NOTES ON COMBINATORIAL VECTOR FIELDS

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1. Introduction

(1) Let Mbe a finite simplicial complex, with K the set of simplices. Ifσ and τ are simplices of M, then write $\sigma^{(p)}$ if $\dim \sigma = p$ and write $\sigma < \tau$ if σ lies in the boundary of τ.

(2) A *combinatorial vector field* on M is a map

$$V : K \to K \cup \{0\}$$

such that
 (a) if $V(\sigma) \neq 0$, then $\dim V(\sigma) + 1$ and $\sigma < V(\sigma)$.
 (b) if $V(\sigma) = \tau \neq 0$, then $V(\tau) = 0$.
 (c) for all $\sigma \in K$, $\#V(\sigma) \leq 1$.
in other words:
 (a) V maps simplices to higher simplices (by one dimension). Also, the smaller simplex is contained in the bigger simplex that V maps it to, that is, σ is always a face of $V(\sigma)$.
 (b) A sink (according to V) cannot be a source. This also means $V \circ V = 0$.
 (c) Each sink can have only one source coming into it (via V).
 (d) Simplices that don't map to higher simplices, must map to 0.

(3) Thus, for every simplex $\sigma^{(p)} \in K$, there are precisely 3 disjoint possibilities:
 (a) σ is a sink/head, that is $\sigma \in \text{Image}(V)$.
 (b) σ is a source/tail, that is $V(\sigma) \neq 0$.
 (c) σ is neither source nor sink, that is $\sigma \notin \text{Image}(V)$ and $V(\sigma) = 0$. In this case, σ is a *rest point of* V *of index* p.

(4) Combinatorial Poincaré-Hopf formula:

$$\chi(M) := \sum_{p=0}^{n} (-1)^p \{\# \text{ of } p\text{-cells}\} = \sum_{p=0}^{n} (-1)^p \{\# \text{ of rest points of index } p\}$$