, h, h), (h, t, h), (h, h, t)\} = 3/8$.
- $P(Y = 3) = P\{(h, h, h)\} = 1/8$.

3.2 Bayes' Theorem Recap

The Posteriori probability is calculated as:

$$Pr(B_i|A) = \frac{Pr(A|B_i)Pr(B_i)}{\sum_{j=1}^{n} Pr(A|B_j)Pr(B_j)}$$

4 Types of Discrete Random Variables

Distribution	PMF Form	Context/Example
Bernoulli	$P(X = 1) = p, P(X = 0) = 1 - p$	Single trial (Success/Failure).
Binomial	$p(i) = \binom{n}{i} p^i (1-p)^{n-i}$	i successes in n independent trials.
Geometric	$P(X = n) = (1-p)^{n-1} p$	Number of trials until first success.
Poisson	$p(i) = e^{-\lambda} \frac{\lambda^i}{i!}$	Number of rare events in a unit.

4.1 Geometric RV Details

An experiment is performed until a success occurs. X denotes the number of trials required. **Example:** Selecting balls from an urn with N white and M black balls (with replacement) until a black ball is drawn. $p = M/(M + N)$.

4.2 Poisson RV Details

Used for "rare events". It can approximate a Binomial RV $B(n, p)$ when n is large and p is small such that np is moderate. **Examples:** Misprints on a page, customers entering a post office, or wrong phone numbers dialed.

5 Independence

Two events A and B are independent if $Pr(A|B) = Pr(A)$ and $Pr(B|A) = Pr(B)$. This implies $Pr(A, B) = Pr(A)Pr(B)$.