Recurrence Relations

Q.1) Solve the recurrence relation

$$a_n = a_{n-1} + 2, \quad n \geq 2, \quad a_1 = 3$$

A.1)
$$a_n = a_{n-1} + 2$$
$$a_{n-1} = a_{n-2} + 2$$
$$a_{n-2} = a_{n-3} + 2$$
$$\ldots$$
$$a_4 = a_3 + 2$$
$$a_3 = a_2 + 2$$
$$a_2 = a_1 + 2$$
$$a_1 = 3$$

Adding vertically, (since 2 are added $(n-1)$ times)

$$a_n = 3 + 2(n-1)$$
$$a_n = 3 + 2n - 2$$
$$a_n = 2n + 1$$

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

real constants $[c_k \neq 0]$

is called.

k^{th} order linear homogeneous recurrence relation with constant coefficients.

appear in first degree (linear relation)

same degree (homogeneous relation)

$a_n = 2a_{n-1} + 3$

$a_n = 2a_{n-1} + 3a_{n-2} + 4$

not homogeneous.

The basic approach for solving homogeneous recurrence relation $f(n) = 0$ is to look for the solutions of the form $\boxed{a_n = r^n}$

Let the given relation be

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$$

$a_n = r^n$ will be a solution of this relation if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots c_k r^{n-k}$$

Dividing by r^{n-k}, we get

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k r^0$$

$$\therefore \boxed{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0}$$

This is called the characteristic equation of the recurrence relation.

The roots of this equation is called characteristic roots.

Q] Solve the recurrence relation

$a_n - 7a_{n-1} + 10a_{n-2} = 0$ with

initial conditions $a_0 = 1$, $a_1 = 6$

A] The given recurrence relation is

$$a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \text{--- (1)}$$

This is a 2^{nd} order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1)

$$\therefore r^n - 7r^{n-1} + 10r^{n-2} = 0$$

$$\therefore r^{n-2}\underline{(r^2 - 7r + 10)} = 0$$

characteristic Equation of (1)

Dividing by r^{n-2}.

$$\cancel{r^{n-2}}(r^2 - 5r - 2r + 10) = 0$$

$$\therefore \cancel{r^{n-2}}(r-2)(r-5) = 0 \quad \therefore r = 2, 5.$$

The roots are real, rational & distinct.

Hence, let the general solution be $a_n = b_1 \cdot 2^n + b_2 \cdot 5^n$

Now, use initial conditions to find b_1 & b_2.

Putting $n=0$, $a_0 = b_1 + b_2 = 1$

$n=1$, $a_1 = 2b_1 + 5b_2 = 6$

Solving these equations, we get $\boxed{\begin{array}{l} b_1 = -1/3 \\ b_2 = \frac{4}{3} \end{array}}$

Hence, the explicit solution of the recurrence relation is

$$a_n = \left(-\frac{1}{3}\right)2^n + \left(\frac{4}{3}\right)5^n$$

Q] Find the solution of the recurrence relation $a_n = \underset{c_1}{6}\, a_{n-1} - \underset{c_2}{11}\, a_{n-2} + \underset{c_3}{6}\, a_{n-3}$ —(1)
with initial conditions $a_0 = 2,\ a_1 = 5,\ a_2 = 15$

A] 3rd order linear homogeneous recurrence relation with constant coefficients.

Solution of (1) be $a_n = r^n$

$$r^n = 6r^{n-1} - 11r^{n-2} + 6r^{n-3}$$

$$r^n - 6r^{n-1} + 11r^{n-2} - 6r^{n-3} = 0$$

$$r^{n-3}\underbrace{(r^3 - 6r^2 + 11r - 6)} = 0$$

∴ The characteristic equation of (1) is

$$(r-1)(r-2)(r-3) = 0 \quad \therefore r = 1, 2, 3$$

The roots are real, rational & distinct

Hence, let the general solution be

$$a_n = b_1 \cdot 1^n + b_2 \cdot 2^n + b_3 \cdot 3^n$$

We now use the initial conditions to find b_1, b_2, b_3

Putting

$n=0,\ a_0=2 \qquad b_1 + b_2 + b_3 = 2$

$n=1,\ a_1=5 \qquad b_1 + 2b_2 + 3b_3 = 5$

$n=2,\ a_2=15 \qquad b_1 + 4b_2 + 9b_3 = 15$

Solving the three equations $b_1 = 1,\ b_2 = -1,\ b_3 = 2$

Hence, explicit solution of the given recurrence relation is $\boxed{a_n = 1 - 2^n + 2\times 3^n}$

Non-homogeneous Recurrence Relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$$

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$$

particular solution

Particular solution depends on the characteristic roots (α) and the nature of $f(n)$

$f(n)$	Particular Solution
If $f(n)$ is constant, $\alpha \neq 1$	A, a constant
If $f(n)$ is constant, $\alpha = 1$ of multiplicity m	$A \cdot n^m$
If $f(n)$ is of form $an+b$, where a, b are constants	$An + B$ where A, B are constants
If $f(n)$ is of form an^2 where a is a constant	$An^2 + Bn + C$
If $f(n)$ is of form a^n	$A a^n$
If $f(n)$ is of the form $a_n e^n$	$(An + B) e^n$

Q] Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

A]
As before the solution of the corresponding homogeneous equation is

$$a_n^{(h)} = A \cdot 2^n + B \cdot 3^n$$

Since $\boxed{f(n) = 7^n}$, we assume the particular solution to be $\boxed{a_n = C \cdot 7^n.}$

Putting this in the given equation,

$$C \cdot 7^n - 5 \cdot C \cdot 7^{n-1} + 6 \cdot C \cdot 7^{n-2} = 7^n$$

$$C \cdot 49 \cdot 7^{n-2} - 5 \cdot C \cdot 7 \cdot 7^{n-2} + 6 \cdot C \cdot 7^{n-2} = 49 \times 7^{n-2}$$

$$\therefore \left(\frac{49 - 35 + 6}{49}\right) C = 1 \quad \therefore \boxed{C = \frac{49}{20}}$$

Hence, the desired solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A \cdot 2^n + B \cdot 3^n + \frac{49}{20} \cdot 7^n$$

Q) Use generating functions to solve the recurrence relation (8)

$$a_n = 3a_{n-1} + 2, \quad a_0 = 1$$

A) Consider $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$\therefore \; 3x\,g(x) = 3a_0 x + 3a_1 x^2 + \cdots + 3a_{n-1} x^n + \cdots$

& $\dfrac{2}{1-x} = 2 + 2x + 2x^2 + \cdots + 2x^n + \cdots$

Subtracting the last two series from the first & noting that $a_0 = 1$, we get

$$g(x) - 3x\,g(x) - \frac{2}{1-x} = (1-2) + (a_1 - 3a_0 - 2)x + (a_2 - 3a_1 - 2)x^2 + \cdots$$

Since $a_0 = 1$ & $a_n - 3a_{n-1} - 2 = 0$, each bracket on the right except the first is zero. Hence,

$$(1-3x)\,g(x) = -1 + \frac{2}{1-x} = \frac{-1+x+2}{1-x} = \frac{1+x}{1-x}$$

$$\therefore \; g(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{2}{1-3x} - \frac{1}{1-x} \quad \text{[By partial fractions]}$$

$$= 2\left[1 + (3x) + (3x)^2 + \cdots\right] - \left[1 + x + x^2 + \cdots\right]$$

$$= 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\left[2 \times \{3^n - 1\}\right] x^n$$

But $g(x) = \sum a_n x^n \quad \therefore \; a_n = 2 \cdot 3^n - 1$

In order to make use of $a_n - 3a_{n-1} - 2 = 0$ we multiply $g(x)$ by 1, $g(x)$ by $3x$ & 2 by $\frac{1}{(1-x)}$ & subtract as above.