State space analysis:

State space analysis is an excellent method for the design and analysis of control systems. The conventional and old method for the design and analysis of control systems is the transfer function method. The transfer function method for design and analysis had many drawbacks.

Drawbacks of transfer function analysis.

- Transfer function is defined under zero initial conditions.
- Transfer function approach can be applied only to linear time invariant systems.
- It does not give any idea about the internal state of the system.
- It cannot be applied to multiple input multiple output systems.
- It is comparatively difficult to perform transfer function analysis on computers.

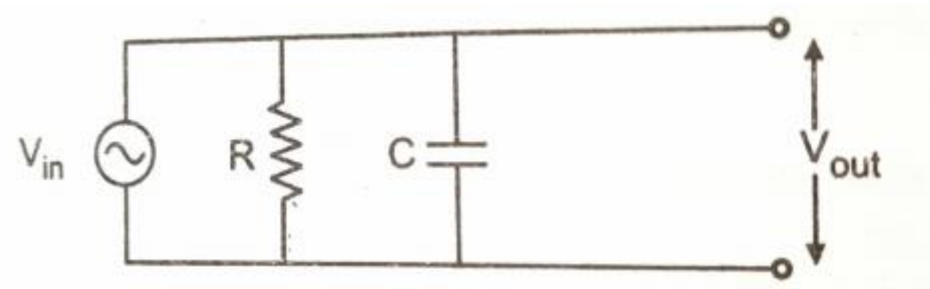
State variable analysis can be performed on any type systems and it is very easy to perform state variable analysis on computers. The most interesting feature of state space analysis is that the state variable we choose for describing the system need not be physical quantities related to the system. Variables that are not related to the physical quantities associated with the system can be also selected as the state variables. Even variables that are immeasurable or unobservable can be selected as state variables.

A state space representation is a mathematical model of a physical system, as a set of input, output and state variables related by first order differential equations. The state space representation (also known as the "time-domain approach") provides a convenient and compact way to model and analyze systems with multiple inputs and outputs. The use of the state space representation is not limited to systems with linear components and zero initial conditions.

Advantages of state variable analysis.

- It can be applied to non-linear system.
- It can be applied to time invariant systems.
- It can be applied to Multiple Input Multiple Output(MIMO) systems.
- Its gives an idea about the internal state of the system.

Dynamic System:



Consider the system shown in figure, to find the output Vout, knowledge of the initial capacitor voltage must be known. Only the information about Vin will not be sufficient to obtain precisely the Vout at any time t>=0. Such systems in which the output is not only dependent on the input but also on the initial conditions are called the system with memory or **Dynamic Systems.**

While in the above network capacitor is replaced by another resistance R1 then the output will be dependent only on the input applied Vin. Such systems in which the output of the system depends only on the input applied at t=0, are called systems with **zero memory or static systems.**

Thus initial conditions affects the system characterization and subsequent behaviour and describe the state of the system at t=t0. So, the state can be regarded as a compact and concise representation of the past history of the system.

State of a dynamic system.

State is the group of variables which summarises the history of the system in order to predict the future values.

The **state** of a system is the minimum set of variables (state variables) whose knowledge at time t=0, along with the knowledge of the inputs at time t≥ t0 completely describes the behaviour of a dynamic system for a time t >t0 .

State variable is a smallest set of variables which fully describes the state of a dynamic system at a given instant of time.

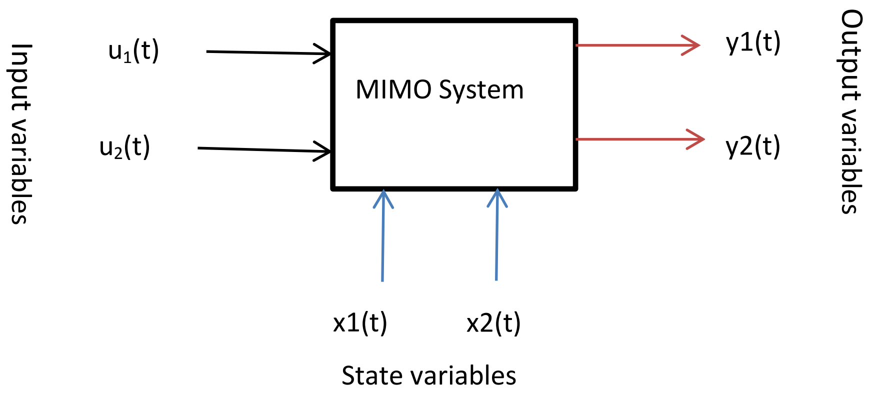
The number of the state variables required is equal to the number of the storage elements present in the system.

Examples − current flowing through inductor, voltage across capacitor

State vector is a vector which contains the state variables as elements.

State model of dynamic systems:

The output equations and state equations together is called state model of a system.



Output equation for two of the outputs can be written as:

$$y_1(t) = C_{11}x_1(t) + C_{12}x_2(t) + d_{11}u_1(t) + d_{12}u_2(t)$$
$$y_2(t) = C_{21}x_1(t) + C_{22}x_2(t) + d_{21}u_1(t) + d_{22}u_2(t)$$

Where c-represents the constants relating the state variables to the outputs
d-represents the constants relating the inputs of a system to the outputs

Matrix form of the output equations:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

So the output equation is

Y(t)=CX(t)+DU(t)
C is the matrix representation of c's and is called as output matrix
D is the matrix representation of d's and is called as transition matrix

State equations (Next state equations) for two of the states and matrix form of state equations can be written as:

$$\frac{dx_1(t)}{dt} = \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_{11}u_1(t) + b_{12}u_2(t)$$

$$\frac{dx_2(t)}{dt} = \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_{21}u_1(t) + b_{22}u_2(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Where
a-represents the constants relating the state variables of the system to the first derivatives of the state variables
b-represents the constants relating the input variables of a system to the first derivatives of the state variables

So the state equation is

$\dot{X}$(t)=AX(t)+BU(t)

A is the matrix representation of a's and is called as system matrix
B is the matrix representation of b's and is called as input matrix

State transition matrix:

State transition matrix is an (nXn) matrix and is designated by ϕ(t), which satisfies the linear homogeneous state equation.

$\frac{dX(t)}{dt} = \dot{X}(t) = AX(t)$ ----1

i,e $\dot{\phi}(t) = A\phi(t)$ so $\phi(t) = X(t)$

•State transition matrix depends only on matrix **A**

•System's response, when it is excited only by its initial condition, **x0,** that is when **u(t)**=0 is called free response

•**ϕ(t)** completely defines the *transition* of the state vector **x(t)**from it's initial state **x(0)**to any new state **x(t). I,e x(t) = ϕ(t)**x(0)

•This is the reason why the matrix **ϕ(t**) is called State transition matrix

Taking the LT of eq1

SX(s)+X(0)=AX(s)

Above eq can be written as

ISX(s)-x(0)=AX(s)
Where I is the identity matrix
ISX(s)-AX(s) = x(0)
X(s)[IS-A] = x(0)
X(s) = $[IS-A]^{-1}$x(0)
Taking inverse LT,
x(t) =L^{-1} $[IS-A]^{-1}$x(0)
we know that **x(t) = ϕ(t)**x(0), Therefore **ϕ(t)** = $[IS-A]^{-1}$

Properties of state transition matrix

1. If t = 0 , then state transition matrix will be equal to an Identity matrix
   ϕ(0)= I
2. Inverse of state transition matrix will be same as that of state transition matrix just by replacing t by -t.

$$\phi^{-1}(t) = \phi(-t)$$

3. If t=t1+t2 , then the corresponding state transition matrix is equal to the multiplication of the two state transition matrices at t=t1 and t=t2.

   ϕ(t1+t2)=ϕ(t1)ϕ(t2)

Examples:

1. Determine the state transition matrix of the given matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Soln:

$$\dot{X}(t) = AX + BU$$

$$y(t) = CX$$

$$\phi(t) = L^{-1}[SI - A]^{-1}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$(SI - A) = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix}$$

$$[SI - A]^{-1} = \frac{\mathrm{Adj}[SI - A]}{|SI - A|}$$

$$\mathrm{Adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\mathrm{Adj}(SI - A) = \mathrm{Adj} \underset{c \qquad d}{\begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix}} = \begin{bmatrix} S-1 & 0 \\ 1 & S-1 \end{bmatrix}$$

$$|SI - A| = \begin{vmatrix} S-1 & 0 \\ -1 & S-1 \end{vmatrix} = (S-1)^2 - 0 = (S-1)^2$$

$$(SI - A)^{-1} = \frac{\begin{bmatrix} S-1 & 0 \\ 1 & S-1 \end{bmatrix}}{(S-1)^2} = \begin{bmatrix} \frac{1}{S-1} & 0 \\ \frac{1}{(S-1)^2} & \frac{1}{S-1} \end{bmatrix}$$

$$\varphi(t) = L^{-1}[SI - A]^{-1} = \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix}$$

Obtaining the Transfer function from the state model:

$$\dot{X}(t) = A X(t) + B U(t) \rightarrow (1)$$

$$y(t) = C X(t) + D U(t) \rightarrow (2)$$

Let L.T for eq (1) B.S.

$$S X(S) - X(0) = A X(S) + B U(S)$$

$$S X(S) - A X(S) = B U(S) + X(0)$$

$$[SI - A] X(S) = X(0) + B U(S)$$

$$X(S) = X(0)[SI-A]^{-1} + B[SI-A]^{-1} U(S)$$

L.T of eq & sub $x(s)$.

$$Y(S) = C X(S) + D U(S)$$

$$= C\left[X(0)[SI-A]^{-1} + B[SI-A]^{-1} U(S)\right] + D U(S)$$

T.F $X(0) = 0$.

$$Y(S) = C\left[B(SI-A)^{-1} U(S)\right] + D U(S)$$

$$= U(S)\left[C(SI-A)^{-1} B + D\right]$$

$$T.F = \boxed{\frac{Y(S)}{U(S)} = C(SI-A)^{-1} B + D}$$

1) Find the transfer function for the system having state model given below.

$$\dot{x} = \begin{bmatrix} 0 & +1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Soln:-

$$\frac{Y(s)}{U(s)} = C[SI - A]^{-1} B + D$$

$$\frac{Y(s)}{U(s)} = C[SI - A]^{-1} B$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$[SI - A] = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$|SI - A| = S(S+3) + 2 = S^2 + 3S + 2$$

$$adj[SI - A] = \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$[SI - A]^{-1} = \frac{\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}}{S^2 + 3S + 2}$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+3 & 1 \\ 2 & s \end{bmatrix}}{s^2+3s+2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 \\ -2 \end{bmatrix}}{s^2+3s+2}$$

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Physical variable model:

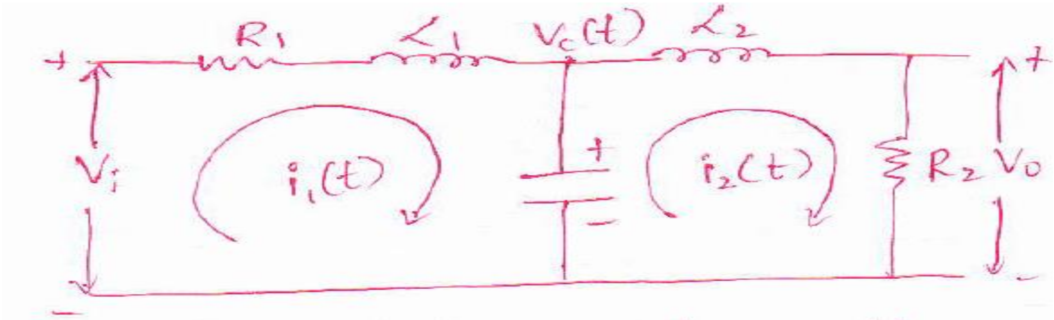
•Here the state variables selected are actual physical variables in the system such as current thru' an inductor or voltage across a capacitor

•Differential equations for the system can be written using KVL or KCL

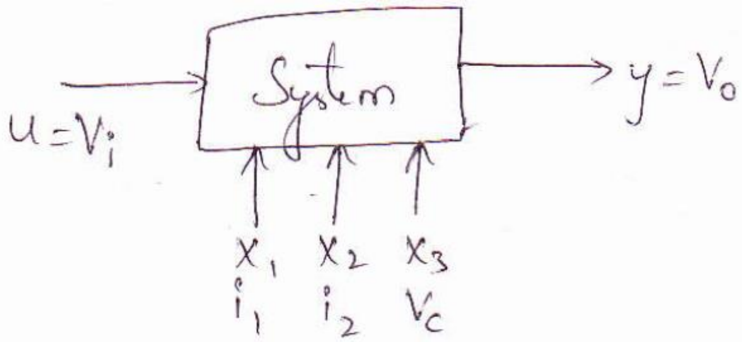
•From these, eq^n for derivative of inductor current or derivative of capacitor voltage can be obtained in terms of inductor current, capacitor voltage and the inputs to the system.

Example 1:

Obtain the state model for the electrical circuit shown in figure, Choose state variables as $i_1(t)$,$i_2(t)$ and $V_c(t)$.



Soln:-



By applying KVL at loop i_1

$$-i_1R_1 - L_1\frac{di_1}{dt} - V_c + V_i = 0$$

$$L_1\frac{di_1}{dt} = -i_1R_1 - V_c + V_i$$

$$\frac{di_1}{dt} = -\frac{i_1R_1}{L_1} - \frac{V_c}{L_1} + \frac{V_i}{L_1}$$

$$\boxed{\dot{x}_1 = -\frac{R_1}{L_1}x_1 - \frac{1}{L_1}x_3 + \frac{1}{L_1}u} \longrightarrow (1)$$

By applying KVL at loop i_2

$$-L_2 \frac{d i_2}{dt} - R_2 i_2 + V_c = 0$$

$$L_2 \frac{d i_2}{dt} = -R_2 i_2 + V_c$$

$$\frac{d i_2}{dt} = \frac{-R_2 i_2}{L_2} + \frac{V_c}{L_2}$$

$$\boxed{\dot{x}_2 = \frac{-R_2}{L_2} x_2 + \frac{1}{L_2} x_3} \longrightarrow 2$$

$$I_C = I_1 - I_2$$

$$\frac{C dV_c}{dt} = x_1 - x_2$$

$$\frac{dV_c}{dt} = \frac{1}{C} x_1 - \frac{1}{C} x_2$$

$$\boxed{\dot{x}_3 = \frac{1}{C} x_1 - \frac{1}{C} x_2} \longrightarrow 3$$

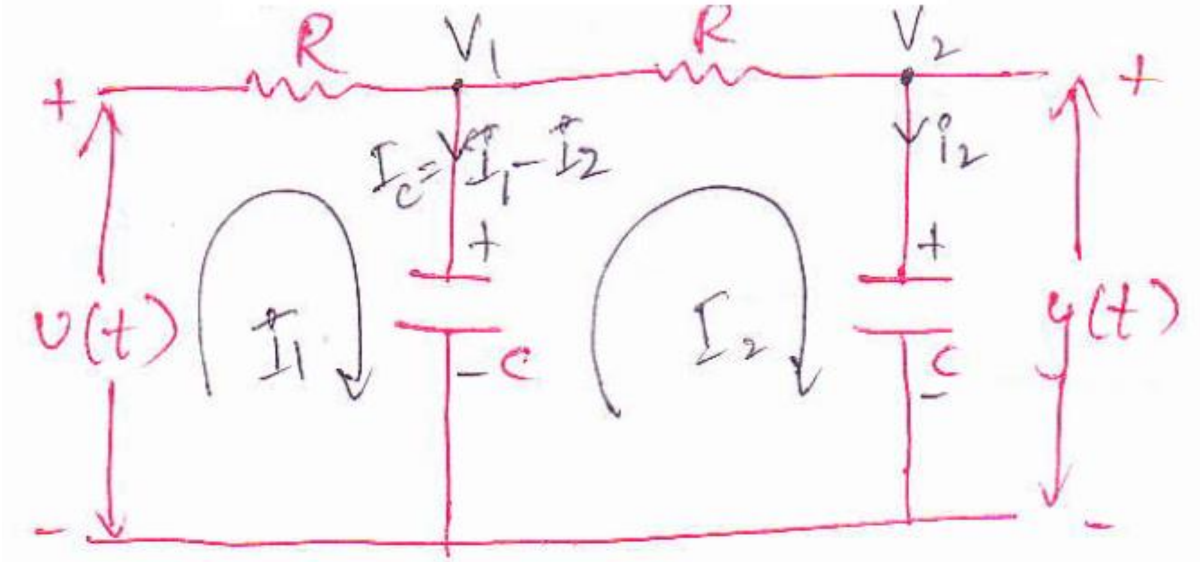
$$y = V_0 = I_2 R_2$$

$$\boxed{y = x_2 R_2}$$

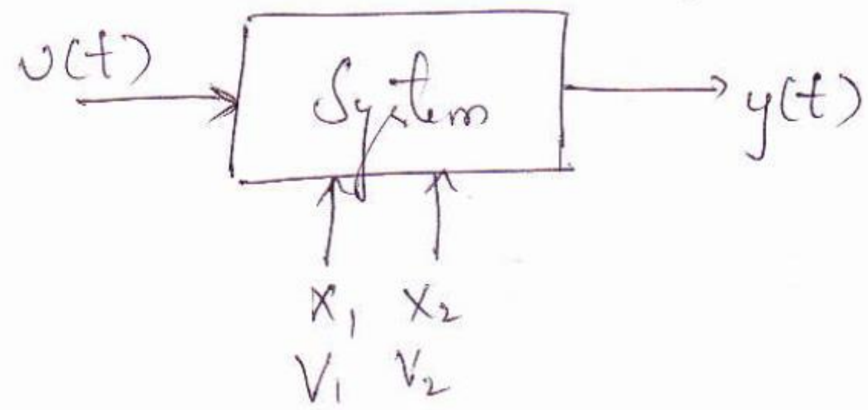
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

Example 2:
Obtain the state model for the electrical circuit shown in figure.



Soln:-



KVL to loop I_1

$$-R\dot{i}_1 - V_1 + u(t) = 0$$

$$R\dot{i}_1 = -V_1 + u(t)$$

$$\dot{i}_1 = \frac{-V_1 + u(t)}{R} \Rightarrow \boxed{\dot{i}_1 = \frac{u}{R} - \frac{x_1}{R}} \rightarrow (1)$$

KVL to loop I_2

$$-R\dot{i}_2 - V_2 + V_1 = 0$$

$$R\dot{i}_2 = V_1 - V_2$$

$$\dot{i}_2 = \frac{V_1 - V_2}{R} \Rightarrow \boxed{\dot{i}_2 = \frac{x_1}{R} - \frac{x_2}{R}} \rightarrow (2)$$

$$\dot{i}_c = \dot{i}_1 - \dot{i}_2$$

$$C\frac{dV_1}{dt} = \frac{u}{R} - \frac{x_1}{R} - \left(\frac{x_1}{R} - \frac{x_2}{R}\right)$$

$$= \frac{u}{R} - \frac{x_1}{R} - \frac{x_1}{R} + \frac{x_2}{R}$$

$$\boxed{\dot{x}_1 = \frac{u}{RC} - \frac{2x_1}{RC} + \frac{x_2}{RC}} \longrightarrow (3)$$

$$i_2 = C\frac{d}{dt}V_2$$

$$C\frac{dV_2}{dt} = \frac{x_1}{R} - \frac{x_2}{R}$$

$$\frac{dV_2}{dt} = \frac{1}{RC}x_1 - \frac{1}{RC}x_2$$

$$\boxed{\dot{x}_2 = \frac{1}{RC}x_1 - \frac{1}{RC}x_2} \longrightarrow (4)$$

$$y = V_o = V_2 = x_2$$

state eqn

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{Rc} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} & 0 \end{bmatrix} u$$

Output Eqn

$$[y] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

Phase Variable model

•For a physical system, let a particular quantity x1 be chosen as state variable. If successive derivatives of x1 are also chosen as state variables, the state model thus obtained is said to be in phase variable form.

In the Phase variable method state variables are obtained from one of the system variable and its derivatives, usually the variable chosen to be the output of the system.

Ex:

$\frac{d^3y(t)}{dt^3}$ +5 $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt}$+3y(t) =u(t)

Consider a SISO system represented by differential equation.

$$Y^n + a_{n-1} Y^{n-1} + a_{n-2} y^{n-2} + \ldots + a_1 Y' + a_0 Y = b_0 U$$

where Y = output, U = input and $Y^n = \frac{dY^n(t)}{dt^n}$

Select the state variables from the lowest order Y(t) term existing in equation.

i.e. $\quad Y(t) = X_1(t)$

and then each successive differentiations of y(t) gives the required state variables.

$$\dot{Y}(t) = \dot{X}_1(t) = X_2(t)$$

$$\ddot{Y}(t) = \ddot{X}_1(t) = \ddot{X}_2(t) = X_3(t)$$

$$\vdots \qquad \vdots$$

$$Y^{n-1}(t) = X_1^{n-1}(t) = X_2^{n-2}(t) = \ldots\ldots = X_{n-1}(t) = X_n(t)$$

Now as the order of system is `n', n state variables are allowed to be selected to form minimal set of variables.

Now state equations become

$$\dot{X}_1(t) = X_2(t)$$
$$\dot{X}_2(t) = X_3(t)$$
$$\vdots \qquad \vdots$$
$$\dot{X}_{n-1}(t) = X_n(t)$$
$$\dot{X}_n(t) = ?$$

$\dot{X}_n(t)$ can be obtained by actually substituting all selected variables into original differential equation i.e.

$$\dot{X}_n(t) + a_{n-1} X_n(t) + a_{n-2} X_{n-1}(t) + \ldots\ldots + a_1 X_2(t) + a_0 X_1(t) = b_0 U(t)$$

$$\dot{X}_n(t) = -a_0 X_1(t) - a_1 X_2(t) - \ldots - a_{n-2} X_{n-1}(t) - a_{n-1} X_n(t) + b_0 U(t)$$

Hence in general the standard state model becomes

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \\ \vdots \\ \vdots \\ \dot{X}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \\ \vdots \\ \vdots \\ \dot{X}_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} U(t)$$

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

Matrix A has a special form :

i) Upper off diagonal i.e. upper parallel row to the main principle diagonal contains all elements as 1.

ii) All other elements except last row are zeros.

iii) Last row consists of the negatives of the coefficients contained by the original differential equation.

Such a form of matrix A is called *Bush Form.* The set of state variables which give matrix A in Bush form is called set of *Phase Variables.*

Example 1:

Obtain a state model for the system described by the differential Eqn given below

$$\frac{d^3y}{dt^3} + \frac{6d^2y}{dt^2} + \frac{11dy}{dt} + 6y = 5u_1 + 10u_2$$

Soln:-

$$y \rightarrow x_1 \longrightarrow (1)$$

$$\dot{y} \rightarrow \dot{x}_1 \rightarrow x_2 \longrightarrow (2)$$

$$\ddot{y} \rightarrow \ddot{x}_1 \rightarrow \dot{x}_2 \rightarrow x_3 \longrightarrow (3)$$

$$\dddot{y} \rightarrow \dddot{x}_1 \rightarrow \ddot{x}_2 \rightarrow \dot{x}_3$$

Given $\frac{d^3y}{dt^3} + \frac{6d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 5u_1 + 10u_2$

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 5u_1 + 10u_2$$

$$\dot{x}_3 + 6x_3 + 11x_2 + 6x_1 = 5u_1 + 10u_2$$

$$\dot{x}_3 = 5u_1 + 10u_2 - 6x_3 - 11x_2 - 6x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Example 2:

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + \frac{7dy(t)}{dt} + 2y(t) = 5u(t)$$

$y \rightarrow x_1 \longrightarrow (1)$

$\dot{y} \longrightarrow \dot{x}_1 \longrightarrow x_2 \longrightarrow (2)$

$\ddot{y} \longrightarrow \ddot{x}_1 \longrightarrow \dot{x}_2 \longrightarrow x_3 \longrightarrow (3)$

$\dddot{y} \longrightarrow \dddot{x}_1 \longrightarrow \ddot{x}_2 \longrightarrow \dot{x}_3$

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

$$\dddot{y} + 4\ddot{y} + 7\dot{y} + 2y = 5u(t)$$

$$\dot{x}_3 + 4x_3 + 7x_2 + 2x_1 = 5u(t)$$

$$\dot{x}_3 = 5u(t) - 4x_3 - 7x_2 - 2x_1 \longrightarrow (4)$$

from Eqn (2), (3) and (4)

State Eqn

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix}$$

Output Eqn

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

(1)

State model from the transfer function

1) Using signal flow graph method
2) Direct decomposition of transfer function
3) Cascade programming (pole-zero form) or (Gallemin's form)
4) parallel programming or canonical form or Foster's form

Canonical form

method is based on partial fraction approach
No of state variable = No of denominator polynomial.

1) Diagonal canonical form (Foster's form) → For non repeated roots
2) Jordan canonical form → for repeated roots.

Ex:-
1) Obtain the state model by using Diagonal canonical form, whose T.F is

$$T.F = \frac{Y(s)}{U(s)} = \frac{s^2+4}{(s+1)(s+2)(s+3)}$$

Partial fraction expansion is

$$\frac{s^2+4}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$(s+1) \to P_1$, $(s+2) \to P_2$, $(s+3) \to P_3$

$$A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) = s^2+4.$$

(2)

After simplification, we get

$A = 2.5 \quad B = -8 \quad C = 6.5$

$$T.F = \frac{\overset{\to r_1}{2.5}}{s+1} - \frac{\overset{\to r_2}{8}}{s+2} + \frac{\overset{\to r_3}{6.5}}{s+3}$$

$r_1, r_2, r_3 \to$ called residues.

No of state variables = No of denominator Polynomials, Here we have P_1, P_2 & $P_3 \to 3$ Polynomials. $\to$ So, state variables are

x_1, x_2 & x_3.

$x_1(s) = \frac{1}{s+1} u(s) \Rightarrow s x_1(s) + X_s(s) = u(s)$ —(1)

$x_2(s) = \frac{1}{s+2} u(s) \Rightarrow s x_2(s) + 2x_2(s) = u(s)$ —(2)

$x_3(s) = \frac{1}{s+3} u(s) \Rightarrow s x_3(s) + 3x_3(s) = u(s)$ —(3)

Apply I.L.T to eqn (1), (2) & (3) we get.

$\frac{dx}{dt} = \dot{x}$

$L.T \frac{dx}{dt} = s X(s)$

$\frac{dx_1}{dt} + x_1(t) = u(t)$

$\dot{x}_1 + x_1(t) = u(t)$

$\dot{x}_2 + 2x_2(t) = u(t)$

$\dot{x}_3 + 3x_3(t) = u(t)$

$$\left.\begin{aligned} \dot{x}_1 &= u(t) - x_1(t) \\ \dot{x}_2 &= u(t) - 2x_2(t) \\ \dot{x}_3 &= u(t) - 3x_3(t) \end{aligned}\right\} \text{state equations}$$

3

O/P equation.

$$y(s) = \frac{r_1 u(s)}{s+p_1} + \frac{r_2 u(s)}{s+p_2} + \frac{r_3 u(s)}{s+p_3} + \cdots$$

$r_1 = 2.5$

$r_2 = -8$

$r_3 = 6.5$

$$\frac{u(s)}{s+p_1} = \frac{u(s)}{s+1} = x_1(s)$$

$$\frac{u(s)}{s+p_2} = \frac{u(s)}{s+2} = x_2(s)$$

$$\frac{u(s)}{s+p_3} = \frac{u(s)}{s+3} = x_3(s)$$

O/P eqn is

$$y(s) = 2.5\, x_1(s) - 8\, x_2(s) + 6.5\, x_3(s)$$

Taking I.L.T

$$y(t) = 2.5\, x_1(t) - 8\, x_2(t) + 6.5\, x_3(t)$$

$$y = \underbrace{\begin{bmatrix} 2.5 & -8 & 6.5 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u.$$

State eqn

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B.} u$$

4

Obtain the state model in Jordan's canonical form of a system whose T.F is

$$T.F = \frac{1}{(s+2)^2 (s+1)} = \frac{Y(s)}{U(s)}$$

It has repeated roots at $s=2$.

Finding partial fraction expansion

$$T.F = \frac{A}{(s+2)^2} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$A(s+1) + B(s+2)(s+1) + C(s+2)^2 = 1$$

$B+C=0$
$A+3B+C=0$
$A+2B+4C=1$

By solving we get
$A=-1 \quad B=-1 \quad C=1$

State variables are x_1, x_2, x_3

$$X_1(s) = \frac{X_2(s)}{s+2}, \quad X_2(s) = \frac{U(s)}{s+2}, \quad X_3(s) = \frac{U(s)}{s+1}$$

$\dot{x}_1 + 2x_1(t) = x_2(t) \qquad \dot{x}_2 = u(t) - 2x_2(t)$

$\dot{x}_1 = x_2(t) - 2x_1(t) \qquad \dot{x}_3 = u(t) - x_3(t)$

O/P eqn is $y(t) = \underset{\Downarrow\ -1}{r_1} x_1(t) + \underset{\Downarrow\ -1}{r_2} x_2(t) + \underset{\Downarrow\ +1}{r_3} x_3(t)$

$$y(t) = -x_1(t) - x_2(t) + 1\,x_3(t)$$

State Model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} -1 & -1 & -1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0$$