UNIT 5

Concept of state variables, Physical variable model, phase variable model, canonical model, obtaining transfer function from state model

Ref:

1. Control Systems by K R Vermah
2. Modern control Engineering by K Ogata
3. State space representation of TFs by Burak Demire

Introduction to state space representation of systems

- Is a time domain method for design and analysis.
- Can be conveniently applied to

  (i)linear and non-linear systems.

  (ii)Time invariant or time varying systems.

  (iii)System with multiple inputs and multiple outputs.

- Also Initial Conditions may be incorporated in the System design

Intro…

- As systems become more complex, representing them with differential equations or transfer functions becomes cumbersome. This is even more true if the system has multiple inputs and outputs.

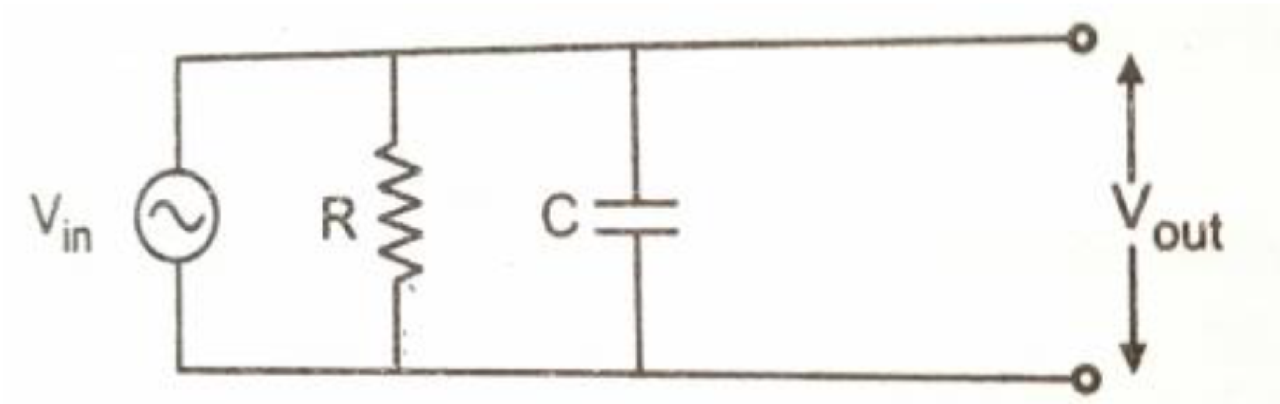
.

Drawbacks of TF method

- Transfer function is defined under zero initial conditions
- Applicable to LTI systems
- SISO systems
- Does not provide the information regarding internal state of the system

- A state space representation is a mathematical model of a physical system, as a set of input, output and state variables related by first order differential equations. The state space representation (also known as the "time-domain approach") provides a convenient and compact way to model and analyze systems with multiple inputs and outputs. The use of the state space representation is not limited to systems with linear components and zero initial conditions

Dynamic System:



Consider the system shown in figure, to find the output Vout, knowledge of the initial capacitor voltage must be known. Only the information about Vin will not be sufficient to obtain precisely the Vout at any time t>=0. Such systems in which the output is not only dependent on the input but also on the initial conditions are called the system with memory or **Dynamic Systems.**

While in the above network capacitor is replaced by another resistance R1 then the output will be dependent only on the input applied Vin. Such systems in which the output of the system depends only on the input applied at t=0, are called systems with **zero memory or static systems.**

Thus initial conditions affects the system characterization and subsequent behaviour and describe the state of the system at t=t0. So, the state can be regarded as a compact and concise representation of the past history of the system.

Concept of the state

- State is the group of variables which summarizes the history of the system in order to predict the future values. The state of a system is the minimum set of variables (state variables) whose knowledge at time $t=0$, along with the knowledge of the inputs at time

  $t \geq t0$ completely describes the behavior of a dynamic

  system for a time $t > t0$
- The concept of the state of a dynamic system refers to a minimum set of variables, known

  as state variables, that fully describe the

  system and its response to any given set of

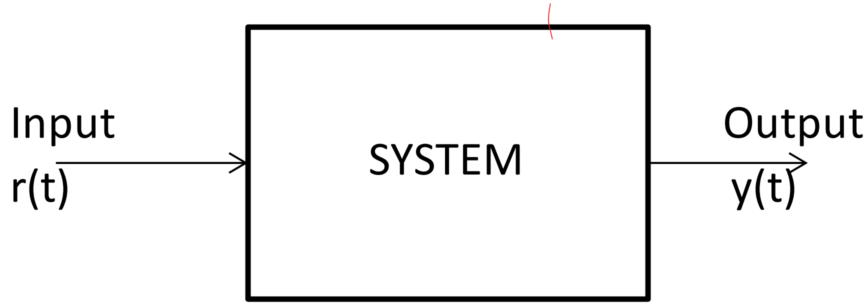
  inputs

Concept of state variables

- **State variable** is a smallest set of variables which fully describes the state of a dynamic system at a given instant of time.
- The number of the state variables required is equal to the number of the storage elements present in the system. i.e. The state variables give information about the internal structure of the system, which the classical methods do not.

- This information is of great significance in the study of the structure and properties of the system, as well as to the solution of high-performance control design problems, such as optimal control, adaptive control, robust control, and pole assignment

State and state variable



The output not only depends on the input applied to the system for $t > t_0$, but also on the initial conditions at time $t = t_0$

$$y(t) = y(t)\big|_{t=t0} + y(t)\big|_{t\geq t0}$$

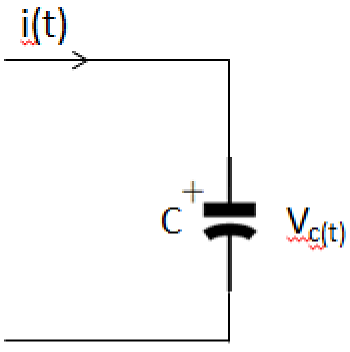
$$= \int_{-\infty}^{t_0} y(t) + \int_{t_0}^{t} y(t) \quad = y(t_0) + \int_0^t y(t)$$

The term $y(t_0)$ is called the state of the system.
The variable that represents this state of the system is called the state variable(For ex: Voltage across a capacitor)

Voltage across a capacitor

Consider the circuit shown in figure

$i(t) = C\,\frac{dv_c(t)}{dt}$



$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$

$= \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt + \frac{1}{C}\int_{t_0}^{t} i(t)dt$

$= V_c(t_0) + V_c(t)$

$V_c(t_0)$ = initial voltage across capacitor

The voltage across capacitor can be taken as a state variable

- State vector is a vector which contains the state variables as elements.
- The output equations and state equations together is called state model of a system.

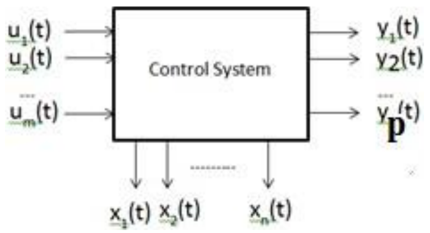
State Model

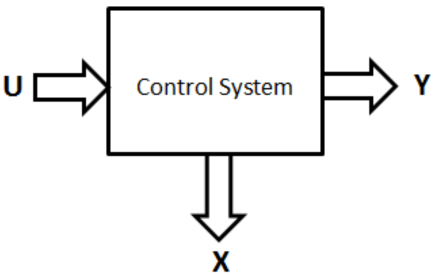
consider a multi-input & multi-output system is having

m inputs $u_1(t)$, $u_2(t)$,.......$u_m(t)$

p outputs $y_1(t)$, $y_2(t)$,.......$y_p(t)$

n state variables $x_1(t)$, $x_2(t)$,.......$x_n(t)$





The different variables may be represented by the vectors as shown below

Input vector U(t) = $\begin{bmatrix} u_1(t) \\ u_2(t) \\ .. \\ u_m(t) \end{bmatrix}$; Output vector Y(t) = $\begin{bmatrix} y_1(t) \\ y_2(t) \\ .. \\ y_p(t) \end{bmatrix}$

State variable vector X(t) = $\begin{bmatrix} x_1(t) \\ x_2(t) \\ .. \\ x_n(t) \end{bmatrix}$

State Equations

The state variable representation can be arranged in the form of n number of first order differential equations as shown below:

$$\frac{dx_1}{dt} = \dot{x}_1 = f1(x_1, x_2, \ldots. x_n ; u_1, u_2 \ldots.. u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f1(x_1, x_2, \ldots. x_n ; u_1, u_2 \ldots.. u_m)$$

……

……

$$\frac{dx_n}{dt} = \dot{x}_n = f1(x_1, x_2, \ldots. x_n ; u_1, u_2 \ldots.. u_m)$$

In vector notation, $\dot{X}(t) = f(X(t), U(t))$

Similarly the output vector $Y(t) = f(X(t), U(t))$

State Model of Linear System

The state model of a system consist of state equation and output equation.

The state equation of a system is a function of state variables and inputs.

For LTI systems, the first derivatives of state variables can be expressed as a linear combination of sate variables and inputs

$\dot{x}_1 = a_{11}x_1+a_{12}x_2+----+a_{1n}x_n + b_{11}u_1+b_{12}u_2+---+b_{1m}u_m$

$\dot{x}_2 = a_{21}x_1+a_{22}x_2+----+a_{2n}x_n + b_{21}u_1+b_{22}u_2+---+b_{2m}u_m$

$\dot{x}_n = a_{n1}x_1+a_{n2}x_2+----+a_{nn}x_n + b_{n1}u_1+b_{n2}u_2+---+b_{nm}u_m$

where the coefficients a_{ij} and b_{ij} are constants

In the matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ .. \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & & b_{1m} \\ b_{21} & & b_{2m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ .. \\ .. \\ u_m \end{bmatrix}$$

$\dot{X}(t) = A\,X(t) + B\,U(t)$……… state equation

where, A is state matrix of size (n×n)

B is the input matrix of size (n×m)

X(t) is the state vector of size (n×1)

U(t) is the input vector of size (m×1)

Output equation

The output at any time are functions of state variables and inputs.

output vector, $Y(t) = f(x(t), U(t))$

Hence the output variables can be expressed as a linear combination of state variables and inputs.

$y_1 = c_{11}x_1 + c_{12}x_2 + \text{----} + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \text{---} + d_{1m}u_m$

$y_2 = c_{21}x_1 + c_{22}x_2 + \text{----} + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \text{---} + d_{2m}u_m$

$y_p = c_{p1}x_1 + c_{p2}x_2 + \text{----} + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \text{---} + d_{pm}u_m$

where the coefficients c_{ij} and d_{ij} are constants

In the matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot\cdot \\ \cdot\cdot \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ \cdot\cdot \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & & d_{1m} \\ d_{21} & \cdots & d_{2m} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot\cdot \\ \cdot\cdot \\ u_m \end{bmatrix}$$

Y(t) = C X(t) + D U(t)………….. output equation

where, C is the output matrix of size (p×n)

D is the transmission matrix of size (p×m)

X(t) is the state vector of size (n×1)

Y(t) is the output vector of size (p×1)

U(t) is the input vector of size (m×1)

Definitions

State: The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time t = t_o (Initial condition), together with the knowledge of input for t ≥ t_0 , completely determines the behavior of the system for any time $t \geq t_0$.

State Variables: A set of variables which describes the system at any time instant are called state variables

State vector: A vector whose elements are the state variables

State space: The n-dimensional space whose co-ordinate axes consists of the x_1 axis, x_2 axis,.... x_n axis, (where x_1 , x_2 ,..... x_n are state variables:) is called a state space.

State space representation of a system

- The state space representation of an LTI system is given by the equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

where **x(t)** is an n-dimensional **state vector**, **u(t)** is an m-dimensional input vector, and **y(t)** is a p-dimensional output vector.
The matrices **A, B, C,** and **D** are time invariant,
and their dimensions are n X n, n X m, p X n, and p X m, respectively.
The initial conditions are at t = 0 and they are given by **x(0)=** $\mathbf{x_0}$

w.r.t previous slide..

- The first equation is called the state equation, the second equation is called the output equation. For an nth order system (i.e., it can be represented by an **nth** order differential equation) with **m** inputs and **p** outputs the size of each of the matrices is as follows:

x is nx1 (n rows by 1 column); **x** is called the state vector, it is a function of time

A is (nxn); **A** is the state matrix, a constant

B is (nxm); **B** is the input matrix, a constant

u is (mx1); **u** is the input, a function of time

C is (pxn); **C** is the output matrix, a constant

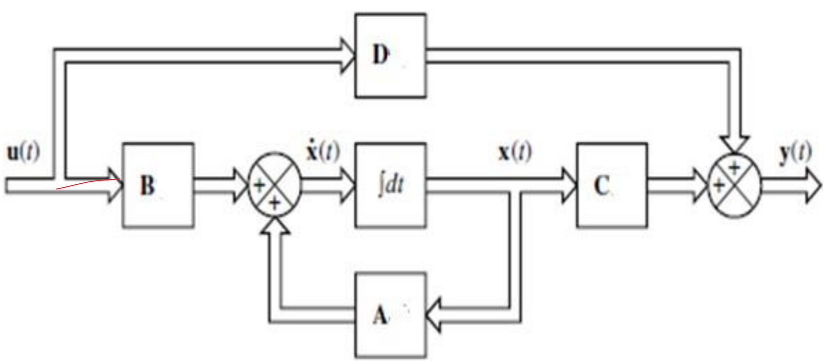
D is (pxm); **D** is the direct transition (or feed through) matrix, a constant

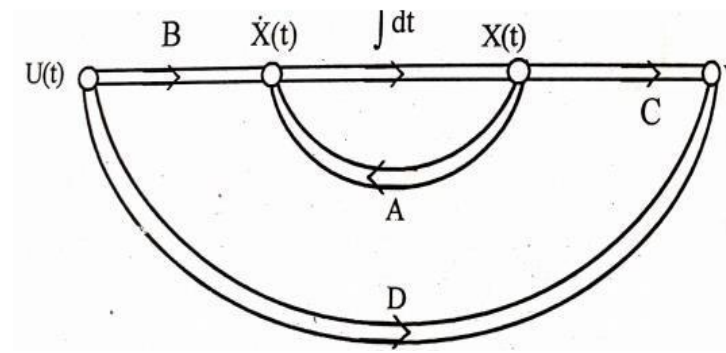
y is (px1); **y** is the output, a function of time

State Model

$\dot{X}(t) = A\,X(t) + B\,U(t)$ state equation

$Y(t) = C\,X(t) + D\,U(t)$ output equation





State transition matrix

- State transition matrix is an (nxn) matrix and is designated by **ϕ(t),** which satisfies the homogeneous equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

i.e.,

$$\dot{\boldsymbol{\phi}}(t) = \mathbf{A}\boldsymbol{\phi}(t)$$

State transition matrix contd..,

- State transition matrix depends only on matrix **A**
- System's response, when it is excited only by its initial condition, $\mathbf{x_0}$, that is when **u(t)**=0 is called free response
- **ϕ(t)** completely defines the *transition* of the state vector **x(t)** from it's initial state **x(0)** to any new state **x(t)**
- This is the reason why the matrix **ϕ(t**) is called

  State transition matrix

Properties of state transition matrix

- The matrix $\boldsymbol{\phi}(\mathbf{t}) = \mathbf{e}^{\mathbf{At}} = \mathbf{L}^{-1}\left[\mathbf{sI} - \mathbf{A}\right]^{-1}$

Is called the state transition matrix

- $\boldsymbol{\phi}(\mathbf{0}) = \mathbf{I}$ is unity matrix or identity matrix

- $\dot{\boldsymbol{\phi}}(t) = \mathbf{A}$ is the system matrix
- The state transition matrix is non singular and its inverse exits for all t
- $\boldsymbol{\phi}(\mathbf{t}) = \left(\mathbf{e}^{\mathbf{At}}\right)^{-1} = \mathbf{e}^{-\mathbf{At}} = \boldsymbol{\phi}(\mathbf{-t})$

Contd..

- An n^{th} order linear physical system can be represented using a state space approach as a single first order matrix differential equation.
- Thus state-space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations.
- A **state space** is the set of all possible configurations of a system.

Advantages of State space representation

- The notation is very compact. Even large systems can be represented by two simple equations.(plus Initial condition).

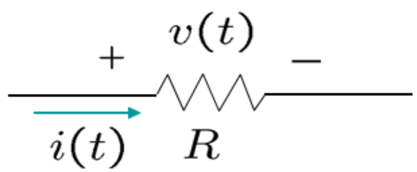
- Because all systems are represented by the same notation, it is very easy to develop general techniques to solve these systems.

- The state equation has a single first order derivative of the state vector and computers can easily simulate first order equations.

State space representation of electrical systems

- To develop a state space system for an electrical system, voltage across capacitors, and current through inductors as state variables

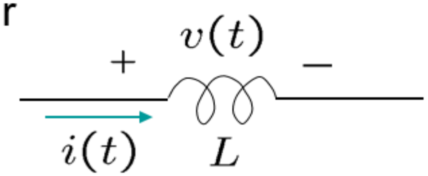
resistor



$$v(t) = Ri(t)$$

$$V(s) = RI(s) \Rightarrow \frac{V(s)}{I(s)} = R$$

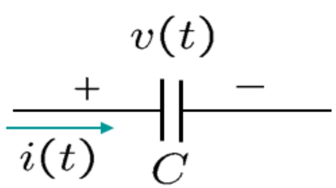
inductor



$$v(t) = L\frac{di(t)}{dt}$$

$$V(s) = LsI(s) \Rightarrow \frac{V(s)}{I(s)} = Ls$$

capacitor



$$i(t) = C\frac{dv(t)}{dt}$$

$$I(s) = CsV(s) \Rightarrow \frac{V(s)}{I(s)} = \frac{1}{Cs}$$

Selection of state variables

- The state variables of a system are not unique.
- There are many choices for a given system

Guide lines:

1. For a physical systems, the number of state variables needed to represent the system must be equal to the number of energy storing elements present in the system
2. If a system is represented by a linear constant coefficient differential equation, then the number of state variables needed to represent the system must be equal to the order of the differential equation
3. If a system is represented by a transfer function, then the number of sate variables needed to represent the system must be equal to the highest power of s in the denominator of the transfer function.

State space Representation using Physical variables

- In state-space modeling of the systems, the choice of sate variables is arbitrary.
- One of the possible choice is physical variables.
- The state equations are obtained from the differential equations governing the system

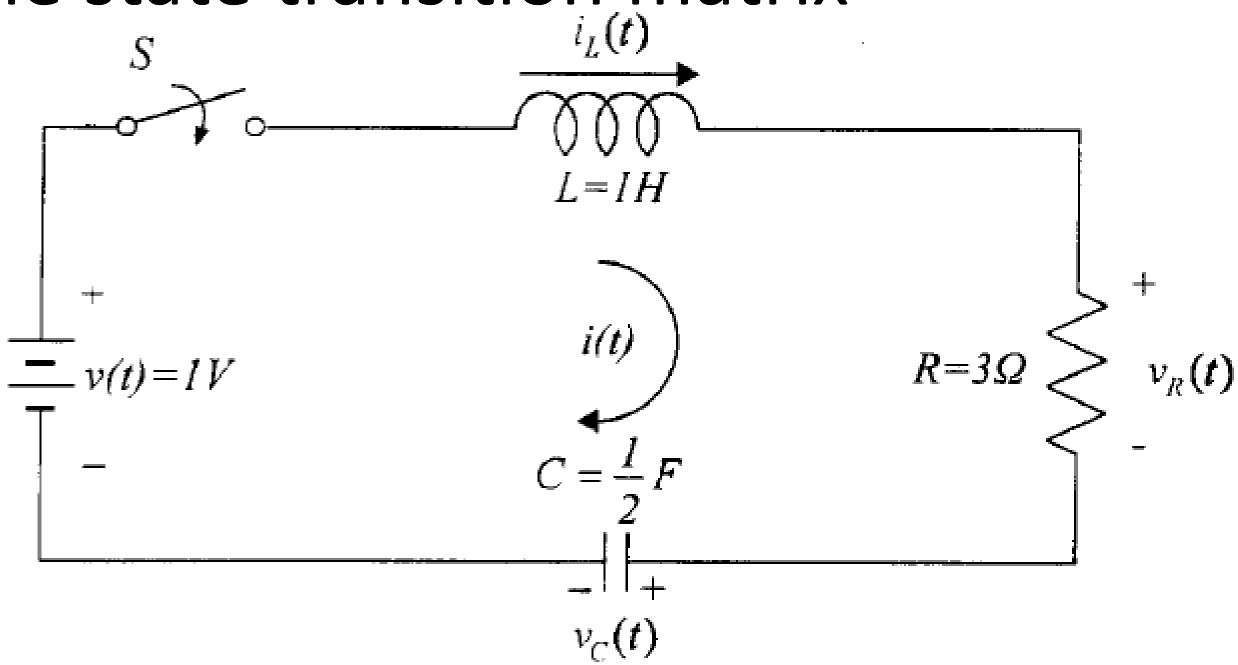
Example-Representation of electrical system

For the n/w shown, with initial conditions

$i_L(0)=1$ and $v_C(0)=0$, Determine

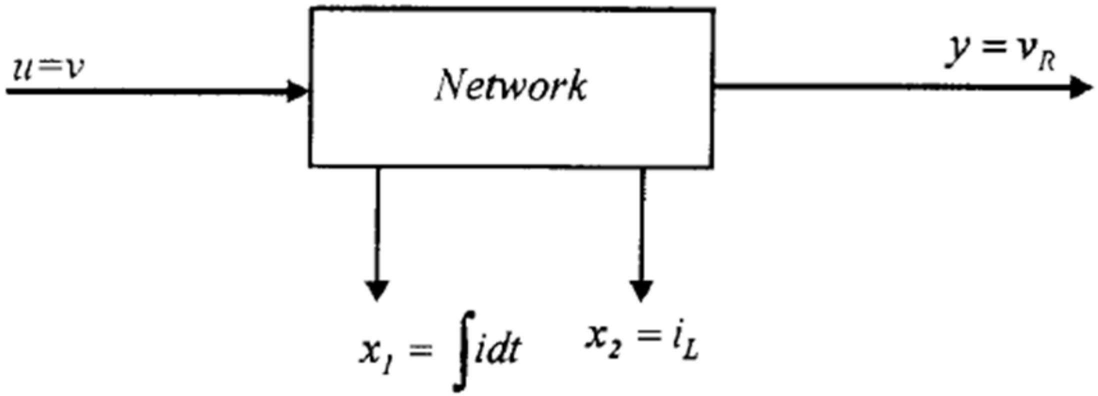
(i) A state space description

(ii) The state transition matrix



(i) The loop equation is

$$\frac{di}{dt} + 3i + 2\int_0^t i\,dt = v(t)$$



The two state variables,

$$x_1(t) = \int_0^t i(t)dt \qquad \text{and} \qquad x_2(t) = \dot{x}_1(t) = i_L(t) = i(t)$$

Using the above definitions, the loop equation can be written in state space as follows: $\dot{x}_2(t) + 3x_2(t) + 2x_1(t) = v(t)$. This equation, combined with the equation $\dot{x}_1(t) = x_2(t)$, yields the following state equations for the given network:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_2(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

with initial condition vector $\mathbf{x}(0) = \mathbf{x}_0 = [0, 1]^{\mathrm{T}}$ and $u(t) = v(t)$.

(ii) To determine the transition matrix $\boldsymbol{\phi}(t)$,

$$\boldsymbol{\phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

Hence

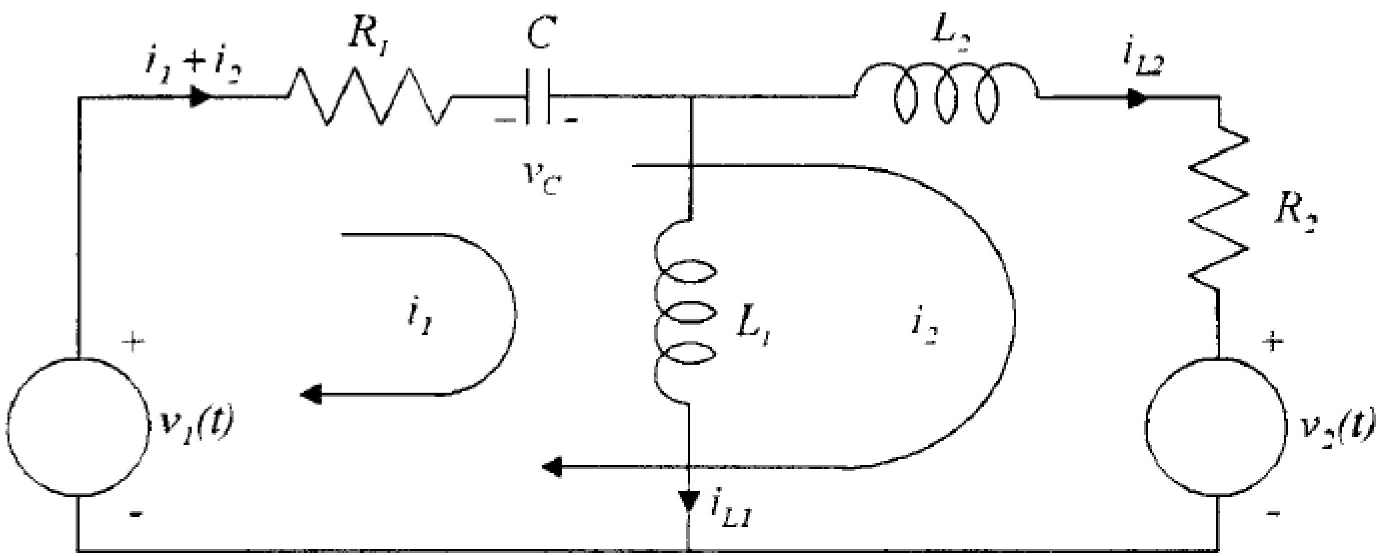
$$\boldsymbol{\phi}(t) = L^{-1}\{\boldsymbol{\phi}(s)\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Physical variable model

- Here the state variables selected are actual physical variables in the system such as current thru' an inductor or voltage across a capacitor
- Differential equations for the system can be written using KVL or KCL
- From these, eqn for derivative of inductor current or derivative of capacitor voltage can be obtained in terms of inductor current, capacitor voltage and the inputs to the system

Example-physical variable model

For the network shown determine the state equations



Equations using KVL

The two loop equations are

$$R_1(i_1 + i_2) + v_c + L_1 \frac{di_1}{dt} = v_1(t)$$

$$R_1(i_1 + i_2) + v_c + L_2 \frac{di_2}{dt} + R_2 i_2 = v_1(t) - v_2(t)$$

Furthermore, we have that

$$C \frac{dv_c}{dt} = i_1 + i_2$$

Differential equations

Hence, the three first-order differential equations that describe the network are

$$\frac{dv_c}{dt} = \frac{1}{C}i_1 + \frac{1}{C}i_2$$

$$\frac{di_1}{dt} = -\frac{1}{L_1}v_c - \frac{R_1}{L_1}i_1 - \frac{R_1}{L_1}i_2 + \frac{1}{L_1}v_1(t)$$

$$\frac{di_2}{dt} = -\frac{1}{L_2}v_c - \frac{R_1}{L_2}\mathrm{i}_1 - \frac{R_1 + R_2}{L_2}i_2 + \frac{1}{L_2}v_1(t) - \frac{1}{L_2}v_2(t)$$

Note that $i_1 = i_{L_1}$ and $i_2 = i_{L_2}$

State equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

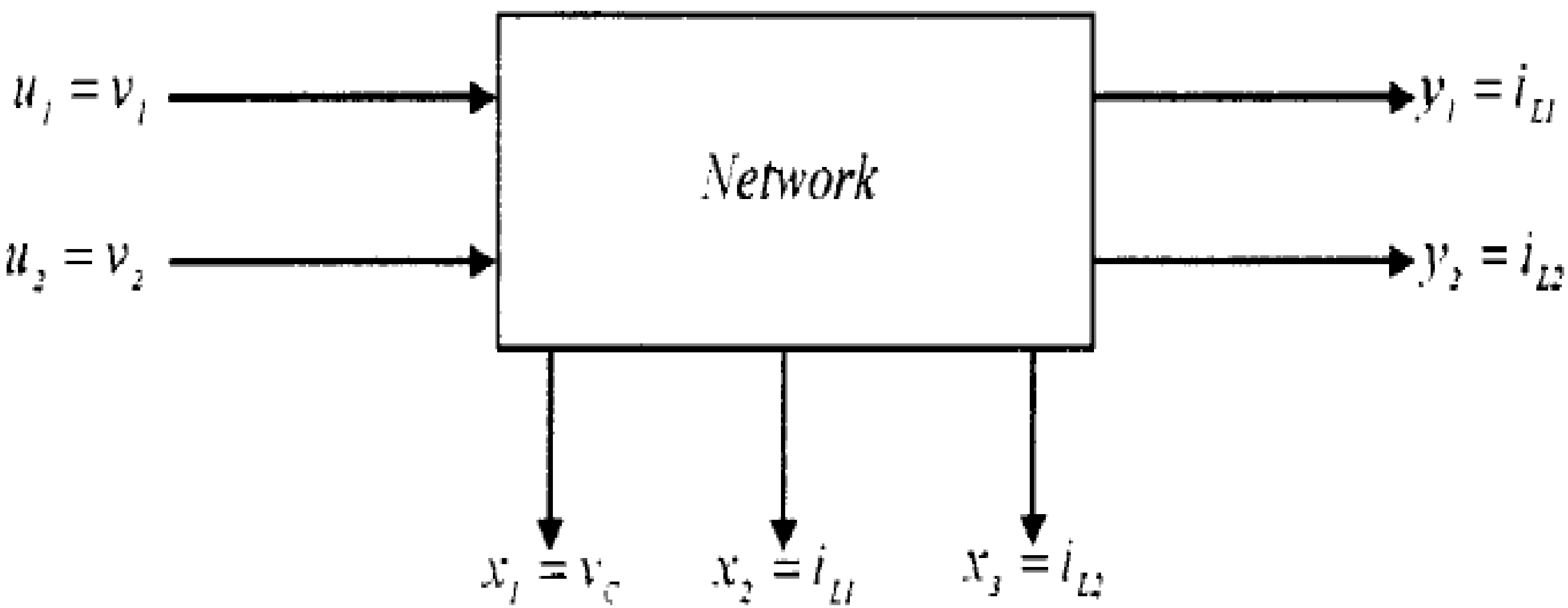
where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} v_c \\ i_{L_1} \\ i_{L_2} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{C} & \frac{1}{C} \\ -\frac{1}{L_1} & -\frac{R_1}{L_1} & -\frac{R_1}{L_1} \\ -\frac{1}{L_2} & -\frac{R_1}{L_2} & -\frac{R_1 + R_2}{L_2} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{L_1} & 0 \\ \frac{1}{L_2} & -\frac{1}{L_2} \end{bmatrix},$$

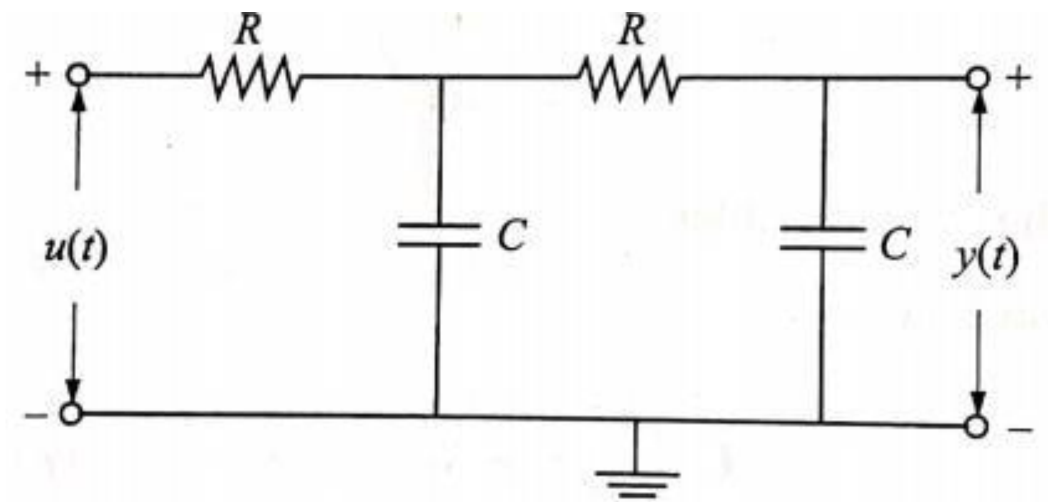
$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inputs, state variables and outputs



Problem

Obtain the state model for a system represented by an electrical system as shown in figure

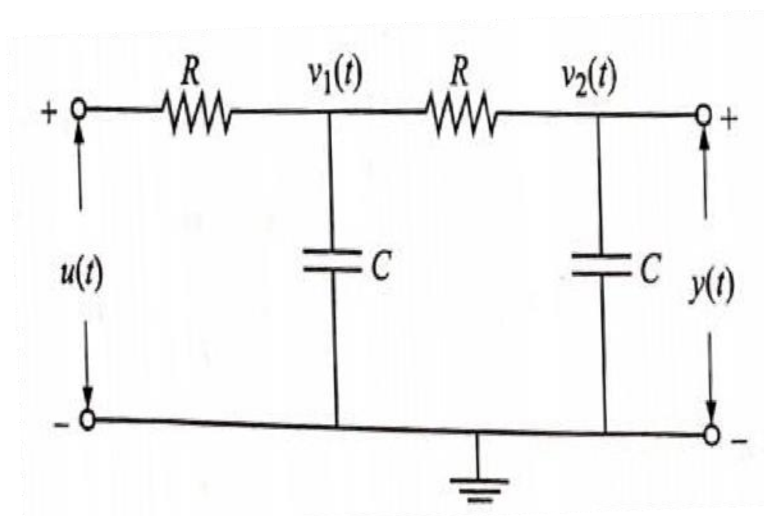


Since there are two energy storage elements present in the system, assume two state variables to describe the system behavior.

Let the two state variables be x_1 and x_2 be related to physical quantities as shown

Let $v_1(t) = x_1(t)$

$v_2(t) = x_2(t)$



Applying KCL at node $v_1(t)$,

$$\frac{v_1(t)-u(t)}{R} + C\,\frac{dv_1(t)}{dt} + \frac{v_1(t)-v_2(t)}{R} = 0$$

$$\Longrightarrow \quad \frac{x_1(t)-u(t)}{R} + C\,\frac{dx_1(t)}{dt} + \frac{x_1(t)-x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \frac{2x_1(t)}{R} - \frac{u(t)}{R} + C\,\dot{x}_1(t) - \frac{x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \dot{x}_1(t) = -\frac{2x_1(t)}{RC} + \frac{x_2(t)}{RC} + \frac{u(t)}{RC} \text{ ------------}(1)$$

Applying KCL at node $v_2(t)$,

$$C\,\frac{dv_2(t)}{dt} + \frac{v_2(t)-v_1(t)}{R} = 0$$

$$\Longrightarrow \quad C\,\frac{dx_2(t)}{dt} + \frac{x_2(t)-x_1(t)}{R} = 0$$

$$\Longrightarrow \quad C\,\dot{x}_2(t) - \frac{x_1(t)}{R} + \frac{x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \dot{x}_2(t) = \frac{x_1(t)}{RC} - \frac{x_2(t)}{RC} \qquad \text{----------(2)}$$

putting 1 and 2 in matrix form,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} [u(t)]$$

This is the state equation

The output of the circuit is given by

$$y(t) = v_2(t)$$

$$= x_2(t)$$

$$= [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This is the output equation