BMS COLLEGE OF ENGINEERING, BENGALURU-19

Autonomous Institute, Affiliated to VTU

DEPARTMENT OF MATHEMATICS

Sem & Branch:	**IV MECHANICAL**	**Subject:**	**HIGHER ENGINEERING MATHEMATICS**	**Sub Code:**	**19MA4BSHEM**
Duration	**75 MINUTES**	**Test Date:**	**20.06.2020**	**Max Marks:**	40

Test No.	Q. No.	SCHEME AND SOLUTIONS	Marks	CO
Test -3		**PART - A**		3
	1	If $f(z)=u(r,\theta)+i\,v(r,\theta)$ is an analytic function, then prove that $v(r,\theta)$ is harmonic function.	**5**	
		Solution:-Given $f(z)=u(r,\theta)+i\,v(r,\theta)$ is an analytic function, by CR equations, $u_r=\frac{1}{r}v_\theta \to (1)$ $\quad and \quad v_r=-\frac{1}{r}u_\theta \to (2) \qquad \to 1\text{M}$ $(2)\Rightarrow v_{rr}=-\left[\frac{1}{r}u_{r\theta}-\frac{u_\theta}{r^2}\right] \qquad \to 2\text{M}$ From $(1), v_\theta = ru_r \Rightarrow v_{\theta\theta}=ru_{\theta r}$ $\therefore \nabla^2 v = v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}=-\frac{1}{r}u_{r\theta}+\frac{u_\theta}{r^2}+\frac{1}{r}\left[-\frac{1}{r}u_\theta\right]+\frac{1}{r^2}ru_{\theta r}=0 \quad \to 2\text{M}$	**5**	
		PART - B		
	2	**(a)** Evaluate $\oint_C \frac{z^4}{(3z+1)^4}dz$, where c is the circle $\lvert z\rvert=1$.	**5**	
		Solution:- Let I = $\oint_C \frac{z^4}{(3z+1)^4}dz=\frac{1}{3^4}\oint_C \frac{z^4}{(z+1/3)^4}dz$ $=\frac{2\pi i}{3^4 3!}f'''\left(-\frac{1}{3}\right) \qquad \to 2\text{M}$ $f'''(z)=24z$ and $f'''\left(-\frac{1}{3}\right)=-8 \qquad \to 2\text{M}$ $\therefore \text{I} = \frac{-8\pi i}{3^5} \qquad \to 1\text{M}$	**5**	
		(b) Find the harmonic conjugate of $u(x,y)=e^{2x}(x\cos 2y - y\sin 2y)$.	**5**	
		Solution:- Given $f(z)=u+iv$ is analytic function to find $v(x,y)$ such that CR equations are satisfied.	**5**	

		$dv = v_x dx + v_y dy$ $= -u_y dx + u_x dy$ $= e^{2x}\left(2x\sin 2y + 2y\cos 2y + \sin 2y\right)dx + e^{2x}\left(2x\cos 2y - 2y\sin 2y + \cos 2y\right)\mathrm{dy} \quad \to 3\mathrm{M}$ $\therefore v(x,y) = (\sin 2y)\int\limits_{y-\text{constant}} (2x+1)e^{2x}dx + (2\,\mathrm{y}\cos 2y)\int\limits_{y-\text{constant}} e^{2x}dx + c$ $= e^{2x}\left(x\sin 2y + y\cos 2y\right) + c \qquad \to 2\mathrm{M}$		
		(c) Find the orthogonal trajectories of the family of curves $-r^3\sin 3\theta = c_1$.	**5**	
		Solution:- Let $u(r,\theta) = -r^3\sin 3\theta$ then $f(z) = u(r,\theta) + i\,v(r,\theta)$ is an analytic function. $dv = v_r dr + v_\theta d\theta$ $= -\frac{1}{r}u_\theta dr + r u_r d\theta \qquad \to 2\mathrm{M}$ RHS is of the form $M(r,\theta)dr + N(r,\theta)d\theta$ and $M_\theta = N_r$, hence exact. $\to 1\mathrm{M}$ $\therefore v(r,\theta) = \int\limits_{\theta-\text{constant}} 3r^2\cos 3\theta dr = r^3\cos 3\theta + c$ The required orthogonal trajectories are $r^3\cos 3\theta = c_2 \qquad \to 2\mathrm{M}$	**5**	
	PART-C			
	3	**(a)** If $f(z) = u + iv$ is an analytic function, then prove that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\left\|\mathrm{Re}\{f(z)\}\right\|^2 = 2\left\|f'(z)\right\|^2$, where $\mathrm{Re}\{f(z)\}$ denotes the real part of $f(z)$. **Solution:-** Let $\phi = \left\|\mathrm{Re}\{f(z)\}\right\|^2 = u^2 \qquad \to 1\mathrm{M}$ Then $\phi_x = 2uu_x$, $\phi_{xx} = 2\left(u_x^2 + uu_{xx}\right) \qquad \to 2\mathrm{M}$ $\therefore \phi_{yy} = 2\left(u_y^2 + uu_{yy}\right) \qquad \to 1\mathrm{M}$ *and* $\nabla^2\phi = 2\left(u_x^2 + uu_{xx} + u_y^2 + uu_{yy}\right) = 2\left(u_x^2 + v_x^2\right) \because u_{xx} + u_{yy} = 0$ *and* $v_x = -u_y$ $= 2\left\|f'(z)\right\|^2 = \mathrm{RHS}. \qquad \to 2\mathrm{M}$ **OR** **(b)** Determine the analytic function $f(z)$ as function of z ,whose imaginary part is $\frac{x-y}{x^2+y^2}$.	**6**	
	3	**Solution:-** $v_x = \frac{\left(x^2+y^2\right) - 2x(x-y)}{\left(x^2+y^2\right)^2}$ *and* $v_y = \frac{-\left(x^2+y^2\right) - 2y(x-y)}{\left(x^2+y^2\right)^2} \quad \to 2\mathrm{M}$ $f'(z) = u_x + iv_x$ $= v_y + i\,v_x$ using CR equations $\to 2\mathrm{M}$ Replacing x by z and y by zero, $f'(z) = \frac{-z^2}{z^4} - i\frac{z^2}{z^4} = \frac{-(1+i)}{z^2}$	**6**	

		$\therefore f(z) = -\int \frac{1+i}{z^2} dz = \frac{1+i}{z} + c \quad \to 2\text{M}$		
	4	**(a)** Find the bilinear transformation which maps the points $\infty, i, 0$ of the Z-plane onto the points -1,-*i*, 1 of the W-plane respectively. Also find the invariant points of the transformation.	7	
		Solution:- The bilinear transform which maps the points $z_1 = \infty, z_2 = i, z_3 = 0$ on to the points $w_1 = -1, w_2 = -i, w_3 = 1$ is, $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z/z_1-1)(z_2-z_3)}{(z-z_3)(z_2/z_1-1)} \quad \to 2\text{M}$ $\Rightarrow \frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{-i}{-z} \quad or \quad \frac{w+1}{1-w} = \frac{1}{z} \Rightarrow w = \frac{1-z}{1+z} \quad \to 3\text{M}$ Invariant points are $z = -1 \pm \sqrt{2}$. $\to 2\text{M}$		
		OR		
		(b) Discuss the transformation $w = z + \frac{a^2}{z}, (z \neq 0)$.		
		Solution $:-$Let $w = u + iv$ and $z = re^{i\theta}$ then $u + iv = re^{i\theta} + \frac{a^2}{r} e^{-i\theta}$ $\Rightarrow u + iv = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta$ $\therefore u = \left(r + \frac{a^2}{r}\right)\cos\theta \to (1) \; and \quad v = \left(r - \frac{a^2}{r}\right)\sin\theta \to (2) \quad \to 1\text{M}$ $Case(i)$: Eliminating θ between (1) and (2), using $\cos^2\theta + \sin^2\theta = 1$ $\frac{u^2}{\left(r + \frac{a^2}{r}\right)^2} + \frac{v^2}{\left(r - \frac{a^2}{r}\right)^2} = 1 \to (3)$ When r is a constant (a circle in the Z- plane with center at the origin) , (3) represents an ellipse in the W-plane. $\to 2\text{M}$ $Case(ii)$: Eliminating r between (1) and (2), using $(A+B)^2 - (A-B)^2 = 4AB$ $\frac{u^2}{(2a\cos\theta)^2} - \frac{v^2}{(2a \sin\theta)^2} = 1 \to (4)$ When θ is a constant (a radial line in the Z- plane passing through the origin), (4) represents a hyperbola in the W-plane. $\to 2\text{M}$ Figures in Z- and W- planes, including both the cases. $\to 2\text{M}$		

	5	**(a)** Evaluate $\oint_C \frac{z}{(z^2+1)(z^2-9)}dz$, where c is the circle $\lvert z\rvert = 2$.	7	
		Solution:- The points $z = i \ and \ -i$ lie inside c:$\lvert z\rvert = 2$. $\therefore \text{I} = \oint_C \frac{f(z)}{(z+i)(z-i)}dz, where \ f(z) = \frac{z}{z^2-9}$ $\rightarrow$ 2M $\frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$ $\rightarrow$ 2M $\therefore \text{I} = \frac{1}{2i}\left[\oint_C \frac{f(z)}{z-i}dz - \oint_C \frac{f(z)}{z+i}dz\right] = \frac{1}{2i}\left[2\pi i f(i) - 2\pi i f(-i)\right]$ $\rightarrow$ 1M $\text{I} = \frac{-\pi i}{5}$ $\rightarrow$ 2M		
		OR		
		(b) Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the triangle having vertices (1, 2), (1, 4) and (3, 2).		
		Solution:- Figure - triangle ABC, $\text{A} \equiv (1,2), \text{B} \equiv (3,2) \ and \ \text{C} \equiv (1,4)$ $\rightarrow$ 1M Along AB:- $y = 2, dy = 0 \ and \ x \rightarrow 1$ to 3 , $\int_{AB} z^3 dz = \int_1^3 (x+2i)^3 dz = \frac{(3+2i)^4}{4} - \frac{(1+2i)^4}{4} \rightarrow (1)$ $\rightarrow$ 1M Along BC:- $y = 5-x, dy = -dx \ and \ x \rightarrow 3$ to 1 , $\int_{BC} z^3 dz = \int_3^1 [x+i(5-x)]^3 (dx - idx) = \frac{(1+4i)^4}{4} - \frac{(3+2i)^4}{4} \rightarrow (2)$ $\rightarrow$ 2M Along CA:- $x = 1, dx = 0 \ and \ y \rightarrow 4$ to 2 , $\int_{CA} z^3 dz = \int_4^2 (1+iy)^3 i dy = \frac{(1+2i)^4}{4} - \frac{(1+4i)^4}{4} \rightarrow (3)$ $\rightarrow$ 1M $\oint_{ABC} z^3 dz = \int_{AB} z^3 dz + \int_{BC} z^3 dz + \int_{CA} z^3 dz = 0,$ using $(1),(2) \ and \ (3)$. Hence Cauchy's theorem is verified. $\rightarrow$ 2M		

Course Outcome:	
CO 3	Demonstrate an understanding of analytic functions and their application to evaluate integrals.

Note:- Award full marks for alternate methods.