2. Controllers & Time Response Analysis

* The time response of a system is the output of the closed loop system as a function of time. The time response of a control system is divided into 2 parts

(a) Transient response. (b) Steady state response.

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The time response $c(t)$ is given by

$$c(t) = c_t(t) + c_{ss}(t)$$

* Transient response is also called as dynamic response of the system.

* Steady state response is simply that part of the total response that remains after the transient has died out. Steady state response can still vary in a fixed pattern such as a sine wave or a ramp function that increases with time.

* In order to analyze the transient & steady state behaviour of control systems, the first step always is to obtain a mathematical model of the system. A complete time response can then be obtained for any input signal, through Laplace inverse transform.

Note: (1) for a deterministic signal, the steady state response can be obtained directly without obtaining the time response expression by use of final value theorem.

(2) Before finding the time response of a system it should be ensured that the system is a stable S/m. If system is unstable we need not proceed with the transient response analysis.

* For the purpose of analysis & design, it is necessary to assume some basic types of test signals so that performance of S/m can be evaluated.

* Some of the standard test signals are as follows:-

(a) An Impulse Signal [Sudden Shock]

(b) A Step Signal [Sudden Change]

(c) A Ramp Signal [Constant velocity]

(d) A Parabolic Signal [Constant Acceleration]

* The nature of the transient response is revealed by any one of the test signal.
* Step signal will be generally used since it can be easily generated.
* Then steady state response is examined for step signal as well as other test signals. (i.e, Ramp, Parabolic).
* Steady state error can be easily ~~be~~ determined by using final value theorem.
* Since a physical control s/m involves energy storage, the o/p of the s/m when subjected to an i/p, cannot follow the input immediately, but exhibits a transient response before a steady state can be reached.
* The transient response often exhibits damped oscillations before reaching a steady state. If the o/p of the s/m at steady state does not exactly agree with input then the s/m is said to have steady state error.
* Steady state error is indicative of the accuracy of the s/m.

Note: In analyzing a control s/m, we must examine transient response & steady state behaviour.

Standard Test Signals:

(1) Step Signal:- The step is a signal whose value changes from one level (usually zero) to another level "A" in zero time. Mathematical representation of the step function is,

$$\boxed{r(t) = A\,u(t)}$$

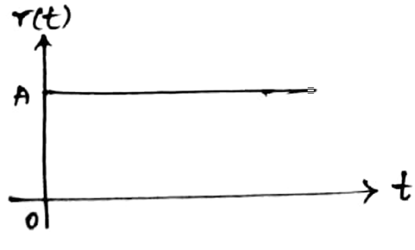
where,

$u(t) = 1 \quad , t>0$

$\quad\;\; = 0 \quad , t<0$

Laplace Transform :

$$\boxed{R(s) = \frac{A}{s}}$$

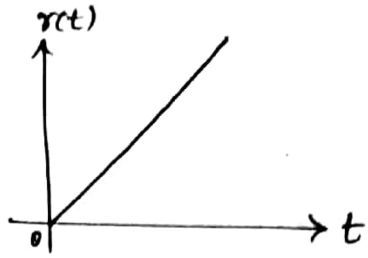


Note: step signal is a very useful test signal since its initial instantaneous jump in amplitude reveals ~~a great~~ system quickness in responding to inputs with abrupt changes.

(2) Ramp Signal:

The ramp signal is a signal which starts at a value of Zero & increases linearly with time.

$$r(t) = At \; ; \; t > 0$$
$$= 0 \; ; \; t < 0$$



Take Laplace Transform

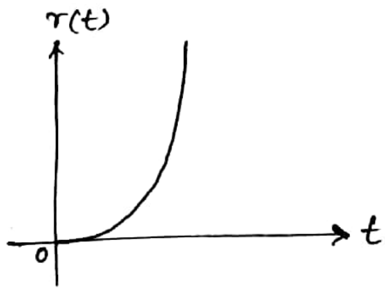
$$R(s) = \frac{A}{s^2}$$

* Ramp Signal is integral of a step signal. The ramp signal has the ability to test how the system would respond to a signal that changes linearly with time.

(3) Parabolic Signal:-

The parabolic function represents a signal that is one order faster than the ramp function.

$$r(t) = \frac{At^2}{2} \; ; \; t > 0$$
$$= 0 \; ; \; t < 0$$



Take Laplace Transform

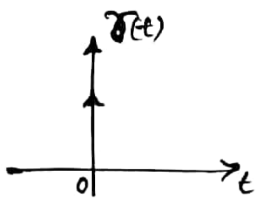
$$R(s) = \frac{A}{s^3}$$

Note: (1) We can have also higher order inputs but it will lead to unstability.

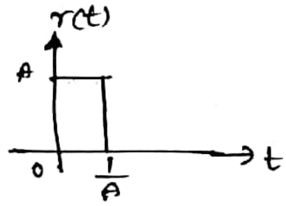
(4) Impulse Signal:-

Impulse is defined as a signal which has zero value everywhere except at $t=0$, where it's magnitude is infinite.

$$\delta(t) = 0 \; ; \; t \neq 0$$
$$= 1 \; ; \; t = 0$$



(Ideal)



Take Laplace Transform

$$\delta(s) = 1$$

Since perfect impulse cannot be achieved practically, it is approximated by a pulse of small width but unit area.

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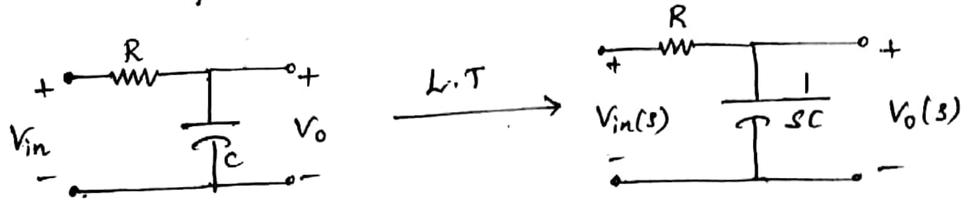
Laplace Transform formula:

function $f(t)$	Laplace Transform $f(s)$
① $f(t) = 1$	$f(s) = \frac{1}{s}$
② $f(t) = e^{at}$	$f(s) = \frac{1}{s-a}$
③ $f(t) = e^{-at}$	$f(s) = \frac{1}{s+a}$
④ $f(t) = t$	$f(s) = \frac{1}{s^2}$
⑤ $f(t) = t^n$	$f(s) = \frac{n!}{s^{n+1}}$
⑥ $f(t) = \sin at$	$f(s) = \frac{a}{s^2+a^2}$
⑦ $\cos at$	$\frac{s}{s^2+a^2}$
⑧ $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
⑨ $e^{at} \cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$
⑩ $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$

Time response of first order Systems:-

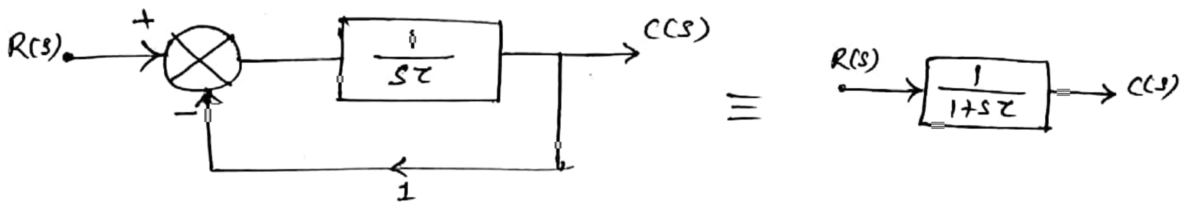
Consider a simple first order s/m eg: a Low pass filter.



$\boxed{\tau = RC}$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{\frac{1}{\cancel{sC}}}{\frac{1+RSC}{\cancel{sC}}} = \frac{1}{1+SRC} = \frac{1}{1+s\tau}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{V_o(s)}{V_{in}(s)} = \frac{1}{1+s\tau}}$$



Now, analyze the S/m responses to different types of input for a first order s/m.

(a) unit-step Response of First order s/m:

WKT: $R(s) = \frac{1}{s}$, ∴ The o/p response is given by

$$C(s) = \frac{1}{s} \cdot \frac{1}{1+s\tau} = \frac{1}{s(s\tau+1)}$$

Apply Partial fractions:

$$C(s) = \frac{A}{s} + \frac{B}{(s\tau+1)}$$

$$C(s) = \frac{A(s\tau+1) + B(s)}{s(s\tau+1)}$$

$A(s\tau+1) + B(s) = 1$

Put $s=0$

$\boxed{A = 1}$

Put $s = -\frac{1}{\tau}$

$B\left(-\frac{1}{\tau}\right) = 1$

$\boxed{B = -\tau}$

$$\therefore C(s) = \frac{1}{s} - \frac{\tau}{s\tau+1}$$

Take inverse L.T

$\boxed{c(t) = 1 - e^{-t/\tau}}$, for $t \geq 0$

* for a unit step response

$$c(t) = 1 - e^{-t/\tau} \quad \text{for } t \geq 0$$

$\therefore$ $e(t) = r(t) - c(t)$

$= 1 - [1 - e^{-t/\tau}]$

where $e(t) \longrightarrow$ Error response of the s/m

$$\boxed{e(t) = e^{-t/\tau}}$$

$\therefore$ The steady state Error is given by

$$e_{ss} = \lim_{t\to\infty} e(t)$$

$$= \lim_{t\to\infty} e^{-t/\tau}$$

$$\boxed{e_{ss} = 0}$$

That means, the first order s/m tracks the unit step input with zero steady state error.

(b) unit - Ramp response of first-order s/m :-

for the unit ramp input, $R(s) = \frac{1}{s^2}$

$$C(s) = \frac{1}{s^2} \cdot \frac{1}{1+s\tau}$$

Apply Partial fraction:

$$\frac{1}{s^2(1+s\tau)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(1+s\tau)}$$

$$1 = A(1+s\tau) + B(s)(1+s\tau) + C(s^2)$$

Put $s=0$

$1 = A$

$\boxed{A=1}$

$s = -\frac{1}{\tau}$

$1 = \frac{C}{\tau^2}$

$\boxed{C = \tau^2}$

$\boxed{s = \frac{1}{\tau}}$

$1 = 2A + \frac{2B}{\tau} + \frac{C}{\tau^2}$

$1 = 2 + \frac{2}{\tau} B + 1$

$-\cancel{2} = \frac{\cancel{2}}{\tau} B$

$\boxed{B = -\tau}$

$$\therefore \quad C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{(1+s\tau)}$$

$$= \frac{1}{s^2} - \tau\frac{1}{s} + \tau \cdot \frac{1}{(s+\frac{1}{\tau})}$$

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Taking Inverse Laplace Transform:

$c(t) = t - \tau + \tau(e^{-t/\tau})$

$$\boxed{c(t) = t - \tau\left[1 - e^{-t/\tau}\right]}$$

$\therefore e(t) = r(t) - c(t)$

$= \cancel{t} - \cancel{t} + \tau(1 - e^{-t/\tau})$

$$\boxed{e(t) = \tau(1 - e^{-t/\tau})}$$

$\therefore$ Steady State Error is given by

$e_{ss} = \lim_{t\to\infty} (\tau(1 - e^{-t/\tau}))$

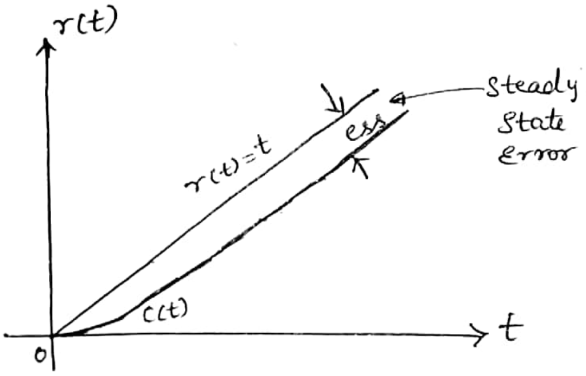
$= \lim_{t\to\infty} \tau - \lim_{t\to\infty} \tau e^{-t/\tau}$

$$\boxed{e_{ss} = \tau}$$

$\therefore$ Steady State Error is dependent on the time constant $\boxed{\tau = RC}$.

Smaller the value of "τ" smaller the steady state error.

Hence reducing the time constant will not only improves the speed of response but also reduces its steady state error to a ramp i/p.



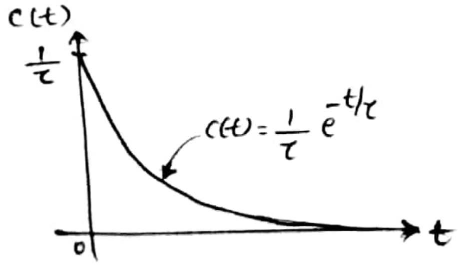
(c) Unit Impulse response of first order system:

$R(s) = 1$ & o/p of the System is given by

$$C(s) = 1 \cdot \frac{1}{1 + s\tau} = \frac{1}{\tau(s + \frac{1}{\tau})}$$

Take inverse Laplace transform:

$$c(t) = \frac{1}{\tau} e^{-t/\tau} \quad \text{for } t \geq 0$$



Note: * c(t) of ramp is given by

$$c(t) = t - \tau + \tau e^{-t/\tau} \quad , \quad t \geq 0$$

differentiating the c(t) w.r.to "t" we get

$$c_1(t) = 1 - e^{-t/\tau} \quad , \quad t \geq 0$$

where $c_1(t)$ is the o/p of unit step signal.

lly differentiating $c_1(t)$ w.r.to "t" we get

$$c_2(t) = \frac{1}{\tau} e^{-t/\tau} \quad , \quad t \geq 0$$

where $c_2(t)$ is the o/p of unit impulse signal.

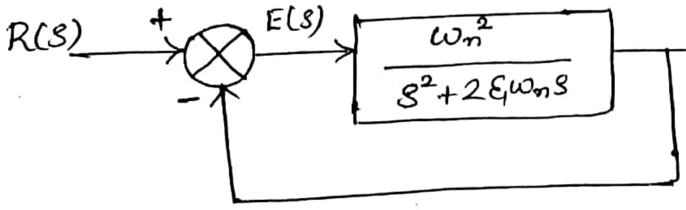
Second - Order Systems:

* Every Practical S/m takes some finite time to reach to its steady state during this Period it oscillates (or) increases exponentially.
* But every S/m has a tendency to oppose the oscillatory behaviour of the system which is called "Damping". And this tendency controls the Nature of the response.
* Damping is measured by a factor (or) a ratio called "Damping Ratio" (ξ). This factor will tell us how much opposition that the s/m have towards oscillatory output.
* If $\xi = 0$ that implies s/m has no opposition for oscillation hence s/m can oscillate freely with "maximum frequency". This frequency of oscillations under
* $\xi = 0$ is called "natural frequency of oscillations" & is denoted by the symbol "ω_n" (rad/sec).

$\xi \rightarrow$ Damping Ratio.

$\omega_n \rightarrow$ undamped Natural frequency.

Consider a second order s/m as shown in the figure below.



The closed loop transfer function $\frac{C(S)}{R(S)}$ of the s/m is given by

$$\boxed{\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}}$$

⇐ This is the standard form representation of the second order s/m.

<u>Note:</u> The dynamic (transient) behaviour of the 2^{nd} order s/m depends on two parameters "ξ & ω_n".

Effect of "ξ" on second order s/m :-

- Consider input applied to the standard 2nd order s/m is "unit step".

$\therefore R(s) = \frac{1}{s}$

WKT

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s + \omega_n^2)} \quad \text{—} (1)$$

finding the roots for $(s^2+2\xi\omega_n s + \omega_n^2) = 0$

$$s = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s = \frac{-\cancel{2}\xi\omega_n \pm \cancel{2}\omega_n\sqrt{\xi^2-1}}{\cancel{2}}$$

$$\therefore \boxed{s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}}$$

$\therefore$ Eq (1) can be written as

$$\boxed{C(s) = \frac{\omega_n^2}{s\left(s+\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)}}$$

Thus roots in the above equation is dependent on damping ratio "ξ".

Consider the following Cases:

Case (1): $\xi = 0$

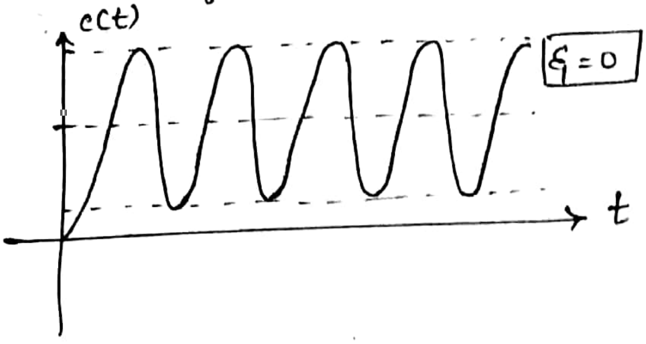
* When $\xi = 0$ the s/m is said to be "underdamped s/m", with sustained oscillatory behaviour at the o/p.

the roots are $\boxed{s = \pm j\omega_n}$ Purely Imaginary

* The response is purely oscillatory & oscillating at a maximum frequency without any opposition. Hence this frequency is called natural frequency of oscillations denoted by "ω_n".

* The response is oscillatory with constant frequency & amplitude.

The nature of response of an undamped s/m is as shown below.



Eg: Pendulum.

Case (ii): when $\boxed{0 < \xi < 1}$

when $0 < \xi < 1$, the s/m is called "under damped s/m."

Note: "ξ & ω_n" cannot be negative. Hence real part is always negative.

The roots are $s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

Since $\xi < 1$ $\therefore$ $\sqrt{\xi^2 - 1}$ can be written as

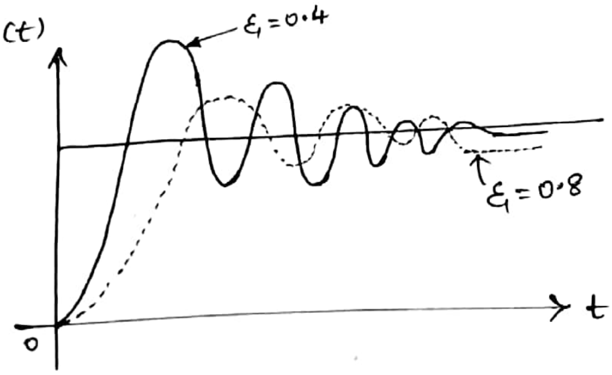
$j\sqrt{1-\xi^2}$

Hence we have written

$$\boxed{s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}}$$

Roots are negative, complex conjugate.

* The response is oscillatory with oscillating frequency $\omega_n\sqrt{1-\xi^2}$ but it has decreasing amplitude.



Eg: Bouncing Ball, Aircraft Landing.

Case (iii):- $\boxed{\xi = 1}$

When $\xi = 1$, the s/m is said to be "Critically damped s/m".

The roots are $\boxed{s = -\omega_n, -\omega_n}$ — Repetative Roots

Both the roots will be $\boxed{s = -\omega_n}$ (since it is a 2nd order s/m).

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)(s+\omega_n)}$$

$$= \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$$

Apply Partial fraction, & take inverse Laplace transform.

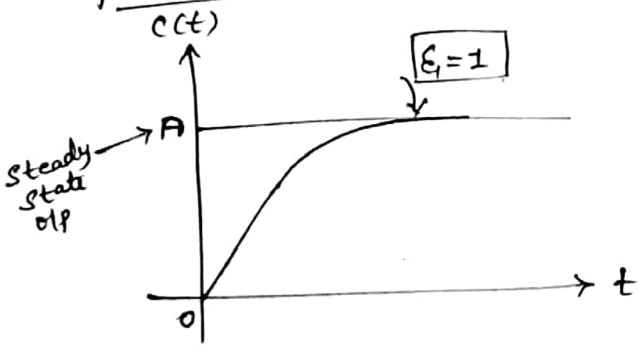
We get $\boxed{c(t) = A + B e^{-\omega_n t} + C t e^{-\omega_n t}}$ where A, B & C will have some value

A $\longrightarrow$ steady state o/p value.

By looking at the above equation we understand that the o/p is purely Exponential & it's settling time will be (Less) when compared to overdamped case hence this s/m is called "Critically damped s/m".

Note: $\boxed{\xi = 1}$ is the critical value of damping ratio because if it is decreased further, roots will become complex conjugate & this is the least value of damping ratio for which roots are real, negative & o/p is exponential.

The response is as shown below:



Similar to
Eg: charging process of a Capacitor.

Case (iv): $\boxed{\xi > 1}$

When $\xi > 1$, the s/m is said to be "overdamped s/m".

The roots are,

$$\boxed{s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}}$$

Thus the roots are real, unequal & negative hence let us say $-K_1$ & $-K_2$ are the roots.

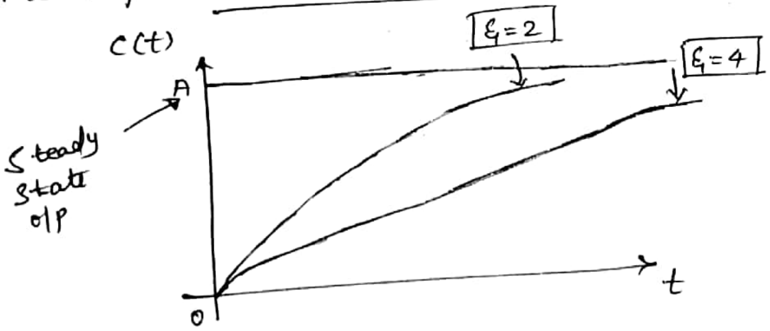
$$\therefore C(s) = \frac{\omega_n^2}{s(s+K_1)(s+K_2)} = \frac{A}{s} + \frac{B}{(s+K_1)} + \frac{C}{(s+K_2)}$$

Taking Inverse L.T:

$$\boxed{c(t) = A + Be^{-K_1 t} + Ce^{-K_2 t}}$$

Thus we can see that the o/p is purely exponential. This means damping (ξ) is very high hence there is no oscillations in the output. Hence such s/m are called "over damped s/m".

The response of such s/m is as shown below:



Eg: Application related to space craft's like missile launching, Robotic's etc. (Lift, JCB etc).

Summarising all the cases:

SL No	Range of ξ	Type of Poles	Nature of response	s/m classification
(1)	$\xi = 0$	Purely Imaginary	oscillations with constant frequency & amplitude.	undamped
(2)	$0 < \xi < 1$	Complex conjugates with negative real part	Damped oscillation	underdamped
(3)	$\xi = 1$	Real, Equal & Negative	Critical & pure exponential	Critically damped
(4)	$\xi > 1$	Real, unequal & Negative	Purely exponential slow & sluggish	overdamped

Derivation of unit step response of a Second order system:-

This derivation is valid for underdamped Sm only ($0<\xi<1$)

Consider the transfer of a 2^{nd} order Sm.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Consider the denominator $(s^2+2\xi\omega_n s+\omega_n^2)=0$

The roots of this

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

Let $\alpha = \xi\omega_n$ and $\omega_d = \omega_n\sqrt{1-\xi^2}$ ⟸ (This shows ω_d is Less than max oscillation freq ω_n).

$\therefore$ $\boxed{s = -\alpha \pm j\omega_d}$

where,

$\omega_d \rightarrow$ Damped oscillation frequency

Let us Consider a unit step input $\boxed{R(s) = \frac{1}{s}}$

$$\therefore C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$C(s) = \frac{A}{s} + \frac{(Bs+C)}{(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$\omega_n^2 = A(s^2+2\xi\omega_n s+\omega_n^2) + (Bs+C)s$$

$$\omega_n^2 = As^2+2\xi\omega_n sA+\omega_n^2 A + Bs^2+Cs$$

$$\omega_n^2 = s^2[A+B] + s[2\xi\omega_n A+C] + \omega_n^2 A$$

Equating the co-efficients:

$A+B=0$	$2\xi\omega_n A+C=0$	$\cancel{\omega_n^2} = \cancel{\omega_n^2}A$
$\boxed{A=-B}$	$C=-2\xi\omega_n$	$\boxed{A=1}$
$\boxed{B=-1}$	$\boxed{C=-2\alpha}$	

$$\therefore C(s) = \frac{1}{s} + \frac{[-s-2\alpha]}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{(s+2\alpha)}{s^2+2\xi\omega_n s+\omega_n^2}$$

Consider $s^2+2\xi\omega_n s+\omega_n^2$

Add & Subtract (α^2)

$$\therefore \underline{s^2} + \underline{2\xi\omega_n s} + \omega_n^2 + \underline{\alpha^2} - \alpha^2 \qquad (\because \alpha = \xi\omega_n)$$

$$\underline{s^2} + \underline{2\alpha s} + \omega_n^2 + \underline{\alpha^2} - \alpha^2$$

$$(s+\alpha)^2 + \omega_n^2 - \xi^2\omega_n^2$$

$$(s+\alpha)^2 + \omega_n^2(1-\xi^2) \qquad (\because \omega_d = \omega_n\sqrt{1-\xi^2})$$

$$(s+\alpha)^2 + \omega_d^2$$

$$\therefore C(s) = \frac{1}{s} - \frac{(s+2\alpha)}{(s+\alpha)^2+\omega_d^2}$$

i.e

$$C(s) = \frac{1}{s} - \frac{(s+\alpha+\alpha)}{(s+\alpha)^2+\omega_d^2} \qquad (\text{Split the LCM})$$

$$C(s) = \frac{1}{s} - \frac{(s+\alpha)}{(s+\alpha)^2+\omega_d^2} - \frac{\alpha}{(s+\alpha)^2+\omega_d^2} \quad \text{——} \; (1)$$

<u>WKT:</u> $e^{-at}\cos bt = \dfrac{(s+a)}{(s+a)^2+b^2}$ & $e^{-at}\sin bt = \dfrac{b}{(s+a)^2+b^2}$

Take Inverse Laplace Transform of Eq (1)

$$c(t) = 1 - e^{-\alpha t}\cos\omega_d t - \left(e^{-\alpha t}\sin\omega_d t\right)\frac{\alpha}{\omega_d}$$

$$= 1 - e^{-\alpha t}\left[\cos\omega_d t + \frac{\alpha}{\omega_d}\sin\omega_d t\right]$$

Substitute α & ω_d value <u>only in Co-Efficients</u>

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\cos\omega_d t\sqrt{1-\xi^2} + \xi\sin\omega_d t\right]$$

WKT: $\sin(\omega_d t + \theta) = \sin\omega_d t \cos\theta + \cos\omega_d t \sin\theta$

$$\therefore \boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\omega_d t + \theta)\right]}$$

Expression of $c(t)$ for a unit step under damped 2nd order System.

where

$$\cos\theta = \xi$$

$$\sin\theta = \sqrt{1-\xi^2}$$

$$\therefore \tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\boxed{\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]}$$

Note:

(1) If the input is given as step size of "A" units, then

$$c(t) = A\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]$$

(2) If S/m is not in standard form, i-e numerator is not ω_n^2 but some other constant "K", then

$$\frac{C(S)}{R(S)} = \frac{K}{S^2 + 2\xi\omega_n S + \omega_n^2}$$

$$\frac{C(S)}{R(S)} = \frac{K}{\omega_n^2}\left[\frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}\right]$$

$$\therefore c(t) = \frac{K}{\omega_n^2}\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]$$

(3) If numerator is a polynomial.

$$\frac{C(S)}{R(S)} = \frac{P(S)}{S^2 + 2\xi\omega_n S + \omega_n^2}$$

for eg i-e $\frac{C(S)}{R(S)} = \frac{(2S+4)}{S^2+10S+64}$

Then we cannot use the above result $c(t)$. But we can find the "ξ & ω_n" by using denominator.

(4) for any other input, other than step the derivation is not applicable.