Unit - 3

Random Variable (Single)
- Discrete R.V. $X = 1, 2, 3$
- Continuous R.V.

Random Variable
- Single R.V
- Double R.V.

- Dist.
- 5%.
- Cont.

- Dist.
- 5%.
- Cont.

Probability Density Fn (PDF) → Continous

Probability Mass Fn (PMF) → Discrete

Q) A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and Variance.
Success is getting 1 or 6.

p (probability of getting success) $= \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$

P.T.O.

Probability Distribution.

$X = 0$ $\quad P(\text{no Success}) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

$X = 1$ $\quad P(\text{getting 1 success}) = {}^3C_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$ 4/9

$X = 2$ $\quad P(\text{getting 2 success}) = {}^3C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{9} = \frac{6}{27}$

$X = 3$ $\quad P(\text{getting 3 success}) = {}^3C_3 \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

X	0	1	2	3
P(X)	8/27	4/9	2/9	1/27

$\Sigma P(x) = 1$, Mandatory Condition.

$P(X) > 0$ always.

Mean $\Rightarrow \sum p(x) \cdot x = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$

Variance $\Rightarrow \sum p(x) \cdot x^2 - (Mean)^2$

Q) Random Variable X has the following probability function →

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(a) Find K
(b) Evaluate $P(x<6)$, $P(x \geq 6)$ & $P(3 < x \leq 6)$
find the minimum value of x, so that $P(X \leq x) > 1/2$.

Sol →

$\sum p(x) = 1 \Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 9k + 10k^2 = 1$

$\Rightarrow 10k^2 + 9k - 1 = 0$

$k = \dfrac{-9 \pm \sqrt{81 + 40}}{20} = \dfrac{-9 \pm \sqrt{121}}{20}$

$= \dfrac{-9 \pm 11}{20} = \dfrac{2}{20}, \dfrac{-20}{20}$

$= 0.1, \cancel{-1}$

$\boxed{k \Rightarrow 1/10}$

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(i) $P(x < 6) = 1 - P(X \geq 6)$

$= 1 - [P(6) + P(7)]$

$= 1 - [2k^2 + 7k^2 + k]$

$= 1 - \left[\frac{9}{100} + \frac{1}{10}\right]$

$= 1 - \left[\frac{19}{100}\right]$

$= \boxed{\frac{81}{100}}$

(ii) $P(x \geq 6) = \boxed{\frac{19}{100}}$

(iii) $P(3 < x \leq 6) \Rightarrow P(4) + P(5) + P(6)$

$\Rightarrow \frac{3}{10} + \frac{1}{100} + \frac{2}{100}$

$= \frac{30 + 1 + 2}{100} = \boxed{\frac{33}{100}}$

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$P(x \le 1) = \frac{1}{10} < \frac{1}{2}$

$P(x \le 2) = 3k = \frac{3}{10}$

$P(x \le 3) = 5k = \frac{1}{2}$

$P(X \le 4) = 8k = \frac{8}{10} > \frac{1}{2}$

∴ Minimum value of x is $\boxed{4}$.

Q) x : 0 1 2 3 4 5 6 7 8

$p(x)$: a 3a 5a 7a 9a 11a 13a 15a 17a

(i) Determine the value of a.

(ii) Find $P(x<3)$, $P(x \ge 3)$, $P(2 \le x < 5)$

(iii) What is the smallest value of x $P(X \le x) > 0.5$.

Mean $= \sum x P(x)$
Variance $= \sum x^2 P(x) - (\text{Mean})^2$

Sol (i) $a = \frac{1}{81}$

(ii) $P(x<3) = 1/9$
$P(x \ge 3) = 8/9$
$P(2 \le x < 5) = 7/27$

(iii) Smallest value is 6.

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Probability density Function ⟷

x is continuous

$$f(x) = \begin{cases} 0 & x<0 \\ \frac{3x(2-x)}{4} & 0 \le x \le 2 \\ 0 & x>2 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\, dx + \int_{0}^{2} \frac{3x(2-x)}{4}\, dx + \int_{2}^{\infty} 0\, dx$$

$$\Rightarrow \int_{0}^{2} \frac{6x}{4} - \frac{3x^2}{4}\, dx$$

$$\Rightarrow \frac{6}{4}\left[\frac{x^2}{2}\right]_0^2 - \frac{3}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$\Rightarrow \frac{\cancel{6}^{3}}{\cancel{4}_{2}}\left[2\right] - \frac{\cancel{3}}{4}\left[\frac{8}{\cancel{3}}\right]$$

$$\Rightarrow 3 - 2 = 1$$

$$P\left(\tfrac{1}{3} \le x \le \tfrac{1}{2}\right) = \int_{1/3}^{1/2} f(x)\, dx$$

$$P\left(\tfrac{1}{3} \le x \le \tfrac{1}{2}\right) = \int_{1/3}^{1/2} \frac{6}{4}x - \frac{3x^2}{4}\,dx$$

$$\Rightarrow \frac{6}{4}\left[\frac{x^2}{2}\right]_{1/3}^{1/2} - \frac{3}{4}\left[\frac{x^3}{3}\right]_{1/3}^{1/2}$$

$$\Rightarrow \frac{6}{4}\left[\frac{1}{8} - \frac{1}{18}\right] - \frac{3}{4}\left[\frac{1}{24} - \frac{1}{81}\right]$$

$$\Rightarrow \frac{6}{4\times 2}\left[\frac{1}{4} - \frac{1}{9}\right] - \frac{3}{48}\left[\frac{1}{\cancel{10}8} - \frac{1}{27}\right]$$

$$\Rightarrow \frac{6}{8}\left[\frac{9-4}{36}\right] - \frac{3}{12}\left[\frac{19}{216}\right]$$

$$\Rightarrow \frac{6}{8}\left[\frac{5}{36}\right] - \frac{\cancel{3}}{12} \times \frac{19}{\cancel{216}\,72}$$

$$\Rightarrow \frac{5}{48} - \frac{19}{864}$$

$$= 0.082175926$$

Cummulative Distribution function
OR
Distribution Function

$$F(x) = \int_{-\infty}^{x} f(x)\, dx = P(X \le x)$$

$F(x) \downarrow$ Distribution Function.

Properties →

1) $\boxed{F'(x) = f(x)}$

2) $F(-\infty) = 0$

3) $F(\infty) = 1$

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Q) If : $f(x) = cx^2 \quad 0 < x < 1$ is pdf. Determine c. Find the Probability that $\frac{1}{3} < x < \frac{1}{2}$

i.e. $P\left(\frac{1}{3} < x < \frac{1}{2}\right)$

$\Rightarrow \int_{-\infty}^{\infty} cx^2\, dx = 1 \Rightarrow \int_0^1 cx^2\, dx = 1$

$\Rightarrow c \cdot \left[\frac{x^3}{3}\right]_0^1 = 1$

$\Rightarrow c\left[\frac{1}{3}\right] = 1$

$\Rightarrow \boxed{c = 3}$

$P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \int_{1/3}^{1/2} 3x^2\, dx$

$\Rightarrow \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_{1/3}^{1/2}$

$\Rightarrow \frac{1}{8} - \frac{1}{27} = \boxed{\frac{19}{216}}$ Ans

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(1) Let X be Continous random variable with PDf given by →

$$f(x) = \begin{cases} kx & 0 \le x < 1 \\ k & 1 \le x < 2 \\ -kx + 3k & 2 \le x < 3 \\ 0 & \text{else where} \end{cases}$$

Determine (1) Const. k. (2) CDF.

Sol → $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{0}^{1} kx\, dx + \int_{1}^{2} k\, dx + \int_{2}^{3} -kx + 3k + \int_{3}^{\infty} 0 = 1$$

$$\Rightarrow \frac{1}{2}k + k - \frac{5k}{2} + 3k = 1$$

$$\Rightarrow \frac{k + 2k - 5k + 6k}{2} = 1$$

$$\Rightarrow 4k = 2$$

$$\Rightarrow \boxed{k = \frac{1}{2}}$$

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CDF ⇒

$$-\infty < x < \infty$$

1) $-\infty < x < 0 \quad \int_{-\infty}^{x} 0\, dt = 0$

2) $0 < x < 1 \quad \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} \frac{1}{2} t\, dt = \frac{x^2}{4}$

3) $1 < x < 2 \quad \int_{-\infty}^{0} 0\, dt + \int_{0}^{1} \frac{1}{2} t\, dt + \int_{1}^{x} \frac{1}{2}\, dt = \frac{x}{2} - \frac{1}{4}$

4) $2 \leq x < 3 \quad \int_{-\infty}^{0} 0\, dt + \int_{0}^{1} \frac{1}{2} t\, dt + \int_{1}^{2} \frac{1}{2}\, dt + \int_{2}^{x} \left(-\frac{1}{2} t + \frac{3}{2}\right) dt$

$$= -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4}$$

5) $3 < x < \infty$

$$\int_{-\infty}^{0} 0\, dt + \int_{0}^{1} \frac{1}{2} t\, dt + \int_{1}^{2} \frac{1}{2}\, dt + \int_{2}^{3} \left(-\frac{1}{2} t + \frac{3}{2}\right) dt + \int_{3}^{\infty} 0\, dt = 1$$

$$\therefore F(x) = \begin{cases} 0 & -\infty \leq x < 0 \\ x^2/4 & 0 \leq x < 1 \\ x/2 - 1/4 & 1 \leq x < 2 \\ -x^2/4 + 3x/2 - 5/4 & 2 \leq x < 3 \\ 1 & 3 \leq x < \infty \end{cases}$$

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1) The diameter ray x of an electric cable is assumed to be a continuous r.v with pdf
$f(x) = 6x(1-x) \quad 0 < x < 1$

(i) Check that above is Pdf.
(ii) obtain CDF
(iii) Compute $P(x \le 1/2 \mid 1/3 \le x \le 2/3)$
(iv) Determine k such that $P(x<k) = P(x>k)$

Sol (i) $\int_0^1 6x(1-x)\,dx = 1$

$\Rightarrow 6\int_0^1 x - x^2\,dx$

$\Rightarrow 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= 6\left[\frac{1}{2} - \frac{1}{3}\right] \Rightarrow 6\left[\frac{3-2}{6}\right] = 1 = RHS$

It is a Pdf.

(ii) Cdf : $-\infty < x < \infty$

1) $-\infty \le x < 0$

$F(x) = \int_{-\infty}^{x} 0\,dt = 0$

2) $0 \leq x < 1$

$$\cancel{F(x) = \int_0^x 6x(1-x)\,dx + \int_{-\infty}^0 0\,dt}$$

$$\cancel{= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]}$$

$$F(x) = \int_{-\infty}^0 0\,dt + \int_0^x 6t(1-t)\,dt$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right] = 3x^2 - 2x^3$$

3) $1 \leq x < \infty$

$F(x) = 1$

$$F(x) = \begin{cases} 0 & \text{for } -\infty \leq x < 0 \\ 3x^2 - 2x^3 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

(iv) $P(x < k) = P(x > k)$

$$\int_0^k 6x(1-x)\,dx = \int_k^1 6x(1-x)\,dx$$

$\Rightarrow$

$$\left[3x^2 - 2x^3\right]_0^k = \left[3x^2 - 2x^3\right]_k^1$$

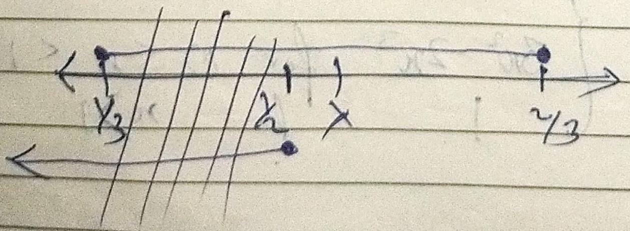
$\Rightarrow$ $3k^2 - 2k^3 = 3 - 2 - 3k^2 + 2k^3$

$\Rightarrow$ $4k^3 - 6k^2 + 1 = 0$

$\boxed{k = 1/2}$, $\frac{1 \pm \sqrt{3}}{2}$

(iii) $P(X \le 1/2 \mid 1/3 \le x \le 2/3)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



$$P(A|B) = \frac{P(1/3 \le x \le 1/2)}{P(1/3 \le x \le 2/3)}$$

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$$\frac{P(1/3 \leq X \leq 1/2)}{P(1/3 \leq X \leq 1/3)} = \frac{\int_{1/3}^{1/2} 6x(1-x)\,dx}{\int_{1/3}^{1/3} 6x(1-x)\,dx} = \boxed{\frac{11}{26}}$$

(i) Prev. due $\Rightarrow$

$$f(x) = \begin{cases} \frac{1}{2}x & 0 \leq x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ -\frac{1}{2}x + \frac{3}{2} & 2 \leq x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(iii) If x_1, x_2, x_3 are three independent observations from X. What is the probability that exactly one of 3 number is larger than 1.5.

Sol $P(x > 1.5) = 1 - P(x < 1.5)$

$$= 1 - \left[\frac{1}{2}\right] = \frac{1}{2}$$

$P(\text{Exactly one of 3 larger than } 1.5) = {}^3C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$$= \frac{3 \times 2!}{1! \times 2!} \times \frac{1}{8} = \boxed{\frac{3}{8}} \text{ Ans}$$

Single R.V.

Discrete RV
- pmf
- Mean, Variance
- $P(2 \le x \le 5)$, $P(x \ge 3)$, $P(x \le 5)$

Continuous R.V.
- pdf → Probability density function.

Q> A random Variable x has the following distribution

x :	1	2	3	4	8	9
$P(x)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$

① Find k ② Find Mean ③ $P(x \ge 3)$ ④ Variance

$\Sigma P(x) = 1$

$\Rightarrow k = \frac{1}{36}$

Mean $= \Sigma x\, p(x) = 1 \times k + 2 \times 3k + 3 \times 5k + 4 \times 7k + 8 \times 9k + 9 \times 11k$
$= 6.138 = \boxed{6.14}$ Ans.

Variance $= \Sigma x^2 p(x) - (\text{Mean})^2 = \boxed{7.77}$ Ans

$P(x \ge 3) = 1 - P(x < 3)$
$= 1 - (P(1) + P(2)) = \boxed{0.889}$ Ans

(1) A random Variable X has the following distribution.

X	0	1	2	3	4	5	6
P(x)	k	3k	5k	7k	9k	11k	13k

Find (1) k (2) $P(x < 4)$ (3)
$P(x \geq 5)$
$P(3 < x \leq 6)$

3) What is the minimum value of x so that $P(x \geq x) > 0.3$.

Sol → 1) $k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$$\boxed{k = \frac{1}{49}}$$

2) $P(x < 4) \rightarrow 1 - P(x \geq 4)$

$1 - (P(4) + P(5) + P(6))$

$\Rightarrow 1 - (9k + 11k + 13k)$

$= \boxed{0.32}$

$P(x \geq 5) \rightarrow P(5) + P(6)$

$= 11k + 13k = \boxed{0.489}$

$P(3 < x \leq 6) \Rightarrow 9k + 11k + 13k$

$= \boxed{0.673}$

3) $P(x \geq 3) \Rightarrow 7k + 9k + 11k + 13k \Rightarrow 0.81$

$P(x \geq 5) \Rightarrow 0.48$.

~~Ans x = 5~~

$\boxed{\text{Min. value of } x = 1}$

Q)

x:	-2	-1	0	1	2	3
$P(x) \Rightarrow$	0.1	k	0.2	$2k$	0.3	k

1) Find k ② Mean ③ Variance

Sol→ $0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$$\boxed{k = \frac{1}{10}}$$

Random Variable → Single, Double; → Discrete, continuous

Q)

$X \backslash Y$	1	2	3	4	5	6	
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$	$8/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$	$10/16$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$	$8/64$
	$3/32$	$3/32$	$11/64$	$13/64$	$6/32$	$16/64$	$\boxed{1}$

1) $P(X \leq 1, Y = 2)$

$\Rightarrow P(0, 2) + P(1, 2)$

$\Rightarrow 0 + 1/16 = \boxed{1/16}$

2) $P(X \leq 1)$

$P(0,1) + P(0,2) + P(0,3) + P(0,4) + P(0,5) + P(0,6) +$
$P(1,1) + P(1,2) + P(1,3) + P(1,4) + P(1,5) + P(1,6)$

$\Rightarrow 0 + 0 + 1/32 + 2/32 + 2/32 + 3/32 + 1/16 + 1/16 + 1/8 + 1/8 + 1/8$

$\Rightarrow \frac{8}{32} + \frac{10}{16}$

$= \frac{8+20}{32} = \boxed{\frac{28}{32}} = \boxed{\frac{7}{8}}$

(iii) $P(Y \leq 3)$

$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$

$= \frac{6+6+11}{64} = \boxed{\frac{23}{64}}$

(iv) $P(X < 3, Y \leq 4)$

$\Rightarrow \frac{3}{32} + \frac{3}{32} + \frac{11}{64} + \frac{13}{64}$

$= \frac{6+6+11+13}{64} = \frac{36}{64} = \boxed{\frac{9}{16}}$

Marginal Distribution of X and Y $\Rightarrow$

$f_X(x) \longrightarrow$ Marginal for X

$f_Y(y) \rightarrow$ Marginal for Y

For Prev. Question $\rightarrow$

$f_X(1) = \frac{10}{16}$ $\quad f_X(0) = 8/16$

$f_Y(1) = 3/32$ etc.

Q) Prev. Ques $\Rightarrow$

Find Condition for Y = 2 given X = 1

Sol $f_{Y/X}(1,2) = \frac{f(1,2)}{f_X(1)} = \frac{1/16}{10/16} = \boxed{\frac{1}{10}}$

Find $f_{X/Y}(2,1) = \frac{f(2,1)}{f_Y(1)} = \frac{1/32}{3/32} = \frac{1}{3}$

Q) A two dimensional r.v x & y have a joint Probability mass function $f(x,y) = \frac{1}{27}(2x+y)$.
where x and y can assume only integer values 0, 1 & 2. Find Marginal distribution for X and Y and conditional distribution of Y for $X = x$.

X \ Y	0	1	2	$f_X(x)$
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
$f_Y(y)$	6/27	9/27	12/27	

X	0	1	2
$f_X(x)$	3/27	9/27	15/27

Y	0	1	2
$f_Y(y)$	6/27	9/27	12/27

Conditional →

X \ Y	0	1	2
0	0	1/3	2/3
1	2/9	1/3	4/9
2	4/15	5/15	6/15

Q) For the joint Probability distribution of two random variable X and Y.

X \ Y	1	2	3	4
1	4/36	3/36	2/36	1/36
2	1/36	3/36	3/36	2/36
3	5/36	1/36	1/36	1/36
4	1/36	2/36	1/36	5/36

Find →

1) Marginal distribution of X and Y.
2) Conditional distribution of X given Y=1 and Conditional distribution of Y given X=2.

Sol →

X given Y=1

X	1	2	3	4
$f_{X\|Y}$	4/11	1/11	5/11	1/11

Y given X=2

Y	1	2	3	4
$f_{Y\|X}$	1/9	1/3	1/3	2/9

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Double Random Variable →
Continuous r.v →

Q> The joint Probability distribution function of a two dimensional r.v (X, Y) is given by →

$$f(x,y) = \begin{cases} 2 & 0<x<1,\ 0<y<x \\ 0 & \text{elsewhere} \end{cases}$$

1) Find the marginal and conditional density function.
2) Check whether X and Y are independent or not.

Sol → i) Marginal for X

$$\boxed{f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy}$$

Marginal for Y

$$\boxed{f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx}$$

$$f(x,y) = \begin{cases} 2 & 0<x<1 \mid 0<y<x \\ 0 & \text{elsewhere} \end{cases}$$

$$f_X(x) = \int_{y=0}^{x} 2\,dy = 2[y]_0^x = 2x \qquad 0<x<1$$

$$f_Y(y) = \int_{x=y}^{1} 2\,dx = 2[x]_y^1 = 2(1-y) \qquad 0<y<1$$

Conditional for X $\rightarrow$

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)} = \frac{\cancel{2}}{\cancel{2}(1-y)} = \frac{1}{1-y}$$

Conditional for Y $\rightarrow$

$$f_{Y/X}(x,y) = \frac{f(x,y)}{f_X(x)} = \frac{\cancel{2}}{\cancel{2}x} = \frac{1}{x}$$

(ii) If $f(x,y) = f_x(x) \cdot f_y(y)$

then x and y are independent.

$$2 \neq 2x \times 2(1-y)$$

X and Y are not independent.

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d) $f(x,y) = \begin{cases} \frac{8}{9}xy & 1 \le x \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$

Find marginal and Continonal.

Sol $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$; $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

$$= \int_{y=x}^{2} \frac{8}{9} xy\, dy$$

$$\Rightarrow \frac{8}{9} x \left[\frac{y^2}{2}\right]_x^2$$

$$= \frac{8}{9} x \left[\frac{4}{2} - \frac{x^2}{2}\right]$$

$$\Rightarrow \frac{4}{9} x \left[4 - x^2\right] \quad 1 \le x \le 2$$

$$f_Y(y) = \int_{x=1}^{y} \frac{8}{9} xy\, dx$$

$$\Rightarrow \frac{8}{9} y \left[\frac{x^2}{2}\right]_1^y = \frac{4}{9} y \left[y^2 - 1\right]$$

$1 \le y \le 2$

Q) Given, $f(x,y) = e^{-(x+y)}$ $x \geq 0$, $y \geq 0$

evaluate (1) $P(x>1)$ (2) $P(x<y \mid x<2y)$

(3) $P(x+y<1)$

Sol (i) $P(x>1) = \int_{y=0}^{\infty}\int_{x=1}^{\infty} e^{-x} e^{-y}\, dx\, dy$

$$= \int_0^{\infty} e^{-y} \left[\int_1^{\infty} e^{-x}\, dx\right] dy$$

$$= \int_0^{\infty} e^{-y} \left[\frac{-e^{-x}}{}\right]_1^{\infty}$$

$$= \int_0^{\infty} e^{-y} \left[-(0 - e^{-1})\right] dy$$

$$= e^{-1} \int_0^{\infty} e^{-y}\, dy$$

$$= e^{-1} \cdot \left[-e^{-y}\right]_0^{\infty}$$

$$= e^{-1} \cdot 1 = \boxed{e^{-1}}$$

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(ii) $P(x<Y \mid x<2Y)$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{P(x<Y)}{P(x<2Y)}$$

$$P(x<Y) = \int_{y=0}^{\infty}\int_{x=0}^{y} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^{\infty} e^{-y}\cdot\left[-e^{-x}\right]_0^{y} dy$$

$$= \int_0^{\infty} e^{-y}\left[-e^{-y}+1\right] dy$$

$$= \int_0^{\infty} -e^{-2y}+e^{-y}\, dy$$

$$\Rightarrow \left[+\frac{e^{-2y}}{2}\right]_0^{\infty} + \left[-(e^{-y})\right]_0^{\infty}$$

$$\Rightarrow \frac{1}{2}\left[0-1\right] + \left[+1\right]$$

$$= \boxed{\frac{1}{2}}$$

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$$P(x < 2y) = \int_{y=0}^{\infty} e^{-y} \int_{x=0}^{2y} e^{-x}\, dx\, dy$$

$$\Rightarrow \int_0^{\infty} e^{-y} \left(-e^{-x}\right)_0^{2y} dy$$

$$= \int_0^{\infty} e^{-y} \left(-e^{-2y} + 1\right) dy$$

$$\Rightarrow \int_0^{\infty} - e^{-3y} + e^{-y}\, dy$$

$$\Rightarrow - \left[\frac{e^{-3y}}{-3}\right]_0^{\infty} + \left[-e^{-y}\right]_0^{\infty}$$

$$= \left[\frac{e^{-3y}}{3}\right]_0^{\infty} + \left(-e^{-y}\right)_0^{\infty}$$

$$\Rightarrow \frac{1}{3}\left[0 - 1\right] + 1$$

$$= \frac{-1}{3} + 1 = \frac{2}{3}$$

$$P(A \mid B) = \frac{1/2}{2/3} = \frac{3}{4}$$

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(iii) $P(X+Y<1) = \int_{0}^{1}\int_{y=0}^{1-x} e^{-(x+y)}\, dy\, dx$

$x=1-y \qquad x=0 \qquad y=0$

$y=0$

$x=1, \ \boxed{x=0}$

$y=\infty$

$x=\infty$

$\Rightarrow \int_{x=0}^{1} e^{-x} \int_{y=0}^{1-x} e^{-y}\, dy\, dx$

$\Rightarrow \int_{0}^{1} e^{-x} \left[-e^{-y}\right]_{0}^{1-x} dx$

$\Rightarrow \int_{0}^{1} e^{-x} \cdot \left[-e^{-(1-x)} + 1\right] dx$

$= \int_{0}^{1} -e^{-1} + e^{-x}\, dx$

$\Rightarrow -e^{-1} \int_{0}^{1} e^{-2x}\, dx + \int_{0}^{1} e^{-x}\, dx$

$\Rightarrow +e^{-1} \left[\frac{e^{-2x}}{-2}\right]_{0}^{1} + \left[\frac{e^{-x}}{-1}\right]_{0}^{1}$

$\Rightarrow +e^{-1}\left[e^{-2}-1\right] + \left[-e^{-1}+1\right] \Rightarrow e^{-3} - e^{-1} - e^{-1} + 1$

$$= \int_0^1 -e^{-1}\, dx + \int_0^1 e^{-x}\, dx$$

$$= -e^{-1}[x]_0^1 + [-e^{-x}]_0^1$$

$$= -e^{-1} + [-e^{-1} + 1]$$

$$= \boxed{1 - 2e^{-1}}$$

Ans

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Q) $f(x) = \begin{cases} \frac{1}{8}(6-x-y) & 0 \le x < 2,\ 2 \le y < 4 \\ 0 & \text{otherwise} \end{cases}$

Find → 1) $P(x<1 \cap Y<3)$

2) $P(X+Y<3)$

3) $P(x<1 \mid Y<3)$

sol (ii) $X+Y<3$

$x = 3-Y$

$Y=2 \qquad Y=4$

$x=1 \qquad X=-1 \; \times$

$$\int_0^1 \int_2^{3-x} \frac{1}{8}(6-x-y)\, dy\, dx.$$

(i) $P(x<1 \cap Y<3) = \int_{x=0}^{1} \int_{y=2}^{3} \frac{1}{8}(6-x-y)\, dy\, dx$

(iii) $P(x<1 \mid Y<3) = \frac{P(x<1 \cap Y<3)}{P(Y<3)}$

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Q) $f(x,y) = 4xy\, e^{-(x^2+y^2)}$; $x \geq 0$, $y \geq 0$

Test whether X and Y are independent. Find the conditional distribution of X given Y = y.

Sol → (i) $f(x,y) = f_X(x) \cdot f_Y(y)$ → Independent Condition

$$f_X(x) = \int_{y=0}^{\infty} f(x,y)\, dy = 4xe^{-x^2} \int_0^{\infty} y \cdot e^{-y^2}\, dy$$

$$f_Y(y) = \int_{x=0}^{\infty} f(x,y)\, dx = 4ye^{-y^2} \int_0^{\infty} x \cdot e^{-x^2}\, dx$$

⇒

Let $y^2 = t$
$2y\, dy = dt$
$y\, dy = \frac{dt}{2}$

$$f_X(x) = 4xe^{-x^2} \int_{t=0}^{\infty} \frac{e^{-t}}{2}\, dt$$

$$\Rightarrow 2xe^{-x^2} \left[-e^{-t}\right]_0^{\infty}$$

$$\Rightarrow 2xe^{-x^2}, \quad x \geq 0$$

$$f_Y(y) = 2ye^{-y^2}, \quad y \geq 0$$

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

$$4xy\,e^{-(x^2+y^2)} = 2x e^{-x^2} \cdot 2y e^{-y^2}$$

$$LHS = RHS$$

Hence, X and Y are Independent.

(ii) Condition distribution of x given $Y = y$

$$f_{X|Y}(x, y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{4xy\,e^{-x^2} \cdot \cancel{e^{-y^2}}}{\cancel{2y\,e^{-y^2}}}$$

$$= \boxed{2x e^{-x^2}}$$

Q) $f(x,y) = \dfrac{9(1+x+y)}{2(1+x)^4(1+y)^4}$ $\quad 0 \le x < \infty$, $0 \le y < \infty$

Find the marginal distribution of X & Y & Conditional distribution of Y given X.

Sol $f_X(x) = \dfrac{9}{2} \displaystyle\int_{y=0}^{\infty} \dfrac{1+x+y}{(1+x)^4(1+y)^4}\, dy$

$= \dfrac{9}{2} \dfrac{1}{(1+x)^4} \displaystyle\int_0^{\infty} \left\{ \dfrac{1}{(1+y)^3} + \dfrac{x}{(1+y)^4} \right\} dy$

$\Rightarrow \dfrac{9}{2} \dfrac{1}{(1+x)^4} \left\{ \dfrac{(1+y)^{-2}}{-2} + \dfrac{x(1+y)^{-3}}{-3} \right\}_0^{\infty}$

$= \dfrac{9}{2} \dfrac{1}{(1+x)^4} \left[\dfrac{1}{2} + \dfrac{x}{3} \right]$

$\Rightarrow \dfrac{9}{4} (1+x)^{-4} + \dfrac{3x}{2}$

$\Rightarrow \boxed{\dfrac{3}{4} \cdot \dfrac{3+2x}{(1+x)^4}}$

$f_Y(y) = \boxed{\dfrac{3}{4} \cdot \dfrac{3+2y}{(1+y)^4}}$

Conditional distribution of Y given X →

$$f_{Y|X}(x,y) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\cancel{9}^{3}(1+x+y)}{\cancel{2}(1+x)^{4}(1+y)^{4}} \times \frac{\cancel{4}^{2}}{\cancel{3}} \frac{(1+x)^{\cancel{4}}}{3+2x}$$

$$\Rightarrow \boxed{\frac{6\,(1+x+y)}{(3+2x)(1+y)^{4}}} \text{ Ans}$$

Q) $f(x) = \begin{cases} \frac{1}{4} & -2 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

1) $P(|x| > 1)$

2) $P(2X+3 > 5)$

Sol) 1) $P(|x|>1) = P(x<-1) + P(x>1)$

$$\Rightarrow \int_{-2}^{-1} \frac{1}{4}\,dx + \int_{1}^{2} f(x)\,dx$$

$$\Rightarrow \frac{1}{4}[x]_{-2}^{-1} + \frac{1}{4}[x]_{1}^{2}$$

$$\Rightarrow \frac{1}{4}[-2+1] + \frac{1}{4}[2-1] \Rightarrow \boxed{\frac{1}{2}} \text{ Ans}$$

(ii) $P(2x+3>5)$

$$P(x>1) = \int_1^{\infty} \frac{1}{4}\, dx$$

$$\Rightarrow \boxed{\frac{1}{4}} \text{ Ans}$$