Generating Functions

Def^n: Let $\{a_n\} = \{a_0, a_1, a_2 \cdots\}$ be a sequence of real numbers. The function

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called the generating function of the sequence $\{a_n\}$.

eg:

power of x used as order of the term

(i) $g(x) = 1 + x + x^2 + x^3 + \cdots + x^n = \dfrac{1 - x^n}{1 - x}$

(ii) $g(x) = 1 + x + x^2 + x^3 + \cdots = (1 - x)^{-1}$

(iii) $g(x) = 1 + 3x + 6x^2 + \cdots + \dfrac{n(n+1)}{2} x^n + \cdots$

coefficient a_r of x^r is the value of the value at r.

Ordinary Generating Function

Let $\{a_n\}$ be a sequence of real numbers & x be a real variable. Then the infinite sum

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is called the <u>ordinary generating function</u>

eg. Find the ordinary generating function of the sequence $\{1, 1, 1, \ldots\}$.

Ordinary generating function is obtained by multiplying each member of the sequence successively by $x^0, x, x^2, x^3 \ldots$ & taking the sum.

$$A(x) = 1\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + 1\cdot x^3 + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots$$

$$= (1-x)^{-1}$$

Exponential Generating Function

Let $\{a_n\}$ be a sequence of real number & x be a real variable. Then the infinite sum

$$\sum_{n=0}^{\infty} \frac{a^n x^n}{n!} = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \cdots$$

is called exponential generating function of the sequence $\{a_n\}$.

eg. find the exponential generating functions of the sequence $\{1, 1, 1, \ldots\}$.

Exponential generating function is obtained by multiplying each member of the sequence successively by $\frac{x^0}{0!}, \frac{x^1}{x!}, \frac{x^2}{2!}, \frac{x^3}{3!} \ldots$ & taking the sum.

$$\therefore A(x) = 1 \cdot \frac{x^0}{0!} + 1 \cdot \frac{x^1}{1!} + 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Q.1) Find the ordinary generating functions of the following sequences.

(i) $1, -1, 1, -1, +1$

(ii) $1, 2, 3, 4$

(iii) $1, 3, 9, 27$

A.1) (i) $x^0 - x^1 + x^2 - x^3 + x^4 - \cdots$

$$= (1+x)^{-1}$$

(ii) $1 + 2x + 3x^2 + 4x^3 + \cdots = (1-x)^{-2}$

(iii) $1 + 3x + 9x^2 + 27x^3 + \cdots = (1-3x)^{-1}$

We have the Binomial Theorem;

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}.a^{n-3}b^3 + \cdots$$

If n is a positive integer, then there are finite number of terms on the r.h.s & infinite number of terms otherwise.

(i) Putting $a=1$, $b=x$ & $n=-1$, we get

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \infty$$

(ii) Putting $a=1$, $b=-x$, $n=-1$, we get

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \infty$$

(iii) Putting $a=1$, $b=-x$, $n=-2$, we get

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \infty$$

(iv) Replacing x by $3x$ in (1), we get

$$(1-3x)^{-1} = 1 + 3x + 9x^2 + 27x^3 + \cdots \infty$$

Q] Find the Ordinary generating function for the sequence.

i) $2, 2, 2, 2 \ldots$

ii) $0, 0, 0, 1, 1, 1, 1$

iii) $3, 3, 3, 3, \ldots$

iv) $6, -6, 6, -6, 6, -6 \ldots$

Q) Find the exponential generating functions of the following sequences.

i) $\{1, 1, 1, 1 \ldots\}$

ii) $\{1, 2, 3, 4 \ldots\}$.

iii) $\{1, 2a, 3a^2, 4a^3, \ldots\}$

iv) $\{0, 1, 0, -1, 0, 1, 0, -1\}$

A)

i) $1 + 1 \cdot x + 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!} + 1 \cdot \frac{x^4}{4!} + \cdots = e^x$

ii) $1 + 2 \cdot x + 3 \cdot \frac{x^2}{2!} + 4 \cdot \frac{x^3}{3!} + \cdots$

iii) $1 + 2a \cdot x + 3a^2 \cdot \frac{x^2}{2!} + 4a^3 \cdot \frac{x^3}{3!} + \cdots$

iv) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$