ALGEBRAIC GEOMETRY NOTES

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1. Affine and projective space

(1) Let k be an algebraically closed field.

(2) Let A_k^n denote *affine n-space*. Define $A_k^n = k^n$.

(3) Let $M = k^{n+1} - \{(0, \dots, 0)\}$. Define the equivalence relation $\sim$ to be $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if $\exists\ r \neq 0$ such that $a_i = rb_i \forall\ i \in \{0, \dots, n\}$. Then projective n-space is $M/\sim$ and is denoted by $\mathbb{P}_k^n$.

(4)
$$\mathbb{P}_k^n = \mathbb{A}_k^n \cup \mathbb{A}_k^{n-1} \cup \cdots \cup \mathbb{A}_k^1 \cup \mathbb{P}_k^0\ ,$$
where $\mathbb{P}_k^0 = \{\text{point}\}$.

(5) Let $P(X_1, \dots, X_n)$ be a polynomial with coefficients in k. Let $V(P)$ and $D(P)$ be subsets of $\mathbb{A}_k^n$ where
$$V(P) = \{\, (a_1, \dots, a_n) \in \mathbb{A}_k^n \colon\ P(a_1, \dots, a_n) = 0 \,\}$$
and
$$D(P) = \{\, (a_1, \dots, a_n) \in \mathbb{A}_k^n \colon\ P(a_1, \dots, a_n) \neq 0 \,\}\ .$$

(6) More generally, let $V(P_1, \dots, P_m) = \bigcap_{i=1}^m V(P_i)$. These are *affine subsets of* $\mathbb{A}_k^n$.

(7) If $m = 1$ then $V(P_1)$ is an *affine hypersurface.*

(8) If $m = 1$ and $\deg(P_1) = 1$ then $V(P_1)$ is an *affine hyperplane.*

(9) Let $Q(X_0, \dots, X_n)$ be a homogeneous polynomial with coefficients in k. Let $V_+(P)$ and $D_+(P)$ be subsets of $\mathbb{P}_k^n$ where
$$V_+(P) = \{\, (a_0 : \dots : a_n) \in \mathbb{P}_k^n \colon\ Q(a_0, \dots, a_n) = 0 \,\}$$
and
$$D_+(P) = \{\, (a_0 : \dots : a_n) \in \mathbb{P}_k^n \colon\ Q(a_0, \dots, a_n) \neq 0 \,\}\ .$$

(10) More generally, let $V_+(Q_1, \dots, Q_m) = \bigcap_{i=1}^m V_+(Q_i)$. These are *projective subsets of* $\mathbb{P}^n_k$.

(11) If $m = 1$ then $V_+(Q_1)$ is a *projective hypersurface.*

(12) If $m = 1$ and $\deg(Q_1) = 1$ then $V_+(Q_1)$ is a *projective hyperplane.*

(13) Projective and affine subsets together are *algebraic subsets.*

(14) Let V be a finite-dimensional k-vector space. $\mathbb{P}(V)$ is the set of all 1-dimensional k-subspaces U of V. This is a coordinate-free definition for projective space.

(15) Let V be an $(n+1)$-dimensional k-vector space. One can identify $\mathbb{P}(V)$ with $\mathbb{P}^n_k$:

$$(a_o : \dots : a_n) \longleftrightarrow \text{ subspace spanned by } a_0 v_0 + \dots + a_n v_n \ ,$$

where $\{v_0, \dots, v_n\}$ is a basis for V.

(16) Coordinate change in $\mathbb{A}^n_k$ can be encoded by an $n \times n$ matrix with entries in k.

(17) Coordinate change in $\mathbb{P}^n_k$ can be encoded by an $(n+1) \times (n+1)$ matrix with entries in k.

(18) The *projective hyperplane at infinity* is $X_0 = 0$ and is thus identified with $\mathbb{P}^{n-1}_k$. The complement of this can be identified with the affine space $\mathbb{A}^n_k$.

(19) *Affine properties* are properties that are invariant under *affine transformations* that is, under maps of the form $\mathbb{A}^n_k \to \mathbb{A}^n_k$. *Projective properties* are analogously defined.

(20) Affine properties include:
- incidence: that a point lies on a line or a line passes through on a point.
- collinearity.
- concurrency: that several lines pass through a common point.
- being an ellipse.
- a line in $\mathbb{A}^2_{\mathbb{R}}$ bisecting a given angle.
- tangency.

(21) Non-examples of affine properties include:
- being a circle.
- two lines in $\mathbb{A}^2_{\mathbb{R}}$ forming a right angle.

(22) Points at infinity are not preserved under a general projective transformation.

(23) **Proposition:** Consider $n+2$ points $\{P_1, \dots, P_{n+2}\} \subset \mathbb{P}^n_k$ no three of which are collinear, as well as another set of points $\{P'_1, \dots, P'_{n+2}\} \subset \mathbb{P}^n_k$ such that no three points of it are collinear. Then, $\exists$ a projective transformation G of $\mathbb{P}^n_k$ *onto* itself, mapping P_i to P'_i, $\forall i \in \{1, \dots, n+2\}$.

(24) **Corollary:** Given $n+2$ points $\{P_1, \dots, P_{n+2}\} \subset \mathbb{P}^n_k$ no three of which are collinear, one can always find a projective transformation mapping P_i to $(0 : \dots : 0 : 1 : 0 : \cdots 0)$ for $i \in \{1, \dots, n+1\}$ and P_{n+2} to $(1 : \dots : 1)$.

(25) **A geometry theorem that has no reasons for being true but still is:** aka. *Theorem of Desargues for projective space over any field.*
Let two triangles ABC and $A'B'C'$ be given in $\mathbb{P}^3_k$, such that $A \neq A'$, $B \neq B'$ and $C \neq C'$. If the lines AA', BB' and CC' pass through the same point O, that is, if O is the *center of perspective* and the two triangles are *perspective* from O, then:
- Lines AB and $A'B'$ intersect in a common point D.
- Lines BC and $B'C'$ intersect in a common point E.
- Lines CA and $C'A'$ intersect in a common point F.
- Points D, E and F are collinear. They pass through the *line of perspective.*