PART-B

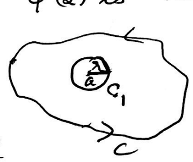
UNIT-5

Cauchy's integral formula :-

Statement :- If $f(z)$ is analytic function inside and on the boundary of a simple closed curve c and the point 'a' lies inside c then $f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$

or $\oint_c \left[\frac{f(z)}{z-a}\right] dz = 2\pi i f(a)$

Proof :- * Let $\phi(z) = \frac{f(z)}{z-a}$, then $\phi(z)$ is not analytic inside c.



* If $c_1 : |z-a| = r$, then $\phi(z)$ is analytic function inside and on the boundary of ring shaped region between c and c_1.

* By cauchy's theorem, (Cor 2)

$$\oint_c \phi(z) dz = \oint_{c_1} \phi(z) dz = \oint_{c_1} \frac{f(z)}{z-a} dz$$

* But $|z-a| = r \Rightarrow z-a = re^{i\theta}$

$$dz = re^{i\theta} i d\theta$$

$$\therefore \oint_c \phi(z) dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} re^{i\theta} i d\theta$$

* In the limiting case as $r \to 0$

$$\oint_c \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = 2\pi i\, f(a).$$

To show that $f^n(a) = \frac{\lfloor n}{2\pi i}\oint_c \frac{f(z)\,dz}{(z-a)^{n+1}}$

we have $f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$, by Cauchy's integral formula

Differentiating w.r.to 'a' on either side

$$f'(a) = \frac{1}{2\pi i}\oint_c -\frac{f(z)}{(z-a)^2}(-1)\,dz = \frac{1}{2\pi i}\oint_c f(z)(z-a)^{-2}\,dz$$

$$\therefore f''(a) = \frac{2}{2\pi i}\oint_c \frac{f(z)}{(z-a)^3}\,dz$$

$||^{ly}$ $f'''(a) = \frac{\lfloor 3}{2\pi i}\oint_c \frac{f(z)}{(z-a)^4}\,dz$

Generalizing $f^n(a) = \frac{\lfloor n}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz$

Examples: E1 Using Cauchy's integral formula

Evaluate $I = \oint_c \frac{e^z\,dz}{(4z+1)^3}$ where $c: |z| = 4$

Solution $I = \oint_c \frac{e^z\,dz}{4^3(z+1/4)^3}$

Singularity of Integrand is $z = -\frac{1}{4}$

lies inside $|z| = 4$

we have $\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i \, f^{n}(a)}{\lfloor n}$

$n = 2$.

$\therefore \quad I = \frac{1}{64} \quad \frac{2\pi i}{\lfloor 2} \quad [f''(z)]_{z=a}$

$f(z) = e^{z} \qquad f''(z) = e^{z}$

$\therefore \quad I = \frac{\pi i}{64} \quad e^{-1/4}$

$\boxed{E2}$ Evaluate $\oint_C \frac{e^{-z} dz}{(z-1)(z-2)^2}$ where $C : |z| = 3$

Solution $z = 1$ and $z = 2$ are the singularities of the integrand, lie inside $|z| = 3$.

Consider $\frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$

$\therefore \quad 1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$

$z = 1 \Rightarrow \quad A = 1$

$z = 2 \Rightarrow \quad C = 1$

$z = 0 \Rightarrow \quad 4A + 2B - C = 1$ or $B = -1$

$I = \oint_C \frac{e^{-z} dz}{(z-1)(z-2)} = \oint_C \frac{e^{-z}}{z-1} dz - \oint_C \frac{e^{-z}}{z-2} dz + \oint_C \frac{e^{-z} dz}{(z-2)^2}$

Using $f^n(a) = \frac{L n}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$ $[n=1, a=2]$

$f(z) = e^{-z}$

$$\oint_C \frac{e^{-z} dz}{(z-2)^2} = 2\pi i \, f'(2) = 2\pi i \; -e^{-2} \quad -①$$

Using $\oint_C \frac{f(z)}{z-a} dz = 2\pi i \, f(a)$

$$\oint_C \frac{e^{-z} dz}{z-1} = 2\pi i \, f(1) = 2\pi i \, e^{-1} \quad -② \; (a=1)$$

$$\oint_C \frac{e^{-z} dz}{z-2} = 2\pi i \, f(2) = 2\pi i \, e^{-2} \quad \rightarrow ③ \; a=2$$

$\therefore$ using ①, ② and ③

$$I = 2\pi i \left[\frac{1}{e} - \frac{1}{e^2} + (-e^{-2}) \right].$$

E③ Evaluate $\oint_C \frac{[\sin(\pi z^2) + \cos(\pi z^2)] dz}{(z-1)^2 (z-2)}$ $C: |z|=3$

Solution $z=1$ and 2 are the singularities of integrand lie inside $C: |z|=3$

Consider $\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$

$\therefore$ $1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$

$z=1 \Rightarrow B=-1$, $z=2 \Rightarrow C=1$

$z=0 \Rightarrow 2A - 2B + C = 1$ & $A=-1$

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{-1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{z-2}$$

$$\therefore \oint_c \frac{f(z)}{(z-1)^2(z-2)} = \oint_c \frac{f(z)}{z-2}\,dz - \oint_c \frac{f(z)\,dz}{z-1} - \oint_c \frac{f(z)\,dz}{(z-1)^2} \quad \text{—(1)}$$

where, $f(z) = \cos(\pi z^2) + \sin(\pi z^2)$

using $\oint \frac{f(z)}{z-a} = 2\pi i\, f(a)$,

$$\oint_c \frac{f(z)}{z-2} = 2\pi i\, f(2) = 2\pi i(\cos 4\pi + \sin 4\pi) = 2\pi i$$

and $\oint_c \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i(\cos \pi + \sin \pi) = -2\pi i$

using $\oint_c \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i\, f^{n}(a)}{n!}$ (when $n=1$), $a=1$

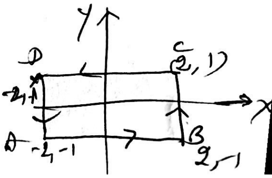
$$\oint_c \frac{f(z)\,dz}{(z-1)^2} = 2\pi i\, f'(1); \quad f'(z) = 2\pi z(\cos \pi z^2 - \sin \pi z^2)$$

$$f'(1) = 2\pi(-1-0)$$

$\therefore$ (1) becomes $\oint_c \frac{(\cos \pi z^2 + \sin \pi z^2)\,dz}{(z-1)^2(z-2)} = 2\pi i + 2\pi i + 2\pi$
$= 2\pi(1+2i)$

$\boxed{E\,4}$ Evaluate $\oint_c \frac{\cos(\pi z)\,dz}{z^2-1}$ where c is the boundary of the rectangle with vertices $2 \pm i$, $-2 \pm i$

Solution; $z = 1, -1$ are the singular points of Integrand and both these points lie inside C



$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

Resolving into partial fraction

$$\therefore I = \oint_C \frac{\cos(\pi z)\,dz}{z^2-1} = \frac{1}{2}\left[\oint_C \frac{\cos(\pi z)\,dz}{\underset{\boxed{a=1}}{z-1}} - \oint_C \frac{\cos(\pi z)\,dz}{\underset{\boxed{a=-1}}{z+1}}\right]$$

using $\oint_C \frac{f(z)}{z-a} = 2\pi i f(a)$

$$I = \frac{1}{2}\left[2\pi i\,(\cos\pi) - 2\pi i\,\cos(-\pi)\right]$$

$$= 0$$