ROOT LOCUS TUTORIAL

Q. Sketch the root locus for a unity feedback system with open-loop transfer function,

$$G(S) = \frac{K(s^2 + 2s + 10)}{(s^2 + 4s + 5)}$$

Also, find the value of K for a damping ratio of 0.5 due to a pair of complex conjugate poles and the corresponding closed-loop transfer function

Solution:

Number of Open-loop poles $n = 2$; $s = -2 \pm j1$;$n=2$
Number of Open-loop zeros $m = 2$; $s = -1 \pm j3$;$m=2$

Step 1 *Starting points*

One branch of the RL starts from $s = -2 + j1$
and the other branch starts from $s = -2 - j1$

Step 2 *Terminating points*

One locus terminates at $s = -1 + j3$
and the other terminates at $s = -1 - j3$

Step 3 *Section of the Real-axis*

As there is no open-loop poles & zeros on the real axis, the real axis is not a part of RL.

Step 4 *Break-away and break-in points*

Since no section of the real axis is a part of the RL, there is no Break-away and break-in points on the real axis.

Step 5 *Asymptotes*

There are no asymptotes as both the branches of the RL do not go to infinity, but terminate at finite open-loop zeros.

Step 6 *Angle of departure from complex poles*

6.1 Angle of departure from s = - 2 + j1

$$\theta_{D1} = 180^0 + Angle\left[\frac{K(s+1+j3)(s+1-j3)}{(s+2+j1)}\right]_{s=-2+j1}$$

$$= 180^0 + Angle\left[\frac{K(-1+j4)(-1-j2)}{j2}\right]$$

$$= 180^0 + \left[104.4^0 + (-116.57^0) - 90^0\right]$$

$$= 77.47^0$$

6.2 Angle of departure from s = - 2 - j1

$$\theta_{D2} = 180^0 + Angle\left[\frac{K(s+1+j3)(s+1-j3)}{(s+2-j1)}\right]_{s=-2-j1}$$

$$= 180^0 + Angle\left[\frac{K(-1+j2)(-1-j4)}{-j2}\right]$$

$$= 180^0 + \left[166.57^0 + (-104.04^0) - (-90^0)\right]$$

$$= -77.47^0$$

Step 7 *Angle of arrival at complex zeros*

7.1 Angle of arrival of RL at s = -1 + j3

$$\theta_{A1} = 180^0 - Angle\left[\frac{K(s+1+j3)}{(s+2+j1)(s+2-j1)}\right]_{s=-1+j3}$$

$$= 180^0 - Angle\left[\frac{K(j6)}{(1+j4)(1+j12)}\right]$$

$$= 180^0 - \left[90^0 - 75.96^0 - 63.43^0\right]$$

$$= -130.6^0$$

7.2 Angle of arrival of RL at s = -1 - j3

$$\theta_{A2} = 180^0 - Angle\left[\frac{K(s+1-j3)}{(s+2+j1)(s+2-j1)}\right]_{s=-1-j3}$$

$$= 180^0 - Angle\left[\frac{K(-j6)}{(1-j2)(1-j4)}\right]$$

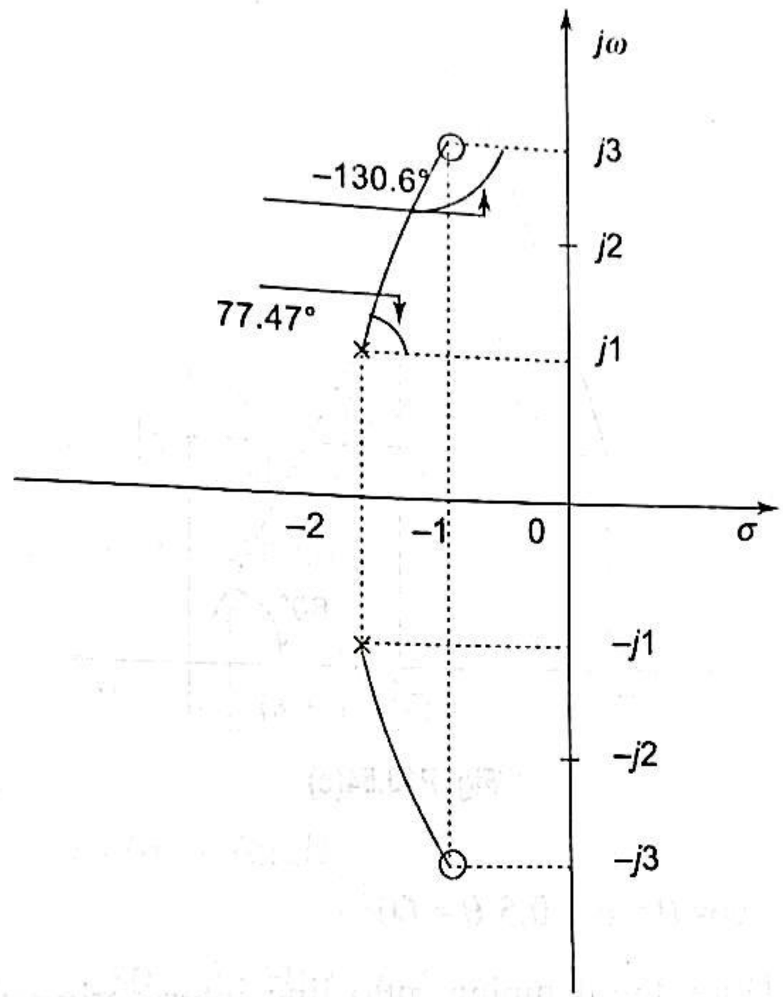
$$= 180^0 - \left[-(-90^0) - (-63.43)^0 - (-75.96)^0\right]$$

$$= 130.6^0$$

Step 8 *Imaginary axis crossing point*

As the RL enters the open-loop zero on a same quadrant itself, it does not cross the imaginary axis.

The salient points are as shown .



Construction

- One root locus starts from the open-loop pole at

  $s = -2 + j1$, at an angle of 77.47^o

  and enters the open-loop zero at

  $s = -1 + j3$, at an angle of -130.6^o

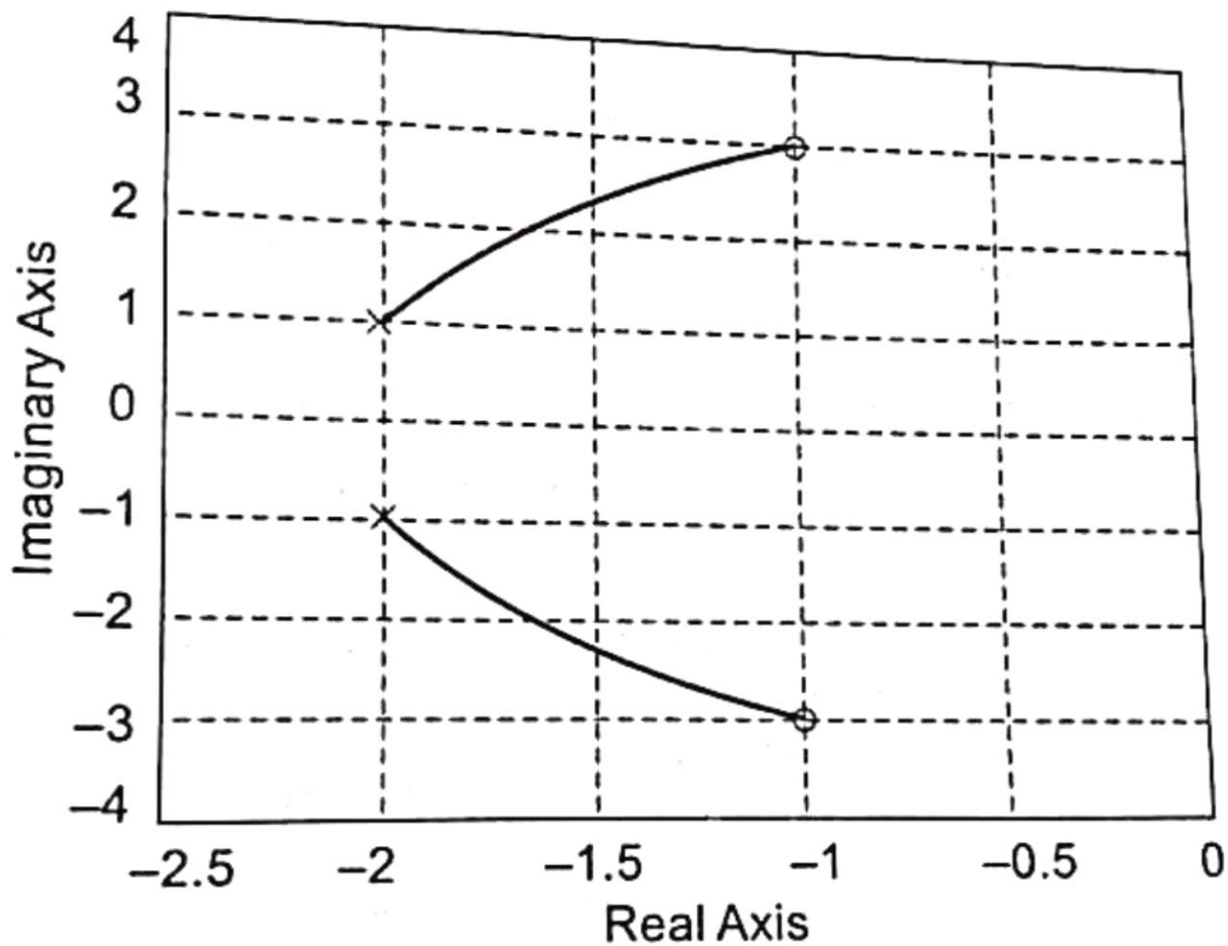
- The second root locus starts from

  $s = -2 - j1$, at an angle of -77.47^o

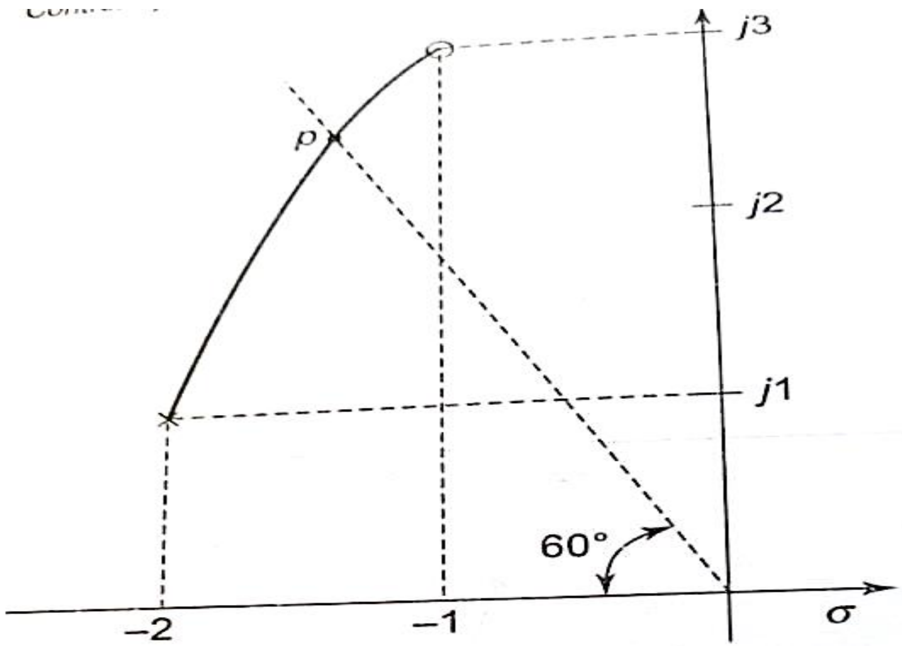
  and terminates at

  $s = -1 - j3$, at an angle of 130.6^o

The complete RL is shown below



For finding the value of K corresponding to a damping ratio of 0.5 , a part of the RL is shown below



$$\cos\theta = \zeta = 0.5$$
$$\theta = 60^{\circ}$$

Draw the damping ratio line intersecting the root locus at point P .

Let the point P be at an angle of 120º.

Since the point is on the root locus it should satisfy the characteristic equation,

$$1+\frac{K(s^2+2s+10)}{(s^2+4s+5)}=0$$

$$s^2+4s+5+K(s^2+2s+10)=0$$

$$(K+1)s^2+(2K+4)s+(10K+5)=0$$

$$(K+1)a^2\angle 240^0 + (2K+4)a\angle 120^0 + (10K+5) = 0$$

$$(K+1)a^2(-0.5 - j0.866) + (2K+4)a(-0.5 + j0.866) + (10K+5) = 0$$

Separating the real and imaginary parts and equating them to zero,

$$-0.5(\mathrm{K}+1)\mathrm{a}^2 \; -(\mathrm{K}+2)\mathrm{a} + 10\mathrm{K} + 5 = 0 \qquad (1)$$

$$-0.866(\mathrm{K}+1)\mathrm{a}^2 + 0.866(2\mathrm{K}+4)\mathrm{a} = 0 \qquad (2)$$

$$\text{From Eq. (2)}, \quad (\mathrm{K}+1)\mathrm{a} = 2\mathrm{K}+4$$

$$\text{or } \mathrm{K} = \frac{4-a}{a-2} \qquad (3)$$

Substituting for K in Eq. (1)

$$-0.5\left(\frac{4-a}{a-2}+1\right)\mathrm{a}^2 - \left(\frac{4-a}{a-2}+2\right)\mathrm{a}+10\left(\frac{4-a}{a-2}\right)+5=0$$

$$or$$

$$\mathrm{a}^2+2.5\mathrm{a}-15=0 \qquad (4)$$

Solving Eq. (4) we get,

a = 2.82

Substituting the value of a in Eq. (3) we get,

K = 1.44

The closed-loop poles are

$$\mathrm{s} = 2.82\angle \pm 120^0 = -1.41 \pm j2.44$$

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{K(s^2+2s+10)}{s^2+4s+5+K(s^2+2s+10)} = \frac{1.44(s^2+2s+10)}{s^2+4s+5+1.44(s^2+2s+10)}$$

$$= \frac{1.44(s^2+2s+10)}{2.44s^2+6.88s+19.4}$$

$$\frac{\mathrm{C(s)}}{\mathrm{R(s)}} = \frac{0.59(\mathrm{s}^2+2\mathrm{s}+10)}{\mathrm{s}^2+2.82\mathrm{s}+7.95}$$