UNIT - 4

PART-B MILNE - THOMSON METHOD

To find $f(z)$ as a function of z, given u or v or $lu \pm mv$

~~TYPE I~~ ~~Given~~

Step 1 $f'(z) = u_x + i v_x$; given $u(x,y)$

Step 2 $= u_x - i u_y$ by CR equations

$= \phi_1(x,y) - i \phi_2(x,y)$

Step 3 Replacing x by z and y by zero

$f'(z) = \phi_1(z,0) - i \phi_2(z,0)$

Step 4 $f(z) = \int f'(z)\,dz + C$

TYPE I Given real part or imaginary part of an analytic function to find $f(z)$ as a function of z.

Example 1 Determine an analytic function $f(z)$ as a function of z, whose real part is $u = e^{-x}\left[(x^2-y^2)\cos y + 2xy \sin y\right]$

Solution: $u_x = e^{-x}\left[2x\cos y + 2y \sin y\right]$

$- e^{-x}\left[(x^2-y^2)\cos y + 2xy \sin y\right]$

$= \phi_1(x,y)$

$$U_y = e^{-x}[-x^2\sin y - 2y\cos y + y^2\sin y + 2xy\cos y + 2x\sin y] = \phi_2(x,y)$$

Step 1 $f'(z) = U_x + iV_x$

Step 2 $= U_x - iU_y$ by CR equations

$= \phi_1(x,y) - i\,\phi_2(x,y)$

Step 3 Replacing x by z and y by zero

$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$

$= e^{-z}(2z) - e^{-z}z^2 - i(0)$ $\because \phi_2(z,0) = 0$

$\therefore f'(z) = e^{-z}2z - e^{-z}z^2$

Step 4 $f(z) = \int f'(z)\,dz + C$

$= 2\int z e^{-z}\,dz - \int z^2 e^{-z}\,dz + C$

$f(z) = 2\left[(z)\left(\frac{e^{-z}}{-1}\right) - \cancel{e^{-z}}\right] - \left[(z^2)\left(\frac{e^{-z}}{-1}\right) - (2z)e^{-z} + 2\cancel{e^{-z}}\right] + C$

$= z^2 e^{-z} + C$

Example 2 If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, find the complex potential as a function of complex variable z.

$$\psi_x = 2x + \frac{(x^2+y^2)\cdot 1 - x\,2x}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\psi_y = -2y + \frac{(x^2+y^2)(0) - x\,2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Step 1 $f'(z) = \phi_x + i\psi_x$

Step 2 $= \psi_y + i\psi_x$ by CR equations $u_x = v_y$

Step 3: ~~Rep~~ Replacing x by z and y by zero

$$f'(z) = 0 + i\left[2z + \frac{-z^2}{(z^2)^2}\right]$$

$$= i\left(2z - \frac{1}{z^2}\right)$$

Step 4 $f(z) = \int f'(z)\,dz + c = i\int \left(2z - \frac{1}{z^2}\right) dz + c$

$$\therefore f(z) = i\left(z^2 + \frac{1}{z}\right) + c'$$

$\boxed{\text{TYPE II}}$ Given $lu \pm mv$, to find $f(z)$ as a function of z, l and m are constants (positive or negative).

Assignment

E(5) Find $f(z) = u + iv$, given $u - v = 2xy + x^2 - y^2 + x - y$

E(6) Find $f(z) = u + iv$ given $2u + v = e^x(\cos y - \sin y)$

(E3) Find the analytic function $f(z) = u+iv$ given

$$u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$$

Solution: $u+v = \frac{\cancel{2}\sin 2x}{\cancel{2}\cosh 2y - \cancel{2}\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Step 1 $u_x + v_x = \frac{2(\cosh 2y - \cos 2x)\cos 2x - (\sin 2x)(\sin(2x)' \times 2)}{(\cosh 2y - \cos 2x)^2}$

$$= \frac{2(\cosh 2y \times \cos 2x) - 2}{(\cosh 2y - \cos 2x)^2} \quad -①$$

$u_y + v_y = \cancel{2}\sin(2x) \times -\frac{\sinh(2y) 2}{(\cosh 2y - \cos 2x)^2}$

Step 2 $-v_x + u_x = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \quad -②$ by CR equations

Step 3

① + ② $\Rightarrow$ $\cancel{2}u_x = \frac{\cancel{2}[\cosh 2y \times \cos 2x - 1] - \cancel{2}\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$

$= \phi_1(x,y)$

① − ② $\Rightarrow$ $\cancel{2}v_x = \frac{\cancel{2}(\cosh 2y \times \cos 2x - 1) + \cancel{2}\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$

$= \phi_2(x,y)$

Step 4 $\rightarrow$ $f'(z) = u_x + iv_x = \phi_1(x,y) + i\phi_2(x,y)$

Step 5 $\rightarrow$ Replacing x by z and y by zero

$$f'(z) = \frac{(\cos 2z - 1)}{(1-\cos 2z)^2} + i\,\frac{(\cos 2z - 1)}{(1-\cos 2z)^2}$$

$$= \frac{-(1-\cos 2z)}{(1-\cos 2z)^2} - i\,\frac{(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-(1+i)}{(1-\cos 2z)} = \frac{-(1+i)}{2\sin^2 z}$$

Step 6 $f(z) = \frac{-(1+i)}{2}\int \operatorname{cosec}^2 z\, dz + c = -\frac{1}{2}(1+i)\cot z + c$

E 4 Find the analytic function $f(z) = u + iv$

if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

and $f(\pi/2) = 0$

Solution ~~Step 1~~ $\cancel{u_x - v_x = \frac{2(\cos x - \cosh y)(\sin x + \cos x) -}{[2(\cos x - \cosh y)]^2}}$

Step 1 $u_x - v_x = \dfrac{(2\cos x - \cosh y)(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{[2(\cos x - \cosh y)]^2}$

$$= \frac{(\sin x - \cos x)\cancel{2}\cosh y - \cancel{2}e^{-y}\sin x + \cancel{2}(\cos^2 x + \sin \ldots}{2\cancel{4}(\cos x - \cosh y)^2} \quad -(\ldots$$

$u_y - v_y = \dfrac{\cancel{2}(\cos x - \cosh y) \times e^{-y} - (\cos x + \sin x - e^{-y})(-\cancel{2}\sinh y)}{2[\cancel{2}(\cos x - \cosh y)]^2}$

Step 2 $-v_x - u_x =$ RHS — ②

using CR equations

① − ② ⇒ $2u_x = \dfrac{(\sin x - \cos x)\cosh y - e^{-y}\sin x + 1 - \{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y\}}{2(\cos x - \cosh y)^2}$

$= \phi_1(x, y)$

① + ② ⇒

$-2v_x = \dfrac{(\sin x - \cos x)\cosh y - e^{-y}\sin x + 1 + (\cos x - \cosh y)e^{-y} + \sinh y(\cos x + \sin x - e^{-y})}{2(\cos x - \cosh y)^2}$

$= \phi_2(x, y)$ say

Steps If $f(z) = u + iv$

$f'(z) = u_x + iv_x = \phi_1(x, y) + \phi_2(x, y)$

Steps Replacing x by z and y by zero

$f'(z) = \dfrac{1}{4(\cos z - 1)^2}\left[(\sin z - \cos z)\cdot 1 - \sin z + 1 - (\cos z - 1)\right]$

$- i\left\{(\sin z - \cos z)\cdot 1 - \sin z + 1 - (\cos z - 1)\right\}\Big]$

$= \dfrac{1}{4(\cos z - 1)^2}\left[(2 - 2\cos z) - i(2 - 2\cos z)\right]$

$= \dfrac{1 - i}{2(\cos z - 1)^2} = \dfrac{1 - i}{2}\,\dfrac{1}{2\sin^2(z/2)} = \dfrac{1 - i}{4}\operatorname{cosec}^2 z/2$

$f(z) = \dfrac{(1-i)}{4} \times \dfrac{-\cot(z/2)}{1/2} = -\dfrac{1-i}{2}\cot(z/2) + c$

$f(\pi/2) = 0 \Rightarrow 0 = -\dfrac{(1-i)}{2}\cot(\pi/4) + c \Rightarrow c = \dfrac{1-i}{2}$

$f(z) = \left(\dfrac{1-i}{2}\right)\{1 - \cot(z/2)\}$.

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