Combinatorics

Course: Discrete Structures

(Course Code : DJ19CEC303)

Detailed Syllabus: (Unitwise)
Unit 3 : Combinatorics

- Combinatorics: Mathematical Induction, Basics of counting - Pigeon-hole principle, **permutations and combinations**, recurrence relations, solving recurrence relations, generating functions.

Probability: Basic probability, conditional probability, Bayes theorem

Permutations

Permutations

- One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order).

  An ordering of elements is called a **permutation** of the elements.

Definition of Permutation

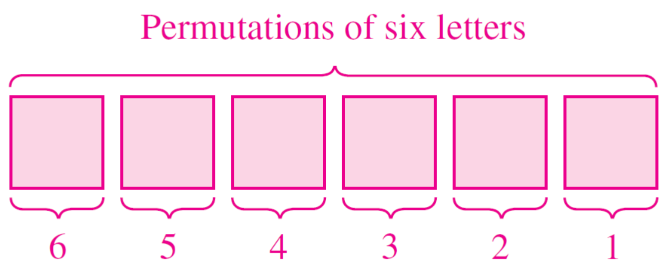
A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example – *Finding the Number of Permutations of n Elements*

- How many permutations of the following letters are possible?
- A B C D E F
- Solution:
- Consider the following reasoning.
- First position: Any of the *six* letters
- Second position: Any of the remaining *five* letters
- Third position: Any of the remaining *four* letters
- Fourth position: Any of the remaining *three* letters
- Fifth position: Either of the remaining *two* letters
- Sixth position: The *one* remaining letter

Example – *Solution*

- So, the numbers of choices for the six positions are as follows.



- The total number of permutations of the six letters is

- $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$

Permutations

Number of Permutations of *n* Elements

The number of permutations of n elements is given by

$$n \cdot (n-1) \cdots 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

Permutations of *n* Elements Taken *r* at a Time

The number of **permutations of n elements taken r at a time** is given by

$$_nP_r = \frac{n!}{(n-r)!}$$

$$= n(n-1)(n-2) \cdots (n-r+1).$$

Permutations

- Using this formula, find the number of permutations of eight horses taken three at a time is

$$
\begin{aligned}
{}_8P_3 &= \frac{8!}{(8-3)!} \\
&= \frac{8!}{5!} \\
&= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}}
\end{aligned}
$$

- $= 336$

Permutations

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdots + n_k.$$

The number of **distinguishable permutations** of the n objects is given by

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!}.$$

Example – *Distinguishable Permutations*

- In how many distinguishable ways can the letters in BANANA be written?

- Solution:
- This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways in which the letters can be written is

$$\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}$$

- = 60.

Example – *Solution*

- The 60 different distinguishable permutations are as follows.

- AAABNN AAANBN AAANNB AABANN
- AABNAN AABNNA AANABN AANANB
- AANBAN AANBNA AANNAB AANNBA
- ABAANN ABANAN ABANNA ABNAAN
- ABNANA ABNNAA ANAABN ANAANB
- ANABAN ANABNA ANANAB ANANBA
- ANBAAN ANBANA ANBNAA ANNAAB

Example 7 – *Solution*

• ANNABA	ANNBAA	BAAANN	BAANAN
• BAANNA	BANAAN	BANANA	BANNAA
• BNAAAN	BNAANA	BNANAA	BNNAAA
• NAAABN	NAAANB	NAABAN	NAABNA
• NAANAB	NAANBA	NABAAN	NABANA
• NABNAA	NANAAB	NANABA	NANBAA
• NBAAAN	NBAANA	NBANAA	NBNAAA
• NNAAAB	NNAABA	NNABAA	NNBAAA

Review Q

- Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

1) Do the words start with P
2) Do all the vowels always occur together
3) Do the vowels never occur together
4) Do the words begin with I and end in P?

Review Ans

- There are 12 letters of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore

- The required number of arrangements
- $= 12!/(3!\,4!\,2!)$
- $= 1663200$

Review Ans [(1) Do the words start with P]

- (1) Let us fix P at the extreme left , we then count the arrangements of the remaining 11 letters. Therefore ,
- The required number of words starting with P are
- = 11!/(3! 4! 2!) = 138600

Review Ans[(2) Do all the vowels always occur together]

- (2)There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object EEEEI for the time being.

- This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2Ds, can be rearranged in 8!/(3! 2!) ways.

- Corresponding to each of these arrangements , the 5 vowels E,E,E,E,I can be rearranged in 5!/4! Ways.

- Therefore, by multiplication principle the number of arrangements
- = 8!/(3! 2!) x 5!/4! = 16800

Review Ans[(3) Do the vowels never occur together]

- (3) The required number of arrangements = the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.

- = 1663200 – 16800 = 1646400

Review Ans[(4) Do the words begin with I and end in P?]

- Let us fix I and P at the extreme ends (I at the left end) and P at the right end . We are left with 10 letters.
- Hence, the required number of arrangements
- = 10!/(3! 2! 4!) = 12600

Combinations

Combinations

- When you count the number of possible permutations of a set of elements, order is important.

  Another method for selecting subsets of a larger set in which order *is not* important.

  Such subsets are called **combinations of *n* elements taken *r* at a time.** For instance, the combinations

- {A, B, C} and {B, A, C}

  are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important.

Combinations

- So, you would count only one of the two sets. A common example of a combination is a card game in which the player is free to reorder the cards after they have been dealt.

Combinations of n Elements Taken r at a Time

The number of **combinations of n elements taken r at a time** is given by

$$_nC_r = \frac{n!}{(n-r)!r!}.$$

Example – *Combinations of n Elements Taken r at a Time*

a. In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

b. A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt, the player may reorder them, so order is not important.)

Example – *Solution*

- **a.** You can find the number of different ways in which the letters can be chosen by using the formula for the number of combinations of five elements taken three at a time, as follows.

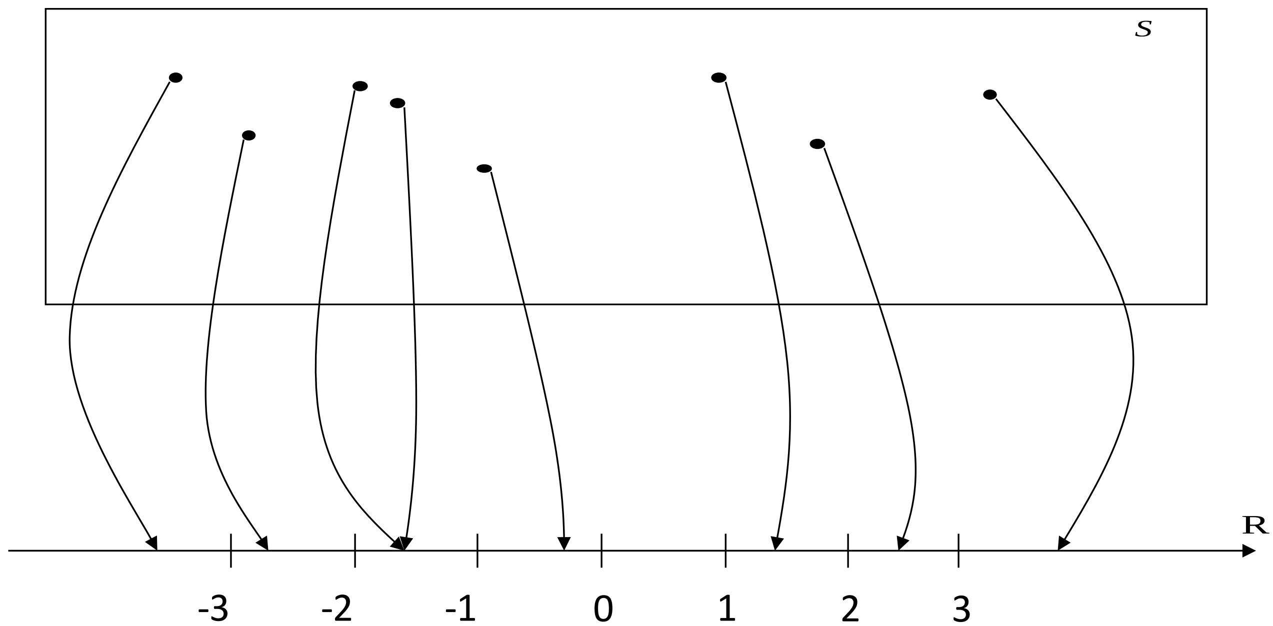
$$_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

- **b.** You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$$_{52}C_5 = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2{,}598{,}960$$

Definition of a Random Variable

- Random variable
  - A numerical value to each outcome of a particular experiment



Two Types of Random Variables

- A **discrete random variable** can assume a countable number of values.
  - Number of steps to the top of the Eiffel Tower*
- A **continuous random variable** can assume any value along a given interval of a number line.
  - The time a tourist stays at the top once s/he gets there

*Believe it or not, the answer ranges from 1,652 to 1,789. See Great Buildings

Two Types of Random Variables

- **Discrete random variables**
  - Number of sales
  - Number of calls
  - Shares of stock
  - People in line
  - Mistakes per page

- **Continuous random variables**
  - Length
  - Depth
  - Volume
  - Time
  - Weight

Probability Sample Questions

[Q1] There are 5 black 7 white balls. Assume we have drawn two balls randomly one by one without any replacement. What will be the probability that both balls are black?

[A.1] Detailed Solution:

Probability of first ball being black = $5/(5 + 7) = 5/12$.

Probability of drawing second ball black is = $4/(4 + 7) = 4/11$.

Now overall probability of both balls being black = $(5/12)\times(4/11) = 20/132$

Q2

[Q2]A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

[A2] Total number of balls = $(2 + 3 + 2) = 7$.

Let S be the sample space.

Then, $n(S)$ = Number of ways of drawing 2 balls out of $7 = 7C_2 = 21$

Let, E = event of drawing 2 balls, none of which is blue

$n(E)$= Number of ways of drawing 2 balls out of $(2+3)$ i.e, 5 balls = $5C_2 = 10$

Then, $P(E) = 10/21$

Q3

[Q3] What is the probability of getting a sum 9 from two throws of a dice?

[A3] In two throws of a dice, $n(S) = (6 \times 6) = 36$.

Let E = event of getting a sum $= \{(3, 6), (4, 5), (5, 4), (6, 3)\}$.

Then, $P(E) = n(E) / n(S) = 4/36 = 1/9$

Probability

Random Experiment

- When a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called random experiments.

Random Experiment

- An experiment is called random experiment if it satisfies the following two conditions:

i) It has more than one possible outcome

ii) It is not possible to predict the outcome in advance

Outcome

A possible result of a random experiment is called its outcome.

Consider the experiment of rolling a die. The outcome of this experiment are 1,2,3,4,5, or 6, if we are interested in the number of dots on the upper face of the dice.

Sample space

- The set of outcomes S = {1,2,3,4,5,6} is called the sample space of the experiment.
- The set of all possible outcomes of a random experiment is called the sample space associated with the experiment.
- Consider the experiment of tossing a coin two times. An associated sample space is S = { HH, HT, TH, TT }

Sample point

Each element of the sample space is called the sample point.

Each outcome of the random experiment is called the sample point.

Event

Any subset E of a sample space S is called an event.

Consider the experiment of rolling a die.

Let A be the event ‘getting a prime number’

Let B be the event ‘getting a odd number’

Probability

- Let S be a sample space and E be an event, such that n(S) = n and n(E) = m. If each outcome is equally likely, then it follows that

- P(E)

= m/n

= (Number of outcomes favourable to E)/(Total possible outcomes)

Probability

- If A and B are any two events, then

- P(A or B) = P(A) + P(B) – P(A and B)

equivalently, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability

- If A and B are mutually exclusive, then
- $P(A \text{ or } B) = P(A) + P(B)$
- If A is any event, then
- $P(\text{not } A) = 1 - P(A)$

Review Q

Q) A bag contains 9 discs of which 4 are read, 3 are blue, and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be

i) Red

ii) Yellow

iii) Blue

iv) Not blue

v) Either red or yellow

Review Ans

There are 9 discs in all so that total number of possible outcomes is 9.

The events A, B, C be defined as

A: ‘the disc drawn is red’

B: ‘the disc drawn is yellow’

C: ‘the disc drawn is blue’

i) The number of red discs = 4 i.e $n(A) = 4$

Hence $P(A) = 4/9$

Review Ans

ii) The number of yellow discs = 2 i.e n(B) = 2

Hence P(B) = 2/9

iii) The number of blue discs = 3 i.e n(C) = 3

Hence P(C) = 3/9 = 1/3

iv) The event ‘not blue’ is ‘not C’.

P(not C) = 1 – P(C) = 1 – 1/3 = 2/3

Review Ans

v) The event ‘either red or yellow’ may be described by the set ‘A’ or ‘C’

Since A and C are mutually exclusive events,

$P(A \text{ or } C) = P(A \cup C) = P(A) + P(C)$

$= 4/9 + 1/3$

$= 7/9$

Review Q

Q)A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man (b) one man (c) two men ?

Review Ans

- The total number of persons = 2 + 2 = 4. Out of these four persons, two can be selected in

$^4C_2$ ways

Combinations of *n* Elements Taken *r* at a Time

The number of **combinations of *n* elements taken *r* at a time** is given by

$$_nC_r = \frac{n!}{(n-r)!r!}.$$

Review Ans (a)

- No men in the committee of two means there will be two women in the committee, Out of the two women , two can be selected in ${}^2C_2$ = 1 way

P(no man) = ${}^2C_2/{}^4C_2 = 1/6$

Review Ans (b)

- One man in the committee means that there is one woman. One man out of 2 can be selected in ${}^2C_1$ ways and one woman out of 2 can be selected in ${}^2C_1$ ways . Together they can be selected in ${}^2C_1$ X ${}^2C_1$ ways.
- P(One man) = ${}^2C_1 \times {}^2C_1 / {}^4C_2$

  $= 2/3$

Review Ans (c)

- Two men can be selected in ${}^{2}C_{2} = 1$ way
- P(Two men) = ${}^{2}C_{2}/{}^{4}C_{2} = 1/6$

Review Q

Q) Find the probability that when a hand of 7 cards is drawn from a well shuffled deck of 52 cards, it contains (i) all/4 kings (ii) 3 kings (iii) atleast 3 kings.

Total number of possible hands = ${}^{52}C_7$

Review Ans (i)

Number of hands with 4 kings = ${}^{4}C_{4} \times {}^{48}C_{3}$ (other 3 cards must be chosen from the rest 48 cards)

P(a hand will have 4 kings) = ${}^{4}C_{4} \times {}^{48}C_{3} / {}^{52}C_{7}$

$= 1/7735$

Review Ans (ii)

Number (hand with 3 kings and 4 non-king cards)

$= {}^{4}C_{3} \text{ x } {}^{48}C_{4}$

P(3 kings) $= {}^{4}C_{3} \text{ x } {}^{48}C_{4} \, / \, {}^{52}C_{7}$

$= 9/1547$

Review Ans (iii)

P(atleast 3 king) = P (3 kings or 4 kings)

= P (3 kings) + P (4 kings)

= 9/1547 + 1/7735

= 46/7735

Review Q

Q) In a relay race, there are 5 teams A,B,C,D and E

a) What is the probability that A,B and C finish first , second and third respectively

b) What is the probability that A,B and C are first three to finish (in any order) (Assume that all finishing orders are equally likely)

Review Ans

If we consider the sample space consisting of all finishing orders in the first three places,

We will have ${}^5P_3 = 5!/(5 - 3)! = 60$

Review Ans (a)

A, B and C finish first, second and third respectively.

There is only one finishing order for this i.e ABC

P(A,B and C finish first, second and third respectively)

= 1/60

Review Ans (b)

A,B and C are the first three finishers. There will be 3! Arrangements for A,B and C.

P(A, B and C are first three to finish) = 3!/60

= 1/10

Conditional Probability

Conditional Probability

- When there are two events from the same sample space, the occurrence of one of the events affects the probability of the other event.

Example

Consider the experiment of tossing three fair coins. The sample space of the experiment is

S = {HHH,HHT,HTH,THH,HTT,THT,TTH,TTT}

Let E be the event ‘at least two heads appear’

Let F be the event ‘first coin shows tail’

E = {HHH,HHT,HTH,THH} ; $P(E) = ½$

F = {THH,THT,TTH,TTT} ; $P(F) = ½$

$E \cap F$ = {THH} ; $P(E \cap F) = 1/8$

Example

If event F has occurred , what is the probability of occurrence of E?

The occurrence of event F reduces the sample space from the set S to its subset F for the event E

F = {THH,THT,TTH,TTT}

Thus probability of E considering F as the sample space = ¼ OR

probability of E given F has occurred = ¼

P(E|F) = conditional probability of E given that F has already occurred = ¼

Example

- Note that the elements of F which favour the event E are the common elements of E and F,

  i.e. the sample points of $E \cap F$

$$P(E|F) = \frac{\text{Number of sample points in } E \cap F}{\text{Number of sample points in } F}$$

$$= n(E \cap F)/n(F) = P(E \cap F)/P(F)$$

{Dividing the numerator n denominator by n(S)}

Conditional Probability

- The conditional probability of an event E,

Given the occurrence of an event F (event F has already occurred) is given by

$P(E|F) = P(E \cap F)/ P(F)$, $P(F) \neq 0$

Q) If $P(A) = 7/13$, $P(B) = 9/13$ and $P(A \cap B)= 4/13$

Evaluate $P(A|B)$

A) 4/9

Example

Q) A family has two children. What is the probability that both the children are boys, given that at least one of them is a boy ?

Example – *Ans*

Let b stand for boy and g for girl. The sample space of the experiment is

S = {(b,b),(g,b),(b,g),(g,g)}

E = ‘both the children are boys’={(b,b)}

F = ‘Atleast one of the child is a boy’={(b,b),(g,b),(b,g)}

$E \cap F = \{(b,b)\}$

$P(F) = 3/4$

$P(E \cap F) = 1/4$

$P(E \mid F) = P(E \cap F) / P(F) = 1/3$

B = > 3

Review Q

A = even number on card

Q) Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number ?

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 4, 6, 8, 10\}$

$B = \{4, 5, 6, 7, 8, 9, 10\}$

Review Q

$F = \{(1,5), (5,1), (2,4), (4,2), (3,3)\}$

$P(F) = \frac{5}{36}$ $\quad P(E|F) = \frac{P(E \cap F)}{P(F)}$

Q) A dice is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once ?

$n(S) = 6 \times 6 = 36$

$E = \{(4,1), (1,4), (2,4), (4,2), (3,4), (4,3), (4,4), (5,4), (4,5), (6,4), (4,6)\}$

$P(E) = \frac{11}{36}$

$E \cap F = \{(2,4), (4,2)\}$

$P(E \cap F) = 2/36$

Multiplication rule of probability

$P(E|F) = P(E \cap F) / P(F)$, $P(F) \neq 0$

$P(E \cap F) = P(F) * P(E|F)$ (A)

$P(F|E) = P(F \cap E) / P(E)$, $P(E) \neq 0$

$P(F|E) = P(E \cap F) / P(E)$ { since $(E \cap F) = (F \cap E)$ }

Thus, $P(E \cap F) = P(E) * P(F|E)$ (B)

$P(E \cap F) = P(E) * P(F|E)$ {$P(F) \neq 0$; $P(E) \neq 0$}

$P(E \cap F) = P(F) * P(E|F)$

Example

E = 1st ball drawn is black = 10/15

F = 2nd ball drawn is black

Q)An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black ?

$$P(E \cap F) = P(E) \times P(F|E)$$

Multiplication Principle

$$= \frac{10}{15} \times \frac{9}{14}$$

$$= 3/7$$

Example

Q)There are 11 tickets in a box bearing numbers 1 to 11. Three tickets are drawn one after the other without replacement. Find the probability that they are drawn in the order bearing

i) even, odd, even number $\frac{4}{33}$

ii) Odd, odd, even number $\frac{5}{33}$

Example

$\frac{16}{22} = 8/11$

Q) In a certain college 4% of the boys and 1% of the girls are taller than 1.8 mts, furthermore 60% of the students are girls. If a student selected at random is taller than 1.8 mts, what is the probability that the student was a boy ?

	Taller than 1.8	less than 1.8	Total
Boys	1~~8~~6	384	400
Girls	6	594	600
Total	22	978	1000

Bayes Theorem

Bayes theorem

- If $E_1, E_2, E_3 \ldots.. , E_n$ are n non-empty events which constitute a partition of sample space S, i.e. $E_1, E_2, E_3 \ldots.. , E_n$ are pairwise disjoint and

  $E_1 \cup E_2 \ldots. \cup E_n = S$ and

  A is any event of nonzero probability, then

$$P(E_i|A) = P(A \cap E_i)/P(A)$$

$$P(A|E_i) = \frac{P(A \cap E_i)}{P(E_i)}$$

$$= P(E_i)P(A|E_i)/P(A)$$ *(by multiplication rule of probability)*

$$P(E_i|A) = P(E_i)P(A|E_i)/\sum_{j=1}^{n} P(E_j)P(A|E_j)$$ *(by theorem of total probability)*

ExampleProblem1

Q1) Bag I contains 3 red and 4 white balls while another Bag II contains 5 red and 6 white balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

Prior probability

A1) Let E_1 be the event of choosing from Bag I.

Let E_2 be the event of choosing from Bag II.

Let A be the event of drawing a red ball.

$P(E_1) = P(E_2) = ½$; $P(AIE_1) = 3/7$; $P(AIE_2) = 5/11$

$$P(E_2IA) = \frac{P(E_2)\,P(AIE_2)}{P(E_1)\,P(AIE_1) + P(E_2)\,P(AIE_2)} = \boxed{35/68}$$

Posterior probability

(red ball from bag I = 3/7)

ExampleProblem2

Q2)Given three identical boxes I, II, III, each containing two coins. In box I, both coins are gold coins. In box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

A2) $P(E_1) = P(E_2) = P(E_3) = 1/3$

$\frac{2}{2}$ $\quad 0 \quad$ $1/2$

Find $P(A|E_1)$,$P(A|E_2)$,$P(A|E_3)$

Using Bayes Theorem, find $P(E_1|A)$ $= \dfrac{1/3 \times 1}{1/3 \times 1 + 1/3 \times 0 + \frac{1}{3} \times 1/2} = \dfrac{2}{3}$

ExampleProblem3

Q3)In a factory which manufactures bolts, machines A,B & C manufacture respectively 25%,35% and 40% of the bolts. Of their outputs, 5,4 & 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B ?

A3)Find $P(B_1)$,$P(B_2)$,$P(B_3)$,$P(EIB_1)$, $P(EIB_2)$, $P(EIB_3)$

Using Bayes Theorem find $P(B_2IE)$

ExampleProblem4

Q4)A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

A4)Find $P(S_1)$,$P(S_2)$,$P(EIS_1)$,$P(EIS_2)$

Using Bayes Theorem, find $P(S_1IE)$