Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

- Set Theory:
- The theory of sets was originated in 1873 by German mathematician G. Cantor who considered a set as a collection or aggregate of definite and distinguishable objects selected by means of some rules or description.
- Set theory is one of the greatest achievements of modern mathematics. Basically all mathematical concepts, methods, and results admit of representation within axiomatic set theory.

Definition: A set is any well-defined collection of objects, called the elements or members of the set. These elements may be anything: Numbers, Points in geometry, Letters, Alphabets, etc. The following are examples of a set:

1. Rivers in India.
2. First-year B. Tech Students in CHARUSAT.
3. Consonants in English alphabets.

- Capital letters A, B, C, ... are ordinarily used to denote sets and lower case letters a, b, c, ... are used to denote elements of sets. Well defined means to decide if a given element belongs to the collection or not. The statement "x is an element of A" or equivalently "x belongs to A" is written as $x \in A$. The statement "x is not an element of A" is written as $x \notin A$.

- **Number Sets:**
  1. Set of all natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \ ...\}$.
  2. Set of all integers $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
  3. Set $\mathbb{Q}$ of all rational numbers of the form p/q, where p and q ($q \neq 0$) are integers.
  4. The set of all real numbers ($\mathbb{R}$).
  5. The set $\mathbb{C}$ of all complex numbers of the form $z = a + ib$, where a and b are real numbers. i.e. $\mathbb{C} = \{z : z = a + ib, a, b \in \mathbb{R}\}$.
- **Representation of a Set:**
  1. Roster or Tabular form.
  2. Rule Method or Set builder form.
- **Roster or Tabular form:** In this form all the elements of the set are listed, the elements being separated by commas and are enclosed within braces.

  eg: A set of binary digits i.e $A = \{0, 1\}$.

- **Rule Method or Set builder form:** In this method a set is defined by specifying a property that elements of the set have in common. The set is then describe as follows: $A = \{x : p(x)\}$.

  eg: The set $A = \{1, 4, 9, 16\}$ can be written as $A = \{x \in \mathbb{N} : x \text{ is square of } n \in \mathbb{N}, n \leq 4\}$.

- **Definition and Examples:**

1. A set with finite number of elements, is called a **finite set.**

   Examples of some finite sets: a. The set of students in a class.

   b. The set of days in a week.

2. A set which contains infinite number of elements, is called an **infinite set.**

   Examples of some infinite sets: a. The set of all integers.

   b. The set of all complex numbers.

3. If A and B are sets such that every element of A is also an element of B, then A is said to be **a subset** of B. (or A is contained in B) and is denoted by $A \subseteq B$.

   eg: {Gandhinagar, Jaipur, Bhopal} is a subset of {x : x is the capital of states of India}.

4.Two sets A and B are said to be **equal** if $A \subset B$ and $B \subset A$.

eg: A = {x ∈ ℕ : x is square of n ∈ ℕ, n≤4} and B={1,4,9,16}, the sets A and B are equal.

5. A set which contains no elements at all is called the **null set** (empty set or void set).

eg: A={x: x is a multiple of 4, x is an odd integer}.

6. A set which has only one element is called a **singleton set.**

eg: S = {a} is a singleton set.

7. Let X be any set. Then set P(X) is the set of all possible subsets of X is called the **power set** of X.

eg: Power set of A = {1, 2, 3} is P(A) = {Φ, A, {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3}}.

8. A non-empty set U of which all the sets under consideration are all subsets of U is called the **Universal set.**

eg: The set of letters in alphabets is the universal from which the letters of any word may be chosen to form the set.

Operations on sets:

1. **Union:** The union of two sets A and B, denoted by $A \cup B$ is the set of all elements which belong to A or to B; that is $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
2. **Intersection:** The intersection of two sets A and B, denoted by $A \cap B$ is the set of all elements which belong to both A and B; that is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
3. **Complements:** Let U be the universal set and A be any subset of U. The complement of A is denoted by A', the set of all elements which belong to U but which do not belong to A; that is $A' = \{x : x \in U \text{ and } x \notin A\}$. If A and B are two sets, the relative complement of B with respect to A or simply, the difference of A and B, denoted by $A-B$, is the set of elements which belong to A but which do not belong to B; that is $A-B = \{x : x \in A \text{ and } x \notin B\}$.
4. **Symmetric Difference**: The symmetric difference of two sets A and B is denoted by $A \Delta B$ and is the set of elements that belong to A or to B, but not to both A and B. It is easy to see that $A \Delta B = (A-B) \cup (B-A) = \{x : x \text{ belongs to exactly one of A and B}\}$.
5. **Cartesian product**: Let A and B be sets. Cartesian product of A and B denoted by $A \times B$ is defined as $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. **eg:** If $A = \{1,2,5\}$ and $B = \{2,4\}$ then $A \times B = \{(1,2),(1,4),(2,2),(2,4),(5,2),(5,4)\}$.

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Operations on sets:

1. **Union:** The union of two sets A and B, denoted by $A \cup B$ is the set of all elements which belong to A or to B; that is $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
2. **Intersection:** The intersection of two sets A and B, denoted by $A \cap B$ is the set of all elements which belong to both A and B; that is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
3. **Complements:** Let U be the universal set and A be any subset of U. The complement of A is denoted by A', the set of all elements which belong to U but which do not belong to A; that is $A' = \{x : x \in U \text{ and } x \notin A\}$. If A and B are two sets, the relative complement of B with respect to A or simply, the difference of A and B, denoted by $A-B$, is the set of elements which belong to A but which do not belong to B; that is $A-B = \{x : x \in A \text{ and } x \notin B\}$.
4. **Symmetric Difference**: The symmetric difference of two sets A and B is denoted by $A\Delta B$ and is the set of elements that belong to A or to B, but not to both A and B. It is easy to see that $A\Delta B = (A-B)\cup(B-A) = \{x : x \text{ belongs to exactly one of A and B}\}$.
5. **Cartesian product**: Let A and B be sets. Cartesian product of A and B denoted by $A\times B$ is defined as $A\times B = \{(a, b) : a \in A \text{ and } b \in B\}$. **eg:** If $A = \{1,2,5\}$ and $B = \{2,4\}$ then $A\times B = \{(1,2),(1,4),(2,2),(2,4),(5,2),(5,4)\}$.

Properties:

1. **Commutative Law:**

   (i) $A \cup B = B \cup A$

   (ii) $A \cap B = B \cap A$

2. **Associative Laws:**

   $(A \cap B) \cap C = A \cap (B \cap C)$

   $(A \cup B) \cup C = A \cup (B \cup C)$

3. **Distributive Laws:**

   1. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
   2. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

4. **De Morgan's Laws:**

   (i) $(A \cap B)' = A' \cup B'$

   (ii) $(A \cup B)' = A' \cap B'$

- **Relation**: Let A and B be two sets. A relation from A to B gives rise to a subset R of A×B. If (a, b) ∈ R we denoted by is 'aRb'.
- **Domain of relation:** Let R is a relation from A to B. Then set {a ∈ A : (a,b) ∈ R} is called the domain of relation R. It is denoted by dom(R).
- **Range of relation:** Let R is a relation from A to B. Then set {b ∈ B : (a,b) ∈ R} is called the range of relation R. It is denoted by ran(R).
- **Function:** Let A and B be two non-empty sets. A function f from A to B is a subset of f ⊆ A×B with property that for each element x in A, there is a unique element y in B such that (x,y) ∈ f. The statement "f is a function from A to B" is usually represented symbolically by $f : A \to B$.
- If f is a function from A to B. Then A is called the **domain** of f and it is denoted by dom f, its members are the first co-ordinates of the ordered pairs belonging to f. The set B is called **co-domain** of f. If (x, y) ∈ f, it is customary to write y = f(x), y is called the **image** of x; and x is **pre-image** of y. The set consisting of all the images of the elements of A under the function f is called the **range** of f. It is denoted by f(A) or R(f), the range of $f = \{f(x) : x \in A\}$.

eg: 1. f(x) = x+3 for x ∈ ℝ represents a function from ℝ to ℝ .

2. Let A={1, 2, 3, 4,5} and B= {1,2,...12} f={(1,1),(2,3),(3,5),(3,7),(4,7),(5,12)},then f is not a function because in f assigns 3 to two different values in B.

3. Let A={1, 2, 3, 4,5} and B= {1,2,...12} g={(1,1),(2,3),(3,5),(3,7),(4,7)} then g is not a function because $5 \in A$ but g does not assign any value to 5.

Classification of Functions:

Function can be classified mainly into two groups.

1. Algebraic function: A function which consist of a finite number of terms of involving powers and roots of the independent variable x and the four fundamental operations addition, subtraction, multiplication and division is called algebraic function. **eg.** $f:[0,\infty) \to [0,\infty), f(x) = \frac{\sqrt{x^2+1}}{x^4+1}, x \in [0,\infty)$.

Three particular cases of algebraic functions are:

- **Polynomial function:** A function of the form $\mathrm{a_0 + a_1x^1 + a_2x^2} + \cdots + \mathrm{a_n}\, x^n$ where n is a positive integer and $\mathrm{a_0, a_1, a_2, \dots a_n}$are real constants and $\mathrm{a_n} \neq 0$is called a polynomial in x of degree n. **eg.**$f(x) = x^2 + 2x + 3$ is a polynomial of degree 2.

- **Rational function:** A function of the form $\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomials in x,$g(x) \neq 0$, is called a rational function,

  eg.$f(x) = \frac{x^2+3x+5}{x+4}$.

- **Irrational function:** A function involving radicals is called irrational functions.

  eg. $f(x) = \sqrt[3]{x} + x + 5$

2. **Transcendental function:** A function which is not algebraic is called Transcendental function.

- **Trigonometric function:** The six functions namely sin x, cos x, tan x, cot x, cosec x and sec x, where x is angle measured in radians are called trigonometric functions.
- **Inverse trigonometric function:** The six functions $\sin^{-1} x$, $\cos^{-1} x$,$\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\mathrm{cosec}^{-1} x$, are called inverse trigonometric function.
- **Power function:** A function f(x) = a^x ,$(a > 0)$ (satisfying the law $a^1 = a$ and$a^x \cdot a^y = a^{x+y}$is called the power function)
- **Logarithmic function:** The inverse of the exponential function is called the logarithmic function. So, if $y = a^x(a > 0, x \in R)$, then $x = \log_a(\boldsymbol{y})$ is called Logarithmic function.

Type of Function:

- **One-to-One Function:** A function f from A to B is one-to-one (or injective) , if for all elements x_1, x_2 in A such that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

  i.e if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

  eg: Let A = {1,2,3}, B = {a, b, c, d} and let f(1) = a, f(2) = c and f(3) = d. Then f is injective since the different elements 1,2,3 in A are assigned to the different elements a, c, d respectively in B.

- **Many-One Function:** A function f from A to B is said to be many-one if two or more elements of A have same image in B.

  eg: Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, x is any real number. Then f is many-one function.

- **Into Function:** A function f from A to B is said to be into if there exists at least one element in B which is not an image of any element in A.

  eg: Let A = {1,2,3},B = {a,b,c,d} and let f(1) = a, f(2) = c and f(3) = d. Then f is into since b is not an image of any element in A.

- **Onto Function:** A function f from A to B is said to be onto (or surjective) if every element of B is an image of some element in A.

  eg: Let f(x) = x+5, x is any real number and f : $\mathbb{R} \rightarrow \mathbb{R}$. Then f is an onto function.

- **Bijective Function:** A function f from A to B is said to be a bijective if f is both one-to-one and onto.

  eg. Let f be a function from A to B, where A={1, 2, 3, 4} and B={a, b, c, d}with f(1)=d, (2)=b, f(3)=c and f(4)=a, then f is a bijective function.

Limit and Continuity:

Limit: Consider a function f : $\mathbb{R} \rightarrow \mathbb{R}$. If f(x) becomes arbitrarily close to a unique number *l* as x approaches a from either side, i.e for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$. This is written as $\lim_{x \to a} f(x) = l$.

Left hand limit of *f* at a: The function f(x) approaches to *l* as x approaches a from left side i.e for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $a - \delta < x < a$. This is written as $\lim_{x \to a^-} f(x) = l$.

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin’s and Taylor’s series expansion.
- L’Hospital’s rule and related applications, Indeterminate forms

- **Onto Function:** A function f from A to B is said to be onto (or surjective) if every element of B is an image of some element in A.

  eg: Let f(x) = x+5, x is any real number and f : $\mathbb{R} \to \mathbb{R}$. Then f is an onto function.

- **Bijective Function:** A function f from A to B is said to be a bijective if f is both one-to-one and onto.

  eg. Let f be a function from A to B, where A={1, 2, 3, 4} and B={a, b, c, d}with f(1)=d, (2)=b, f(3)=c and f(4)=a, then f is a bijective function.

Limit and Continuity:

Limit: Consider a function f : $\mathbb{R} \to \mathbb{R}$. If f(x) becomes arbitrarily close to a unique number l as x approaches a from either side, i.e for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$. This is written as $\lim_{x \to a} f(x) = l$.

Left hand limit of *f* at a: The function f(x) approaches to l as x approaches a from left side i.e for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $a - \delta < x < a$. This is written as $\lim_{x \to a^-} f(x) = l$.

Right hand limit of *f* at a: The function f approaches l as x approaches a from left side **if** given $\varepsilon > 0$ there exists δ > 0 such that $|f(x) - l| < \varepsilon$ whenever $a < x < a + \delta$. This is written as $\lim_{x \to a^+} f(x) = l$.

Working rules of Limit: Let a and b be real numbers, f and g be functions $f, g\colon \mathbb{R} \to \mathbb{R}$ with the following limits: $\lim_{x \to a} f(x) = l_1$ and $\lim_{x \to a} g(x) = l_2$

Scalar multiple: $\lim_{x \to a} [\, b\, f(x)] = b l_1$

Sum or difference: $\lim_{x \to a} [\, f(x) \pm g(x)] = l_1 \pm l_2$.

Product: $\lim_{x \to a} [\, f(x) \cdot\ g(x)] = l_1 \cdot l_2$.

Quotient: $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{l_1}{l_2}$, where $l_2 \neq 0$.

Power: $\lim_{x \to a} [f(x)]^n = l^n$.

Limits of Polynomial and Rational Functions:

1. If p is a polynomial function and c is a real number, then $\lim_{x \to c} p(x) = p(c)$.

2. If r is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$ then $\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$.

Some Important Limits

1. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
2. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
3. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$
4. $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$
5. $\lim_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1$
6. $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$
7. $\lim_{x \to 0} \frac{sinx}{x} = 1$

8. $\lim_{x \to 0} \frac{tanx}{x} = 1$

9. $\lim_{x \to 0} cosx = 1$

10. If K is a constant function, then $\lim_{x \to a} K = K$.

Working rules of Continuity of Function: Consider a function f : $\mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$.

- If $\lim_{x \to a^-} f(x) = f(a)$, then we say that f is left continuous at a.
- If $\lim_{x \to a^+} f(x) = f(a)$, then we say that f is right continuous at a.
- If $\lim_{x \to a} f(x) = f(a)$, then we say that f is continuous at a.

Note that if f is continuous at a, then it is left as well as right continuous at a.

If f is not continuous at a, then f is said to be **discontinuous at x = a,** and a is called a point of discontinuity.

Differentiability of Functions: Consider function f(x) defined on a closed interval [a, b]. Let $c \in (a, b)$. Then the function f(x) is said to be differentiable at x = c, if following is $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. The limit is called the derivative of f(x) with respect to x at c and it is denoted by $\frac{df}{dx}$ or $f'(x)$ at x=c or $\left(\frac{df}{dx}\right)_{x=c}$ or $f'(c)$.

Remark: $f'(a)$ is the slope of the tangent line of the curve $y = f(x)$ at x=a.

The derivative of $\frac{df}{dx}$ w.r.t. x is called the second order derivative of f w.r.t x and is denoted by $\frac{d^2f}{dx^2}$.

Note:

1. If $f: (a, b) \to R$ is differentiable at $x \in (a, b)$, then it is continuous at x.
2. If $f: (a, b) \to R$ is continuous at $x \in (a, b)$, then it may not be differentiable at x. For example: $f(x) = |x|$ is continuous on $\mathbb{R}$ but it is not differentiable at x=0.

Formulas for differentiation:

1. $\frac{d(c)}{dx} = 0$, where c is any constant.
2. $\frac{d(x^n)}{dx} = nx^{n-1}$, where n is any rational number.
3. $\frac{d(e^x)}{dx} = e^x$
4. $\frac{d(a^x)}{dx} = a^x \log_e a$, where a>0.
5. $\frac{d(\log_e a)}{dx} = \frac{1}{x}$.
6. $\frac{d(sinx)}{dx} = cosx$
7. $\frac{d(cosx)}{dx} = -sinx$
8. $\frac{d(tanx)}{dx} = \sec^2 x$
9. $\frac{d(cotx)}{dx} = -cosec^2x$
10. $\frac{d(secx)}{dx} = secx\ tanx$

11. $\frac{d(cosecx)}{dx} = -cosecx\ cotx$

12. $\frac{d(sinhx)}{dx} = coshx$

13. $\frac{d(coshx)}{dx} = sinhx$

14. $\frac{d\left(\sin^{-1} x\right)}{dx} = \frac{1}{\sqrt{1-x^2}}$

15. $\frac{d\left(cos^{-1} x\right)}{dx} = \frac{-1}{\sqrt{1-x^2}}$

16. $\frac{d\left(\tan^{-1} x\right)}{dx} = \frac{1}{1+x^2}$

17. $\frac{d\left(cot^{-1} x\right)}{dx} = \frac{-1}{1+x^2}$

18. $\frac{d\left(cosec^{-1} x\right)}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

19. $\frac{d\left(\sec^{-1} x\right)}{dx} = \frac{1}{x\sqrt{x^2-1}}$

Rules of differentiation: Suppose u and v are functions of x.

1. $\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
2. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
3. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, v \neq 0$

Derivative of the function of a function (Derivative of composition of functions):

If y is function of u and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

This rule is known as the chain rule.

Note: If y=f(t) and x=g(t), where t is a parameter, then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)}$, where $g'(t) \neq 0$

Evaluate : $\lim\limits_{x \to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{x^2}$.

Solution:

$$= \lim_{x \to 0} \frac{\sqrt{2}-\sqrt{1+cosx}}{x^2} \cdot \frac{\sqrt{2}+\sqrt{1+cos\ }}{\sqrt{2}+\sqrt{1+cosx}}$$

$$= \lim_{x \to 0} \frac{2-(1+cos\ )}{x^2(\sqrt{2}+\sqrt{1+cos\ })}$$

$$= \lim_{x \to 0} \frac{1-cos}{x^2(\sqrt{2}+\sqrt{1+cos\ })}$$

$$= \lim_{x \to 0} \frac{(1-cos\ )(1+cos\ )}{x^2(\sqrt{2}+\sqrt{1+cosx})(1+cos\ )}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2(\sqrt{2}+\sqrt{1+cos\ })(1+cosx)}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{2}+\sqrt{1+cos\ })(1+cos\ )} \cdot \lim_{x \to 0} \frac{\sin^2 x}{x^2} = \frac{1}{\left(\sqrt{2}+\sqrt{2}\right)(1+1)} \times 1 = \frac{1}{4\sqrt{2}}$$

Evaluate : $\lim_{x \to 0} \frac{xe^x - log(x+e)^x}{x^2}$

Solution:$\lim_{x \to 0} \frac{xe^x - log(x+e)^x}{x^2} = \lim_{x \to 0} \frac{xe^x - x\log(x+e)}{x^2}$

$$= \lim_{x \to 0} \frac{e^x - \log(x+e)}{x}$$

$$= \lim_{x \to 0} \frac{(e^x - 1) - [\log(x+e) - 1]}{x}$$

$$= \lim_{x \to 0} \frac{(e^x - 1) - \text{lo}\ \left(\frac{x+e}{e}\right)}{x}$$

$$= \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) - \lim_{x \to 0} log\left\{\left(1 + \frac{x}{e}\right)^{\frac{e}{x}}\right\}^{\frac{1}{e}}$$

$$= 1 - loge^{\frac{1}{e}}$$

$$= 1 - \frac{1}{e}$$

Thank You...

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin’s and Taylor’s series expansion.
- L’Hospital’s rule and related applications, Indeterminate forms

Evaluate : $\lim\limits_{x\to 0}\frac{xe^x-log(x+e)^x}{x^2}$

Solution:$\lim\limits_{x\to 0}\frac{xe^x-log(x+e)^x}{x^2} = \lim\limits_{x\to 0}\frac{xe^x-x\log(x+e)}{x^2}$

$$=\lim_{x\to 0}\frac{e^x-\log(x+e)}{x}$$

$$=\lim_{x\to 0}\frac{(e^x-1)-[\log(x+e)-1]}{x}$$

$$=\lim_{x\to 0}\frac{(e^x-1)-\log\left(\frac{x+e}{e}\right)}{x}$$

$$=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)-\lim_{x} log\left\{\left(1+\frac{x}{e}\right)^{\frac{e}{x}}\right\}^{\frac{1}{e}}$$

$$=1-loge^{\frac{1}{e}}$$

$$=1-\frac{1}{e}$$

Evaluate :Discuss continuity of the function $f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ at x=0.

Solution: First calculate right hand limit

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{|x|}{x} = \lim\limits_{x\to 0^+} \frac{x}{x}$ =1 $(\because For\ x > 0, |x| = x)$……(1)

And now calculate left hand limit

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{|x|}{x} = \lim\limits_{x\to 0^-} \frac{-x}{x}$ = -1 $(\because For\ x < 0, |x| = -x)$……(2)

From (1) and (2)

$\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$

$\therefore \lim\limits_{x\to 0} f(x)$ does not exist.

$\therefore$ f is not continuous.

Example : Let $f(x) = \begin{cases} ax + b, & x > 1 \\ 6, & x = 1 \\ 5ax - b, & x < 1 \end{cases}$. If f is continuous at x=1, find a and b.

Solution:

As f is continuous at x=1, we have

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x) = f(1)$$

Taking $\lim_{x\to1^+} f(x) = \mathrm{f}(1)$, we get

$$\lim_{x\to1^+} (ax + b) = \mathrm{f}(1) = 6$$

i.e $(a + b) = 6$ ....(1)

Taking $\lim_{x\to1^-} f(x) = \mathrm{f}(1)$, we get

$$\lim_{x\to1^+} (5ax - b) = \mathrm{f}(1) = 6$$

i.e $(5a - b) = 6$ ....(2)

Solving eq.(1) and eq.(2), we get

a=2 and b=4.

Example : Find $\frac{d^2y}{dx^2}$ for $x = acos^3 t, y = bsin^3 t$.

Solution:

Here $\quad x = \mathrm{acos}^3 t$

$$\therefore \frac{dx}{dt} = 3\,\mathrm{acos}^2 t\,(-sint)$$
$$= -3\,\mathrm{a}\sin t\cos^2 t$$

$$y = \mathrm{b}\sin^3 t$$
$$\therefore \frac{dy}{dt} = 3b\sin^2 t\cos t$$

$$Now\ \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy/dt}}{\mathrm{dx/dt}} = \frac{3\mathrm{b}\sin^2 \mathrm{t}\cos \mathrm{t}}{-3\,\mathrm{a}\sin \mathrm{t}\cos^2 \mathrm{t}} = -\frac{b}{a}\mathrm{tant}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

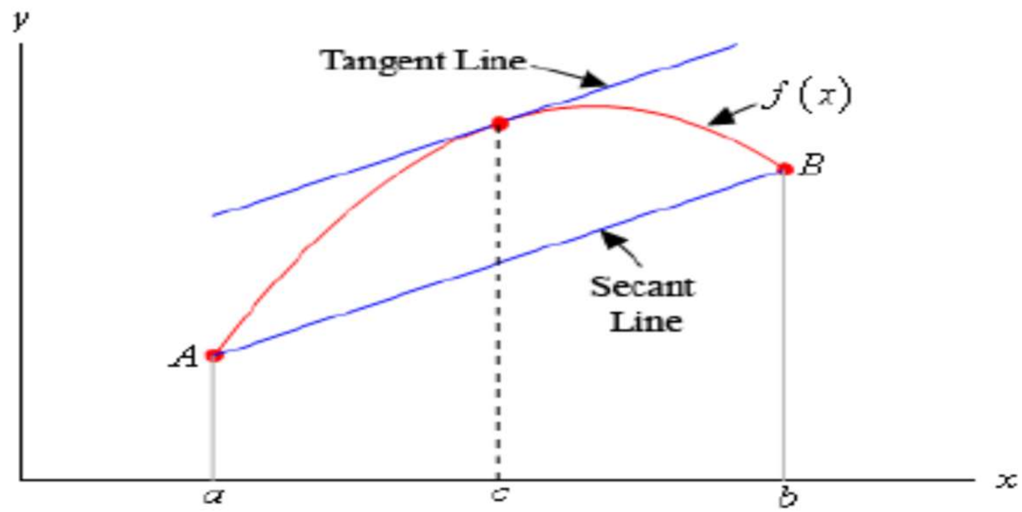
$$= \frac{d}{dt}\left[-\frac{b}{a}\ tant\right]\cdot\frac{1}{-3asint\cos^2 t} = \frac{b\ sec^2\ t}{3a^2sint\cos^2\ t}$$

Mean Value Theorem:

Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that derivative of f exists on the open interval (a, b). Then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The Mean value theorem says that there is a tangent line to the curve whose slope is same as the slop of the line joining the points $A(a, f(a))$ and $B(b, f(b))$ in below figure.



Example: Check whether the Mean Value Theorem can be applied to the function $2x^2 - 7x + 10$ on the closed interval [2,5]. If so, find a value of c which satisfies the Mean value theorem in (2,5).

Solution:

Here $f(x) = 2x^2 - 7x + 10$ is defined for all values of x in the closed interval [2, 5]. Clearly $f(x) = 2x^2 - 7x + 10$ is continuous at all values of x in the interval [2, 5].

We compute the derivative of the function f, which is given by $f'(x) = 4x - 7$

$f'(x) = 4x - 7$ is defined for all values in (2, 5).Thus, f is differentiable at each point of x in the interval (2, 5).

Thus, f(x) satisfies all condition of Mean Value Theorem.

From Mean Value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(2)}{5-2} = \frac{25-04}{3} = 21/3 \qquad \begin{pmatrix} \because \mathrm{f(a)} = \mathrm{f(2)} = 4 \\ \mathrm{f(b)} = \mathrm{f(5)} = 25 \end{pmatrix}$$

And for the derivative $f'(x) = 4x - 7$,

$\therefore f'(c) = 4c - 7$

$\therefore {}^{21}/_{3} = 4c - 7$ Therefore $c = \frac{7}{2} \in (2,5)$.

Example: Check whether the Mean Value Theorem can be applied to the function $4t^3 - 8t^2 + 7t - 2$ on the closed interval [0, 5]. If so, find a value of *c* which satisfies the Mean value theorem in (0, 5).

Solution:

Here $f(t) = 4t^3 - 8t^2 + 7t - 2$ is defined for all values of x in the closed interval [0, 5].
Clearly $f(t) = 4t^3 - 8t^2 + 7t - 2$ is continuous at all values of *x* in the interval [0, 5].

We compute the derivative of the function *f*, which is given by

$$f'(t) = 12t^2 - 16t + 7$$

$f'(t) = 12t^2 - 16t + 7$ is defined for all values in (0, 5).Thus, f is differentiable at each point of *x* in the interval (0, 5).

Thus, f(t) satisfies all condition of Mean Value Theorem.

From Mean Value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(0)}{5-0} = \frac{333}{5} = 67 \quad \begin{pmatrix} \because f(a) = f(0) = -2 \\ f(b) = f(5) = 333 \end{pmatrix}$$

And for the derivative $f'(t) = 12t^2 - 16t + 7$,

$\therefore f'(c) = 12c^2 - 16c + 7 \qquad \therefore 67 = 12c^2 - 16c + 7$ Therefore $c = 3 \in (0,5)$.

Example: Check whether the Mean Value Theorem can be applied to the function $f(x) = logx$ on the closed interval [1, e]. If so, find a value of c which satisfies the Mean value theorem in (1, e).**(H.W.)**

Solution:

$f(x) = logx$ is a logarithmic function . Hence it is continuous in [1,e] and differentiable in (1,e). Hence by Mean Value Theorem, there exist a value $c \in (1, e)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(e) - f(1)}{e - 1} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}$$

$$\because f(a) = f(1) = 0$$
$$f(b) = f(e) = 1$$

And for the derivative $f'(x) = logx$,

$\therefore f'(c) = \frac{1}{c}$

$\therefore \frac{1}{e-1} = \frac{1}{c}$ Therefore $c = e - 1 \in (1, e)$.

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

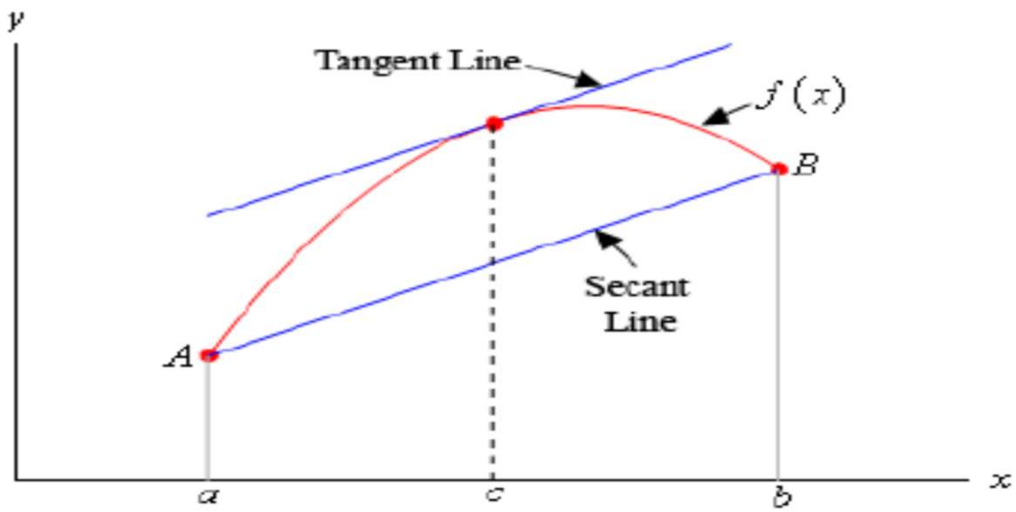
- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin’s and Taylor’s series expansion.
- L’Hospital’s rule and related applications, Indeterminate forms

Mean Value Theorem:

Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that derivative of f exists on the open interval (a, b). Then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The Mean value theorem says that there is a tangent line to the curve whose slope is same as the slop of the line joining the points $A(a, f(a))$ and $B(b, f(b))$ in below figure.



Example: Check whether the Mean Value Theorem can be applied to the function $2x^2 - 7x + 10$ on the closed interval [2,5]. If so, find a value of c which satisfies the Mean value theorem in (2,5).

Solution:

Here $f(x) = 2x^2 - 7x + 10$ is defined for all values of x in the closed interval [2, 5]. Clearly $f(x) = 2x^2 - 7x + 10$ is continuous at all values of x in the interval [2, 5].

We compute the derivative of the function f, which is given by $f'(x) = 4x - 7$

$f'(x) = 4x - 7$ is defined for all values in (2, 5). Thus, f is differentiable at each point of x in the interval (2, 5).

Thus, f(x) satisfies all condition of Mean Value Theorem.

From Mean Value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(2)}{5-2} = \frac{25-04}{3} = 21/3 \qquad \begin{pmatrix} \because f(a) = f(2) = 4 \\ f(b) = f(5) = 25 \end{pmatrix}$$

And for the derivative $f'(x) = 4x - 7$,

$\therefore f'(c) = 4c - 7$

$\therefore {}^{21}/_{3} = 4c - 7$ Therefore $c = \frac{7}{2} \in (2,5)$.

Example: Check whether the Mean Value Theorem can be applied to the function $4t^3 - 8t^2 + 7t - 2$ on the closed interval [0, 5]. If so, find a value of *c* which satisfies the Mean value theorem in (0, 5).

Solution:

Here $f(t) = 4t^3 - 8t^2 + 7t - 2$ is defined for all values of x in the closed interval [0, 5].
Clearly $f(t) = 4t^3 - 8t^2 + 7t - 2$ is continuous at all values of *x* in the interval [0, 5].

We compute the derivative of the function *f*, which is given by

$$f'(t) = 12t^2 - 16t + 7$$

$f'(t) = 12t^2 - 16t + 7$ is defined for all values in (0, 5).Thus, f is differentiable at each point of *x* in the interval (0, 5).

Thus, f(t) satisfies all condition of Mean Value Theorem.

From Mean Value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(0)}{5-0} = \frac{333}{5} = 67 \quad \begin{pmatrix} \because f(a) = f(0) = -2 \\ f(b) = f(5) = 333 \end{pmatrix}$$

And for the derivative $f'(t) = 12t^2 - 16t + 7$,

$\therefore f'(c) = 12c^2 - 16c + 7 \quad \therefore 67 = 12c^2 - 16c + 7$ Therefore $c = 3 \in (0,5)$.

Example: Check whether the Mean Value Theorem can be applied to the function $f(x) = logx$ on the closed interval [1, e]. If so, find a value of c which satisfies the Mean value theorem in (1, e).**(H.W.)**

Solution:

$f(x) = logx$ is a logarithmic function . Hence it is continuous in [1,e] and differentiable in (1,e). Hence by Mean Value Theorem, there exist a value $c \in (1, e)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(e) - f(1)}{e - 1} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}$$

$$\because f(a) = f(1) = 0$$
$$f(b) = f(e) = 1$$

And for the derivative $f'(x) = logx$,

$\therefore f'\ (c) = \frac{1}{c}$

$\therefore \frac{1}{e-1} = \frac{1}{c}$ Therefore $c = e - 1 \in (1, e)$.

Example: Using Mean Value Theorem for the function $f(x) = \tan^{-1} x, x \in R$, prove that $\frac{x}{1+x^2} < \tan^{-1} x < x$

Solution:

Here $f(x) = \tan^{-1} x, x \in R$.

Clearly $f(x) = \tan^{-1} x,$ is continuous in [0, x] and differentiable in (0,x).

From Mean Value theorem, there exists at least one value $c \in (0, x)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(x)-f(0)}{x-0} = \frac{\tan^{-1} x - 0}{x-0}$$

And for the derivative $f'(x) = \frac{1}{1+x^2}$

$$\tan^{-1} x = \frac{x}{1+c^2}$$

Now $0 < c < x \Rightarrow 0 < c^2 < x^2 \Rightarrow 1 < c^2 + 1 < x^2 + 1$

$$\Rightarrow 1 > \frac{1}{c^2+1} > \frac{1}{x^2+1}$$

$$\Rightarrow \frac{x}{1+x^2} < \frac{x}{1+c^2} < x \quad (\because x > 0)$$

$$\Rightarrow \frac{x}{1+x^2} < \tan^{-1} x < x$$

Example: Check whether the Mean Value Theorem can be applied to the function $f(x) = \sin x$ on the closed interval $\left[0, \frac{\pi}{2}\right]$. If so, find a value of c which satisfies the Mean value theorem in $\left(0, \frac{\pi}{2}\right)$. **(H.W.)**

Solution:

Clearly $f(x) = \sin x,$ is continuous in $\left[0, \frac{\pi}{2}\right]$ and differentiable in $\left(0, \frac{\pi}{2}\right)$.

From Mean Value theorem, there exists at least one value $c \in (0, \frac{\pi}{2})$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f\left(\frac{\pi}{2}\right)-f(0)}{\frac{\pi}{2}-0} = \frac{1-0}{\frac{\pi}{2}-0} = \frac{2}{\pi} \quad \text{.......(1)}$$

$$\left(\begin{array}{c} \because f(a) = f\left(\frac{\pi}{2}\right) = 1 \\ f(b) = f(0) = 0 \end{array}\right)$$

And for the derivative $f'(x) = cosx$

From(1) we get,

$$f'(c) = cosc = \frac{2}{\pi} \quad \therefore c = \cos^{-1}\left(\frac{2}{\pi}\right).$$

Example: Check whether the Mean Value Theorem can be applied to the function $f(x) = \frac{6}{x} - 3$ on the closed interval [-1,2]. If so, find a value of c which satisfies the Mean value theorem in (-1,2).

Solution:

Here $f(x) = \frac{6}{x} - 3$, x ∈ [-1,2].

Since the function is not continuous on the given interval, it doesn't satisfy the conditions of the Mean Value Theorem.

Local Maxima and Minima:

- **Maximum value of a function:** If the value of a function f(x) at x=a is maximum in the small interval (a-h, a+h) then we say that f(x) is maximum at x=a.

  The following two conditions must be satisfied for a function f(x) to be maximum at x=a.

  $f'(a) = 0$ (Necessary condition)

  $f''(a) < 0$ (Sufficient condition).

- **Minimum value of a function:** If the value of a function f(x) at x=a is minimum in the small interval (a-h, a+h) then we say that f(x) is minimum at x=a.

  The following two conditions must be satisfied for a function f(x) to be minimum at x=a.

  $f'(a) = 0$ (Necessary condition)

  $f''(a) > 0$ (Sufficient condition).

- **Stationary points**: The point at which a function obtains its maximum or minimum values is called a stationary points. $f'(x) = 0$ is a necessary condition for obtaining stationary points.

Working rules for finding Maxima and Minima:

1. For a given function y=f(x), obtain $\frac{dy}{dx}$ or $f'(x)$.
2. Take $f'(x) = 0$ and solve this equation to find the roots. Let the roots be a, b, c….
3. Obtain the second derivative $f''(x)$ of f.
4. Substitute the roots a, b, c,… in $f''(x)$ one by one. Suppose x=a is substitute in $f''(x)$.

    I. If $f''(a) < 0$, then f is maximum at x=a and the maximum value of f(x) is f(a).

    II. If $f''(a) > 0$, then f is minimum at x=a and the minimum value of f(x) is f(a).

    III. If $f''(a) = 0$, then we can not draw any conclusion about the maximum and minimum value of f at x=a.

Example: Show that the function $y = x^x$ is minimum at $x = e^{-1}$.

Solution: Here $y = x^x$

$\therefore \log y = x\, logx$

Differentiating w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = logx + 1$$

$$\frac{dy}{dx} = x^x(logx + 1)$$

$$\frac{d^2y}{dx^2} = x^x \frac{d}{dx}(logx + 1) + (logx + 1)\frac{d}{dx}(x^x) = x^x\left(\frac{1}{x}\right) + (logx + 1)^2 x^x$$

Now taking $\frac{dy}{dx} = 0$, we have

$x^x(logx + 1) = 0 \quad \therefore x^x = 0 \; or \; logx + 1 = 0$

$x = 0$ is not possible. Hence $logx = -1 \qquad \therefore x = e^{-1}$

$$\frac{d^2y}{dx^2} = x^{x-1} + (logx + 1)^2 x^x = \frac{1}{e}^{\left(\frac{1}{e}-1\right)} + (log\frac{1}{e} + 1)^2 \frac{1}{e}^{\frac{1}{e}}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=e^{-1}} = \frac{1}{e}^{\frac{1}{e}}.e+0=\frac{1}{e}^{\frac{1}{e}}.e>0$$

Example: Show that the function $y = sinx\,(1 + cosx)$ is maximum when $x = \frac{\pi}{3}$.**(H.W.)**

Solution: Consider $y = sinx\,(1 + cosx)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = cosx(1 + cosx) + sinx(-sinx) \ldots\ldots (1)$$

$$= cosx + \cos^2 x - \sin^2 x = cosx + cos2x$$

$$\frac{d^2y}{dx^2} = -sinx - 2sin2x \ldots\ldots.(2)$$

From (2) at $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} = -sin\frac{\pi}{3} - 2sin\frac{2\pi}{3}$

$$= \frac{-\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} < 0$$

Hence the function $y = sinx\,(1 + cosx)$ is maximum value at $x = \frac{\pi}{3}$.

Example: Investigate maxima and minima of the function given by $f(x) = \frac{logx}{x}$ in$(0, \infty)$.

Solution:

Let $y = \frac{logx}{x} = x^{-1} \log x$.

$\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{x}^{-1} \frac{1}{\mathrm{x}} + (-\mathrm{x}^{-2})\mathrm{logx} = \mathrm{x}^{-2}(1 - \mathrm{logx})$

$= \frac{1}{x^2}(1 - logx)$

For maxima and minima, $\frac{dy}{dx} = 0$

$\therefore \frac{1}{x^2}(1 - logx) = 0 \quad \therefore 1 - logx = 0$

$\therefore logx = 1 \qquad \therefore x = e$

$\frac{d^2y}{dx^2} = -2x^{-3}(1 - logx) + x^{-2}\left(-\frac{1}{x}\right) \qquad = -x^{-3}[2(1 - logx) + 1]$

At $x = e, \frac{d^2y}{dx^2} = -\frac{1}{e^3} < 0$

Therefore the Function is maximum at $x = e$ and the maximum value of y is $\frac{1}{e}$.

Example: Find the extreme values of the function $f(x) = 2x^3 - 9x^2 + 12x + 1$.**(H.W.)**

Solution:

Here

$f'(x) = 6x^2 - 18x + 12,$

$f''(x) = 12x - 18$

For maxima or minima,

$$f'(x) = 6x^2 - 18x + 12 = 0 \Rightarrow x = 1,2$$

When x=1, $f''(1) = -6 < 0$

Therefore f(x) is maximum at $x = 1$ and its maximum value is 6.

When $x = 2$, $f''(2) = 6 > 0$.

Therefore f(x) is minimum at $x = 2$ and its minimum value is 5.

Thank You...

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Example: P is the perimeter of a rectangle, show that its area is maximum when it is a square.

Solution: Let P be the fixed perimeter of a rectangle with length x and height y, so that

$P = 2x + 2y$ and the area $A = xy$.

We can write area as a function of x by solving $P = 2x + 2y$ for y and substituting in area

$$P = 2x + 2y \Rightarrow 2y = P - 2x \Rightarrow y = \frac{P}{2} - x$$

$$\Rightarrow A(x) = x\left(\frac{P}{2} - x\right) = \frac{Px}{2} - x^2$$

Differentiating w.r.t x, we get

$$\frac{dA}{dx} = \frac{P}{2} - 2x \ldots\ldots.(1)$$

Again differentiating (1) w.r.t x, we get

$$\frac{d^2A}{dx^2} = -2 \ldots\ldots.(2)$$

For maxima or minima, $\frac{dA}{dx} = 0 \Rightarrow \frac{P}{2} - 2x = 0 \Rightarrow x = \frac{P}{4}$

From (2) at $x = \frac{P}{4}$, we have

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{P}{4}} = -2 < 0$$

Therefore $A(x)$ is maximum at $x = \frac{P}{4}$ and its maximum value is $\frac{P^2}{16}$.

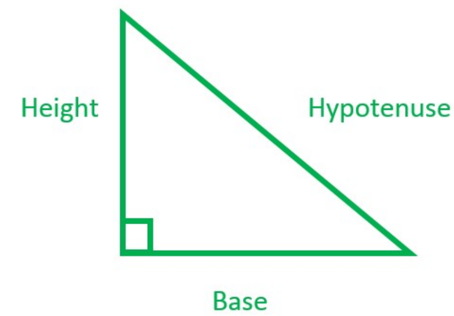
Also $y = \frac{P}{2} - x = \frac{\mathrm{P}}{2} - \frac{\mathrm{P}}{4} = \frac{\mathrm{P}}{4}$.

This implies that rectangle have the same length and height. So it is actually a square.

Example: Prove that among all the right angle triangles of given hypotenuse, the isosceles triangle has maximum area. **Or** Prove that area of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Solution: Let us assume the given hypotenuse is h, and one of the two remaining sides a is x. Hence the remaining last side b is $\sqrt{h^2 - x^2}$ (using pythagorus theorem)

Area of the right angle triangle is



$$A(x) = \frac{1}{2}ab = \frac{1}{2}x\sqrt{h^2 - x^2}$$

Differentiating w.r.t x, we get

$$\frac{dA}{dx} = \frac{1}{2}\sqrt{h^2 - x^2} + \frac{x}{2} \cdot \frac{(-2x)}{2\sqrt{h^2-x^2}} = \frac{1}{2}\sqrt{h^2 - x^2} - \frac{x^2}{2\sqrt{h^2-x^2}} \ldots\ldots.(1)$$

Again differentiating (1) w.r.t x, we get

$$\frac{d^2A}{dx^2} = -\frac{2x}{4\sqrt{h^2-x^2}} - \frac{1}{2}\left(\frac{2x\left(\sqrt{h^2-x^2}\right) - x^2\frac{(-2x)}{2\sqrt{h^2-x^2}}}{\left(\sqrt{h^2-x^2}\right)^2}\right) = \frac{-x}{2\sqrt{h^2-x^2}} - \frac{1}{2}\left(\frac{2xh^2-2x^3+x^3}{\left(\sqrt{h^2-x^2}\right)^3}\right)$$

$$= \frac{-x}{2\sqrt{h^2-x^2}} - \frac{1}{2}\left(\frac{2xh^2-x^3}{\left(\sqrt{h^2-x^2}\right)^3}\right) = -\frac{3xh^2-2x^3}{2\left(\sqrt{h^2-x^2}\right)^3} \ldots\ldots.(2)$$

For maxima or minima, $\frac{dA}{dx} = 0$

$\therefore \frac{1}{2}\sqrt{h^2 - x^2} - \frac{x^2}{2\sqrt{h^2-x^2}} = 0 \ \therefore h^2 - x^2 - x^2 = 0 \ \therefore 2x^2 = h^2 \ \therefore x = \frac{h}{\sqrt{2}}.$

From (2) at $x = \frac{h}{\sqrt{2}}$, we have

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\frac{3\frac{h}{\sqrt{2}}h^2 - 2\left(\frac{h}{\sqrt{2}}\right)^3}{2\left(\sqrt{h^2-\left(\frac{h}{\sqrt{2}}\right)^2}\right)^3} = -\frac{\frac{2h^3}{\sqrt{2}}}{\frac{h^3}{\sqrt{2}}} = -2 < 0$$

Therefore $A(x)$ is maximum at $x = \frac{h}{\sqrt{2}}$ and its maximum value is $\frac{h^2}{4}$.

Hence, the sides of the right angle triangle are $a = x = \frac{h}{\sqrt{2}}$ and

$$b = \sqrt{h^2 - x^2} = \sqrt{h^2 - \frac{h^2}{2}} = \frac{h}{\sqrt{2}} = a$$

This implies that the triangle is isosceles.

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Successive Differentiation:

Successive Differentiation is the process of differentiating a given function successively times and the results of such differentiation are called successive derivatives. The nth order derivative of function $y = f(x)$ is denoted by y_n or $y^{(n)}$ or $f_n(x)$ or $f^{(n)}(x)$.

Some standard results on nth order derivative:

1) If $y = e^{ax+b}$, then $y_n = a^n e^{ax+b}$.

Proof: Here, we have $y = e^{ax+b}$. Next, taking derivative of y with respect to x gives

$y_1 = ae^{ax+}$.

Taking derivative of y_1 with respect to x gives

$y_2 = a^2 e^{ax+b}$.

Taking derivative of y_2 with respect to x gives

$y_3 = a^3 e^{ax+b}$.

Similarly taking nth order derivative of y with respect to x gives $y_n = a^n e^{ax+b}$.

2) If $y = a^{bx}$, then $y_n = b^n(loga)^n a^{bx}$.

Proof: Here, we have $y = a^{bx}$. Next, taking derivative of y with respect to x gives

$y_1 = b(loga)a^{bx}$.

Taking derivative of y_1 with respect to x gives

$y_2 = b^2(loga)^2 a^{bx}$.

Taking derivative of y_2 with respect to x gives

$y_3 = b^3(loga)^3 a^{bx}$.

Similarly taking n^th^ order derivative of y with respect to x gives

$y_n = b^n(loga)^n a^{bx}$.

3) If $y = \sin(ax + b)$, then $y_n = a^n sin\left(ax + b + \frac{n\pi}{2}\right)$.

Proof: Here, we have y $= \sin(ax + b)$. Next, taking derivative of y with respect to x gives

$y_1 = a\ cos(ax + b) = a\ sin\left(ax + b + \frac{\pi}{2}\right)$.

Taking derivative of y_1 with respect to x gives

$$y_2 = a^2 cos\left(ax + b + \frac{\pi}{2}\right) = a^2 sin\left(ax + b + \frac{2\pi}{2}\right).$$

Taking derivative of y_2 with respect to x gives

$$y_3 = a^3 cos\left(ax + b + \frac{2\pi}{2}\right) = a^3 sin\left(ax + b + \frac{3\pi}{2}\right).$$

Similarly taking nth order derivative of y with respect to x gives

$$y_n = a^n sin\left(ax + b + \frac{n\pi}{2}\right).$$

4)If $y = cos(ax + b)$, then $y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right)$.

Proof: Here, we have y $=$ cos$(ax + b)$. Next, taking derivative of y with respect to x gives

$$y_1 = -a\ sin(ax + b) = a\ cos\left(ax + b + \frac{\pi}{2}\right).$$

Taking derivative of y_1 with respect to x gives

$$y_2 = -a^2 sin\left(ax + b + \frac{\pi}{2}\right) = a^2 cos\left(ax + b + \frac{2\pi}{2}\right).$$

Taking derivative of y_2 with respect to x gives

$$y_3 = -a^3 sin\left(ax + b + \frac{2\pi}{2}\right) = a^3 cos\left(ax + b + \frac{3\pi}{2}\right).$$

Similarly taking nth order derivative of y with respect to x gives

$$y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right).$$

5)If $y = e^{ax}\sin(bx + c)$, then $y_n = r^n e^{ax} sin(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$.

Proof: Here, we have y $= e^{ax}\sin(bx + c)$. Next, taking derivative of y with respect to x gives

$$y_1 = ae^{ax}\sin(bx + c) + be^{ax}\cos(bx + c)$$
$$= e^{ax}[a\sin(bx + c) + b\cos(bx + c)].$$

Put $a = rcos\theta$ and $b = rsin\theta$, so that $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore y_1 = e^{ax}[rcos\theta \sin(bx + c) + rsin\theta \cos(bx + c)]$$
$$= re^{ax}[cos\theta \sin(bx + c) + sin\theta \cos(bx + c)]$$
$$= re^{ax}\sin(bx + c + \theta). \quad (\because \sin(A + B) = sinAcosB + cosAsinB)$$

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Successive Differentiation:

Successive Differentiation is the process of differentiating a given function successively times and the results of such differentiation are called successive derivatives. The nth order derivative of function $y = f(x)$ is denoted by y_n or $y^{(n)}$ or $f_n(x)$ or $f^{(n)}(x)$.

Some standard results on nth order derivative:

1) If $y = e^{ax+b}$, then $y_n = a^n e^{ax+b}$.

Proof: Here, we have $y = e^{ax+b}$. Next, taking derivative of y with respect to x gives

$y_1 = ae^{ax+b}$.

Taking derivative of y_1 with respect to x gives

$y_2 = a^2 e^{ax+b}$.

Taking derivative of y_2 with respect to x gives

$y_3 = a^3 e^{ax+b}$.

Similarly taking nth order derivative of y with respect to x gives $y_n = a^n e^{ax+b}$.

2) If $y = a^{bx}$, then $y_n = b^n(loga)^n a^{bx}$.

Proof: Here, we have $y = a^{bx}$. Next, taking derivative of y with respect to x gives

$y_1 = b(loga)a^{bx}$.

Taking derivative of y_1 with respect to x gives

$y_2 = b^2(loga)^2 a^{bx}$.

Taking derivative of y_2 with respect to x gives

$y_3 = b^3(loga)^3 a^{bx}$.

Similarly taking n^th order derivative of y with respect to x gives

$y_n = b^n(loga)^n a^{bx}$.

3) If $y = \sin(ax + b)$, then $y_n = a^n sin\left(ax + b + \frac{n\pi}{2}\right)$.

Proof: Here, we have y $= \sin(ax + b)$. Next, taking derivative of y with respect to x gives

$y_1 = a\ cos(ax + b) = a\ sin\left(ax + b + \frac{\pi}{2}\right)$.

Taking derivative of y_1 with respect to x gives

$$y_2 = a^2 cos\left(ax + b + \frac{\pi}{2}\right) = a^2 sin\left(ax + b + \frac{2\pi}{2}\right).$$

Taking derivative of y_2 with respect to x gives

$$y_3 = a^3 cos\left(ax + b + \frac{2\pi}{2}\right) = a^3 sin\left(ax + b + \frac{3\pi}{2}\right).$$

Similarly taking nth order derivative of y with respect to x gives

$$y_n = a^n sin\left(ax + b + \frac{n\pi}{2}\right).$$

4)If $y = cos(ax + b)$, then $y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right)$.

Proof: Here, we have y $=$ cos$(ax + b)$. Next, taking derivative of y with respect to x gives

$$y_1 = -a\ sin(ax + b) = a\ cos\left(ax + b + \frac{\pi}{2}\right).$$

Taking derivative of y_1 with respect to x gives

$$y_2 = -a^2 sin\left(ax + b + \frac{\pi}{2}\right) = a^2 cos\left(ax + b + \frac{2\pi}{2}\right).$$

Taking derivative of y_2 with respect to x gives

$$y_3 = -a^3 sin\left(ax + b + \frac{2\pi}{2}\right) = a^3 cos\left(ax + b + \frac{3\pi}{2}\right).$$

Similarly taking n^th order derivative of y with respect to x gives

$$y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right).$$

5)If $y = e^{ax}\sin(bx + c)$, then $y_n = r^n e^{ax} sin(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$.

Proof: Here, we have y $= e^{ax}\sin(bx + c)$. Next, taking derivative of y with respect to x gives

$$y_1 = ae^{ax}\sin(bx + c) + be^{ax}\cos(bx + c)$$
$$= e^{ax}[a\sin(bx + c) + b\cos(bx + c)].$$

Put $a = rcos\theta$ and $b = rsin\theta$, so that $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore y_1 = e^{ax}[rcos\theta\sin(bx + c) + rsin\theta\cos(bx + c)]$$
$$= re^{ax}[cos\theta\sin(bx + c) + sin\theta\cos(bx + c)]$$
$$= re^{ax}\sin(bx + c + \theta). \quad (\because \sin(A + B) = sinAcosB + cosAsinB)$$

Taking derivative of y_1 with respect to x gives

$$y_2 = rae^{ax}\sin(bx+c+\theta) + rbe^{ax}\cos(bx+c+\theta)$$

$$= re^{ax}[a\sin(bx+c+\theta) + b\cos(bx+c+\theta)]$$

$$(\because a = rcos\theta \text{ and } b = rsin\theta)$$

$$= re^{ax}[rcos\theta\sin(bx+c+\theta) + rsin\theta\cos(bx+c+\theta)]$$

$$= r^2e^{ax}[cos\theta\sin(bx+c+\theta) + sin\theta\cos(bx+c+\theta)]$$

$$= r^2e^{ax}\sin(bx+c+2\theta)\,.\ (\because \sin(A+B) = sinAcosB + cosAsinB)$$

Similarly, we have

$$y_3 = r^3e^{ax}sin(bx+c+3\theta),$$

$$y_4 = r^4e^{ax}sin(bx+c+4\theta).$$

In general, the n[th] order derivative of y is given by $y_n = r^ne^{ax}sin(bx+c+n\theta)$, where $r = \sqrt{a^2+b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$.

6)If $y = e^{ax}\cos(bx + c)$, then $y_n = r^n e^{ax} cos(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$.

Proof: Here, we have $y = e^{ax}\cos(bx + c)$. Next, taking derivative of y with respect to x gives

$$y_1 = ae^{ax}\cos(bx + c) - be^{ax}\sin(bx + c)$$
$$= e^{ax}[a\cos(bx + c) - b\sin(bx + c)].$$

Put $a = rcos\theta$ and $b = rsin\theta$, so that $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore y_1 = e^{ax}[rcos\theta\cos(bx + c) - rsin\theta\sin(bx + c)]$$
$$= re^{ax}[cos\theta\cos(bx + c) - sin\theta\sin(bx + c)]$$
$$= re^{ax}\cos(bx + c + \theta).$$
$$(\because \cos(A + B) = cosAcosB - sinAsinB)$$

Taking derivative of y_1 with respect to x gives

$$y_2 = rae^{ax}\cos(bx + c + \theta) - rbe^{ax}\sin(bx + c + \theta)$$
$$= re^{ax}[a\cos(bx + c + \theta) - b\sin(bx + c + \theta)]$$
$$= re^{ax}[rcos\theta\cos(bx + c + \theta) - rsin\theta\sin(bx + c + \theta)] (\because a = rcos\theta \text{ and } b = rsin\theta)$$
$$= r^2e^{ax}\cos(bx + c + 2\theta).$$

Similarly, we have

$y_3 = r^3 e^{ax} cos(bx + c + 3\theta)$,

$y_4 = r^4 e^{ax} cos(bx + c + 4\theta)$.

In general, the nth order derivative of y is given by $y_n = r^n e^{ax} cos(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$.

7) If $y = log(ax + b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$.

Proof: Here, we have y $= log(ax + b)$. Next, taking derivative of y with respect to x gives

$y_1 = \frac{a}{ax+b}$.

Taking derivative of y_1 with respect to x gives

$y_2 = \frac{a^2(-1)}{(ax+b)^2}$.

Taking derivative of y_2 with respect to x gives

$$y_3 = \frac{a^3(-1)(-2)}{(ax+b)^3} = \frac{a^3(-1)^2(1\times 2)}{(ax+b)^3} = \frac{a^3(-1)^2(2!)}{(ax+b)^3}.$$

Taking derivative of y_3 with respect to x gives

$$y_4 = \frac{a^4(-1)(-2)(-3)}{(ax+b)^4} = \frac{a^4(-1)^3(1\times 2\times 3)}{(ax+b)^4} = \frac{a^4(-1)^3(3!)}{(ax+b)^4}.$$

Similarly, we have $y_5 = \frac{a^5(-1)^4(4!)}{(ax+b)^5}$.

In general, the nth order derivative of y is given by $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$.

8) If $y = (ax+b)^m$, then $y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, & if\ m > n > 0 \\ 0, \ if \ \ n > m > 0 \\ n!\, a^n, \ if \ \ m = n \\ \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}, & if\ m = -1. \end{cases}$

Proof: Here, we have y $= (ax+b)^m$.

Case 1: $m > n > 0$

Next, taking derivative of y with respect to x gives

$y_1 = am(ax+b)^{m-1}$.

Taking derivative of y_1 with respect to x gives

$y_2 = a^2 m(m-1)(ax+b)^{m-2}$.

Similarly, we have

$y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$,

$y_4 = a^4 m(m-1)(m-2)(m-3)(ax+b)^{m-4}$.

In general, n^{th} order derivative of y is given by

$$y_n = a^n m(m-1)\ldots\big(m-(n-1)\big)(ax+b)^{m-n}$$

$$= a^n m(m-1)\ldots(m-n+1) \times \frac{(m-n)!}{(m-n)!}(ax+b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}.$$

Case 2: If $n > m > 0$, then one of the terms of $m(m-1)\ldots(m-n+1)$ in y_n will be zero. Therefore, $y_n = 0$.

For example, n=5 and m=3 gives

$m(m-1)\ldots(m-n+1)$

$= 3 \times 2 \times 1 \times 0 \times -1$

=0.

Case 3 : $m = n$

Here, we have y $= (ax + b)^n$.

Next, taking derivative of y with respect to x gives

$y_1 = an(ax + b)^{n-1}$.

Taking derivative of y_1 with respect to x gives

$y_2 = a^2 n(n-1)(ax + b)^{n-2}$.

Similarly, we have

$y_3 = a^3 n(n-1)(n-2)(ax + b)^{n-3}$,

$y_4 = a^4 n(n-1)(n-2)(n-3)(ax + b)^{n-4}$.

In general, n^th^ order derivative of y is given by

$$\begin{aligned} y_n &= a^n n(n-1)\ldots\big(n-(n-1)\big)(ax+b)^{n-n} \\ &= a^n n(n-1)\ldots(n-(n-2))\big(n-(n-1)\big) \\ &= a^n n(n-1)(n-2)\ldots 3\times 2\times 1 \\ &= a^n n!\,. \end{aligned}$$

Case 3 : m = -1

Here, we have y $= (ax + b)^{-1}$.

Next, taking derivative of y with respect to x gives

$y_1 = a(-1)(ax + b)^{-2}$.

Taking derivative of y_1 with respect to x gives

$y_2 = a^2(-1)(-2)(ax + b)^{-3} = a^2(-1)^2(2!)(ax + b)^{-3}$.

Similarly, we have

$y_3 = a^3(-1)(-2)(-3)(ax + b)^{-4} = a^3(-1)^3(3!)(ax + b)^{-4}$,

$y_4 = a^4(-1)(-2)(-3)(-4)(ax + b)^{-5} = a^4(-1)^4(4!)(ax + b)^{-5}$.

In general, n^{th} order derivative of y is given by $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$.

9) If $y = sinhax$, then $y_n = \begin{cases} a^n sinhx, if\ n\ is\ even \\ a^n coshx, if\ n\ is\ odd \end{cases}$.

Proof : Here, we have y $= sinhax$. We know that $sinhax = \frac{e^{ax} - e^{-a}}{2}$ and cos$hax = \frac{e^{ax} + e^{-a}}{2}$.

Taking derivative of y with respect to x gives

$$y_1 = \frac{d}{dx}(sinhax) = \frac{d}{dx}\left(\frac{e^{ax} - e^{-ax}}{2}\right) = \frac{ae^{ax} - (-a)e^{-a}}{2} = a\left(\frac{e^{ax} + e^{-ax}}{2}\right) = acoshax.$$

Taking derivative of y_1 with respect to x gives

$$y_2 = \frac{d}{dx}(acoshax) = a\frac{d}{dx}\left(\frac{e^{ax} + e^{-ax}}{2}\right) = a\left[\frac{ae^{ax} + (-a)e^{-ax}}{2}\right] = a^2\left(\frac{e^{ax} - e^{-ax}}{2}\right) = a^2 sinhax.$$

Similarly,

$y_3 = a^3 coshax,$

$y_4 = a^4 sinhax.$

Thus, In general, nth order derivative of y is given by $y_n = \begin{cases} a^n sinhx, if\ n\ is\ even \\ a^n coshx, if\ n\ is\ odd \end{cases}$.

Example 1: Find the n[th] order derivative of the function $y = \frac{x}{2x+5}$.

Solution: Let $y = \frac{x}{2x+}$. We rewrite y as follows

$$y = \frac{1}{2}\left(\frac{2x}{2x+5}\right)$$

$$= \frac{1}{2}\left(\frac{2x+5-5}{2x+5}\right)$$

$$= \frac{1}{2}\left(1 - \frac{-5}{2x+5}\right)$$

$$= \frac{1}{2} - \frac{5}{4x+10}.$$

We know that n[th] order derivative of constant and $\frac{1}{ax+b}$ is zero and $y_n = \frac{(-1)^n n! a^n}{(ax+\ )^{n+1}}$.

Thus, taking n[th] order derivative of y, we get $y_n = (-5)\frac{(-1)^n n! 4^n}{(4x+10)^{n+1}}$.

Example 2: Find the n[th] order derivative of the function $y = e^x sinxcos2x$.

Solution: Let $y = e^x sinxcos2x$. We know that $2sinAcosB = \sin(A+B) + \sin(A-B)$.
The substitution of $A=x$ and $B=2x$ in above formula yields

$$2sinxcos2x = sin3x + \sin(-x)$$

$$= sin3x - \sin x, \ (\because sin(-x) = -sinx).$$

Thus, we have

$$y = \frac{e^x}{2}(sin3x - \sin x)$$

$$= \frac{e^x sin3x}{2} - \frac{e^x \sin x}{2}.$$

On making use of n[th] order derivative of function $e^{ax}\sin(bx+c)$, we get n[th] order derivative of y as

$$y_n = \frac{1}{2}\left[10^{\frac{n}{2}} e^x sin(3x + ntan^{-1}3) - 2^{\frac{n}{2}} e^x sin(x + ntan^{-1}1)\right],$$

$$= \frac{e^x}{2}\left[10^{\frac{n}{2}} sin(3x + ntan^{-1}3) - 2^{\frac{n}{2}} sin(x + ntan^{-1}1)\right].$$

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Example 1: Find the n^th^ order derivative of the function $y = \frac{x}{2x+5}$.

Solution: Let $y = \frac{x}{2x+5}$. We rewrite y as follows

$$y = \frac{1}{2}\left(\frac{2x}{2x+5}\right)$$
$$= \frac{1}{2}\left(\frac{2x+5-5}{2x+5}\right)$$
$$= \frac{1}{2}\left(1 - \frac{-5}{2x+5}\right)$$
$$= \frac{1}{2} - \frac{5}{4x+10}.$$

We know that nth order derivative of constant and $\frac{1}{ax+b}$ is zero and $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$.

Thus, taking nth order derivative of y, we get $y_n = (-5)\frac{(-1)^n n! 4^n}{(4x+10)^{n+1}}$.

Example 2: Find the nth order derivative of the function $y = e^x sinxcos2x$.

Solution: Let $y = e^x sinxcos2x$. We know that $2sinAcosB = \sin(A+B) + \sin(A-B)$.

The substitution of $A=x$ and $B=2x$ in above formula yields

$$2sinxcos2x = sin3x + \sin(-x)$$

$$= sin3x - \sin x, (\because sin(-x) = -sinx).$$

Thus, we have

$$y = \frac{e^x}{2}(sin3x - \sin x)$$

$$= \frac{e^x sin3x}{2} - \frac{e^x \sin x}{2}.$$

On making use of nth order derivative of function $e^{ax}\sin(bx+c)$, we get nth order derivative of y as

$$y_n = \frac{1}{2}\left[10^{\frac{n}{2}} e^x sin(3x + ntan^{-1}3) - 2^{\frac{n}{2}} e^x sin(x + ntan^{-1}1)\right],$$

$$= \frac{e^x}{2}\left[10^{\frac{n}{2}} sin(3x + ntan^{-1}3) - 2^{\frac{n}{2}} sin(x + ntan^{-1}1)\right].$$

Example 3: Show that the n^th order derivative of the function $y = \frac{x}{(x-1)(2x+3)}$ is $y_n = \frac{(-1)^n n!}{5}\left[\frac{1}{(x-1)^{n+1}} + \frac{3(2)^n}{(2x+3)^{n+1}}\right]$.

Solution: Let $y = \frac{x}{(x-1)(2x+3)}$. Using Partial fractions, one can rewrite the function y as follows

$$\frac{x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$$

$$\therefore x = A(2x+3) + B(x-1) ----(1)$$

In (1), the substitution x=1 gives $1 = A5 + 0 \Rightarrow A = \frac{1}{5}$,

In (1), the substitution x=$\frac{-3}{2}$ gives $\frac{-3}{2} = 0 + B\left(-\frac{3}{2} - 1\right) \Rightarrow B = \frac{3}{5}$,

Thus, We have $y = \frac{x}{(x-1)(2x+3)} = \frac{1}{5(x-1)} + \frac{3}{5(2x+3)}$.

On making use of n^th order derivative of $\frac{1}{ax+b}$ gives $y_n = \frac{(-1)^n n!}{5}\left[\frac{1}{(x-1)^{n+1}} + \frac{3(2)^n}{(2x+3)^{n+1}}\right]$.

Example 4: Find the n^{th} order derivative of the function $y = \frac{x^n - 1}{x - 1}$.

Solution: Here, we have $y = \frac{x^n - 1}{x - 1}$.

For n= 1, we have y=1.

For n=2, we have $y = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1$.

For n=3, we have $y = \frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2+x+1)}{x-1} = x^2 + x + 1$.

For n=4, we have y= $\frac{x^4 - 1}{x - 1} = \frac{(x-1)(x^3+x^2+x+1)}{x-1} = x^3 + x^2 + x + 1$.

In general $y = \frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + x^2 + x + 1$.

Thus, y is a polynomial of degree n-1. We know that $y = (ax + b)^m$, then $y_n = 0$ for n>m>0.

Therefore, the of n^{th} order derivative of $y = \frac{x^n - 1}{x - 1}$ is $y_n = 0$.

Example 5: Find the n^{th} order derivative of the function $y = sin2xcos2x$.**(H.W.)**

Solution: Let $y = sin2xcos2x$. We know that $2sinAcosB = \sin(A+B) + \sin(A-B)$.

The substitution of A=2x and B=2x in above formula yields $2sinxcoz2x = sin4x$

Thus, we have $y = \frac{sin4x}{2}$.

On making use of n^{th} order derivative of function $\sin(ax+b)$, we get n^{th} order derivative of y

as $y_n = \frac{4^n}{2} sin\left(4x + \frac{n\pi}{2}\right)$.

Example 6: Find the nth order derivative of the function $y = \frac{2}{(x-1)(x-2)}$ **.(H.W.)**

Solution: Let $y = \frac{2}{(x-1)(x-2)}$. Using Partial fractions, one can rewrite the function y as follows

$$\frac{2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\therefore 2 = A(x-2) + B(x-1) ---- -(1)$$

In (1), the substitution x=1 gives 2= $A(-1) + 0 \Rightarrow A = -2$,

In (1), the substitution x=2 gives $2 = 0 + B(1) \Rightarrow B = 2$.

Thus, We have $y = \frac{2}{(x-1)(x-2)} = \frac{-2}{(x-1)} + \frac{2}{(x-2)}$.

On making use of nth order derivative of $\frac{1}{ax+b}$ gives $y_n = 2(-1)^n n! \left[\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$.

Leibnitz rule for the n^th order derivatives of product of two functions:

If u and v are functions of x such that their n^th derivatives exist, then the n^th derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n.$$

Example 1: Find the n^th order derivative of $y = x^3 \log(2x+1)$.

Solution : Here, we have $y = x^3 \log(2x+1)$ and

Leibnitz's theorem

If u and v are functions of x such that their n^th derivatives exist, then the n^th derivative of their product is given by

$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n$.--------(1)

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Leibnitz rule for the n^{th} order derivatives of product of two functions:

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n.$$

Example 1: Find the n^{th} order derivative of $y = x^3 \log(2x + 1)$.

Solution : Here, we have $y = x^3 \log(2x + 1)$ and

Leibnitz's theorem

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n .\text{--------}(1)$$

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n.\text{--------}(1)$$

Next, taking $u = \log(2x+1)$and $v = x^3$ in (1) gives

$$y_n = (x^3 \log(2x+1))_n$$

$$= (x^3)\left[\frac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n}\right] + \binom{n}{1}(3x^2)\left[\frac{(-1)^{n-2}(n-2)!2^{n-1}}{(2x+1)^{n-1}}\right]$$

$$+ \binom{n}{2}(6x)\left[\frac{(-1)^{n-3}(n-3)!2^{n-2}}{(2x+1)^{n-2}}\right] + \binom{n}{3}(6)\left[\frac{(-1)^{n-4}(n-4)!2^{n-3}}{(2x+1)^{n-3}}\right] + 0$$

$$= (x^3)\left[\frac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n}\right] + (n)(3x^2)\left[\frac{(-1)^{n-2}(n-2)!2^{n-1}}{(2x+1)^{n-1}}\right]$$

$$+ \left[\frac{n(n-1)}{2}\right](6x)\left[\frac{(-1)^{n-3}(n-3)!2^{n-2}}{(2x+1)^{n-2}}\right] + \left[\frac{n(n-1)(n-2)}{6}\right](6)\left[\frac{(-1)^{n-4}(n-4)!2^{n-3}}{(2x+1)^{n-3}}\right]$$

$$= (x^3)\left[\frac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n}\right] + (n)(3x^2)\left[\frac{(-1)^{n-2}(n-2)!2^{n-1}}{(2x+1)^{n-1}}\right]$$

$$+ n(n-1)(3x)\left[\frac{(-1)^{n-3}(n-3)!2^{n-2}}{(2x+1)^{n-2}}\right] + n(n-1)(n-2)\left[\frac{(-1)^{n-4}(n-4)!2^{n-3}}{(2x+1)^{n-3}}\right].$$

This is the required n$^{\text{th}}$ order derivative of $y = x^3 \log(2x+1)$.

Example 2: Find the n^th^ derivative of the function $y = x^3e^{3x}$.**(H.W.)**

Solution : Here, we have $y = x^3e^{3x}$ and

Leibnitz's theorem

If u and v are functions of x such that their n^th^ derivatives exist, then the n^th^ derivative of their product is given by

$(uv)_n = u_nv + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + \cdots + uv_n$.--------(1)

Next, taking $u = e^{3x}$and $v = x^3$ in (1) gives

we have

$$
\begin{aligned}
y_n &= (x^3e^{3x})_n \\
&= (x^3)(3^ne^{3x}) + \binom{n}{1}(3x^2)(3^{n-1}e^{3x}) + \binom{n}{2}(6x)(3^{n-2}e^{3x}) + \binom{n}{3}(6)(3^{n-3}e^{3x}) + 0 \\
&= (x^3)(3^ne^{3x}) + (n)(3x^2)(3^{n-1}e^{3x}) \\
&\qquad + \left[\frac{n(n-1)}{2}\right](6x)(3^{n-2}e^{3x}) + \left[\frac{n(n-1)(n-2)}{6}\right](6)(3^{n-3}e^{3x}) \\
&= (x^3)(3^ne^{3x}) + (n)(3x^2)(3^{n-1}e^{3x}) + n(n-1)(3x)(3^{n-2}e^{3x}) \\
&\qquad + n(n-1)(n-2)(3^{n-3}e^{3x}).
\end{aligned}
$$

This is the required n^th^ order derivative of $y = x^3e^{3x}$.

Example 3: If $y = tan^{-1}x$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$.

Solution : Here, we have $y = tan^{-1}x$.

On taking derivative to this function with respect to x gives

$$y_1 = \frac{1}{1+x^2},$$

$$\therefore (1+x^2)y_1 \;\; = 1,$$

$$\therefore (1+x^2)y_1 \;\; - 1 = 0.$$

On differentiating n times by Leibnitz's theorem

If u and v are functions of x such that their n[th] derivatives exist, then the n[th] derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1}v_1 + \binom{n}{2} u_{n-2}v_2 + \cdots + uv_n,$$

we have

$$((1+x^2)y_1)_n + (1)_n = 0$$

$((1+x^2)y_1)_n + (1)_n$=0

$$\therefore (1+x^2)y_{n+1} + \binom{n}{1}(2x)y_n + \binom{n}{2}(2)y_{n-1} + 0 = 0$$

$$\therefore (1+x^2)y_{n+1} + n(2x)y_n + \left[\frac{n(n-1)}{2}\right](2)y_{n-1} = 0$$

$$\therefore (1+x^2)y_{n+1} + n(2x)y_n + n(n-1)y_{n-1} = 0$$

Thus, we get the required expression in the form

$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0 = 0.$$

Example 4: If $y = sinlog(x^2 + 2x + 1)$, show that

$$(1+x)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0.$$

Solution : Here, we have

$y = sinlog(x^2 + 2x + 1)$,

$\therefore y = sin[log(x+1)^2]$

$\qquad = sin[2\log(x+1)]$.

On taking derivative to this function with respect to x gives

$$y_1 = cos2\left[\log(x+1) \times 2 \times \frac{1}{x+1}\right]$$

$\therefore (1+x)y_1 \ = 2cos2\log(x+1)$

$\therefore (1+x)y_1 \ - 2cos2\log(x+1) = 0$.

On differentiating again with respect to x, we get

$$(1+x)y_2 + y_1 + 2sin2\log(x+1) \ \times 2 \times \frac{1}{x+1} = 0$$

$\therefore (1+x)^2 y_2 + (x+1)y_1 + 4y = 0.$

($\because$Multiply by $x+1$)

On differentiating n times by Leibnitz's theorem

If u and v are functions of x such that their n^th derivatives exist, then the n^th derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n,$$

we have

$$\left((1+x)^2 y_2\right)_n + ((1+x)y_1)_n + (4y)_n = 0$$

$$\therefore (1+x)^2 y_{n+2} + \binom{n}{1}(2(1+x))y_{n+1} + \binom{n}{2}(2)y_n + 0$$
$$+(1+x)y_{n+1} + \binom{n}{1}(1)y_n + \ 0 \ + 4y_n = 0$$

$$\therefore (1+x)^2 y_{n+2} + n(2(1+x))y_{n+1} + \left(\frac{n(n-1)}{2}\right)(2)y_n$$
$$+(1+x)y_{n+1} + n(1)y_n + 4y_n = 0$$

$$\therefore (1+x)^2 y_{n+2} + n(2(1+x))y_{n+1} + n(n-1)y_n + (1+x)y_{n+1} + n(1)y_n + 4y_n = 0$$

Thus, we get the required expression in the form

$$(1+x)^2 y_{n+2} + (1+x)(2n+1)y_{n+1} + (n^2+4)y_n = 0.$$

Example 5: If $y = \sqrt{\frac{1+x}{1-x}}$, show that

$(1-x^2)y_n - [2(n-1)x+1]y_{n-1} - (n-1)(n-2)y_{n-2} = 0$, where $n \geq 2$.

Solution : Here, we have $y = \sqrt{\frac{1+x}{1-x}}$,

$y^2 = \frac{1+x}{1-x}$ --------(1)

(∵ Taking square both sides)

$\therefore y^2 = \frac{1+x}{1-x} \times \frac{1+x}{1+x}$,

(∵ Divide and multiply by 1+x)

$$\therefore y^2 = \frac{(1+x)^2}{1-x^2},$$

$\therefore y^2(1-x^2) = (1+x)^2$.

On taking derivative to this function with respect to x gives

$(1-x^2)2yy_1 + (-2x)y^2 = 2(1+x)$,

$\therefore (1-x^2)2yy_1 + (-2x)y^2 = 2y^2(1-x)$ (∵ Using equation (1))

$(1-x^2)2yy_1 \ + (-2x)y^2 = 2y^2(1-x)$ (Using equation (1))

$\therefore (1-x^2)2y_1 \ + (-2x)y = 2y(1-x)$

($\because$ Divide by y)

$\therefore (1-x^2)2y_1 \ - 2xy = 2y - 2xy$

$\therefore (1-x^2)2y_1 \ - 2y = 0$

$\therefore (1-x^2)y_1 \ - y = 0.$

($\because$ Divide by 2)

On differentiating n times by Leibnitz's theorem

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n,$$

we have

$((1-x^2)y_1)_n - (y)_n$=0

$$\therefore (1-x^2)y_{n+1} + \binom{n}{1}(-2x))y_n + \binom{n}{2}(-2)y_{n-1} - y_n = 0$$

$((1-x^2)y_1)_n-(y)_n=0$

$$\therefore (1-x^2)y_{n+1}+\binom{n}{1}(-2x))y_n+\binom{n}{2}(-2)y_{n-1}-y_n=0$$

$$\therefore (1-x^2)y_{n+1}+n(-2x))y_n+\left[\frac{n(n-1)}{2}\right](-2)y_{n-1}-y_n=0$$

$$\therefore (1-x^2)y_{n+1}+n(-2x))y_n-n(n-1)y_{n-1}-y_n=0$$

$$\therefore (1-x^2)y_{n+1}+(-2nx-1)y_n-n(n-1)y_{n-1}=0$$

$$\therefore (1-x^2)y_{n+1}-(2nx+1)y_n-n(n-1)y_{n-1}=0.$$

On Replacing n by n-1 in this last expression, we get

$$\therefore (1-x^2)y_n-[2(n-1)x+1]y_{n-1}-(n-1)(n-2)y_{n-2}=0.$$

Which is the required expression.

Example 6: If $y = (sin^{-1}x)^2$, show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$.

Solution : Here, we have $y = (sin^{-1}x)^2$.**(H.W.)**

On taking derivative to this function with respect to x gives $y_1 = \frac{2(sin^{-1}x)}{\sqrt{1-x^2}}$,

$\therefore \left(\sqrt{1 - x^2}\right)y_1 \;\; = 2sin^{-1}x$,

(∵ Take square both sides)

$\therefore (1 - x^2)y_1^2 = 4(sin^{-1}x)^2$,

$\therefore (1 - x^2)y_1^2 = 4y$,

$(\because y = (sin^{-1}x)^2)$

$\therefore (1 - x^2)y_1^2 - 4y$=0.

On taking derivative to this function with respect to x gives

$(1 - x^2)2y_1y_2 + (-2x)y_1^2 - 4y_1$=0,

$\therefore (1 - x^2)y_2 - xy_1 - 2$=0. (∵ Divide by $2y_1$)

On differentiating n times by Leibnitz's theorem

If u and v are functions of x such that their n^th^ derivatives exist, then the n^th^ derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n,$$

we have

$$\left((1-x^2)y_2\right)_n - (xy_1)_n - (2)_n = 0$$

$$\therefore (1-x^2)y_{n+2} + \binom{n}{1}(-2x)y_{n+1} + \binom{n}{2}(-2)y_n + 0$$

$$-\left[(x)y_{n+1} + \binom{n}{1}(1)y_n + 0\right] - 0 = 0$$

$$\therefore (1-x^2)y_{n+2} - 2nxy_{n+1} + \left[\frac{n(n-1)}{2}\right](-2)y_n + 0 - (x)y_{n+1} - ny_n = 0$$

$$\therefore (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n + 0 - xy_{n+1} - ny_n = 0$$

Thus, we get the required expression in the form

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0.$$

Example 7: If $y = (cos^{-1}x)^2$, show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0$.

Solution : Here, we have $y = (cos^{-1}x)^2$.**(H.W.)**

On taking derivative to this function with respect to x gives $y_1 = \frac{2(cos^{-1}x)}{\sqrt{1-x^2}}$,

$\therefore \left(\sqrt{1 - x^2}\right)y_1 \; = 2cos^{-1}x$,

($\because$ Take square both sides)

$\therefore (1 - x^2)y_1^2 = 4(cos^{-1}x)^2$,

($\because y = (cos^{-1}x)^2$)

$\therefore (1 - x^2)y_1^2 = 4y$,

$\therefore (1 - x^2)y_1^2 - 4y$=0.

On taking derivative to this function with respect to x gives

$(1 - x^2)2y_1y_2 + (-2x)y_1^2 - 4y_1$=0,

$\therefore (1 - x^2)y_2 - xy_1 - 2$=0. ($\because$ Divide by $2y_1$)

On differentiating n times by Leibnitz's theorem

If u and v are functions of x such that their n^th derivatives exist, then the n^th derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n,$$

we have

$$\left((1-x^2)y_2\right)_n - (xy_1)_n - (2)_n = 0$$

$$\therefore (1-x^2)y_{n+2} + \binom{n}{1}(-2x)y_{n+1} + \binom{n}{2}(-2)y_n + 0$$

$$-\left[(x)y_{n+1} + \binom{n}{1}(1)y_n + 0\right] - 0 = 0$$

$$\therefore (1-x^2)y_{n+2} - 2nxy_{n+1} + \left[\frac{n(n-1)}{2}\right](-2)y_n + 0 - (x)y_{n+1} - ny_n = 0$$

$$\therefore (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n + 0 - xy_{n+1} - ny_n = 0$$

Thus, we get the required expression in the form

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0.$$

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin’s and Taylor’s series expansion.
- L’Hospital’s rule and related applications, Indeterminate forms

Power series expansion of a function: Maclaurin' and Taylor's series expansion

In this section, we shall discuss infinite series representation of a function, especially in the form of a power series. Such series expansions are useful to approximate the function numerically by polynomials e.g. functions such as *sin x*, *log x*, e^x, etc. For that we shall use Taylor's series and Maclaurin's series expansions in the powers of some variable.

Definition: If $c_1, c_2, c_3, \ldots.$ and *a* are constants, then an infinite series expression of the form $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \cdots$ is called a power series in $(x-a)$.

If we put $a = 0$ then $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$ is called a power series in *x*.

Taylor's series:

If *f(x)* possess derivatives of all orders at the point *a*, then

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots \quad \text{\_\_\_\_ (i)}$$

Result-1: Replacing *x* by *a+h* in Taylor's series (i), we get another form of the series.

$f(a+h) = f(a) + \frac{h^1}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \cdots$, which is known as Taylor's series expansion of the function $f(x)$ in the neighborhood of the point *a*.

Result-2: If we put $a = 0$ in Taylor's series (i), then we get
$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$, which known as Maclaurin's series expansion of the function $f(x)$.

Example : Use Taylor's series to find the expansion of $\log_e x$ in powers of $(x - 1)$. find the value of $\log 1.1$. (C.W.)

Solution: Here a =1

Let $f(x) = \log_e x \Rightarrow f(1) = 0$

$\therefore f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$f^{iv}(x) = -\frac{6}{x^4} \Rightarrow f^{iv}(1) = -6$ and so on….

Substitute these values in Taylor's series

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a) + \cdots$$

$$f(x) = f(1) + \frac{(x-1)^1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f^{iv}(1) + \cdots$$

We get

$$\log_e x = 0 + \frac{(x-1)^1}{1!}(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \cdots$$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Now put $x = 1.1$, *we get*

$$\log_e 1.1 = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

$$= 0.1 - 0.005 + 0.0003 - 0.00002 + \cdots = 0.09532$$

Ex: Expand $\tan^{-1} x$ in powers of $x - \frac{\pi}{4}$. (H.W.)

Solution: Here $a = \frac{\pi}{4}$

Let $f(x) = \tan^{-1} x \Rightarrow f\left(\frac{\pi}{4}\right) = \tan^{-1}\frac{\pi}{4}$

$\therefore f'(x)= \frac{1}{1+x^2} \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\frac{\pi^2}{16}}$

$f''(x)=-\frac{2x}{(1+x^2)^2} \Rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{\pi}{2\left(1+\frac{\pi^2}{16}\right)^2}$

and so on….

Substitute these values in Taylor's series

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$\tan^{-1} x = f\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^1}{1!} f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots$$

$$\tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \frac{x-\frac{\pi}{4}}{1+\frac{\pi^2}{16}} - \frac{\pi\left(x-\frac{\pi}{4}\right)^2}{4\left(1+\frac{\pi^2}{16}\right)^2} + \cdots$$

Example: Expand $\log\tan\left(\frac{\pi}{4}+x\right)$ in powers of x using the Taylor's series. (C.W.)

Solution: Here we use

$$f(a+h)=f(a)+\frac{h^1}{1!}f'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^n(a)+\cdots$$

Let $a=\frac{\pi}{4}$ and $h=x$.

$\therefore f(a+x)=\log\tan\left(\frac{\pi}{4}+x\right)$

$\Rightarrow f(x)=\log\tan x \quad \Rightarrow f\left(\frac{\pi}{4}\right)=0$

$\Rightarrow f'(x)=\frac{sec^2 x}{\tan}=\frac{1+tan^2 x}{\tan x}=\cot x+\tan x$

$f'\left(\frac{\pi}{4}\right)=2$

$f''(x)=-cosec^2x+sec^2x \ \Rightarrow f''\left(\frac{\pi}{4}\right)=0$

$f'''(x)=2cosec^2xcot\ x+2sec^2xtan\ x \ \Rightarrow f''\ \left(\frac{\pi}{4}\right)=8$

$$\therefore \log\tan\left(\frac{\pi}{4}+x\right)=f\left(\frac{\pi}{4}\right)+\frac{x^1}{1!}f'\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right)+\cdots$$

$$=2x+\frac{4}{3}x^3+\ldots\ldots$$

Ex: Obtain the Maclaurin's series of $e^{a\sin^{-1}x}$. C.W.

Solution: Let $y = e^{asin^{-1}x} \Rightarrow y(0) = 1$

Differentiating with respect to *x*, then

$y_1 = \frac{a}{\sqrt{1-x^2}} e^{asin^{-1}x} \Rightarrow y_1(0) = a$

$\therefore (1 - x^2){y_1}^2 = a^2y^2$

$\Rightarrow (1 - x^2)2y_1y_2 - 2x{y_1}^2 = 2a^2y\, y_1$

$\Rightarrow (1 - x^2)y_2 - xy_1 - a^2y = 0$

$\Rightarrow y_2(0) = a^2$

Using Leibnitz's Theorem we get

$(1 - x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2) - \{xy_{n+1} + y_n(1)\} - a^2y_n = 0$

$(1 - x^2)y_{n+2} - x(2n + 1)y_{n+1} - (a^2 + n^2)y_n = 0$

$\therefore\ y_{n+2}(0) = (a^2 + n^2)\, y_n(0)\,, n \geq 1$

$y_3(0) = (a^2 + 1)\, y_1(0) = a(a^2 + 1)$

$y_4(0) = (a^2 + 2^2)\, y_2(0) = a^2(a^2 + 2^2)$ and so on..

Substituting these values in Maclaurin's series

$$y(x) = y(0) + \frac{x^1}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \cdots + \frac{x^n}{n!} y^n(0) + \cdots$$

We get

$$e^{asin^{-1}x} = 1 + a\frac{x^1}{1!} + a^2\frac{x^2}{2!} + a(a^2 + 1)\frac{x^3}{3!} + a^2(a^2 + 2^2)\frac{x^4}{4!} + \cdots$$

Example: Obtain the Maclaurin's series of $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$. C.W

Solution: Let $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \Rightarrow y(0) = 0$

$(1 - x^2)y^2 = (\sin^{-1} x)^2$

Differentiating with respect to *x*, then

$\Rightarrow (1 - x^2)2yy_1 - 2xy^2 = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} = 2y$

$\Rightarrow (1 - x^2)y_1 - xy = 1$

$\Rightarrow y_1(0) = 1$

Using Leibnitz's Theorem we get

$\{(1 - x^2)y_{n+1} + ny_n(-2x) + \frac{n(n-1)}{2!}y_{n-1}(-2)\} - \{xy_n + ny_{n-1}(1)\} = 0$

$(1 - x^2)y_{n+1} - x(2n + 1)y_n - n^2y_{n-1} = 0$

Put x=0, we get

$$y_{n+1}(0) = n^2\, y_{n-1}(0)\,,\, n \geq 1$$

$$\therefore\ y_2\ (0) = 0$$

$$y_3(0) = 4$$

$y_4(0) = 0, y_5(0) = 64$ and so on..

Substituting these values in Maclaurin's series

$$\mathrm{y}(x) = y(0) + \frac{x^1}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) \ldots$$

We get

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = 2x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \cdots$$

Ex: Obtain the Maclaurin's series of $\log\sec x$. H.W.

Solution:

Let $f(x) = \log\sec x \Rightarrow f(0) = 0$

$\therefore f'(x) = \frac{1}{\sec x}\sec x \tan x \Rightarrow f'(x) = \tan x \Rightarrow f'(0) = 0$

$f''(x) = sec^2 x \Rightarrow f''(0) = 1$

$f'''(x) = 2sec^2 x \tan x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = 4sec^2 x\, tan^2 x + 2sec^4 x \Rightarrow f^{iv}(0) = 2$

and so on….

Substitute these values in Maclaurin's series

$$f(x) = f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$$\log\sec x = f(0) + \frac{x^1}{1!}(0) + \frac{x^2}{2!}(1) + \ldots..$$

$$\log\sec x = \frac{x^2}{2!} + \frac{x^4}{4!}2 + \ldots\ldots$$

$$\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \ldots\ldots$$

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

INDETERMINATE FORMS

Indeterminate forms , L'Hospital's rule and related applications:

Some limits can be determined the following rules:

$q + \infty = \infty$ if $q \neq -\infty$

$q \times \infty = \infty$ if $q > 0$

$q \times \infty = -\infty$ if $q < 0$

$\frac{q}{\infty} = 0$ if $q \neq \infty$ and $q \neq -\infty$

$\infty^q = 0$ if $q < 0$

$\infty^q = \infty$ if $q > 0$

$q^\infty = 0$ if $0 < q < 1$

$q^\infty = \infty$ if $q > 1$

$q^{-\infty} = \infty$ if $0 < q < 1$

$q^{-\infty} = 0$ if $q > 1$

Here we will discuss seven types of indeterminate forms;

$\frac{0}{0}$, (ii) $\frac{\infty}{\infty}$, (iii) $0 \times \infty$, (iv) $\infty - \infty$, (v) 1^∞, (vi) 0^0, (vii) ∞^0.

If limit is of any one of the above form, then it can be evaluated by using L'Hospital's Rule.

(a) L'Hospital's Rule for the indeterminate form $\left(\frac{0}{0}\right)$:

If $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$; provided the limit exists.

Remark:

Suppose that functions f, g are *n* times differentiable and

$f(a) = f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0.$

$g(a) = g'(a) = g''(a) = \cdots = g^{(n-1)}(a) = 0.$

Suppose that $g^{(n)}(a) \neq 0$. Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$.

Ex: Evaluate $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}-x^2-2}{sin^2x- x^2}$. C.W.

Solution: The limit is an indeterminate form of type $\left(\frac{0}{0}\right)$.

Thus by L'Hospital's rule, we get

$$\lim_{x\to 0} \frac{e^x+e^{-x}-x^2-2}{sin^2x- x^2} = \lim_{x\to 0} \frac{e^x-e^{-x}-2x}{\sin 2x-2x} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0} \frac{e^x+e^{-x}-2x}{2cos\ 2x-2} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0} \frac{e^x-e^{-x}}{-4sin\ 2x} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0} \frac{e^x+e^{-x}}{-8cos\ 2x} = -\frac{1}{4}$$

Ex: If $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\ \sin x} = 2$, find the values of a, b, c. C.W.

Solution: We have

$$L = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\ \sin x}$$

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \quad \left(\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right) \qquad \dots \dots (1)$$

Here the limit of denominator is 0. thus the limit of numerator must also be 0.

$$\therefore a - b + c = 0 \qquad \dots \dots (2)$$

Now applying L'Hospital's rule in (1), we get

$$L = \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{2x} \qquad \dots \dots (3)$$

Since the limit exist, we must have

$$a - c = 0 \qquad \dots \dots (4)$$

Again applying L'Hospital's rule in (3), we get

$$L = \lim_{x \to 0} \frac{ae^x + b\,cos\,x + ce^{-x}}{2}$$

$$L = \frac{a+b+c}{2}$$

But given that L=2

$$\therefore \frac{a+b+c}{2} = 2 \Rightarrow a + b + c = 4 \qquad \ldots\ldots.(5)$$

Solving equations (2), (4) and (5), we get

$a = 1, b = 2, c = 1.$

Evaluate $\lim\limits_{x\to\frac{\pi}{2}} \frac{\log \sin x}{(\pi-2x)^2}$. H.W.

Solution: The limit is of the form $\left(\frac{\mathbf{0}}{\mathbf{0}}\right)$. Thus by L'Hospital's rule

$$\lim_{x\to\frac{\pi}{2}} \frac{\log \mathrm{si}}{(\pi-2x)^2} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}\cos x}{2(\pi-2x)(-2)}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cot x}{(-4)(\pi-2x)} \left(\frac{\mathbf{0}}{\mathbf{0}}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-cos^2 x}{(-4)(-2)}$$

$$= -\frac{1}{8}$$

(b) L'Hospital's Rule for the indeterminate form $\left(\frac{\infty}{\infty}\right)$ or $\left(\frac{-\infty}{\infty}\right)$ or $\left(\frac{\infty}{-\infty}\right)$:

If $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$; provided the limit on the right exists.

Remark:

Suppose that functions f, g are *n* times differentiable and

$f(a) = f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = \infty.$

$g(a) = g'(a) = g''(a) = \cdots = g^{(n-1)}(a) = \infty.$

Suppose that $g^{(n)}(a) \neq 0$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$.

Example: Evaluate $\lim_{x\to\infty} \frac{x(\log x)^3}{1+x+x^2}$. C.W.

Solution: The limit is of the form $\left(\frac{\infty}{\infty}\right)$. Thus by L'Hospital's rule

$$\lim_{x\to\infty} \frac{x(\log x)^3}{1+x+x^2} = \lim_{x\to\infty} \frac{(\log x)^3+3x(\log x)^2\frac{1}{x}}{1+2x}$$

$$= \lim_{x\to\infty} \frac{(\log x)^3+3(\log x)^2}{1+2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{3(\log x)^2\frac{1}{x}+6(\log x)\frac{1}{x}}{2} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{3(\log x)^2+6(\log x)}{2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{6(\log x)\frac{1}{x}+6\frac{1}{x}}{2}$$

$$= \lim_{x\to\infty} \frac{6\log x+6}{2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{6\frac{1}{x}}{2} = \lim_{x\to\infty} \frac{3}{x} = 0$$

Example: Evaluate $\lim_{x\to 0}\frac{\log(\sin x)}{\log(\tan x)}$. H.W.

Solution: The limit is of the form $\left(\frac{\infty}{\infty}\right)$. Thus by L'Hospital's rule

$$\lim_{x\to 0}\frac{\log(\sin x)}{\log(\tan x)}=\lim_{x\to 0}\frac{\frac{1}{\sin x}}{\frac{1}{tan\ x}}$$

$$=\lim_{x\to 0}\frac{\tan x}{\sin}$$

$$=\lim_{x\to 0}\frac{\tan x}{x}\cdot\frac{x}{\sin x}$$

$$=1$$

Example: Evaluate $\lim_{x \to \infty} \frac{x^n}{e^x}$. H.W

Solution : The limit is of the form $\left(\frac{\infty}{\infty}\right)$. Thus by L'Hospital's rule

$$\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} \frac{nx^{n-1}}{e^x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{n(n-1)x^{n-2}}{e^x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{n!}{e^x} \quad \text{(Differentiating n times)}$$

$$= 0$$

Following are the indeterminate forms which can be reduced to either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form by simple transformations.

$0 \cdot \infty$ form

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then we can write

$\lim_{x\to a} f(x) \,.\, g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$ which can be solved using L'Hospital's rule.

Example: Evaluate $\lim_{x\to\frac{\pi}{2}} cosx.\, logtanx$. C.W

Solution: The limit is of the form$(0.\infty)$. Thus by L'Hospital's rule

$$\lim_{x\to\frac{\pi}{2}} cosx.\, logtanx = \lim_{x\to\frac{\pi}{2}} \frac{logtanx}{\sec x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\tan x} sec^2 x}{\sec x \tan x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sec x}{tan^2 x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sec x \tan x}{2\tan x\, sec^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{1}{2\sec x} = 0$$

Example: Prove that $\lim_{x\to 1}(1+sec\pi x)tan\frac{\pi x}{2}=0.$ **(Home work)**

Solution: The limit is of the form$(0.\infty)$. Thus by L'Hospital's rule

$$L.H.S.=\lim_{x\to 1}(1+sec\pi x)tan\frac{\pi x}{2}$$

$$=\lim_{x\to 1}\frac{1+sec\pi x}{\cot\frac{\pi x}{2}} \qquad \left(\frac{\mathbf{0}}{\mathbf{0}}\right)$$

$$=\lim_{x\to 1}\frac{sec\pi x.\cot\pi x\ (\pi)}{-co\qquad {}^{2}\frac{\pi x}{2}.(\frac{\pi}{2})}$$

$$=0$$

$(\infty - \infty)$ **form:**

To evaluate the limits of the type $\lim_{x\to a}[f(x) - g(x)]$, when $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, we reduce the expression in the form of $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ by taking LCM or by rearranging the terms and then apply L'Hospital's rule.

Example: Evaluate $\lim_{x\to 1}\left[\frac{1}{logx} - \frac{x}{x-1}\right]$. $(\infty - \infty)$ C.W.

Solution: The limit is of the form$(\infty - \infty)$. Thus

$$\lim_{x\to 1}\left[\frac{1}{logx} - \frac{x}{x-1}\right] = \lim_{x\to 1}\left[\frac{(x-1)-xlog\, x}{(x-1)\,\mathrm{lo}}\right] \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\left[\frac{1-log\, x-1}{(x-1)\,\frac{1}{x}+\log x}\right]$$

$$= \lim_{x\to 1}\left[\frac{-xl \quad x}{(x-1)\, +x\log \quad}\right] \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\left[\frac{-log\, x-1}{1+\log x+1}\right] = -\frac{1}{2}$$

$0^0, \infty^0, 1^\infty$ form (Exponential indeterminate forms)

$\lim_{x\to a} f(x)^{g(x)}$ is called an indeterminate of the type $\mathbf{0^0}$ if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ Or type $\infty^{\mathbf{0}}$ if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ Or type $\mathbf{1^\infty}$ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$.

To evaluate this kind of limit, let $l = \lim_{x\to a} f(x)^{g(x)}$ (0^0).

So $\log l = \lim_{x\to a} g(x) . \log f(x)$ which is of the form $(\infty. 0)$.

Then we can solve it by above method of the form $(\infty. 0)$

Example: Evaluate $\lim_{x\to\frac{\pi}{2}} (\cos x)^{\frac{\pi}{2}-x}$. $\quad (0^0)$ C.W

Solution: The limit is of the form(0^0). Thus

Let $y = \lim_{x\to\frac{\pi}{2}} (\cos x)^{\frac{\pi}{2}-x}$

$$\therefore log\ y = \lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \log\cos x \qquad (0.\infty)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\log\cos x}{1/\left(\frac{\pi}{2}-x\right)} \qquad \left(\frac{\infty}{\infty}\right)$$

Example: Evaluate $\lim\limits_{x \to \frac{\pi}{2}} (\cos x)^{\frac{\pi}{2} - x}$. $\quad (0^0)$

$$logy = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\cos}(-\sin x)}{1 \Big/ \left(\frac{\pi}{2} - x\right)^2}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)^2 (-\sin x)}{\cos x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)^2 (-\cos x) + 2\left(\frac{\pi}{2} - x\right) \sin}{-\sin x}$$

$$logy = 0$$

$$\therefore y = e^0 = 1$$

Example: Evaluate $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$. C.W

Solution: The limit is of the form (1^{∞}). Thus

$$\text{Let } y=\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$$

$$log\ y=\lim_{x\to 0}\frac{\log\left(\frac{a^x+b^x+c^x}{3}\right)}{x}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\frac{3}{a^x+b^x+c^x}\left\{\frac{a^x loga+\ ^x logb+c^x logc}{3}\right\}}{1}$$

$$=\lim_{x\to 0}\frac{a^x loga+b^x logb+c^x logc}{a^x+b^x+c^x}$$

$$=\frac{1}{3}\{\log a+\log b+\log c\}=\log(abc)^{1/3}$$

$$\therefore\ y=(abc)^{1/3}$$

Example: Evaluate $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x}$ H.W

Solution: The limit is of the form $(\infty^{\mathbf{0}})$. Thus

Let $y=\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos}$

$$\therefore \log y=\lim_{x\to 0}(1-\cos x)\log\frac{1}{x}$$

$$=\lim_{x\to 0}\left[2sin^2\frac{x}{2}\log\frac{1}{x}\right]$$

$$=-2\lim_{x\to 0}\frac{\log x}{cosec^2\frac{x}{2}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$=-2\lim_{x\to 0}\frac{\frac{1}{x}}{-2cosec^2\frac{x}{2}\cot\frac{x}{2}\cdot\frac{1}{2}}$$

$$=2\lim_{x\to 0}\frac{sin^3\frac{x}{2}}{x\cos\frac{x}{2}}$$

Example: Evaluate $\lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x}$ H.W

$$\log y = 2\lim_{x\to 0}\left(\frac{sin\frac{x}{2}}{\frac{x}{2}}\right)^{3}\frac{x^{3}}{8}.\frac{1}{\cos\frac{x}{2}}$$

$$\log y = \frac{1}{4}\lim_{x\to 0}\frac{x^{3}}{\text{co}\ \frac{x}{2}} = 0$$

$$\therefore y = e^{0} = 1$$

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Tests of convergence of series:

Some concepts to discuss for sequences:

- An ordered set of real numbers $a_1, a_2, a_3, \dots \dots ..$ is called a **sequence** and is denoted by $\{a_n\}$. If the number of terms is unlimited, then the sequence is said to be an infinite sequence and a_n is its nth term.

  e.g. $\{3,5,7,9, \dots ..\}$, Here nth term $u_n=2n+1$.
- **Limit**: A sequence is said to tend to a limit l, if for every ε>0, a natural number n0 can be found such that $|a_n - l| < \varepsilon \; for \; all \; n \geq n_0$.
- **Convergence:** If a sequence $\{a_n\}$ has a finite limit, it is called a convergent sequence. If the limit of sequence $\{a_n\}$ does not tend to a finite number, it is said to be divergent.

  e.g. $\left\{1, \frac{1}{4}, \frac{1}{9}, \dots .., \frac{1}{n^2}, \dots ..\right\}$ is a convergent sequence.

  $\{3,5,7, \dots ., (2n+1), \dots ..\}$ is a divergent sequence.
- **Bounded sequence:** A sequence $\{a_n\}$ is said to be bounded, if there exists a number k > 0 such that $|a_n| < k$ for every n.
- A sequence $\{a_n\}$ is called **increasing** if an ≤ an+1 for all n
- A sequence $\{a_n\}$ is called **decreasing** if an ≥ an+1 for all n

Monotonic sequence: A sequence $\{a_n\}$ is called monotonic if it is either increasing or decreasing.

e.g. $\{1,4,7,10, \ldots\ldots\}$ is a monotonic sequence.

$\left\{1, \frac{1}{4}, \frac{1}{9}, \ldots\ldots, \frac{1}{n^2}, \ldots\ldots\right\}$ is also a monotonic sequence.

$\{1, -1, 1, -1, \ldots \ldots\}$ is not a monotonic sequence.

A monotonic sequence always tends to a limit, finite or infinite.

A sequence which is monotonic and bounded is convergent.

Series:

Definition: If $u_1, u_2, u_3, \ldots \ldots\ldots, u_n, \ldots \ldots$ is an infinite sequence of real numbers, then $u_1 + u_2 + u_3 + \ldots + u_n + \cdots$ is called an infinite series. An infinite series is denoted by $\sum_{n=1}^{\infty} u_n$, the sum of its first n terms is denoted by $s_n = \sum_{j=1}^{n} u_j$ and it is known as partial sum of n terms.

e.g. 1+3+5+7+….. is an infinite series.

Convergent and divergent series:

Consider the infinite series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots \ldots\ldots + u_n + \cdots$ and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + \cdots \ldots\ldots + u_n$.

- If s_n tends to a **finite number** as $n \to \infty$, then the series $\sum_{n=1}^{\infty} u_n$ is said to be convergent.

  e.g. Test the convergence of the series $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots \frac{1}{(2n-1).(2n+1)}$

Solution: Let $s_n = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots \frac{1}{(2n-1).(2n+1)}$

$$= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{(2n+1)}\right).$$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{(2n+1)}\right) = \frac{1}{2}$, which is a finite quantity.

the series is convergent.

- If s_n tends to **infinity** as $n \to \infty$, then the series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

  e.g. Test the convergence of the series $1 + 2 + 3 + \cdots$.

Solution: Let $s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$, the series is divergent.

SOME STANDARD LIMITS

- $\lim_{n\to\infty} \frac{logn}{n} = 0.$
- $\lim_{n\to\infty} \frac{1}{logn} = 0.$
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$
- $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a.$
- $\lim_{n\to\infty} (1+n)^{1/n} = e.$
- $\lim_{n\to\infty} (n)^{\frac{1}{n}} = 1.$
- $\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty.$
- $\lim_{n\to 0} e^n = 1.$
- $\lim_{n\to\infty} x^n = 0$ if $|x| < 1.$
- $\lim_{n\to\infty} x^n = \infty$ if $|x| > 1.$
- $\lim_{n\to\infty} n. x^n = 0$ if $|x| < 1.$
- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all values of $x.$
- $\lim_{n\to\infty} \left[\frac{(n!)}{n}\right]^{1/n} = \frac{1}{e}.$

Example-1: $1+4+9+16+\cdots$ C.W.

Solution: We are given that $1+4+9+16+\cdots$

So, $S_n = 1+2^2+3^2+4^2+\cdots+n^2 = \frac{n}{6}(n+1)\,(2n+1)$

$\lim_{n\to\infty} s_n \to \infty$

Hence the given series is divergent.

Example-2 $4-9+5+4-9+5+4-9+5+\cdots$ C.W.

Solution: Here, $S_{3n} = 0$, $S_{3n+1} = 4$, $S_{3n+2} = -5$

As S_n does not tend to an unique limit, the series is oscillatory.

General properties of infinite series:

The nature (convergence or divergence) of an infinite series does not change:

- by multiplication of all terms by a non-zero number *k*.
- by addition or deletion of a finite number of terms.

If two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent, then $\sum_{n=1}^{\infty}(u_n + v_n)$ is also convergent.

Necessary condition for convergence

If a series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim\limits_{n\to\infty} u_n = 0.$

Note: The converse of this result need not be true.

Consider, for instance, the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + .. + \frac{1}{\sqrt{n}} + \cdots.$

Since the term go on descending,

$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + .. + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + .. + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}}$

So, $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \sqrt{n} \longrightarrow \infty.$

Thus the series is divergent even though $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{\mathrm{n}}} = 0.$

$\lim\limits_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for the convergence of $\sum_{n=1}^{\infty} u_n$.

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

General properties of infinite series:

The nature (convergence or divergence) of an infinite series does not change:

- by multiplication of all terms by a non-zero number *k*.
- by addition or deletion of a finite number of terms.

If two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent, then $\sum_{n=1}^{\infty}(u_n + v_n)$ is also convergent.

Necessary condition for convergence

If a series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim\limits_{n\to\infty} u_n = 0$.

Note: The converse of this result need not be true.

Consider, for instance, the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + .. + \frac{1}{\sqrt{n}} + \cdots$.

Since the term go on descending,

$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + .. + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + .. + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}}$

So, $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \sqrt{n} \longrightarrow \infty$.

Thus the series is divergent even though $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{\mathrm{n}}} = 0$.

$\lim\limits_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for the convergence of $\sum_{n=1}^{\infty} u_n$.

We have simple test/ zero test for divergence from above:
If $\lim_{n\to\infty} u_n \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ does not converge.

Geometric series:
Consider geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

$$\therefore s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

a) When $|r| < 1$,

$$\lim_{n\to\infty} r^n = 0 \therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \text{ is finite.}$$

Hence the series is **convergent**.

b) When $r > 1$,

$$\lim_{n\to\infty} r^n \to \infty \therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} \to \infty$$

Hence the series is **divergent**.

c) When $r = 1$ $s_n = a + a + a + \cdots + a(\text{n times}) = na$

$$\therefore \lim_{n\to\infty} s_n \to \infty$$

Hence the series is **divergent**.

Note: Geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is

i. **convergent if $|r| < 1$**
ii. **divergent if $r \geq 1$**
iii. **oscillatory if $r \leq -1$**

The sum of the Geometric series is $S = \frac{a}{1-r}$.

Example-3: $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\cdots$ C.W.

Solution: We are given that $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\cdots$

So, $S_n=1+\frac{3}{4}+\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^4+\cdots\left(\frac{3}{4}\right)^{n-1}$

Which is in Geometric series having common ration $\mathrm{r}=\frac{3}{4}$ < 1 and a=1 .

$$S_n=\frac{\left(1-\left(\frac{3}{4}\right)^n\right)}{1-\frac{3}{4}}=4\left[1-\left(\frac{3}{4}\right)^n\right]$$

$$\therefore \lim_{n\to\infty} s_n=\lim_{n\to\infty} 4\left[1-\left(\frac{3}{4}\right)^n\right]$$

$$\lim_{n\to\infty} s_n=4$$

Hence the series is convergent.

Example-4: $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. H.W.

Solution: Given series is geometric series with $r = \frac{1}{2} < 1$.

Here $S_n = \frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 2\left[1 - \left(\frac{1}{2}\right)^n\right]$

$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} 2\left[1 - \left(\frac{1}{2}\right)^n\right]$

$\lim_{n\to\infty} s_n = 2$

Hence the series is convergent.

Example-5: The water treatment plant removes one mth of the impurity in the first stage. In each succeeding stage, the amount of impurity removed is one-mth of the removed in the preceding stage. Show that if *m*=2, the water can be made as pure as you like, but if *m*=3, at least one-half of the impurity will remain no matter how many stages are used.

Solution: The total of impurity P removed after infinite number of stage (really it is not possible to wait for that long time) is given by the series,

$P = \sum_{n=1}^{\infty} \frac{1}{m^n}$

If we take *m=2* then $P = \sum_{n=1}^{\infty} \frac{1}{2^n}$, which is geometric series with $r = \frac{1}{2}$

So, the series is convergent and its value is $P = \frac{1/2}{1-1/2} = 1.$

Thus it completely removes the impurity.

Example-5: The water treatment plant removes one mth of the impurity in the first stage. In each succeeding stage, the amount of impurity removed is one-mth of the removed in the preceding stage. Show that if *m*=2, the water can be made as pure as you like, but if *m*=3, at least one-half of the impurity will remain no matter how many stages are used.

Now if we take *m=3* then $P = \sum_{n=1}^{\infty} \frac{1}{3^n}$. which is geometric series with $r = \frac{1}{3}$

So, the series is convergent and its value is $P = \frac{1/3}{1-1/3} = \frac{1}{2}$.

It will remain at least one-half of the impurity.

- It is not always possible to find the partial sum S_n for every series easily. Thus, it becomes necessary to use other tests for series with all terms positive.
- Using these tests, we can discuss the convergence/divergence of series.

***p*- Series test or Hyper harmonic series test:**

The *p*- series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is

i. convergent if $p>1$

ii. divergent if $p \leq 1$.

<u>Comparison Test:</u>

1. If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ (i.e. $u_n > 0$ and $v_n > 0$ for all n) are such that

   - $\sum_{n=1}^{\infty} v_n$ converges and

   $- u_n \leq v_n$, for all *n*,

   then $\sum_{n=1}^{\infty} u_n$ also converges.

2. If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are such that

   - $\sum_{n=1}^{\infty} v_n$ diverges and

   - $u_n \geq v_n$, for all *n*,

   then $\sum_{n=1}^{\infty} u_n$ also diverges.

3. **Limit form**

   If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are such that $\lim_{n\to\infty} \frac{u_n}{v_n} =$ finite and nonzero, then

   $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converge or diverge together.

Note:

1. The comparison test requires convergence or divergence of another series $\sum_{n=1}^{\infty} v_n$, which is known as "Auxiliary series"
2. To check convergence of auxiliary series, generally we use geometric series test or p-series test.
3. In general consider $v_n = \frac{1}{n^p}$ which can be obtained from u_n ,

where

p = (the highest power n in denominator of u_n) - (the highest power n in numerator of u_n)

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

General properties of infinite series:

The nature (convergence or divergence) of an infinite series does not change:

- by multiplication of all terms by a non-zero number *k*.
- by addition or deletion of a finite number of terms.

If two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent, then $\sum_{n=1}^{\infty}(u_n + v_n)$ is also convergent.

Necessary condition for convergence

If a series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

Note: The converse of this result need not be true.

Consider, for instance, the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + .. + \frac{1}{\sqrt{n}} + \cdots$.

Since the term go on descending,

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + .. + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + .. + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}}$$

So, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sqrt{n} \longrightarrow \infty$.

Thus the series is divergent even though $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{\mathrm{n}}} = 0$.

$\lim_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for the convergence of $\sum_{n=1}^{\infty} u_n$.

We have simple test/ zero test for divergence from above:

If $\lim_{n\to\infty} u_n \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ does not converge.

Geometric series:

Consider geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

$$\therefore s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

a) When $|r| < 1$,

$\lim_{n\to\infty} r^n = 0 \therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$ is finite.

Hence the series is **convergent**.

b) When $r > 1$,

$\lim_{n\to\infty} r^n \to \infty \therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} \to \infty$

Hence the series is **divergent**.

c) When $r = 1$ $s_n = a + a + a + \cdots + a(\text{n times}) = na$

$\therefore \lim_{n\to\infty} s_n \to \infty$

Hence the series is **divergent**.

Note: Geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is

i. **convergent if $|r| < 1$**
ii. **divergent if $r \geq 1$**
iii. **oscillatory if $r \leq -1$**

The sum of the Geometric series is $S = \frac{a}{1-r}$.

Example-3: $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\cdots$ C.W.

Solution: We are given that $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\cdots$

So, $S_n = 1+\frac{3}{4}+\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^4+\cdots\left(\frac{3}{4}\right)^{n-1}$

Which is in Geometric series having common ration $\mathrm{r}=\frac{3}{4}$ < 1 and a=1 .

$$S_n = \frac{\left(1-\left(\frac{3}{4}\right)^n\right)}{1-\frac{3}{4}} = 4\left[1-\left(\frac{3}{4}\right)^n\right]$$

$$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} 4\left[1-\left(\frac{3}{4}\right)^n\right]$$

$$\lim_{n\to\infty} s_n = 4$$

Hence the series is convergent.

Example-4: $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. H.W.

Solution: Given series is geometric series with $r = \frac{1}{2} < 1$.

Here $S_n = \frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 2\left[1 - \left(\frac{1}{2}\right)^n\right]$

$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} 2\left[1 - \left(\frac{1}{2}\right)^n\right]$

$\lim_{n\to\infty} s_n = 2$

Hence the series is convergent.

Example-5: The water treatment plant removes one m^{th} of the impurity in the first stage. In each succeeding stage, the amount of impurity removed is one-m^{th} of the removed in the preceding stage. Show that if *m*=2, the water can be made as pure as you like, but if *m*=3, at least one-half of the impurity will remain no matter how many stages are used.

Solution: The total of impurity P removed after infinite number of stage (really it is not possible to wait for that long time) is given by the series,

$P = \sum_{n=1}^{\infty} \frac{1}{m^n}$

If we take *m=2* then $P = \sum_{n=1}^{\infty} \frac{1}{2^n}$, which is geometric series with $r = \frac{1}{2}$

So, the series is convergent and its value is $P = \frac{1/2}{1-1/2} = 1.$

Thus it completely removes the impurity.

Example-5: The water treatment plant removes one m^{th} of the impurity in the first stage. In each succeeding stage, the amount of impurity removed is one-m^{th} of the removed in the preceding stage. Show that if *m*=2, the water can be made as pure as you like, but if *m*=3, at least one-half of the impurity will remain no matter how many stages are used.

Now if we take *m=3* then $P = \sum_{n=1}^{\infty} \frac{1}{3^n}$. which is geometric series with $r = \frac{1}{3}$

So, the series is convergent and its value is $P = \frac{1/3}{1-1/3} = \frac{1}{2}$.

It will remain at least one-half of the impurity.

- It is not always possible to find the partial sum S_n for every series easily. Thus, it becomes necessary to use other tests for series with all terms positive.
- Using these tests, we can discuss the convergence/divergence of series.

***p*- Series test or Hyper harmonic series test:**

The *p*- series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is

i. convergent if $p>1$

ii. divergent if $p \leq 1$.

<u>Comparison Test:</u>

1. If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ (i.e. $u_n > 0$ and $v_n > 0$ for all n) are such that

   - $\sum_{n=1}^{\infty} v_n$ converges and

   $- u_n \leq v_n$, for all *n*,

   then $\sum_{n=1}^{\infty} u_n$ also converges.

2. If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are such that

   - $\sum_{n=1}^{\infty} v_n$ diverges and

   - $u_n \geq v_n$, for all *n*,

   then $\sum_{n=1}^{\infty} u_n$ also diverges.

3. **Limit form**

   If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are such that $\lim_{n\to\infty} \frac{u_n}{v_n} =$ finite and nonzero, then

   $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converge or diverge together.

Note:

1. The comparison test requires convergence or divergence of another series $\sum_{n=1}^{\infty} v_n$, which is known as "Auxiliary series"
2. To check convergence of auxiliary series, generally we use geometric series test or p-series test.
3. In general consider $v_n = \frac{1}{n^p}$ which can be obtained from u_n ,

where

p = (the highest power n in denominator of u_n) - (the highest power n in numerator of u_n)

Example-6: Test the convergence of the series $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\cdots$. C.W.

Solution: Here $u_n=\frac{2n-1}{n(n+1)(n+2)}$

Let $v_n=\frac{1}{n^2}$

$$\therefore \frac{u_n}{v_n}=\frac{n^2(2n-1)}{n(n+1)(n+2)}$$

$$=\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=2$ which is finite

$\therefore$ Both series are either divergent or convergent together.

But $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent.

(as per *p*-series test , here p=2>1)

$\therefore$ The given series is convergent , by comparison test.

Example-7: Test the convergence of the series$\sum_{n=1}^{\infty}\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$.
H.W.

Solution: Here $u_n=\sqrt{n^3+1}-\sqrt{n^3}$

$$=\left(\sqrt{n^3+1}-\sqrt{n^3}\right)\times\frac{\sqrt{n^3+1}+\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$=\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$$

Let $v_n=\frac{1}{n^{3/2}}$

$$\therefore \frac{u_n}{v_n}=\frac{n^{3/2}}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$=\frac{1}{\sqrt{1+\frac{1}{n^3}}+1}$$

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{2}$ which is finite

Example-7: Test the convergence of the series $\sum_{n=1}^{\infty}\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$.
H.W.

$\therefore$ Both series are either divergent or convergent together.

But $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

(as per *p*-series test , here p=3/2>1)

$\therefore$ The given series is convergent, by comparison test.

Example-8: Test the convergence of the series

$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$. C.W.

Solution: Here $u_n = \frac{n+1}{n^p}$

Let $v_n = \frac{1}{n^{p-1}}$

$$\therefore \frac{u_n}{v_n} = \frac{(n+1)n^{p-1}}{n^p}$$

$$= 1 + \frac{1}{n}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = 1$ which is finite

$\therefore$ Both series are either divergent or convergent together.

But $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ is convergent if *p-1>1 (p>2)* and divergent if *p-1≤1 (p ≤ 2)* . (as per *p*-series test)

$\therefore$ The given series is convergent if *p>2* and divergent if *p ≤ 2*, by comparison test.

Example-9: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n} sin\left(\frac{1}{n}\right)$. C.W.

Solution: Here $u_n = \frac{1}{n} sin\left(\frac{1}{n}\right)$

$$u_n = \frac{1}{n}\left(\frac{1}{n} - \frac{1}{3!n^3} + \frac{1}{5!n^5} - \cdots\right) \quad \left(\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \frac{1}{n^2} - \frac{1}{3!n^4} + \frac{1}{5!n^6} - \cdots$$

Let $v_n = \frac{1}{n^2}$

$$\therefore \frac{u_n}{v_n} = 1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots$$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = 1$ which is finite

$\therefore$ Both series are either divergent or convergent together.

But $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(as per *p*-series test, here p=2>1)

$\therefore$ The given series is convergent , by comparison test.

Example-10: Test the convergence of the series $\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+\cdots$. H.W.

Solution: Here $u_n=\frac{1}{2n(2n+2)}$

Let $v_n=\frac{1}{n^2}$

$$\therefore \frac{u_n}{v_n}=\frac{n^2}{2n(2n+2)}$$

$$=\frac{1}{4\left(1+\frac{2}{2n}\right)}$$

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=1/4$ which is finite

$\therefore$ Both series are either divergent or convergent together.

But $\sum_{n=1}^{\infty} v_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent.

(as per *p*-series test , here p=2>1)

$\therefore$ The given series is convergent , by comparison test.

Example-11: Test the convergence of the series $\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\cdots+\frac{n+1}{n^2}$ H.W.

Solution: Here $u_n=\frac{n+1}{n^2}$

Let $v_n=\frac{1}{n}$

$$\therefore \frac{u_n}{v_n}=\frac{\left(\frac{n+1}{n^2}\right)}{\left(\frac{1}{n}\right)}$$

$$=\frac{n+1}{n}$$

$\therefore \lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{n+1}{n}=1$ which is finite

∴ Both series are either divergent or convergent together.

But $\sum_{n=1}^{\infty} v_n=\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

(as per *p*-series test , here p=1)

∴ The given series is divergent , by comparison test.

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin’s and Taylor’s series expansion.
- L’Hospital’s rule and related applications, Indeterminate forms

D'Alembert's Ratio Test:

If $\sum_{n=1}^{\infty} u_n$ is a positive term series and $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then

- $\sum_{n=1}^{\infty} u_n$ is convergent if $l < 1$.
- $\sum_{n=1}^{\infty} u_n$ is divergent if $l > 1$.
- If $l = 1$, the ratio test fails. i.e., no conclusion can be drawn about the convergence or divergence of the series.

Example-12: Test the convergence of the series $\frac{1}{10}+\frac{2!}{10^2}+\frac{3!}{10^3}+\cdots$ .C.W.

Solution: Here $u_n=\frac{n!}{10^n}$

$$\therefore u_{n+1}=\frac{(n+1)!}{10^{n+1}}$$

$$\therefore \frac{u_{n+1}}{u_n}=\frac{(n+1)!}{10^{n+1}}\frac{10^n}{n!}$$

$$=\frac{n+1}{10}$$

$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n}\to\infty$ which is greater than 1.

$\therefore$ The given series is divergent , by D'Alembert's ratio test.

Example-13: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ .C.W.

Solution: Here $u_n = \frac{n^2+1}{5^n}$

$$\therefore\ u_{n+1} = \frac{(n+1)^2+1}{5^{n+1}}$$

$$\therefore\ \frac{u_{n+1}}{u_n} = \frac{(n+1)^2+1}{5^{n+1}} \frac{5^n}{n^2+1}$$

$$= \frac{(1+1/n)^2+1}{5\,(1+1/n^2)}$$

$\therefore\ \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{5}$ which is less than 1.

$\therefore$ The given series is convergent , by D'Alembert's ratio test.

Example-14: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$.C.W.

Solution: Here $u_n = \frac{3^n n!}{n^n}$

$$\therefore u_{n+1} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{n^n}{3^n n!}$$

$$= \frac{3n^n}{(n+1)^n} = \frac{3(n+1)^{-n}}{n^{-n}}$$

$$\therefore \frac{u_{n+1}}{u_n} = 3\left(1+\frac{1}{n}\right)^{-n}$$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 3e^{-1}$ which is greater than 1.

$\therefore$ The given series is divergent , by D'Alembert's ratio test.

Example-15: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{e^{-n}}$.H.W.

Solution: Here $u_n = \frac{n}{e^{-n}} = ne^n$

$$\therefore u_{n+1} = (n+1)e^{(n+1)}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)e^{(n+1)}}{ne^n}$$

$$= \left(\frac{n+1}{n}\right) e$$

$$\therefore \frac{u_{n+1}}{u_n} = \left(1 + \frac{1}{n}\right) e$$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = e$ which is greater than 1.

$\therefore$ The given series is divergent , by D'Alembert's ratio test.

Example-16: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n.2^n(n+1)!}{3^n n!}$.H.W.

Solution: Here $u_n = \frac{n.2^n(n+1)!}{3^n n!}$

$$\therefore u_{n+1} = \frac{(n+1).2^{n+1}(n+2)!}{3^{n+1}(n+1)!}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1).2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \frac{3^n n!}{n.2^n(n+1)!}$$

$$= \frac{(n+1).2^n.2\ (n+2)(n+1)!}{3^n.\,3(n+1)!} \frac{3^n n!}{n.2^n(n+1)!}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{2}{3}\frac{(n+1)}{n}$$

$\therefore \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{2}{3}$ which is less than 1.

$\therefore$ The given series is convergent , by D'Alembert's ratio test.

Example-17: Test the convergence of the series

$\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+\cdots$ .H.W.

Solution: Here $u_n=\frac{n}{1+2^{-n}}$

$$\therefore\ u_{n+1}=\frac{n+1}{1+2^{-(n+1)}}$$

$$\therefore\ \frac{u_{n+1}}{u_n}=\frac{n+1}{1+2^{-(n+1)}}\frac{1+2^{-n}}{n}$$

$$=\frac{n+1}{n}\frac{1+2^{-n}}{1+2^{-(n+1)}}$$

$$\therefore\ \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=1\ .$$

As the limit is one, the D'Alembert's ratio test fails.

Now let $v_n=\frac{1}{n^{-1}}$

$$\therefore\ \frac{u_n}{v_n}=\frac{1}{1+2^{-n}}$$

Example-17: Test the convergence of the series

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \cdots$$ .H.W.

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = 1$ (finite)

$\therefore$ Both series are either divergent or convergent together.

But $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{-1}}$ is divergent.

(as per *p*-series test , here p=-1)

$\therefore$ The given series is divergent , by comparison test.

Cauchy's Root Test:

If $\sum u_n$ is a positive term series and $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$, then

- $\sum u_n$ is convergent if *l* < 1.
- $\sum u_n$ is divergent if *l* > 1.
- If *l* = 1, the root test fails. i.e., no conclusion can be drawn about the convergence or divergence of the series.

Thank You…

Department of Mathematical Sciences,
P. D. Patel Institute of Applied Sciences,
Charotar University of Science and Technology, Changa

Unit I: Higher Order Derivatives and Applications

Unit I : Higher Order Derivatives and Applications

- Set theory and Function
- Limit, Continuity, Differentiability for function of single variable and its uses. Mean Value Theorem, Local Maxima and Minima
- Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic etc.
- Leibnitz rule for the n^{th} order derivatives of product of two functions
- Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test. Power series expansion of a function: Maclaurin's and Taylor's series expansion.
- L'Hospital's rule and related applications, Indeterminate forms

Cauchy's Root Test:

If $\sum u_n$ is a positive term series and $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$, then

- $\sum u_n$ is convergent if *l* < 1.
- $\sum u_n$ is divergent if *l* > 1.
- If *l* = 1, the root test fails. i.e., no conclusion can be drawn about the convergence or divergence of the series.

Example-18: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.C.W.

Solution: Here $u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

$$\therefore (u_n)^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \frac{1}{e} < 1$$

$\therefore$ The given series is convergent , by Cauchy root test.

Example-19: Test the convergence of the series

$\sum_{n=1}^{\infty}\left(\frac{\log n}{1000}\right)^n$ .C.W.

Solution: Here $u_n = \left(\frac{\log n}{1000}\right)^n$

$$\therefore (u_n)^{\frac{1}{n}} = \left(\frac{\log}{1000}\right)$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{\log n}{1000}\right) \to \infty$$

which is greater than 1.

$\therefore$ The given series is divergent, by Cauchy root test.

Example-20: Test the convergence of the series$\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ .H.W.

Solution: Here $u_n = \frac{n^{10}}{10^n}$

$$\therefore\ (u_n)^{\frac{1}{n}} = \left[\frac{n^{10}}{10^n}\right]^{\frac{1}{n}}$$

$$= \frac{n^{10/n}}{10}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{10/n}}{10}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \frac{1}{10} < 1$$

$\therefore$ The given series is convergent by Cauchy root test.

Example-21: Test the convergence of the series $\sum\left(\frac{n}{n+1}\right)^{n^2}$ H.W.

Solution: Here $u_n=\left(\frac{n}{n+1}\right)^{n^2}$

$$\therefore (u_n)^{\frac{1}{n}}=\left(\frac{n}{n+1}\right)^{n}$$

$$=[1+1/n]^{-n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=\lim_{n\to\infty}[1+1/n]^{-n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=\frac{1}{e}<1$$

$\therefore$ The given series is convergent, by Cauchy root test.

Example-22: Test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$ .H.W.

Solution: Here $u_n=\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-n}$

$$\therefore (u_n)^{\frac{1}{n}}=\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-1}$$

$$=\left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{-1}$$

$$=\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=[e-1]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=\frac{1}{e-1}<1$$

$\therefore$ The given series is convergent , by Cauchy root test.

Alternating series:

An infinite series with alternate positive and negative terms is called an alternating series.

Leibnitz's test for alternating series:

An alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, where $u_n > 0$ is convergent if

- $\{u_n\}$ is strictly decreasing, i.e., $u_{n+1} < u_n$ for all *n*.
- $\lim_{n\to\infty} u_n = 0.$

Remark:

If $\lim_{n\to\infty} u_n \neq 0$, then the series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ is an oscillating series.

Example-23: Show that the series

$\frac{1}{2^3} - (1+2)\frac{1}{3^3} + (1+2+3)\frac{1}{4^3} - \cdots$ is convergent. C.W.

Solution: Given series $\sum_{n=1}^{\infty}(-1)^{n+1}(1+2+3+\cdots+n)\frac{1}{(n+1)^3}$ is an alternating series

$$u_n = (1+2+3+\cdots+n)\frac{1}{(n+1)^3}$$

$$= \frac{n(n+1)}{2(n+1)^3} = \frac{n}{2(n+1)^2}$$

Clearly $u_{n+1} = \frac{n+1}{2(n+2)^2} < \frac{n}{2(n+1)^2} = u_n$

Here each term is less than the preceding term of the series.

Example-23: Show that the series

$$\frac{1}{2^3} - (1+2)\frac{1}{3^3} + (1+2+3)\frac{1}{4^3} - \cdots$$ is convergent.C.W.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{2\left(1+\frac{1}{n}\right)^2} = 0$$

∴ By Leibnitz test, the given series is convergent.

Example-24: Test the convergence of the series$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2n-1}$.
C.W.

Solution: Given series $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{n}{2n-1}$ is an alternating series

Here $u_n = \frac{n}{2n-1}$

$u_{n+1} = \frac{n+1}{2n+1}$

$\therefore u_{n+1} - u_n = \frac{n+1}{2n+1} - \frac{n}{2n-} = -\frac{1}{4n^2-1} < 0$

$\therefore$ each term is less than the preceding term of the series.

And $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0.$

$\therefore$ The given series is an oscillatory by Leibnitz test.

Example-25: Test the convergence of the series

$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$ H.W.

Solution: Given series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$ is an alternating series

Here $u_n = \left(\sqrt{n+1}-\sqrt{n}\right)$

$$= (\sqrt{n+1}-\sqrt{n}) \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Clearly $u_{n+1} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} < \frac{1}{\sqrt{n+1}+\sqrt{n}} = u_n$

$\therefore$ each term is less than the preceding term of the series.

And $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$

$\therefore$ The given series is convergent by Leibnitz test.

Example-26: Test the convergence of the series

$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ \ldots$ .H.W.

Solution: Given series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n!}$ is an alternating series

Here $u_n=\frac{1}{n!}$

$$u_{n+1}=\frac{1}{(n+1)!}$$

$\therefore\ u_{n+1}-u_n=\frac{1}{(n+1)!}-\frac{1}{n!}<0$

$\therefore$ each term is less than the preceding term of the series.

And $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{1}{n!}=0.$

$\therefore$ The given series is convergent by Leibnitz test.

Thank You…