School of Engineering and Applied Science (SEAS), Ahmedabad University

CSE 400: Fundamentals of Probability in Computing

Lecture - 6 scribe

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Lecture 6: Discrete Random Variables, Expectation, and Problem Solving

1 Overview

This lecture revisits the concept of **random variables**, introduces and formalizes **independent events**, and then develops **types of discrete random variables**, including **Bernoulli**, **Binomial**, and **Geometric** random variables.

The lecture also reviews **probability mass functions (PMFs)**, includes multiple **worked examples**, and concludes with a **recap of Bayes' Theorem** and its applications to structured probability problems.

2 Definitions and Notation

2.1 Random Variable (RV)

A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

The distribution of a random variable can be visualized using a bar diagram:

- The x-axis represents the values the random variable can take.
- The height of the bar at value a is the probability $\Pr[X = a]$.
- Each probability is computed by evaluating the probability of the corresponding event in the sample space.

2.2 Discrete Random Variable

A random variable is said to be discrete if it can take on at most a countable number of possible values.

Properties:

- Countable support
- Probabilities assigned to single values
- Each possible value has strictly positive probability
- Characterized by a Probability Mass Function (PMF)

2.3 Continuous Random Variable (Contrast)

A continuous random variable has:

- Uncountable support
- Probabilities assigned to intervals of values
- Each exact value has probability zero
- Characterized by a Probability Density Function (PDF)

(This distinction is shown visually in the lecture slides using distribution diagrams.)

3 Main Results

3.1 Independent Events

3.1.1 Definition (Two Events)

Two events A and B are independent if:

$$\Pr(A \mid B) = \Pr(A) \quad \text{and} \quad \Pr(B \mid A) = \Pr(B)$$

Equivalently,

$$\Pr(A, B) = \Pr(A)\Pr(B)$$

3.1.2 Definition (Three Events – Mutual Independence)

Three events A, B, C are mutually independent if:

$$\begin{aligned}\Pr(A, B) &= \Pr(A)\Pr(B) \\ \Pr(A, C) &= \Pr(A)\Pr(C) \\ \Pr(B, C) &= \Pr(B)\Pr(C) \\ \Pr(A, B, C) &= \Pr(A)\Pr(B)\Pr(C)\end{aligned}$$

3.2 Probability Mass Function (PMF)

Let X be a discrete random variable with range finite or countably infinite

$$R_X = \{x_1, x_2, x_3, \dots\}$$

The Probability Mass Function (PMF) of X is defined as:

$$P_X(x_k) = \Pr(X = x_k), \quad k = 1, 2, 3, \dots$$

Since X must take one of its possible values:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

4 Derivations and Proofs

4.1 PMF Normalization Condition

Given a discrete random variable X:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

This follows directly from the fact that the events $\{X = x_k\}$ form a partition of the sample space.

4.2 PMF with Parameter λ

Given:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

Using normalization:

$$\sum_{i=0}^{\infty} p(i) = 1 \Rightarrow c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Since:

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

We obtain:

$$ce^{\lambda} = 1 \Rightarrow c = e^{-\lambda}$$

5 Worked Examples

5.1 Example 1: Independent Events – Auditorium

Let:

- Event A: Row 20 is selected
- Event B: Seat 15 is selected

Assume each row has an equal number of seats.

Question: Can event B give any new information about the likelihood of event A?

Answer: No. Hence, A and B are independent.

5.2 Example 2: Communication Network

A communication network has nodes A, B, C, D and links a_1, a_2, a_3, a_4. Each link is available with probability p, independently.

A message can be sent from A to D if there exists a path of available links.

The probability is computed by evaluating unions and intersections of independent path events using:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

5.3 Example 3: Modified Communication Network

An additional link from B to D is added. Each link is independently available with probability p.

The probability of being able to send a message from A to D is recomputed using independence assumptions and unions of multiple paths.

5.4 Example 4: True or False (Independence)

Suppose events A and B are independent.

(a) Is A independent of $\bar{B}$?

(b) Is $\bar{A}$ independent of B?

Each part requires either:

- A proof using the definition of independence, or
- A counter example.

6 Random Variable Examples

6.1 Tossing 3 Fair Coins

Let Y be the number of heads.

Possible values: $\{0, 1, 2, 3\}$

$$\Pr(Y = 0) = \frac{1}{8}$$
$$\Pr(Y = 1) = \frac{3}{8}$$
$$\Pr(Y = 2) = \frac{3}{8}$$
$$\Pr(Y = 3) = \frac{1}{8}$$

Check:

$$\sum_{i=0}^{3} \Pr(Y = i) = 1$$

7 Standard Discrete Random Variables

7.1 Bernoulli Random Variable

Experiment: Outcome is either Success or Failure.
Define:

$$X = \begin{cases} 1, & \text{if Success} \\ 0, & \text{if Failure} \end{cases}$$

PMF:

$$\Pr(X = 1) = p, \quad \Pr(X = 0) = 1 - p, \quad p \in (0, 1)$$

Applications:

- Single coin toss
- Randomly chosen person being Indian with probability p
- Email being spam with probability p

7.2 Binomial Random Variable

Experiment:

- n independent trials
- Each trial results in success with probability p

Define:

$$X = \text{number of successes in } n \text{ trials}$$

Notation:

$$X \sim B(n, p)$$

PMF:

$$p(i) = \binom{n}{i} p^i (1 - p)^{n-i}, \quad i = 0, 1, \ldots, n$$

Applications:

- Number of correct answers in a multiple-choice test
- Number of defective items in a sample of size n

7.3 Geometric Random Variable

Experiment:

- Independent trials
- Success probability p
- Trials continue until the first success

Define:

$$X = \text{number of trials required for success}$$

This random variable is known as a Geometric Random Variable.
(The PMF is introduced but derived in a later lecture.)

8 Bayes' Theorem (Recap)

Using:

$$\Pr(A, B) = \Pr(B \mid A) \Pr(A)$$

8.1 Bayes' Formula

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_j \Pr(A \mid B_j)\Pr(B_j)}$$

- $\Pr(B_i)$: Prior probability
- $\Pr(B_i \mid A)$: Posterior probability

8.2 Bayes' Example 1: Auditorium

An auditorium has 30 rows.

- Row 1 has 11 seats
- Row 30 has 40 seats

A row is selected uniformly. A seat is selected uniformly within the chosen row.
Compute:

$$\Pr(\text{Seat 15} \mid \text{Row 20})$$

$$\Pr(\text{Row 20} \mid \text{Seat 15})$$

Bayes' theorem is applied step-by-step using total probability.

8.3 Bayes' Example 2: Communication System

Binary data (0 or 1) is transmitted.

The receiver:

- Sometimes detects 0 as 1
- Sometimes detects 1 as 0

Given conditional probabilities of detection errors, Bayes' theorem is used to compute posterior probabilities of transmitted symbols.

9 Summary

- Random variables map outcomes to numerical values
- Discrete random variables are described using PMFs
- Independence requires product-form joint probabilities
- Bernoulli, Binomial, and Geometric RVs model common experiments
- PMFs must sum to 1
- Bayes' Theorem allows updating probabilities based on observed evidence

All examples illustrate structured probabilistic reasoning for computation-focused problems.