4.17 RANDOM VARIABLE

Let E be a random experiment and S be a sample space associated with it. A function X assigning to every element $s \in S$, a real number, $X(s)$ is called a random variable.

A random variable is any numerical quantity whose value will be determined by the outcome of a random experiment and which have a specific range and a definable probability associated with each value.

In other words, if the numerical values assumed by a variable are the result of some chance factors, so that a particular value cannot be exactly predicted in advance, the variable is then called a *random variable*. A random variable is also called '*chance variable*' or '*stochastic variable*'. Random variables are denoted by capital letters, usually, from the last part of the alphabet, for instance, X, Y, Z, etc.

4.17.1 Definition

A real valued function, X, defined on a sample space, S of a random experiment, E, is called a random variable which assigns to each sample point (or to every outcome of the random experiment E) $s \in S$ one and only one real number $X(s) = x$. The domain of the random variable X is the sample space S and its range, A is the non empty set of real numbers such that

1. The set $[X(s) \leq x]$ is an event for any real number x.

2. There corresponds a well defined unique probability of the events, "X assumes the value x" and "X assumes any value in the interval".

e.g., If three coins are tossed, then the sample space contains 8 sample points. Let the random variable X denote "the number of heads". Then X is a real valued function over S with a space A which has four elements 0, 1, 2, 3, 4 as explained below:

Sample Point	*X(s)*	*X = no. of heads*	*Probability*
TTT	X(TTT)	0	$\frac{1}{8}$
HTT, THT TTH	X(HTT), X(THT), X(TTH)	1	$\frac{3}{8}$
THH, HTH, HHT	X(THH), X(HTH), X(HHT)	2	$\frac{3}{8}$
HHH	X(HHH)	3	$\frac{1}{8}$

Thus A the space of X is a set of real numbers given by

$$A = \{x : x = X(s), s \in S\}.$$

In this example, we have

$$S = \{TTT, HTT, THT, TTH, THH, HTH, HHT, HHH\}$$

and

$$A = \{0, 1, 2, 3\}.$$

If S has elements which are themselves real numbers then $X(s) = s$ and $A = S$. As in case of rolling a die $S = \{1, 2, 3, 4, 5, 6\} = A$.

4.18 TYPES OF RANDOM VARIABLES

There are two types of random variables:

(1) Discrete Random Variable. A *discrete random variable* is one which can assume only isolated values. For example,

(*i*) the number of heads in 4 tosses of a coin is a discrete random variable as it cannot assume values other than 0, 1, 2, 3, 4.

(*ii*) the number of aces in a draw of 2 cards from a well shuffled deck is a random variable as it can take the values 0, 1, 2 only.

(2) Continuous Random Variable. A *continuous random variable* is one which can assume any value within an interval, *i.e.* all values of a continuous scale. For example (*i*) the weights (in kg) of a group of individuals, (*ii*) the heights of a group of individuals.

4.19 PROBABILITY FUNCTION (OR PROBABILITY MASS FUNCTION)

Let $x_1, x_2, x_3, \ldots$ be the values of a discrete random variable X and let $p_1, p_2, p_3, \ldots, p_i > 0, i = 1, 2, 3, \ldots$ be the corresponding probabilities. That is $P(X = x_i) = p(x_i)$ or simply p_i.

A function $f(x)$ or $p(x)$ defined by

$$P(X = x) = p(x) = \begin{cases} p(x_i) \text{ or } p_i \text{ where } x = x_i, i = 1, 2, \ldots \\ 0 \text{ otherwise} \end{cases}$$

is called the probability function of the (discrete) random variable X.

The probability function $p(x)$ yields the probability that the random variable X assumes any particular value x in its range.

The probability function (or often called frequency function) of the random variable X satisfies the following conditions:

(*i*) $p(x_i) \geq 0$, (*ii*) $\Sigma\, p(x_i) = 1$.

Example. *A random variable X takes values 1, 2, 3, ... with probability mass function* $\dfrac{\lambda^r}{r!}$, $r = 1, 2, 3, \ldots \infty$. *Find the value of* λ.

Sol. $\displaystyle\sum_{r=1}^{\infty} \frac{\lambda^r}{r!}$ should be 1. Thus,

$$\sum_{r=1}^{\infty} \frac{\lambda^r}{r!} = 1$$

$$\Rightarrow \quad \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots = 1$$

$$\Rightarrow \quad e^{\lambda} - 1 = 1$$

$$\Rightarrow \quad e^{\lambda} = 2$$

$$\lambda = \log_e 2.$$

4.20 PROBABILITY DISTRIBUTION OF A DISCRETE RANDOM VARIABLE

A table or a formula listing all possible values that a random variable can take on together with the respective probabilities, is called a probability distribution of the random variable.

In other words, the set of ordered pairs $[x_i, p(x_i)]$ is called the probability distribution of a discrete random variable X provided $p(x_i) \geq 0$ and $\Sigma\, p(x_i) = 1$.

e.g. (*i*) Let X = The number of points appearing in a throw of a die. Then the probability distribution of X is given by the probability function

$$p(x) = \frac{1}{6},\ x = 1, 2, 3, 4, 5, 6$$
$$= 0 \text{ otherwise.}$$

This probability distribution may also be expressed as follows:

X:	1	2	3	4	5	6
$p(x) = P(X = x)$:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(*ii*) Suppose a coin is tossed three times. Then the distribution of the number of heads is

$X = x$:	0	1	2	3
$P(X = x) = p(x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Note. In general, both terms probability function and probability distribution are used interchangeably.

4.21 MEAN AND VARIANCE OF RANDOM VARIABLES

Let
$$X : x_1,\quad x_2,\quad x_3, \ldots\ldots, x_n$$
$$P(X) : p_1,\quad p_2,\quad p_3, \ldots\ldots, p_n$$
be a discrete probability distribution.

We denote the *mean* by μ and define $\mu = \dfrac{\Sigma p_i x_i}{\Sigma p_i} = \Sigma p_i x_i$ $\qquad (\because\ \Sigma p_i = 1)$

Other names for the mean are *average or expected value E(X)*.

We denote the *variance* by σ^2 and define $\sigma^2 = \Sigma p_i (x_i - \mu)^2$

If μ is not a whole number, then $\sigma^2 = \Sigma p_i x_i^2 - \mu^2$

Standard deviation $\sigma = +\sqrt{\text{Variance}}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Five defective bulbs are accidentally mixed with twenty good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution of the number of defective bulbs, if four bulbs are drawn at random from this lot.*

Sol. Let X denote the number of defective bulbs in 4. Clearly X can take the values 0, 1, 2, 3 or 4.

Number of defective bulbs = 5

Number of good bulbs = 20

Total number of bulbs = 25

P(X = 0) = P (no defective) = P (all 4 good ones)

$$= \frac{^{20}C_4}{^{25}C_4} = \frac{20 \times 19 \times 18 \times 17}{25 \times 24 \times 23 \times 22} = \frac{969}{2530}$$

$$P(X = 1) = P\text{ (one defective and 3 good ones)} = \frac{{}^5C_1 \times {}^{20}C_3}{{}^{25}C_4} = \frac{1140}{2530}$$

$$P(X = 2) = P\text{ (2 defectives and 2 good ones)} = \frac{{}^5C_2 \times {}^{20}C_2}{{}^{25}C_4} = \frac{380}{2530}$$

$$P(X = 3) = P\text{ (3 defectives and 1 good one)} = \frac{{}^5C_3 \times {}^{20}C_1}{{}^{25}C_4} = \frac{40}{2530}$$

$$P(X = 4) = P\text{ (all 4 defectives)} = \frac{{}^5C_4}{{}^{25}C_4} = \frac{1}{2530}$$

$\therefore$ The probability distribution of the random variable X is

X:	0	1	2	3	4
P(X):	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

Example 2. *A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and the variance of the number of successes.*

Sol. Let X denote the number of success. Clearly X can take the values 0, 1, 2 or 3.

Probability of success $= \frac{2}{6} = \frac{1}{3}$; Probability of failure $= 1 - \frac{1}{3} = \frac{2}{3}$

$$P(X = 0) = P\text{ (no success)} = P\text{ (all 3 failures)} = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

$$P(X = 1) = P\text{ (one success and 2 failures)} = {}^3C_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$$

$$P(X = 2) = P\text{ (two successes and one failure)} = {}^3C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$$

$$P(X = 3) = P\text{ (all 3 successes)} = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$\therefore$ The probability distribution of the random variable X is

X:	0	1	2	3
P(X):	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

To find the mean and variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	$\frac{8}{27}$	0	0
1	$\frac{12}{27}$	$\frac{12}{27}$	$\frac{12}{27}$
2	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{24}{27}$
3	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{9}{27}$
		1	$\frac{5}{3}$

Mean $\mu = \Sigma p_i x_i = 1$

Variance $\sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}$.

Example 3. *A random variable X has the following probability function:*

Values of X,	*x:*	0	1	2	3	4	5	6	7
	p(x):	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (A.K.T.U. 2023)

(ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(3 < X \leq 6)$

(iii) Find the minimum value of x so that $P(X \leq x) > \frac{1}{2}$.

Sol. (*i*) Since $\sum_{x=0}^{7} p(x) = 1$, we have

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow \quad 10k^2 + 9k - 1 = 0 \quad \Rightarrow \quad (10k - 1)(k + 1) = 0$$

$$\Rightarrow \quad k = \frac{1}{10} \qquad [\because \; p(x) \geq 0]$$

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + \cdots\cdots + P(X = 5)$

$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) = 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

$$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6) = 3k + k^2 + 2k^2 = \frac{3}{10} + \frac{3}{100} = \frac{33}{100}$$

(*iii*) $P(X \leq 1) = k = \frac{1}{10} < \frac{1}{2}$; $\qquad P(X \leq 2) = k + 2k = \frac{3}{10} < \frac{1}{2}$

$P(X \leq 3) = k + 2k + 2k = \frac{5}{10} = \frac{1}{2}$; $\qquad P(X \leq 4) = k + 2k + 2k + 3k = \frac{8}{10} > \frac{1}{2}$

$\therefore$ The maximum value of x so that $P(X \leq x) > \frac{1}{2}$ is 4.

TEST YOUR KNOWLEDGE

1. Find the probability distribution of the number of doublets in four throws of a pair of dice.
2. Two bad eggs are mixed accidently with 10 good ones. Find the probability distribution of the number of bad eggs in 3, drawn at random, without replacement, from this lot.
3. A die is tossed twice. Getting a number greater than 4 is considered a success. Find the variance of the probability distribution of the number of successes.
4. Two cards are drawn simultaneously from a well-shuffled deck of 52 cards. Compute the variance for the number of aces.
5. An urn contains 4 white and 3 red balls. Three balls are drawn, with replacement, from this urn. Find μ, σ^2 and σ for the number of red balls drawn.

6. A random variable X has the following probability distribution:

Values of X, x :	0	1	2	3	4	5	6	7	8
$p(x)$:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(*i*) Determine the value of a. (*ii*) Find $P(X < 3)$, $P(X \geq 3)$, $P(2 \leq X < 5)$

(*iii*) What is the smallest value of x for which $P(X \leq x) > 0.5$?

7. Find the standard deviation for the following discrete distribution:

x :	8	12	16	20	24
$p(x)$:	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$.

8. A random variable x has the following probability function:

x :	0	1	2	3
$f(x)$:	$3k$	$2k$	$2k$	k

Determine:

(*i*) k (*ii*) mean of the distribution. (*A.K.T.U. 2022*)

Answers

1.

X :	0	1	2	3	4
P(X) :	$\frac{625}{1296}$	$\frac{500}{1296}$	$\frac{150}{1296}$	$\frac{20}{1296}$	$\frac{1}{1296}$

2.

X :	0	1	2
P(X) :	$\frac{12}{22}$	$\frac{9}{22}$	$\frac{1}{22}$

3. $\frac{4}{9}$ **4.** $\frac{400}{2873}$ **5.** $\frac{9}{7}, \frac{36}{49}, \frac{6}{7}$

6. (*i*) $a = \frac{1}{81}$ (*ii*) $\frac{1}{9}, \frac{8}{9}, \frac{7}{27}$ (*iii*) 5 **7.** $2\sqrt{5}$.

8. (*i*) $\frac{1}{8}$ (*ii*) $\frac{9}{8}$

4.22 CONTINUOUS DISTRIBUTIONS

So far we have dealt with discrete distributions *i.e.*, the distribution in which the variate takes a finite set of values. But when we deal with variates like temperature, heights and weights, we find that variates can take an infinite set of values in a given interval, say, $a \leq x \leq b$. Such variates are called continuous variates and their distributions are accordingly known as *continuous distributions*.

$$y = \frac{1}{2} \sin x \qquad (0 \leq x \leq \pi)$$

is an example of a continuous distribution, as x can assume all values lying between 0 and π.

4.23 PROBABILITY DENSITY FUNCTION

Let the probability of the variate x falling in the infinitesimal interval $(x - \frac{1}{2}dx, x + \frac{1}{2}dx)$ be expressed in the form $f(x)\,dx$, where $f(x)$ is a continuous function of x. Then $f(x)$ is called the *probability density function* or simply *density function*. The continuous curve $y = f(x)$ is called the *probability density curve* or briefly the *probability curve*.

Symbolically it is expressed as

$$P(x - \frac{1}{2}\,dx \le X \le x + \frac{1}{2}dx) = f(x)\,dx.$$

The interval of the variate may be finite or infinite. A function defined only for a finite interval say $f(x) = \phi(x)$ when $a \le x \le b$ can be put in the following form

$$\begin{aligned} f(x) &= 0, && x < a \\ f(x) &= \phi(x), && a \le x \le b \\ f(x) &= 0, && x > b. \end{aligned}$$

The density function possesses the following two properties:

(1) $f(x) \ge 0$ for every x, as negative probability has no meaning,

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$, this corresponds to the fact that the probability of an event that is sure to happen is equal to unity.

For a density function, $f(x)$, the probability that the variate X falls in any interval (p, q) is given by

$$P(p \le X < q) = \int_p^q f(x)\,dx.$$

Note 1. Any positive function of a variate x can be changed to give a probability density function $f(x)$ by multiplying it by a constant which will make the total area under the curve $y = f(x)$ equal to unity. For example we know that

$$\int_0^2 x(2 - x)\,dx = \frac{4}{3},$$

hence if we multiply both sides by $\frac{3}{4}$, we get

$$\int_0^2 \frac{3}{4}x\,(2 - x)\,dx = 1.$$

Hence a probability density function can be formed as given by

$$f(x) = \begin{cases} 0, & x < 0 \\ \dfrac{3x\,(2 - x)}{4}, & 0 \le x \le 2 \\ 0, & x > 2. \end{cases}$$

Note 2. Whenever $f(x)$ is constant throughout its interval, the variable is said to have a rectangular distribution of same probability.

ILLUSTRATIVE EXAMPLES

Example 1. *If the function $f(x)$ is defined by $f(x) = ce^{-x}$, $0 \le x \le \infty$, find the value of c which changes $f(x)$ to a probability density function.* (A.K.T.U. 2021)

Sol. In order that $f(x)$ may be density function, we should have

(a) $f(x) \ge 0$ for every x (b) $\int_{-\infty}^{+\infty} f(x)\, dx = 1.$

Since e^{-x} is always positive for the values of x lying between 0 and ∞, the condition will be satisfied if $c \ge 0$.

The second condition will be satisfied if

$$\int_0^{\infty} ce^{-x}\, dx = 1$$

i.e., if

$$\left[-ce^{-x}\right]_0^{\infty} = 1$$

i.e., if

$$c = 1.$$

Example 2. *If $f(x)$ has probability density cx^2, $0 < x < 1$, determine c and find the probability that $\frac{1}{3} < x < \frac{1}{2}$ i.e., $P\left(\frac{1}{3} < x < \frac{1}{2}\right)$.*

Sol. $f(x)$ will have a probability density if $\int_0^1 cx^2 dx = 1$

i.e.,

$$\left[\frac{1}{3}cx^3\right]_0^1 = 1, \text{ i.e., } c = 3.$$

$$P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} 3x^2 dx = \left[x^3\right]_{\frac{1}{3}}^{\frac{1}{2}} = \left(\frac{1}{8} - \frac{1}{27}\right) = \frac{19}{216}.$$

4.24 VARIOUS MEASURES FOR CONTINUOUS PROBABILITY DISTRIBUTIONS

Let $f_X(x)$ or $f(x)$ be the p.d.f. of a random variable X where X is defined from a to b. Then,

(*i*) **Arithmetic Mean** $= \int_a^b x\, f(x)\, dx$ ...(1)

(*ii*) **Harmonic Mean.** Harmonic mean H is given by: $\frac{1}{H} = \int_a^b \frac{1}{x} . f(x)\, dx$ ...[1(*a*)]

(*iii*) **Geometric Mean.** Geometric mean G is given by: $\log G = \int_a^b \log x .\, f(x)\, dx$ ...[1(*b*)]

(*iv*) v_r (about origin) $= \int_a^b x^r . f(x)\, dx$ ...(2)

μ_r' (about the point $x = A$) $= \int_a^b (x - A)^r . f(x)\, dx$ ...[2(*a*)]

and μ_r (about mean) $= \int_a^b (x - \text{mean})^r . f(x)\, dx$ ...[2(*b*)]

In particular, from (1) and (2), we have

$$\nu_1 \text{ (about origin)} = Mean = \int_a^b x\, f(x)\, dx \quad \text{and} \quad \nu_2 = \int_a^b x^2 f(x)\, dx$$

Hence, $$\mu_2 = (\nu_2 - \nu_1^2) = \int_a^b x^2 f(x)\, dx - \left(\int_a^b x\, f(x)\, dx\right)^2 \qquad ...[2(c)]$$

From (2), on putting $r = 3$ and 4 respectively, we get the values of ν_3 and ν_4 and consequently the moments about mean can be obtained and hence β_1 and β_2 can be computed.

(*v*) **Median.** Median is the point which divides the entire distribution in two equal parts. In case of continuous distribution, median is the point which divides the total area into two equal parts. Thus if M is the median, then

$$\int_a^M f(x)\, dx = \int_M^b f(x)\, dx = \frac{1}{2} \qquad ...(3)$$

Thus solving $$\int_a^M f(x)\, dx = \frac{1}{2} \text{ or } \int_M^b f(x)\, dx = \frac{1}{2} \qquad ...[3(a)]$$

for M, we get the value of median.

(*vi*) **Mean Deviation.** Mean deviation about the mean μ_1' is given by:

$$\text{M.D.} = \int_a^b |x - \text{mean}|\, f(x)\, dx \qquad ...(4)$$

In general, mean deviation about an average 'A' is given by:

$$\text{M.D. about 'A'} = \int_a^b |x - A|\, f(x)\, dx \qquad ...[4(a)]$$

(*vii*) **Quartiles and Deciles.** Q_1 and Q_3 are given by the equations:

$$\int_a^{Q_1} f(x)\, dx = \frac{1}{4} \text{ and } \int_a^{Q_3} f(x)\, dx = \frac{3}{4} \qquad ...(5)$$

$$D_i,\ i^{th} \text{ decile is given by: } \int_a^{D_i} f(x)\, dx = \frac{i}{10};\ i = 1, 2, ..., 9 \qquad ...[5(a)]$$

(*viii*) **Mode.** Mode is the value of x for which $f(x)$ is maximum. Mode is thus the solution of $f'(x) = 0$ and $f''(x) < 0$, provided it lies in $[a, b]$. ...(6)

4.25 CUMULATIVE DISTRIBUTION FUNCTION

If $F(x) = \int_{-\infty}^{x} f(x)\, dx = P(X \le x)$, then the function F($x$) is the probability that the value of the variate X will be $\le x$.

$$F(b) = P(X \le b).$$

and $$F(b) - F(a) = \int_a^b f(x)\, dx = P(a \le X \le b).$$

F(x) is called the cumulative distribution function of x or simply the *distribution function*. The cumulative distribution function has the following properties:

(1) $F'(x) = f(x) \ge 0$, so that F(x) is a non decreasing function. This means that $dF(x) = f(x)\, dx$. This is known as *probability differential of* X.

(2) $F(-\infty) = 0$

(3) $F(\infty) = 1$.

(4) $F(x)$ is a continuous function of x on the right.

Example 3. *The distribution function of a random variable X is given by*

$$F(x) = \begin{cases} 1-(1+x)e^{-x}, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases},$$

find the corresponding density function of random variable X.

Sol. The probability density function $= f(x) = \dfrac{d}{dx}\{F(x)\}$

$$f(x) = -e^{-x} + (1+x)e^{-x}$$

$$= \begin{cases} xe^{-x}, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0. \end{cases}$$

Example 4. *A random variable x has the density function*

$$f(x) = k.\frac{1}{1+x^2}, -\infty < x < \infty.$$

Determine k and the distribution function.

Sol. It will be a density function if

$$\int_{-\infty}^{\infty} k.\frac{1}{1+x^2}\, dx = 1$$

i.e., $\qquad k.\pi = 1$ giving $k = 1/\pi$

$$F(x) = \int \frac{1}{\pi}.\frac{1}{1+x^2}\, dx = \frac{1}{\pi}\tan^{-1} x + c$$

But $F(-\infty)$ should be zero for distribution function

$$\therefore \qquad \frac{1}{\pi}\left(-\frac{\pi}{2}\right) + c = 0 \text{ giving } c = \frac{1}{2}.$$

$$\therefore \qquad F(x) = \frac{1}{\pi}\tan^{-1} x + \frac{1}{2} \text{ for } -\infty < x < \infty.$$

Example 5. *The diameter, say X, of an electric cable, is assumed to be a continuous random variable with p.d.f.: $f(x) = 6x(1-x)$, $0 \leq x \leq 1$*

(i) Check that the above is a p.d.f.,

(ii) Obtain an expression for the c.d.f. of X.,

(iii) Compute $P\left(X \leq \frac{1}{2} \,\middle|\, \frac{1}{3} \leq X \leq \frac{2}{3}\right)$, and

(iv) Determine the number k such that $P(X < k) = P(X > k)$.

Sol. (*i*) Since $\int_0^1 f(x)\, dx = \int_0^1 6x(1-x)\, dx = 6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$, $f(x)$ is a p.d.f.

$$(ii)\ \mathrm{F}(x) = \begin{cases} 0, & \text{if } x \le 0 \\ \int_0^x 6t\,(1-t)dt = (3x^2 - 2x^3), & 0 < x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$$

$$(iii)\ \mathrm{P}\left(\mathrm{X} \le \frac{1}{2} \,\middle|\, \frac{1}{3} \le \mathrm{X} \le \frac{2}{3}\right) = \frac{\mathrm{P}\left(\frac{1}{3} \le \mathrm{X} \le \frac{1}{2}\right)}{\mathrm{P}\left(\frac{1}{3} \le \mathrm{X} \le \frac{2}{3}\right)} = \frac{\int_{1/3}^{1/2} 6x\,(1-x)\,dx}{\int_{1/3}^{2/3} 6x\,(1-x)\,dx} = \frac{11/54}{13/27} = \frac{11}{26}$$

(*iv*) We have $\mathrm{P}(\mathrm{X} < k) = \mathrm{P}(\mathrm{X} > k)$

$$\Rightarrow \quad \int_0^k 6x(1-x)dx = \int_k^1 6x\,(1-x)\,dx$$

or
$$3k^2 - 2k^3 = 3(1-k^2) - 2\,(1-k^3)$$

$$\Rightarrow \quad 4k^3 - 6k^2 + 1 = 0$$

$$\Rightarrow \quad k = \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2}.$$

The only admissible value of k in the given range is $\frac{1}{2}$. Hence the value of k is $\frac{1}{2}$.

Example 6. *Let X be a continuous random variable with p.d.f. given by:*

$$f(x) = \begin{cases} kx, & 0 \le x < 1 \\ k, & 1 \le x < 2 \\ -kx + 3k, & 2 \le x < 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Determine the constant k, (ii) Determine F(x), the c.d.f., and

(iii) If x_1, x_2 and x_3 are three independent observations from X, what is the probability that exactly one of these three numbers is larger than 1.5?

Sol. (*i*) Since $f(x)$ in the p.d.f. of X, we have:

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = 1$$

$$\Rightarrow \quad \int_0^1 k\,x\,dx + \int_1^2 k\,dx + \int_2^3 (-kx + 3k) = 1$$

$$\therefore \quad \left[k\frac{x^2}{2}\right]_0^1 + [kx]_1^2 + \left[-k.\frac{x^2}{2} + 3kx\right]_2^3 = 1 \Rightarrow k = \frac{1}{2}.$$

(*ii*) For any x such that $-\infty < x < 0$; $\mathrm{F}(x) = \int_{-\infty}^{x} 0\,.\,dt = 0$

For any x, where $0 \le x < 1$; $\mathrm{F}(x) = \int_{-\infty}^{0} 0\,.\,dt + \int_0^x \frac{t}{2}dt = \frac{x^2}{4}$

For x, where $1 \le x < 2$, $\mathrm{F}(x) = \int_{-\infty}^{0} 0\,.\,dt + \int_0^1 \frac{t}{2}dt + \int_1^x \frac{1}{2}dt = \frac{2x-1}{4}$

For any x, where $2 \le x < 3$,

$$F(x) = \int_{-\infty}^{0} 0\,.\,dt + \int_{0}^{1} \frac{t}{2} dt + \int_{1}^{2} \frac{1}{2} dt + \int_{2}^{x} \left(-\frac{t}{2} + \frac{3}{2}\right) dt.$$

$$= \frac{1}{4} + \left(1 - \frac{1}{2}\right) + \left(-\frac{x^2}{4} + \frac{3x}{2} - 2\right) = -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4}$$

For any x, where $3 \le x < \infty$

$$F(x) = \int_{-\infty}^{0} 0.\,dt + \int_{0}^{1} \frac{t}{2} dt + \int_{1}^{2} \frac{1}{2} dt + \int_{2}^{3} \left(-\frac{t}{2} + \frac{3}{2}\right) dt + \int_{3}^{x} 0.\,dt$$

$$= \frac{1}{4} + \left(1 - \frac{1}{2}\right) + \left(-\frac{9}{4} + \frac{9}{2} + 1 - 3\right) = 1$$

Hence the distribution function F(x) is given by:

$$F(x) = \begin{cases} 0, & \text{for } -\infty \le x < 0 \\ \dfrac{x^2}{4}, & \text{for } 0 \le x < 1 \\ \dfrac{2x - 1}{4}, & \text{for } 1 \le x < 2 \\ -\dfrac{x^2}{4} + \dfrac{3x}{2} - \dfrac{5}{4}, & \text{for } 2 \le x < 3 \\ 1, & \text{for } 3 \le x < \infty \end{cases}$$

(*iii*) The probability that X is larger than 1.5 is given by:

$$P(X > 1.5) = 1 - P\,(X < 1.5) = 1 - P(1.5) = \left(1 - \frac{3-1}{4}\right) = \frac{1}{2}.$$

$\therefore$ The probability that X is not larger than 1.5 = $P(X < 1.5) = 1 - \frac{1}{2} = \frac{1}{2}$.

Hence out of three numbers x_1, x_2 and x_3, the probability that exactly one is larger than 1.5 is:

$$3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{8}.$$

Example 7. *A petrol pump is supplied with petrol once a day. If its daily volume of sales (X) in thousands of litres is distributed by: $f(x) = 5\,(1 - x)^4$, $0 \le x \le 1$, what must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0.01?*

Sol. Let the capacity of the tank (in '000 of litres) be 'a' such that

$$P(X \ge a) = 0.01$$

$$\Rightarrow \quad \int_{a}^{1} f(x)\,dx = 0.01$$

$$\Rightarrow \quad \int_{a}^{1} 5(1 - x)^4\,dx = 0.01$$

$$\Rightarrow \quad (1 - a)^5 = \frac{1}{100}$$

$$\Rightarrow \quad a = 1 - \left(\frac{1}{100}\right)^{1/5} = 1 - 0.3981 = 0.6019$$

Hence the capacity of the tank = 0.6019 × 1,000 litres = 601.9 litres.

Example 8. *A continuous random variable X has a p.d.f. $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that (i) $P(X \le a) = P(X > a)$, and (ii) $P(X > b) = 0.05$.*

Sol. (*i*) Since $P(X \le a) = P(X > a)$, each must be equal to $\frac{1}{2}$, because total probability is always unity.

$$\therefore \quad P(X \le a) = \frac{1}{2}$$

$$\Rightarrow \quad \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \quad 3\int_0^a x^2\,dx = \frac{1}{2}$$

$$\Rightarrow \quad 3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$\Rightarrow \quad a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

(*ii*)
$$P(X > b) = 0.05$$

$$\Rightarrow \quad \int_b^1 f(x)\,dx = 0.05$$

$$\Rightarrow \quad 3\left|\frac{x^3}{3}\right|_b^1 = \frac{1}{20}$$

$$\Rightarrow \quad 1 - b^3 = \frac{1}{20}$$

$$\Rightarrow \quad b = \left(\frac{19}{20}\right)^{\frac{1}{3}}.$$

Example 9. (*a*) *A random variable X is distributed at random between the values 0 and 1 so that its probability density function is: $f(x) = kx^2(1 - x^3)$, where k is a constant. Find the value of k. Using this value of k, find its mean and variance.*

(*b*) *A variable X is distributed at random between the values 0 and 4 and its probability density function is given by: $f(x) = kx^3(4 - x)^2$. Find the value of k, the mean and standard deviation of the distribution.*

Sol. (*a*) Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$,

$$k\int_0^1 (x^2 - x^5)\,dx = 1 \Rightarrow k\left|\frac{x^3}{3} - \frac{x^6}{6}\right|_0^1 = 1 \Rightarrow k = 6.$$

$$\text{Mean} = \mu_1' = \int_{-\infty}^{\infty} x\,f(x)\,dx = 6\int_0^1 (x^3 - x^6)\,dx = 6\left|\frac{x^4}{4} - \frac{x^7}{7}\right|_0^1 = \frac{9}{14}$$

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 6\int_0^1 (x^4 - x^7)\,dx = 6\left|\frac{x^5}{5} - \frac{x^8}{8}\right|_0^1 = \frac{9}{20}$$

$$\therefore \quad \text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = \left\{\frac{9}{20} - \left(\frac{9}{14}\right)^2\right\} = \frac{9}{245}.$$

(b) Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$, $k\int_0^4 x^3(4-x)^2\,dx = 1 \Rightarrow k = \frac{15}{1024}$

$$\text{Mean} = \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx = k\int_0^4 x^4(4-x^2)\,dx = \frac{16}{7}$$

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx = k\int_0^4 x^5(4-x)^2\,dx = \frac{40}{7}$$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = \left\{\frac{40}{7} - \left(\frac{16}{7}\right)^2\right\} = \frac{24}{49}$$

$\therefore$ Standard deviation $= \frac{2\sqrt{6}}{7}$.

Example 10. *The time one has to wait for a bus at a downtown bus stop is observed to be random phenomenon (X) with the following probability density function:*

$$f_X(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{1}{9}(x+1), & \text{for } 0 \le x < 1 \\ \frac{4}{9}\left(x - \frac{1}{2}\right), & \text{for } 1 \le x < \frac{3}{2} \\ \frac{4}{9}\left(\frac{5}{2} - x\right), & \text{for } \frac{3}{2} \le x < 2 \\ \frac{1}{9}(4-x), & \text{for } 2 \le x < 3 \\ \frac{1}{9}, & \text{for } 3 \le x < 6 \\ 0, & \text{for } x \ge 6 \end{cases}$$

Let the events A and B be defined as follows:

A: One waits between 0 to 2 minutes inclusive.

B: One waits between 0 to 3 minutes inclusive.

Show that (a) $P(B|A) = \frac{2}{3}$, *(b)* $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$.

Sol. (a) $\quad P(A) = P(X \le 2) = \int_0^2 f(x)\,dx$

$$= \int_0^1 \frac{1}{9}(x+1)\,dx + \int_1^{3/2} \frac{4}{9}\left(x - \frac{1}{2}\right)dx + \int_{3/2}^2 \frac{4}{9}\left(\frac{5}{2} - x\right)dx$$

$$= \frac{1}{2}. \text{ (on simplification).}$$

$$P(A \cap B) = P[(0 \le X \le 2) \cap (1 \le X \le 3)] = P(1 \le X \le 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^{3/2} \frac{4}{9}\left(x - \frac{1}{2}\right)dx + \int_{3/2}^2 \frac{4}{9}\left(\frac{5}{2} - x\right)dx$$

$$= \frac{4}{9}\left|\frac{x^2}{2} - \frac{x}{2}\right|_1^{3/2} + \frac{4}{9}\left|\frac{5}{2}x - \frac{x^2}{2}\right|_{3/2}^2 = \frac{1}{3} \text{ (on simplification)}$$

$$\therefore \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}.$$

(*b*) $\bar{A} \cap \bar{B}$ means that waiting time is more than 3 minutes.

$$\therefore \quad P(\bar{A} \cap \bar{B}) = P(X > 3) = \int_3^\infty f(x)\,dx = \int_3^6 f(x)\,dx + \int_6^\infty f(x)\,dx$$

$$= \int_3^6 \frac{1}{9}\,dx = \frac{1}{9}\left| x \right|_3^6 = \frac{1}{3}$$

TEST YOUR KNOWLEDGE

1. For continuous random variable X if

$$f(x) = \frac{3}{4}(x^2 + 1), \quad 0 \le x \le 1.$$

then,

(*i*) Verify that $f(x)$ is a probability distribution function.

(*ii*) Find λ such that $P(X \le \lambda) = P(X > \lambda)$. (*A.K.T.U. 2023*)

2. (*i*) Find the constant k so that function $f(x)$ defined as follows be a density function:

$$f(x) = \begin{cases} 1/k, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

(*ii*) Find the value of y_0 so that the function $f(x)$ defined as follows be a density function:

$$f(x) = y_0 e^{-x/\sigma}, \qquad 0 \le x \le \infty.$$

3. If $f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$; find the probability $P\left(\frac{1}{4} \le x \le \frac{1}{2}\right)$.

4. If $f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3 + 2x), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$

prove that it is a density function. Find the probability that a variate having this density will fall in the interval $2 \le x \le 3$.

5. Define distribution function. Verify that the following is a distribution function:

$$F(x) = \begin{cases} 0, & x < -a \\ \frac{1}{2}\left(\frac{x}{a} + 1\right), & -a \le x \le a \\ 1, & x > a \end{cases}$$

6. What do you understand by probability distribution?

7. Find the mean, variance and the coefficients β_1, β_2 of the distribution: $dF = kx^2 e^{-x}\,dx = 1, 0 < x < \infty$.

8. Show that the symmetrical distribution

$$f(x) = \frac{2a}{\pi}\left(\frac{1}{a^2 + x^2}\right), \quad -a \le x \le a$$

represents a probability density function. Also show that

$$\mu_2 = \frac{a^2}{\pi}(4 - \pi) \text{ and } \mu_4 = a^4\left(1 - \frac{8}{3\pi}\right).$$

9. The amount of bread (in hundreds of pounds) x that a certain bakery is able to sell in a day is found to be a numerical valued random phenomenon with a probability function specified by the p.d.f. $f(x)$ is given by

$$f(x) = \begin{cases} kx, & 0 \le x < 5 \\ k(10 - x), & 5 \le x < 10 \\ 0, & \text{Otherwise} \end{cases}$$

Find the value of k such that $f(x)$ is a probability density function.

10. The kms X (in thousands of kms) which car owners get with a certain kind of tyre is a random variable having probability density function:

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$$

Find the probabilities that one of these tyres will last (*i*) atmost 10,000 kms, (*ii*) anywhere from 16,000 to 24,000 kms, and (*iii*) at least 30,000 miles.

[**Hint.** (*i*) $P(X \le 10) = \int_0^{10} f(x)\, dx$, (*ii*) $P(16 \le X \le 24)$, (*iii*) $P(X \ge 30)$]

11. If the probability density function is $f(x) = \begin{Bmatrix} kx^3, & \text{if } 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{Bmatrix}$, find the value of '$k$'. Also, find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$. (*A.K.T.U. 2022*)

Answers

1. (*ii*) $\lambda = 0.59607$ 2. (*i*) $b - a$ (*ii*) $y_0 = \frac{1}{\sigma}$ 3. $\frac{3}{16}$ 4. $\frac{4}{9}$

7. $k = \frac{1}{2}$, mean = 3, variance = 3, $\beta_1 = \frac{4}{3}$, $\beta_2 = 5$

9. $\frac{1}{25}$ 10. (*i*) 0.3935 (*ii*) 0.1481 (*iii*) 0.2231

11. $k = \frac{4}{81}$, $P\left(\frac{1}{2} < x < \frac{3}{2}\right) = \frac{5}{81}$.

4.26 JOINT PROBABILITY DISTRIBUTION

Let X and Y be two random variables defined on the same sample space.

Case 1. If X and Y are discrete random variables then a probability function

$$P(X = x_i, Y = y_j) = P(x_i, y_j) \text{ or } p_{ij}$$

that yields the probability that X will assume a particular value x_i while at the same time Y assumes a particular value y_j called a joint probability function of X and Y and has the following properties:

(*i*) $p(x_i, y_j) \ge 0$ (*ii*) $\sum_i \sum_j p(x_i, y_j) = 1$.

Case 2. If X and Y are continuous random variables then the probability density function

$$P\left[x - \frac{dx}{2} \le X \le x + \frac{dx}{2}, y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right] = f(x, y)$$

that yields the probability that the point (x, y) lies in the infinitesimal rectangular region of area $dx\, dy$ is called the joint probability density function of X and Y and satisfies (*i*) $f(x, y) \geq 0$ for all (x, y) in the given range.

(*ii*) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy$ exists and is equal to 1.

(*iii*) $P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y)\, dx\, dy$

The probability distribution defined in terms of joint probability function or joint probability density function is called a joint probability distribution.

4.27 MARGINAL PROBABILITY DISTRIBUTION

Let X and Y be two discrete random variables defined on the same sample space, such that

$$P(X = x_i) = p_i \text{ and } P(Y = y_j) = p_j'; \; i = 1, ..., m, j = 1, ..., n$$

and

$$P(X = x_i, Y = y_j) = p_{ij}$$

then their joint occurrence have $m \times n$ distinguished pairs (x_i, y_j) which can be arranged in a rectangular array with their probabilities.

As the variable X assumes a definite value x_i, it is acccompained by one of the n values $y_1, y_2, ..., y_n$ of Y. *i.e.*, $X = x_i$ can occur in n mutually exclusive ways ; therefore

$$P(X = x_i) = P(X = x_i, Y = y_1) + P(X = x_i, Y = y_2) + \cdots + P(X = x_i, Y = y_n)$$

or $p = p_{i1} + p_{i2} + ... + p_{in} = \sum_{j=1}^{n} p_{ij} = g(x_i)$ say, which is called marginal probability of X for $X = x_i$.

and similarly for fixed y_j

$$p_j' = \sum_{i=1}^{n} p_{ij} = h(y_j) \text{ say,}$$

which is called marginal probability of Y for $Y = y_j$.

i.e., by adding the probabilities p_{ij} ... in individual rows and columns, we obtain the marginal probabilities of X and Y which are identical to the individual probabilities of X and Y respectively.

The set of values of a random variable X together with the marginal probabilities is called the *marginal distribution* of that random variable. In fact, marginal distribution is simply the probability distribution of the random variable.

Let X and Y be continuous random variables.

If (X, Y) has the joint density function f(x, y), then the marginal probability density function of X is

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

and marginal probability density function of Y is

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

The corresponding probability distribution is called *Marginal Probability Distribution.*

4.28 CONDITIONAL PROBABILITY DISTRIBUTION

Suppose $P(X = x_i, Y = y_j) = p_{ij}$, $i = 1, 2,......, m, j = 1, 2,....., n$ is the joint probability distribution of two discrete random variables, $p_j = P(Y = y_j)$ the probability distribution of Y and $p_i = P(X = x_i)$ the probability distribution of X.

The conditional probability of the event "$Y = y_j$, given that $X = x_i$ (with $p_i > 0$)" is

$$\frac{P(X = x_i, Y = y_j)}{P(X = x_i)}$$

i.e.,

$$P(Y = y_j | X = x_i) = \frac{p_{ij}}{p_i} = h(y_j/x_i), \text{ say.}$$

The function $h(y_j/x_i) = \dfrac{p_{ij}}{p_i}$ $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ *that yields the conditional probabilities of the random variable Y is called the conditional distribution of Y for given X, and* $g(x_i/y_j) = P(X = x_i/Y = y_j) = \dfrac{p_{ij}}{p_j}$ is the conditional probability distribution of X for given Y.

If (X, Y) has the joint probability density function f(x, y), then the conditional probability density function of X for given Y is defined for any y such that h(y) > 0 and is equal to

$$g(x/y) = \frac{f(x, y)}{h(y)}$$

and the conditional p.d.f. of Y for given X is defined for any x such that g(x) > 0 and is equal to

$$h(y/x) = \frac{f(x, y)}{g(x)}.$$

ILLUSTRUTIVE EXAMPLES

Example 1. *The joint probability distribution of two random variables X and Y is given by :* $P(X = 0, Y = 1) = \dfrac{1}{3}$, $P(X = 1, Y = -1) = \dfrac{1}{3}$, *and* $P(X = 1, Y = 1) = \dfrac{1}{3}$.

Find (i) Marginal distributions of X and Y, and (ii) the conditional probability distribution of X given Y = 1.

Sol. (*i*) $P(X = -1) = \sum_y P(X = -1, Y = y)$

$$= P(X = -1, Y = -1) + P(X = -1, Y = 0) + P(X = -1, Y = 1) = 0$$

Similarly $P(X = 0) = \dfrac{1}{3}$ and $P(X = 1) = \dfrac{2}{3}$

Y \ X	−1	0	1	Marginal Y
−1	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Marginal (X)	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Thus

Marginal distribution of X is:				Marginal distribution of Y is:			
Values of X, x :	-1	0	1	Values of Y, y :	-1	0	1
$P(X = x)$:	0	$\frac{1}{3}$	$\frac{2}{3}$	$P(Y = y)$:	$\frac{1}{3}$	0	$\frac{2}{3}$

(*ii*) The conditional probability distribution of X given Y is:

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}. \text{ Now}$$

$$P(X = -1 \mid Y = 1) = \frac{P(X = -1, Y = 1)}{P(Y = 1)} = 0,$$

$$P(X = 0 \mid Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$P(X = 1 \mid Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Thus the conditional distribution of X given Y = 1 is :

Values of $X = x$	-1	0	1
$P(X = x \mid Y = 1)$	0	$\frac{1}{2}$	$\frac{1}{2}$

Example 2. *For the adjoining bivariate probability distribution of X and Y, find:*
(i) $P(X \le 1, Y = 2)$, *(ii)* $P(X \le 1)$,
(iii) $P(Y \le 3)$, *and* *(iv)* $P(X < 3, Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Sol. The marginal distributions are given below:

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma\, p(x) = 1$ $\Sigma\, p(y) = 1$

(*i*) $P(X \leq 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) = 0 + \frac{1}{16} = \frac{1}{16}$

(*ii*) $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(*iii*) $P(Y \leq 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(*iv*) $P(X < 3, Y \leq 4) = P(X = 0, Y \leq 4) + P(X = 1, Y \leq 4) + P(X = 2, Y \leq 4)$

$$= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right) = \frac{9}{16}.$$

Example 3. *For the joint probability distribution of two random variables X and Y given below:*

X \ Y	1	2	3	4	Total
1	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{8}{36}$
4	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{9}{36}$
Total	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	1

Find (i) the marginal distributions of X and Y, and (ii) conditional distribution of X given the value of Y = 1 and that of Y given the value of X = 2.

Sol. (*i*) The marginal distribution of X is defined as:

$$P(X = x) = \sum_y P(X = x, Y = y)$$

$$\therefore \quad P(X = 1) = \sum_y P(X = 1, Y = y)$$

$$= P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 1, Y = 3) + P(X = 1, Y = 4)$$

$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}.$$

Similarly, $P(X = 2) = \sum_y P(X = 2, Y = y) = \frac{9}{36}$; $P(X = 3) = \sum_y P(X = 3, Y = y) = \frac{8}{36}$

and $$P(X = 4) = \sum_y P(X = 4, Y = y) = \frac{9}{36}.$$

Similarly, we can obtain the marginal distribution of Y.

Marginal Distribution of X

Values of X, x	1	2	3	4
$P(X = x)$	$\frac{10}{36}$	$\frac{9}{36}$	$\frac{8}{36}$	$\frac{9}{36}$

Marginal Distribution of Y

Values of Y, y	1	2	3	4
$P(Y = y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$

(*ii*) **The conditional probability function of X given Y is defined as follows:**

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}. \text{ Therefore}$$

$$P(X = 1 \mid Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{4/36}{11/36} = \frac{4}{11}$$

$$P(X = 2 \mid Y = 1) = \frac{P(X = 2, Y = 1)}{P(Y = 1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

$$P(X = 3 \mid Y = 1) = \frac{P(X = 3, Y = 1)}{P(Y = 1)} = \frac{5/36}{11/36} = \frac{5}{11}$$

$$P(X = 4 \mid Y = 1) = \frac{P(X = 4, Y = 1)}{P(Y = 1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Hence the conditional distribution of X given Y = 1 is:

x :	1	2	3	4
$P(X = x \mid Y = 1)$:	$\frac{4}{11}$	$\frac{1}{11}$	$\frac{5}{11}$	$\frac{1}{11}$

Similarly, we can obtain the conditional distribution of Y for X = 2 as given below:

y :	1	2	3	4
$P(Y = y \mid X = 2)$:	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{9}$

Example 4. *Two discrete random variables X and Y have the joint probability density function:*

$$p_{XY}(x, y) = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!}, \; y = 0, 1, 2, ..., x \,;\, x = 0, 1, 2, ...,$$

where λ, p are constants with $\lambda > 0$ and $0 < p < 1$.

Find (i) The marginal probability density functions of X and Y.

(ii) The conditional distribution of Y for a given X and of X for a given Y.

Sol. (*i*) The marginal *p.m.f.* of X is given by:

$$p_X(x) = \sum_{y=0}^{x} p(x, y) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} = \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} \frac{x!\, p^y (1-p)^{x-y}}{y!(x-y)!}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} {}^xC_y p^y (1-p)^{x-y} = \frac{\lambda^x e^{-\lambda}}{x!}[p + (1-p)]^x = \frac{\lambda^x e^{-\lambda}}{x!}, \; x = 0, 1, 2, ...$$

which is the probability function of a Poisson distribution with parameter λ.

$$p_Y(y) = \sum_{x=y}^{\infty} p(x, y) = \sum_{x=y}^{\infty} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} = \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!}$$

$$[\because \; y = 0, 1, 2, ..., x \;\Rightarrow\; x \le y \;\Rightarrow\; x \ge y]$$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} e^{\lambda(1-p)} = \frac{e^{-\lambda p} (\lambda p)^y}{y!}; \; y = 0, 1, 2, ...$$

which is the probability function of a Poisson distribution with parameter λp.

(*ii*) The conditional distribution of Y for given X is:

$$P_{Y|X}(y \mid x) = \frac{p_{XY}(x, y)}{p_X(x)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y} x!}{y!(x-y)!\lambda^x e^{-\lambda}} = \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y}$$

$$= {}^xC_y p^y (1-p)^{x-y}, x \geq y \quad i.e., \quad y = 0, 1, 2, ..., x.$$

The conditional probability distribution of X for given Y is:

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} \cdot \frac{y!}{e^{-\lambda p}(\lambda p)^y} \qquad \text{[c.f. Part (i)]}$$

$$= \frac{e^{-\lambda q}(\lambda q)^{x-y}}{(x-y)!}; \; q = 1-p, x \geq y \; i.e., x = y, y+1, y+2, ...$$

Example 5. *If X and Y are two random variables having joint density function:*

$$f(x, y) = \begin{cases} \frac{1}{8}(6-x-y); 0 \leq x < 2, 2 \leq y < 4 \\ 0, \text{otherwise} \end{cases}$$

Find (i) $P(X < 1 \cap Y < 3)$, (ii) $P(X + Y < 3)$, and (iii) $P(X < 1 \mid Y < 3)$.

Sol. We have

(*i*) $P(X < 1 \cap Y < 3) = \int_{-\infty}^{1} \int_{-\infty}^{3} f(x, y)\, dx\, dy = \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\, dx\, dy = \frac{3}{8}$

(*ii*) $P(X + Y < 3) = \int_0^1 \int_2^{3-x} \frac{1}{8}(6-x-y)\, dx\, dy = \frac{5}{24}$

(*iii*) $P(X < 1 \mid Y < 3) = \frac{P(X<1 \cap Y<3)}{P(Y < 3)} = \frac{3/8}{5/8} = \frac{3}{5}$

$$\left[\text{From part (i) and } P(Y < 3) = \int_0^2 \int_2^3 \frac{1}{8}(6-x-y)\, dx\, dy = \frac{5}{8}\right]$$

Example 6. *Suppose that two-dimensional continuous random variable (X, Y) has joint p.d.f. given by* $f(x, y) = \begin{cases} 6x^2 y, 0 < x < 1, 0 < y < 1 \\ 0, \text{elsewhere} \end{cases}$.

(*i*) *Verify that* $\int_0^1 \int_0^1 f(x, y)\, dx\, dy = 1$.

(*ii*) *Find* $P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right)$, $P(X + Y < 1)$, $P(X > Y)$ *and* $P(X < 1 \mid Y < 2)$.

Sol. (*i*) $\int_0^1 \int_0^1 f(x, y)\, dx\, dy = \int_0^1 \int_0^1 6x^2 y\, dxdy = \int_0^1 6x^2 \left| \frac{y^2}{2} \right|_0^1 dx = \int_0^1 3x^2\, dx = \left| x^3 \right|_0^1 = 1$

(*ii*) $P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right) = \int_0^{3/4} \int_{1/3}^{1} 6x^2 y\, dxdy + \int_0^{3/4} \int_1^2 0 . dxdy$

$$= \int_0^{3/4} 6x^2 \left| \frac{y^2}{2} \right|_{1/3}^{1} dx = \frac{8}{9} \int_0^{3/4} 3x^2\, dx = \frac{8}{9} \left| x^3 \right|_0^{3/4} = \frac{3}{8}.$$

$$P(X+Y<1) = \int_0^1 \int_1^{1-x} 6x^2 y\, dxdy = \int_0^1 6x^2 \left| \frac{y^2}{2} \right|_0^{1-x} dx = \int_0^1 3x^2(1-x)^2 dx = \frac{1}{10}$$

$$P(X>Y) = \int_0^1 \int_0^x 6x^2 y\, dxdy = \int_0^1 3x^2 \left| y^2 \right|_0^x dx = \int_0^1 3x^4 dx = \frac{3}{5}.$$

$$P(X<1 \mid Y<2) = \frac{P(X<1 \cap Y<2)}{P(Y<2)}$$

where $$P(X<1 \cap Y<2) = \int_0^1 \int_0^1 6x^2 y\, dxdy + \int_0^1 \int_1^2 0 \,.\, dxdy = 1$$

and $$P(Y<2) = \int_0^1 \int_0^2 f(x,y)\, dxdy = \int_0^1 \int_0^1 6x^2 y\, dxdy + \int_0^1 \int_1^2 0 \,.\, dxdy = 1$$

$\therefore$ $$P(X<1 \mid Y<2) = \frac{P(X<1 \cap Y<2)}{P(Y<2)} = 1.$$

Example 7. *The joint probability density function of a two-dimensional random variable (X, Y) is given by:*

$$f(x,y) = \begin{cases} 2; 0<x<1, 0<y<x; \\ 0, \text{elsewhere} \end{cases}$$

(i) Find the marginal density functions of X and Y.

(ii) Find the conditional density function of Y given X = x and conditional density function of X given Y = y.

(iii) Check for independence of X and Y.

Sol. Evidently $f(x, y) \geq 0$ and $\int_0^1 \int_0^x 2\, dxdy = 2\int_0^1 x\, dx = 1.$

(*i*) The marginal *p.d.f.*'s of X and Y are given by

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = \int_0^x 2dy = 2x, \; 0<x<1 \\ 0, \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_y^1 2dx = 2(1-y), 0<y<1 \\ 0, \text{elsewhere} \end{cases}$$

(*ii*) The conditional density function of Y given X, $(0<x<1)$ is

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}, \; 0<y<x.$$

The conditional density function of X given Y, $(0<y<1)$ is

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)}, \; y<x<1$$

(*iii*) Since $f_X(x)\, f_Y(y) = 2(2x)(1-y) \neq f_{XY}(x, y)$, X and Y are not independent.

Example 8. *The joint p.d.f. of two random variables X and Y is given by*

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}; \; 0 \leq x < \infty, \; 0 \leq y < \infty$$

Find the marginal distributions of X and Y, and the conditional distribution of Y for X = x.

Sol. Marginal *p.d.f.* of X is given by:

$$f_X(x) = \int_0^\infty f(x, y)\, dy = \frac{9}{2(1+x)^4}\int_0^\infty \frac{(1+y)+x}{(1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)^4}\int_0^\infty \{(1+y)^{-3} + x(1+y)^{-4}\}\, dy$$

$$= \frac{9}{2(1+x)^4}\left(\left|\frac{-1}{2(1+y)^2}\right|_0^\infty + x\left|\frac{-1}{3(1+y)^3}\right|_0^\infty\right)$$

$$= \frac{9}{2(1+x)^4}\cdot\left(\frac{1}{2}+\frac{x}{3}\right) = \frac{3}{4}\cdot\frac{3+2x}{(1+x)^4};\ 0 < x < \infty$$

Since $f(x, y)$ is symmetric in x and y, the marginal *p.d.f.* of Y is given by:

$$f_Y(y) = \int_0^\infty f(x, y)\, dx = \frac{3}{4}\cdot\frac{3+2y}{(1+y)^4};\ 0 < y < \infty$$

The conditional distribution of Y for X = x is given by:

$$f_{XY}(Y = y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\times\frac{4(1+x)^4}{3(3+2x)} = \frac{6(1+x+y)}{(1+y)^4(3+2x)};\ 0 < y < \infty.$$

Example 9. *Joint distribution of X and Y is given by: $f(x, y) = 4xy\, e^{-(x^2+y^2)}$; $x \ge 0, y \ge 0$. Test whether X and Y are independent. For the above joint distribution, find the conditional density of X given Y = y.*

Sol. Marginal density of X is given by

$$f_X(x) = \int_0^\infty f_{XY}(x, y)\, dy = \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dy = 4x\, e^{-x^2}\int_0^\infty y\, e^{-y^2}\, dy$$

$$= 4x\, e^{-x^2}.\int_0^\infty e^{-t}.\frac{dt}{2} = 2x.e^{-x^2}\left|-e^{-t}\right|_0^\infty$$

$$\therefore \qquad f_X(x) = 2x\, e^{-x^2};\ x \ge 0$$

Similarly, the marginal *p.d.f.* of Y is given by:

$$f_Y(y) = \int_0^\infty f_{XY}(x, y)\, dx = 2y\, e^{-y^2};\ y \ge 0$$

Since $f_{XY}(x, y) = f_X(x) . f_Y(y)$, X and Y are independently distributed. The conditional distribution of X for given Y is given by :

$$f_{X\mid Y}(X = x \mid Y = y) = \frac{f(x, y)}{f_Y(y)} = 2x\, e^{-x^2};\ x \ge 0.$$

Example 10. *Let X and Y be jointly distributed with p.d.f.:*

$$f_{XY}(x, y) = \begin{cases} \dfrac{1}{4}(1+xy), |x| < 1, |y| < 1 \\ 0, \text{otherwise} \end{cases}$$

Show that X and Y are not independent but X^2 and Y^2 are independent.

Sol. $f_X(x) = \int_{-1}^{1} f(x, y)\, dy = \frac{1}{4}\left| y + \frac{xy^2}{2} \right|_{-1}^{1} = \frac{1}{2}, -1 < x < 1;$

Similarly, $f_Y(y) = \int_{-1}^{1} f(x, y)\, dx = \frac{1}{2}, -1 < y < 1$

Since $f_{X, Y}(x, y) \neq f_X(x) f_Y(y)$, X and Y are not independent. However,

$$P(X^2 \leq x) = P(|X| \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} f_X(x)\, dx = \sqrt{x} \qquad ...(1)$$

$$P(X^2 \leq x \cap Y^2 \leq y) = P(|X| \leq \sqrt{x} \cap |Y| \leq \sqrt{y})$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(u, v)\, dv \right] du = \sqrt{x}\, \sqrt{y}$$

$$= P(X^2 \leq x) \,.\, P(Y^2 \leq y) \qquad \text{[From (1)]}$$

Hence, X^2 and Y^2 are independent.

TEST YOUR KNOWLEDGE

1. Find the mean and variance of p.d.f.

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

2. Find the mean and variance of exponential distribution $f(x) = \frac{1}{b} e^{-x/b}; x > 0, b > 0$

3. A continuous random variable X has the probability density function:

$$f(x) = A + Bx, \; 0 \leq x \leq 1.$$

If the mean of the distribution is $\frac{1}{2}$, find A and B.

4. For the probability density function: $f(x) = cx^2(1 - x), 0 < x < 1$, find
   (*i*) the constant c, and (*ii*) mean.

5. The distribution of a continuous random variable X in range $(-3, 3)$ is given by p.d.f.:

$$f(x) = \begin{cases} \frac{1}{16}(3 + x)^2, & -3 \leq x \leq -1 \\ \frac{1}{16}(6 - 2x^2), & -1 \leq x \leq 1 \\ \frac{1}{16}(3 - x)^2, & 1 \leq x \leq 3 \end{cases}$$

(*i*) Verify that the area under the curve is unity.
(*ii*) Find the mean and variance of the above distribution.

6. The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the probability density function $f(x)$ as:

$$f(x) = \begin{cases} A\, e^{-x/5}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(*a*) Find the value of A that makes $f(x)$ a p.d.f.
(*b*) What is the probability that the number of minutes that she will talk over the phone is: (*i*) more than 10 minutes, (*ii*) less than 5 minutes, and (*iii*) between 5 and 10 minutes.

7. (*a*) Find k so that $f(x, y) = kxy$, $1 \le x \le y \le 2$ will be a probability density function.

(*b*) If $f(x, y) = \begin{cases} e^{-(x+y)}, & x \ge 0, y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$ is the joint probability density function of random variables X and Y, find

(*i*) $P(X < 1)$, (*ii*) $P(X > Y)$, and (*iii*) $P(X + Y < 1)$.

8. The joint probability density function of the two-dimensional variable (X, Y) is of the form:

$$f(x, y) = \begin{cases} e^{-(x+y)}, & 0 \le y < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

(*i*) Determine the constant k.

(*ii*) Find the conditional probability density function $f_1(x \mid y)$.

(*iii*) Compute $P(Y \ge 3)$.

9. Two-dimensional random variable (X, Y) have the joint density function

$$f(x, y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal and conditional distributions.

10. (*a*) The joint probability density function of the two-dimensional random variable (X, Y) is given by:

$$f(x, y) = \begin{cases} \frac{8}{9}xy, & 1 \le x \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(*i*) Find the marginal density functions of X and Y.

(*ii*) Find the conditional density function of Y given $X = x$, and conditional density function of X given $Y = y$.

(*b*) Let $f(x_1, x_2) = \begin{cases} A(x_1x_2 + e^{x_1}); & 0 < (x_1\ x_2) < 1 \\ 0, & \text{elsewhere} \end{cases}$ be the joint p.d.f. of (X_1, X_2). Determine A.

11. The joint probability density function of the two dimensional random variable (X, Y) is given by :

$$f(x, y) = \begin{cases} x^3y^3/16, & 0 \le x \le 2, 0 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the marginal densities of X and Y. Also, find the cumulative distribution functions for X and Y.

12. (*a*) If the joint distribution function of X and Y is given by

$$F_{XY}(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)}; & x > 0, y > 0 \\ 0; & \text{elsewhere} \end{cases}$$

(*i*) Find the marginal densities of X and Y.

(*ii*) Are X and Y independent?

(*iii*) Find $P(X \le 1 \cap Y \le 1)$ and $P(X + Y \le 1)$.

(*b*) Given $f(x, y) = e^{-(x+y)}$, $0 \le x < \infty$, $0 \le y < \infty$. Are X and Y independent? Also, find $P(X > 1)$.

13. A two-dimensional random variable (X, Y) have a bivariate distribution given by :

$P(X = x, Y = y) = \dfrac{x^2 + y}{32}$, for $x = 0, 1, 2, 3$ and $y = 0, 1$. Find the marginal distributions of X and Y.

14. A two-dimensional random variable (X, Y) have a joint probability mass function:

$$p(x, y) = \frac{1}{27}(2x + y),$$ where x and y can assume only the integer values 0, 1 and 2.

Find the conditional distribution of Y for X = x.

Answers

1. Mean = 4, Variance = 80

2. Mean = b, Variance = b^2

3. A = 1, B = 0

4. (*i*) $c = 12$, (*ii*) Mean = 3/5

5. (*ii*) Mean = 0, Variance = 1

6. (*a*) $A = \frac{1}{5}$ (*b*) (*i*) $\frac{1}{e^2}$ (*ii*) $\frac{e-1}{e}$ (*iii*) $\frac{e-1}{e^2}$

7. (*a*) $k = 8/9$ (*b*) (*i*) $1 - \frac{1}{e}$ (*ii*) $\frac{1}{2}$ (*iii*) $1 - \frac{2}{e}$

8. (*i*) $k = 1$ (*ii*) $f_1(x \mid y) = e^{-x}$ (*iii*) e^{-3}

9. (*i*) $f_X(x) = \begin{cases} 4x(1-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$, $f_Y(y) = 4y^3, 0 < y < 1$

(*ii*) $f_{X|Y}(x \mid y) = \frac{2x}{y^2}, 0 < x < y, 0 < y < 1$

$f_{Y|X}(y \mid x) = \frac{2y}{1-x^2}, x < y < 1, 0 < x < 1$

10. (*a*) (*i*) $f_X(x) = \begin{cases} \frac{4}{9}x(4-x^2), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$ (*ii*) $f_{X|Y}(x \mid y) = \frac{2x}{y^2 - 1}, \quad 1 \le x \le y$

$f_Y(y) = \frac{4}{9}y(y^2 - 1), 1 \le y \le 2$ $f_{Y|X}(y \mid x) = \frac{2y}{4 - x^2}, x \le y \le 2.$

(*b*) $A = \frac{4}{4e - 3}$.

11. $f_X(x) = \frac{x^3}{4}, 0 \le x \le 2,$ $f_Y(y) = \frac{y^3}{4}, 0 \le y \le 2;$ $F_X(x) = \begin{cases} 0, x < 0 \\ x^4/16, 0 \le x \le 2, \\ 1, x > 2 \end{cases}$ $F_Y(y) = \begin{cases} 0, y < 0 \\ y^4/16, 0 \le y \le 2 \\ 1, y > 2 \end{cases}$

12. (*a*) (*i*) $f_X(x) = e^{-x}, x \ge 0; f_Y(y) = e^{-y}, y \ge 0$ (*ii*) Yes

(*iii*) $P(X \le 1 \cap Y \le 1) = \left(1 - \frac{1}{e}\right)^2$, $P(X + Y \le 1) = 1 - \frac{2}{e}$

(*b*) Yes; $\frac{1}{e}$.

13.

X / *Y*	*0*	*1*	*2*	*3*	*Marginal distribution of Y, P (Y = y)*
0	0	1/32	1/8	9/32	7/16
1	1/32	1/16	5/32	5/16	9/16
Marginal distribution of X, P(X = x)	1/32	3/32	9/32	19/32	1

14. Conditional distribution of Y for X = x.

Y \ X	0	1	2
0	0	1/3	2/3
1	2/9	3/9	4/9
2	4/15	5/15	6/15

4.29 MATHEMATICAL EXPECTATION OR EXPECTED VALUE OF A RANDOM VARIABLE

Once we have constructed the probability distribution for a random variable, we often want compute the mean or expected value of the random variable. The *expected value* of a discre random variable is a weighted average of all possible values of the random variable, where th weights are the probabilities associated with the corresponding values.

If x, denotes a discrete random variable which can assume the values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$ where $p_1 + p_2 + ... + p_n = 1$, the *mathematical expectation* of x or simply the *expectation of* X, denoted by E(x), is defined as

$$E(x) = p_1x_1 + p_2x_2 + ... + p_nx_n = \sum_{i=1}^{n} p_ix_i = \Sigma px, \Sigma p = 1.$$

Let $\phi(x)$ be a function of the variate x so that it takes the values $\phi(x_1), \phi(x_2), ...$ When x takes the values $x_1, x_2, ...$; and if $p_1, p_2, ...,$ be the respective probabilities, then the expected value of the function $\phi(x)$ is defined as

$$E[\phi(x)] = p_1\phi(x_1) + p_2\phi(x_2) + ... + p_n\phi(x_n), \Sigma p = 1$$

If $\phi(x) = x^r$, then

$$E(x^r) = p_1x_1^r + p_2x_2^r + ... + p_nx_n^r.$$

This is defined as the r^{th} moment of the discrete probability distribution about $x = 0$ and is denoted by $\mu r'$. This $\mu r'$ is the expected value of the r^{th} power of the variate.

Also $\mu r = E(x - \bar{x})^r = \Sigma p_i(x_i - \bar{x})^r$

In particular $\mu_1' = E(x) = p_1x_1 + p_2x_2 + ... = \sum_i p_ix_i$

Here if p_i is replaced by $\frac{f_i}{N}$ where $\Sigma f_i = N$, then

$$E(x) = \frac{\Sigma fx}{N}, \text{ which is the mean.}$$

$\therefore$ E(x) represents the mean.

Just as in the case of frequency, this moment is called the *mean value* of the varia of the distributions. More generally it is known as the *expected value* or the *expectation* variate x. The relation

$$\mu_2 = E[\{x - E(x)\}^2] = E(x^2) - [E(x)]^2.$$

is called the *variance* of the distribution of x and is denoted by *var* (x). The relati