UNIT 4:POLAR PLOTs

Frequency domain analysis:Introduction

- One of the most commonly used test functions for a circuit or system is the sine wave.
- Reason: Any signal going into a circuit can be represented by a sum of sine waves of varying frequency and amplitude (often an infinite sum)
- Arbitrarily complex functions can be represented using only these very simple function(Fourier Series)
- The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as signals throughout the system, is sinusoidal in the steady-state; it differs form the input waveform only in amplitude and phase..
- The amount of difference is a function of the input frequency

- Output is also a sinusoidal signal having the same frequency as the input. But the magnitude and phase angle of output is different.

Magnitude is given by $A|G(j\omega)|$ and phase angle is given by $\delta = |\underline{G(j\omega)}$

- Consider input $r(t)$ to be purely sinusoidal, i.e $r(t) = A \sin \omega t$

  We get the steady state output as

  $C_{ss}(t) = A\,|G(j\omega)| \sin(\omega t + \delta)$

- To get the frequency response means to sketch the variation in magnitude and phase angle of $G(j\omega)$, when ω is varied from 0 to ∞.

- If the transfer function of a system is $T(s)$, the frequency response function is obtained by simply replacing ‘s’ by ‘$j\omega$’ i.e. $T(j\omega)$.

Advantages of Frequency domain

- The transfer function can be experimentally determined.
- GM and PM can be estimated.
- It can be extended for non-linear systems.
- It can be used to design lag or lead compensators for steady state requirements and transient response specifications.

Frequency domain methods:

Variations in magnitude and phase angle of $G(j\omega)$ as input frequency is varied from 0 to ∞ can be obtained by a number of methods. The commonly used methods to sketch frequency response are,

1. Polar plot
2. Bode plot
3. Nyquist plot

1.Polar plot:

- ✓ In polar plot, the magnitude of $G(j\omega)$ is plotted against the phase angle of $G(j\ \omega)$ for various values of ω, in a polar Co-ordinate system.

- ✓ Thus , polar plot is the locus of tips of phasors of various magnitude plotted at the corresponding phase angles for different values of frequencies from 0 to ∞.

Example

For example consider, $G(s) = \dfrac{100}{s+5}$

$$G(j\omega) = \frac{100}{j\omega + 5} = \frac{100}{\sqrt{25+\omega^2}} \angle -\tan^{-1}\frac{\omega}{5}$$

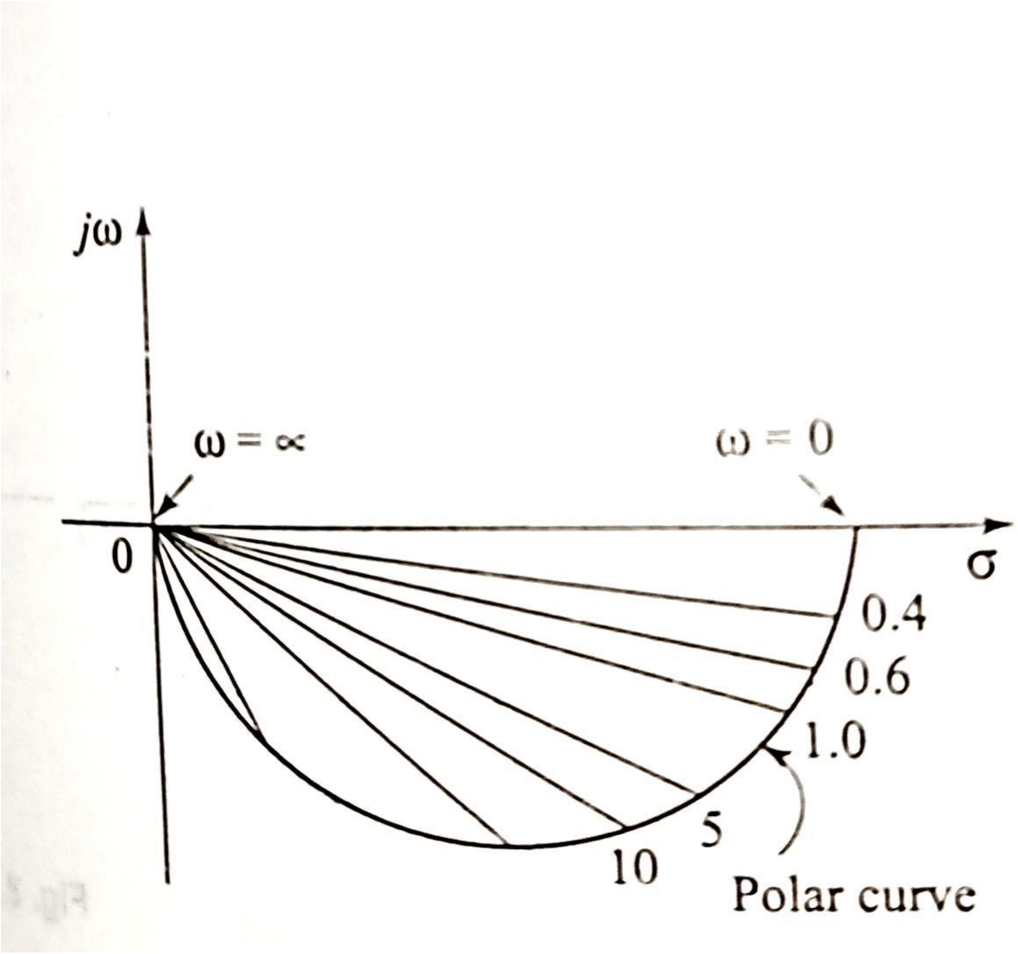
The magnitude is, $G(\omega) = \dfrac{100}{\sqrt{25+\omega^2}}$ and the phase angle is, $\phi(\omega) = -\tan^{-1}\dfrac{\omega}{5}$

The values of $G(\omega)$ and $\phi(\omega)$ for different values of ω are tabulated as shown below.

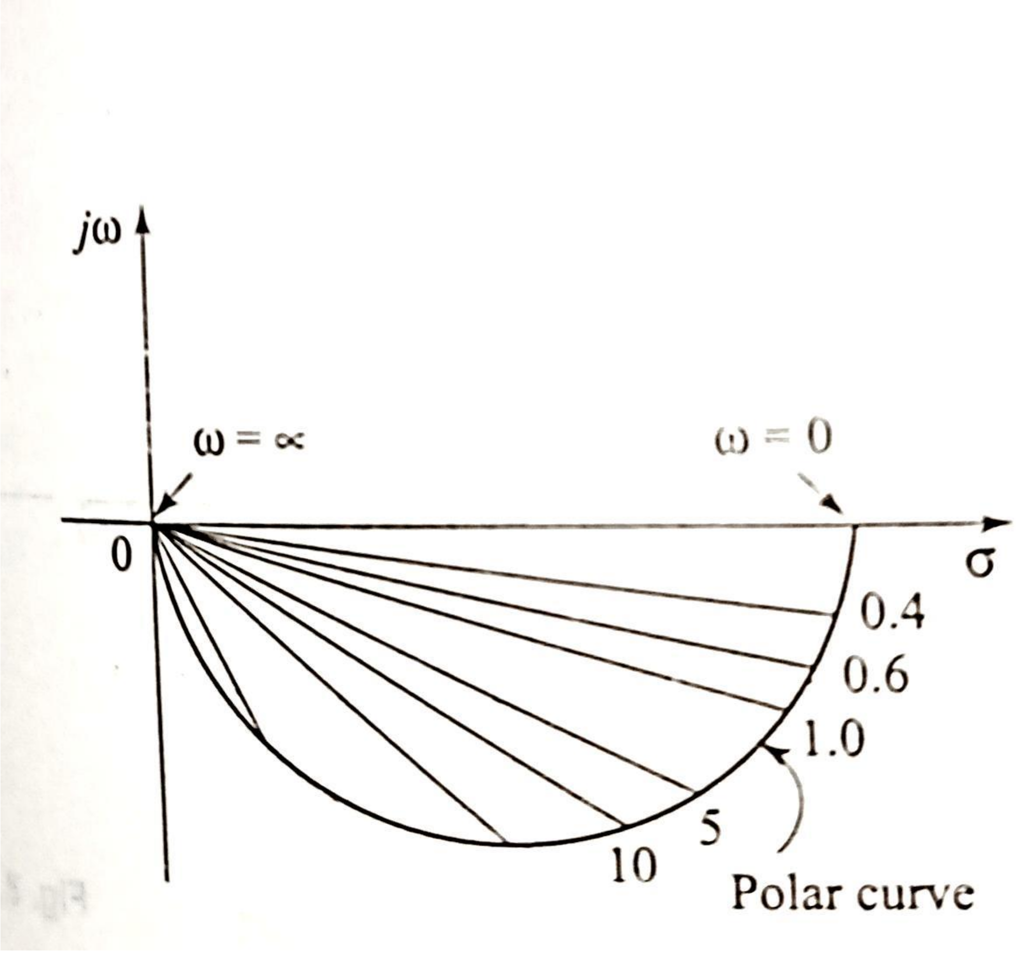
ω radians	$G(\omega)$	$\phi(\omega)$, degrees
0	20	0
0.2	19.984	−2.27
0.4	19.363	−4.57
0.6	19.8575	−6.84
0.8	19.7488	−9.09
1.0	19.6116	−11.31
1.5	19.1565	−16.70
2.0	18.5695	−21.80
4.0	15.6174	−38.66
5.0	14.1421	−45
10.0	8.9943	−63.43
20.0	4.8507	−75.96
—	—	—
—	—	—
∞	0	−90

Vectors are drawn at the respective angles from the real axis. The locus is then drawn which i
polar plot.

Polar curve



Polar curve



- The polar plot of sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) verses the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity.
- Therefore it is the locus of $|G(j\omega)|\angle G(j\omega)$ as ω is varied from zero to infinity.
- In the polar plot the magnitude of G(jω) is plotted as the distance from the origin while phase angle is measured from positive real axis.
- Positive angle is taken in anticlockwise direction.

Properties of Polar Plots

- For transfer function, $G(s)=1/s^n\,(s+a)$, polar curve starts at an angle $-n(90^o)$ and terminates at origin at $\omega=\infty$ through an angle $-(n+1)(90^o)$.
- For every additional factor of the form $1/(s+b)$, polar plot enters the origin through an additional angle of -90^o.
- Polar plots of $M(j\omega)=A+G(j\omega)$ where $A=x+jy$ is identical to polar plot of $G(j\omega)$ but coordinates of origin is shifted to $-A$.
- Polar plot of a transfer function of an LTI system exhibits conjugate symmetry , i.e $M(j\omega)$ for $-\infty< \omega <0$ is the mirror image of $M(j\omega)$ for $0< \omega < \infty$.

Steps to draw Polar Plot

- Step 1: Determine the T.F G(s)
- Step 2: Put s=jω in the G(s)
- Step 3: At ω=0 & ω=∞ find $|G(j\omega)|$ by $\lim_{\omega \to 0} |G(j\omega)|$ & $\lim_{\omega \to \infty} |G(j\omega)|$
- Step 4: At ω=0 & ω=∞ find $\angle G(j\omega)$ by $\lim_{\omega \to 0} \angle G(j\omega)$ & $\lim_{\omega \to \infty} \angle G(j\omega)$
- Step 5: Rationalize the function G(jω) and separate the real and imaginary parts
- Step 6: Put Re [G(jω)]=0, determine the frequency at which plot intersects the Im axis and calculate intersection value by putting the above calculated frequency in G(jω)

Steps to draw Polar Plot

- Step 7: Put Im [G(jω)]=0, determine the frequency at which plot intersects the real axis and calculate intersection value by putting the above calculated frequency in G(jω)
- Step 8: Sketch the Polar Plot with the help of above information

Polar Plot for Type 0 System

- Let $G(s)=\dfrac{K}{(1+sT_1)(1+sT_2)}$

- Step 1: Put s=jω

$$G(j\omega)=\frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$
$$=\frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}}\angle-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2$$

- Step 2: Taking the limit for magnitude of G(jω)

$$\lim_{\omega\to 0}\left|G(j\omega)\right| = \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+j(\omega T_2)^2}} = K$$

$$\lim_{\omega\to \infty}\left|G(j\omega)\right| = \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+j(\omega T_2)^2}} = 0$$

- Step 3: Taking the limit of the Phase Angle of G(jω)

$$\lim_{\omega\to 0}\angle G(j\omega) = \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = 0$$

$$\lim_{\omega\to \infty}\angle G(j\omega) = \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -180$$

- Step 4: Separate the real and Im part of G(jω)

$$G(j\omega)=\frac{K(1-\omega^2T_1T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}-j\frac{K\omega(T_1+T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}$$

- Step 5: Put Re [G(jω)]=0

$$\frac{K(1-\omega^2T_1T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}=0\Rightarrow\omega=\frac{1}{\sqrt{T_1T_2}}\ \ \&\ \omega=\infty$$

$So\ \ When$

$$\omega=\frac{1}{\sqrt{T_1T_2}}\Rightarrow G(j\omega)=\frac{K\sqrt{T_1T_2}}{T_1+T_2}\angle-90^0$$

$$\&\ \ \omega=\infty\qquad\Rightarrow G(j\omega)=0\angle-180^0$$

- Step 6: Put Im [G(jω)]=0

$$\frac{K\omega(T_1+T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}=0 \Rightarrow \omega=0 \ \& \ \pm\infty$$

So When

$$\omega=0 \Rightarrow G(j\omega)=K\angle 0^0$$

$$\omega=\infty \Rightarrow G(j\omega)=0\angle 180^0$$

$$\mathbf{G(s)} = \frac{\mathbf{K}}{(\mathbf{1}+\mathbf{sT_1})(\mathbf{1}+\mathbf{sT_2})}$$

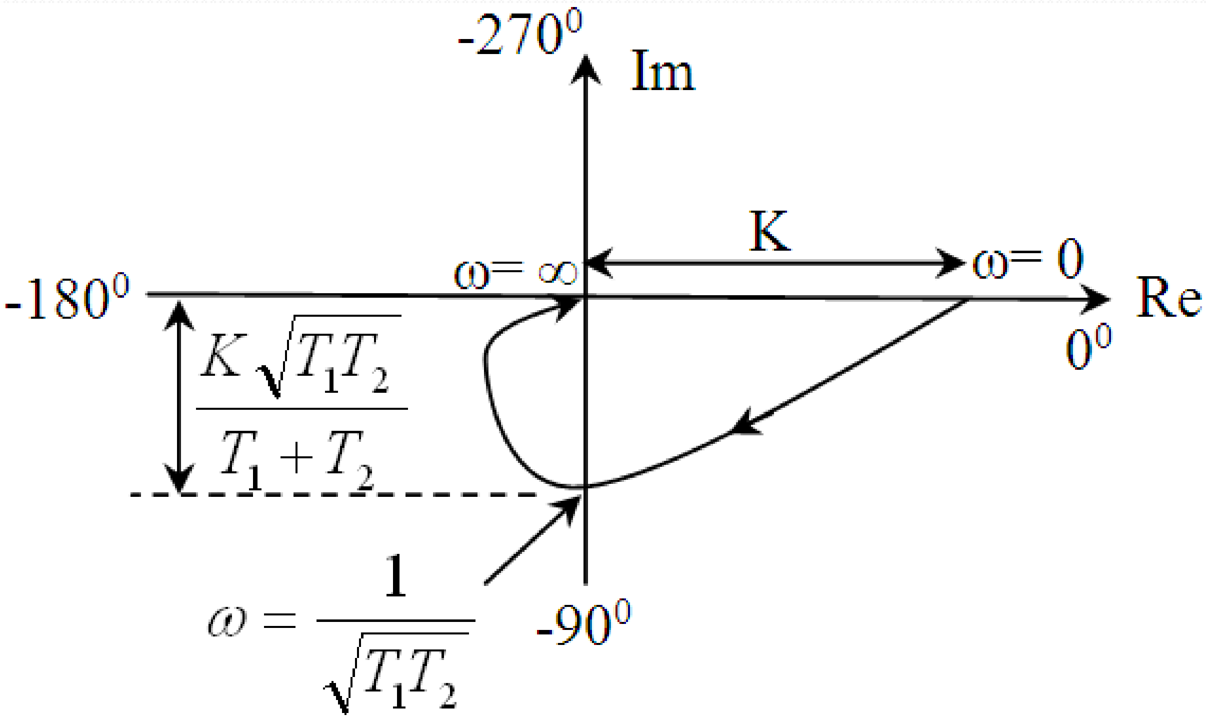
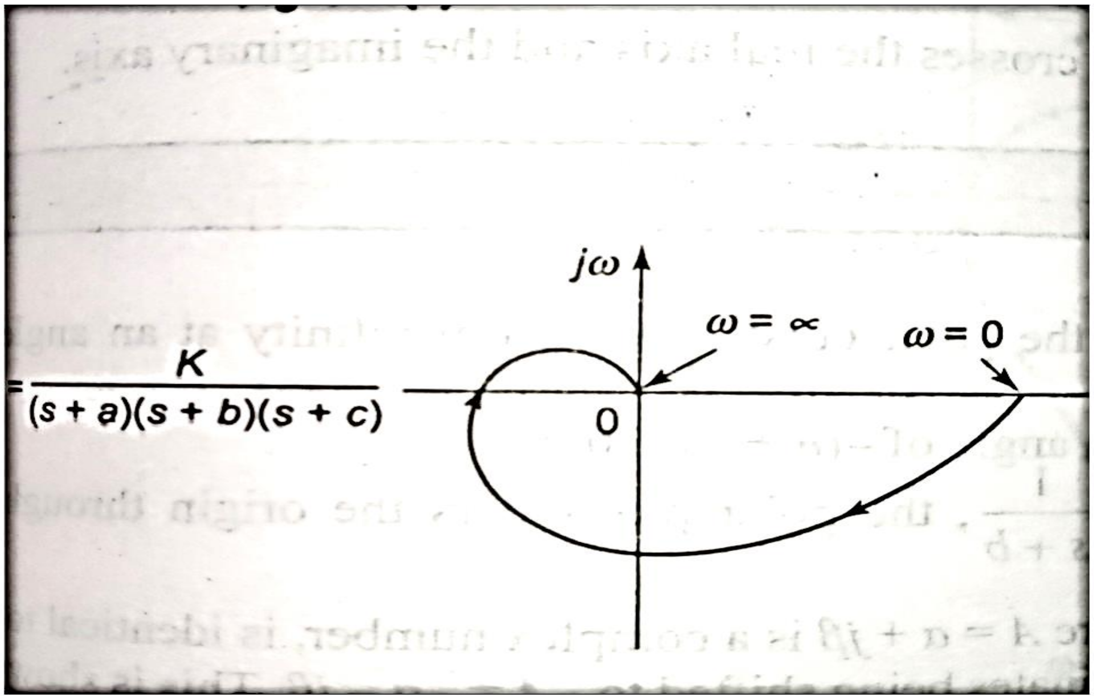
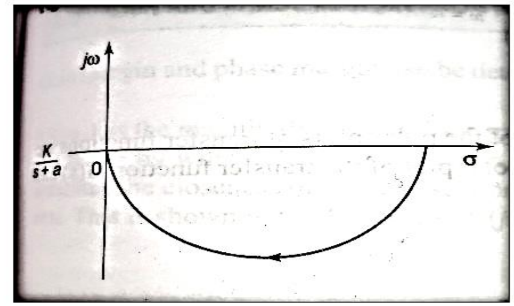
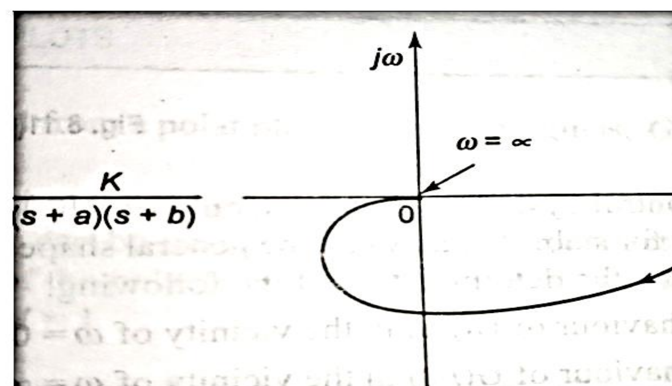


Figure 1: Polar Plot of Type 0 System

Polar curves for certain typical TFs







Type 1 System

- Consider Type 1 system with transfer function
  $G(s)=1 / s(1+Ts)$

The frequency domain transfer function is

$G(j\omega)=1 / j(1+jT\ \omega)$

$=1+j0 / (0+j\ \omega)(1+jT\omega)$

$|G(j\omega)| =1 / \omega\ (1+ \omega^2T^2)^{.5} =M$

$Arg(G(j\omega))=\tan^{-1}(0/1) / (\tan^{-1}(\omega /0).\tan^{-1}(\omega\ T/1))$

$\Phi=-90^o-\tan^{-1} \omega T$

- When $\omega = 0$

$M = \infty$

$\Phi = -90^o$

- When $\omega = \infty$

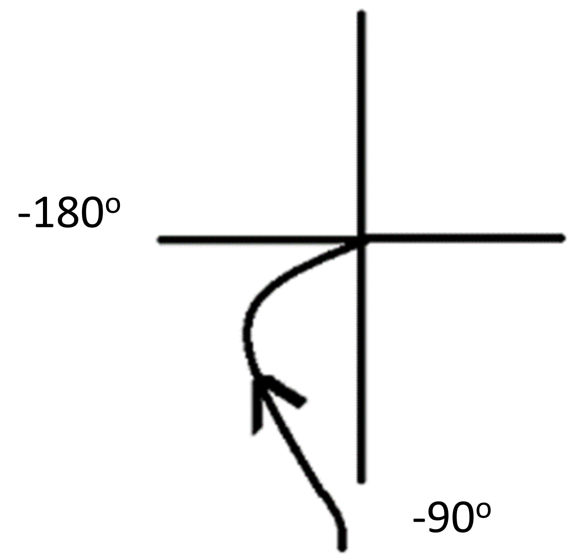
-180^o

$M = 0$

$\Phi = -180^o$

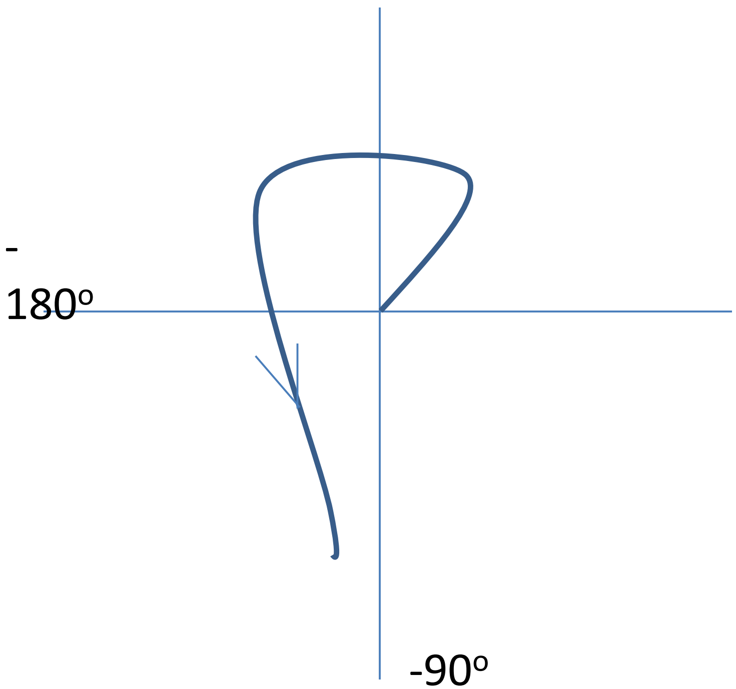
Rotation of plot=

$-180^o - (-90^o) = -90^o$ clockwise

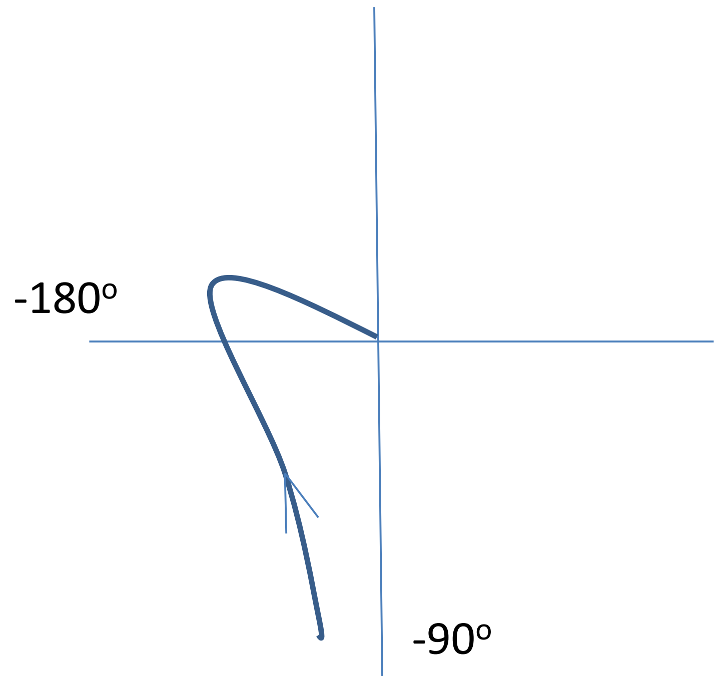


Typical Type1 system TFs

$G(s)=K/s(s+a)(s+b)(s+c)$



$G(s)=K/s(s+a)(s+b)$



Type 2 System

- Consider Type 2 system with transfer function

  $G(s)=1 / s^2(1+Ts)$

The frequency domain transfer function is

$G(j\omega)\ 1 / j\omega.j\omega.(1+jT\omega)$

$=1+j0 / (0+j\omega)\ (0+j\omega)(1+jT\omega)$

$|G(jw)|=1 /\omega^2\times(1+\omega^2T^2)^{.5} =M$

$Arg(G(j\omega))=\tan^{-1}(0/1) / (\tan^{-1}(\omega/0).\ (\tan^{-1}(\omega/0).\tan^{-1}(\omega T/1))$

$=0^o / 90^o.90^o.\tan^{-1}\omega T$

$\Phi=-180^o-\tan^{-1}\omega T$

- When $\omega=0$

$M=\infty$

$\Phi= -180^o$

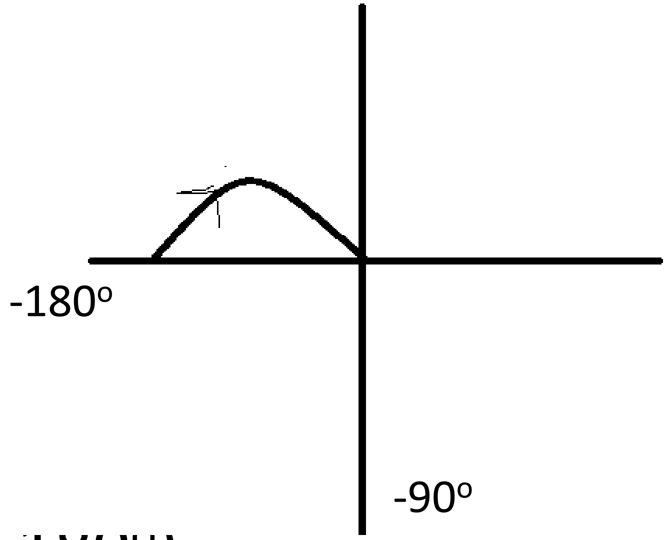
- When $\omega=\infty$

$M=0$

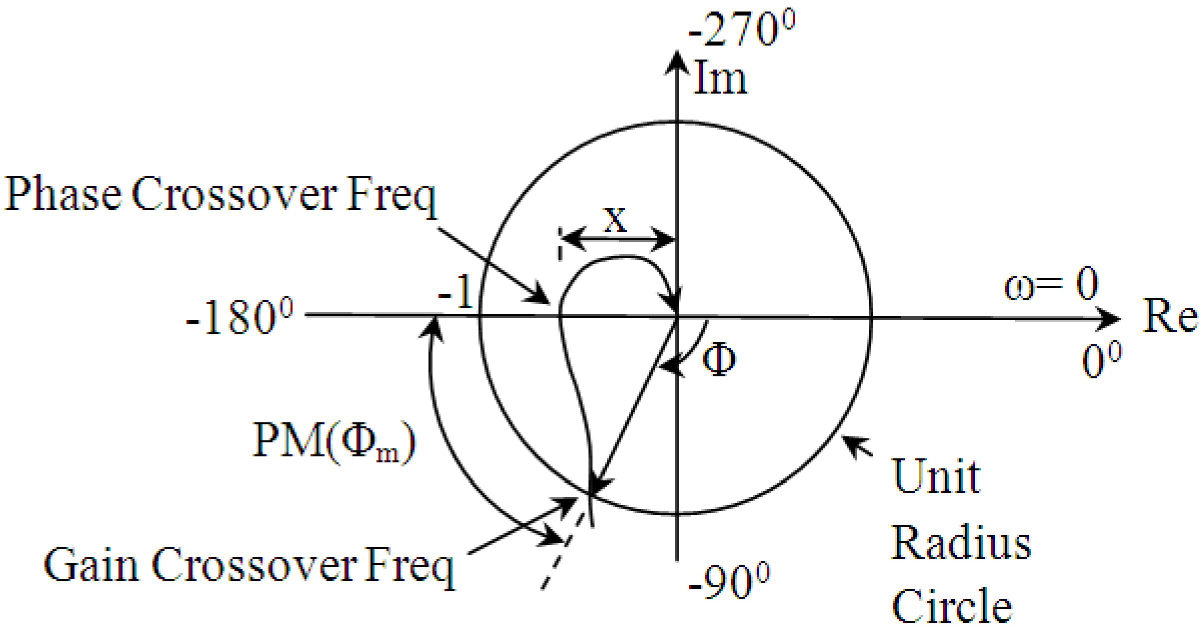
$\Phi= -270^o$

Rotation of plot$= -270^o-(-180^o)$

$= -90^o$clockwise



Gain Margin, Phase Margin & Stability



- **Phase Crossover Frequency (ω_p) :** The frequency where a polar plot intersects the –ve real axis is called phase crossover frequency

- **Gain Crossover Frequency (ω_g) :** The frequency where a polar plot intersects the unit circle is called gain crossover frequency

  So at ω_g $|G(j\omega)| = Unity$

- **Phase Margin (PM):**
  - Phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability (marginally stabile)

$$\Phi_m = 180^0 + \Phi$$

Where

$$\Phi = \angle G(j\omega_g)$$

if $\Phi_m > 0 \Rightarrow +PM$ (Stable System)

if $\Phi_m < 0 \Rightarrow -PM$ (Unstable System)

GM

- **Gain Margin (GM):**
  - The gain margin is the reciprocal of magnitude $|G(j\omega)|$ at the frequency at which the phase angle is -180^0.

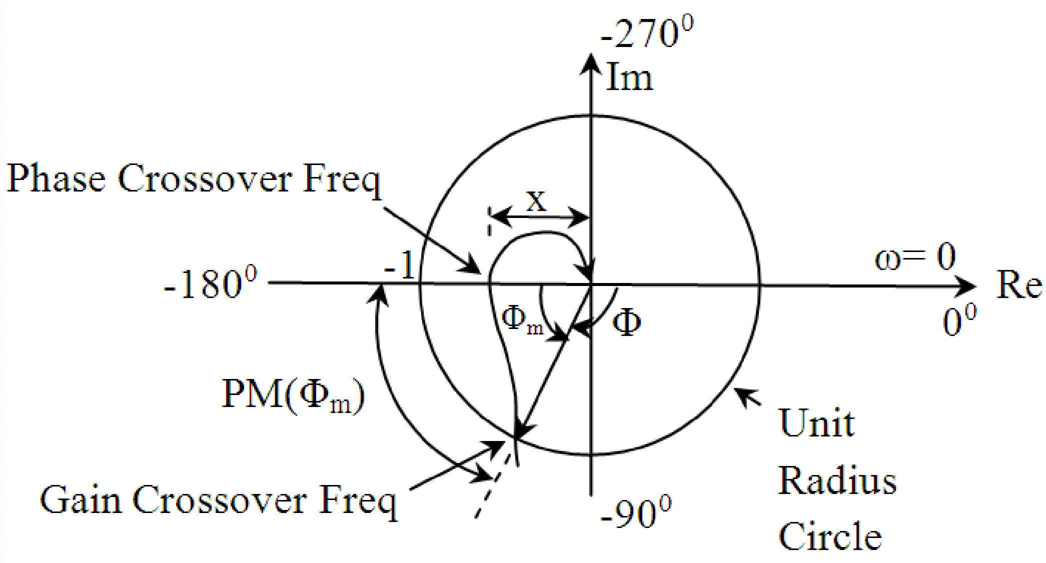
$$GM = \frac{1}{|G(j\omega c)|} = \frac{1}{x}$$

In terms of dB

$$GM\ in\ dB = 20\log_{10}\frac{1}{|G(j\omega c)|} = -20\log_{10}|G(j\omega c)| = -20\log_{10}(x)$$

Stability

- **Stable:** If critical point (-1+j0) is outside the plot as shown, Both GM & PM are +ve

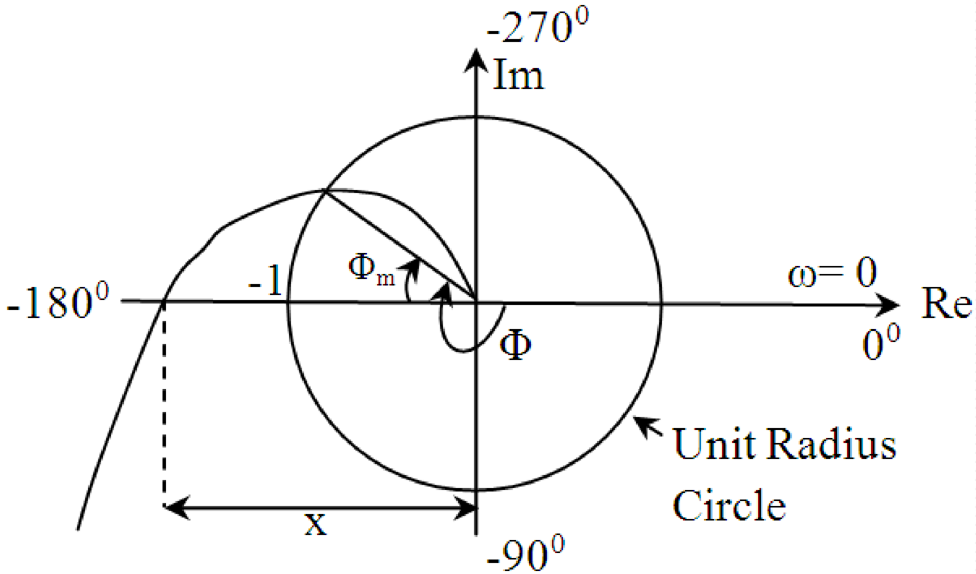


Stable system: +GM, +PM

$$GM = 20\log_{10}(1/x)\ dB$$

$$\omega gc < \omega pc$$

- **Unstable:** If critical point (-1+j0) is within the plot as shown, Both GM & PM are negative .

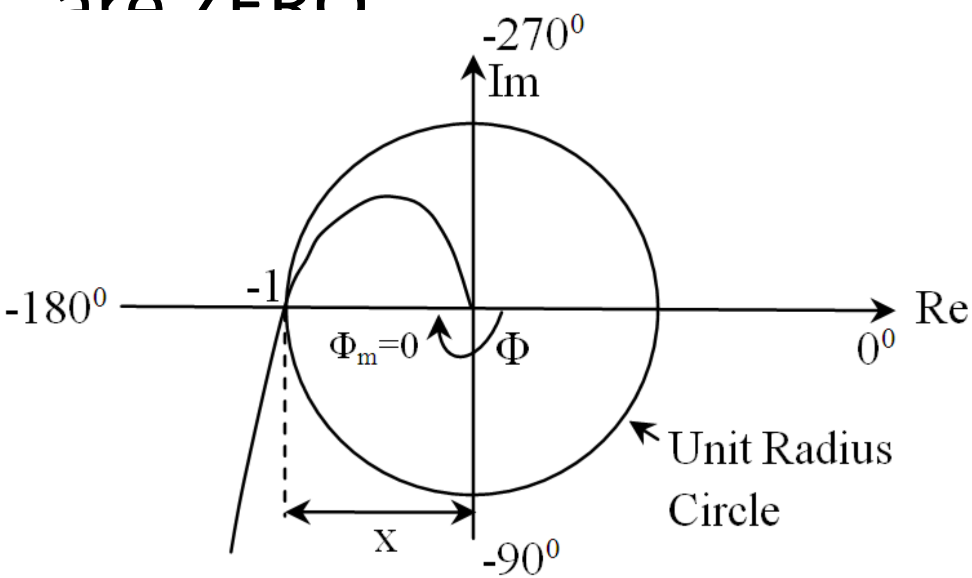


Unstable system: -GM, -PM

$$GM=20\log_{10}(1/x)\ dB$$

$\omega gc > \omega pc$

- **Marginally Stable System:** If critical point (-1+j0) is on the plot as shown, Both GM & PM are ZERO



Marginally stable system: GM=0 dB, PM=0^0

$GM=20\log_{10}(1$

$\omega gc=\omega pc$

IllustrativeExample

For Unity feedback system with

$$G(s) = \frac{40}{(s+4)(s^2+4s+8)}$$

Sketch the polar plot. Find GM and PM

Solution

Magnitude:

When $\omega=0$, it is $40/32=1.25$

When $\omega=\infty$, it is 0

Angle:

When $\omega=0$, it is 0

When $\omega=\infty$, it is -270^o

Point of intersection with Im & Real Axis

Point of intersection with Im Axis:

Equating Re{G(jω)}=0,

Re{(4-jω)(8-ω^2 - 4jω)}=0

We get ω1 =√4, take -2 (C1)

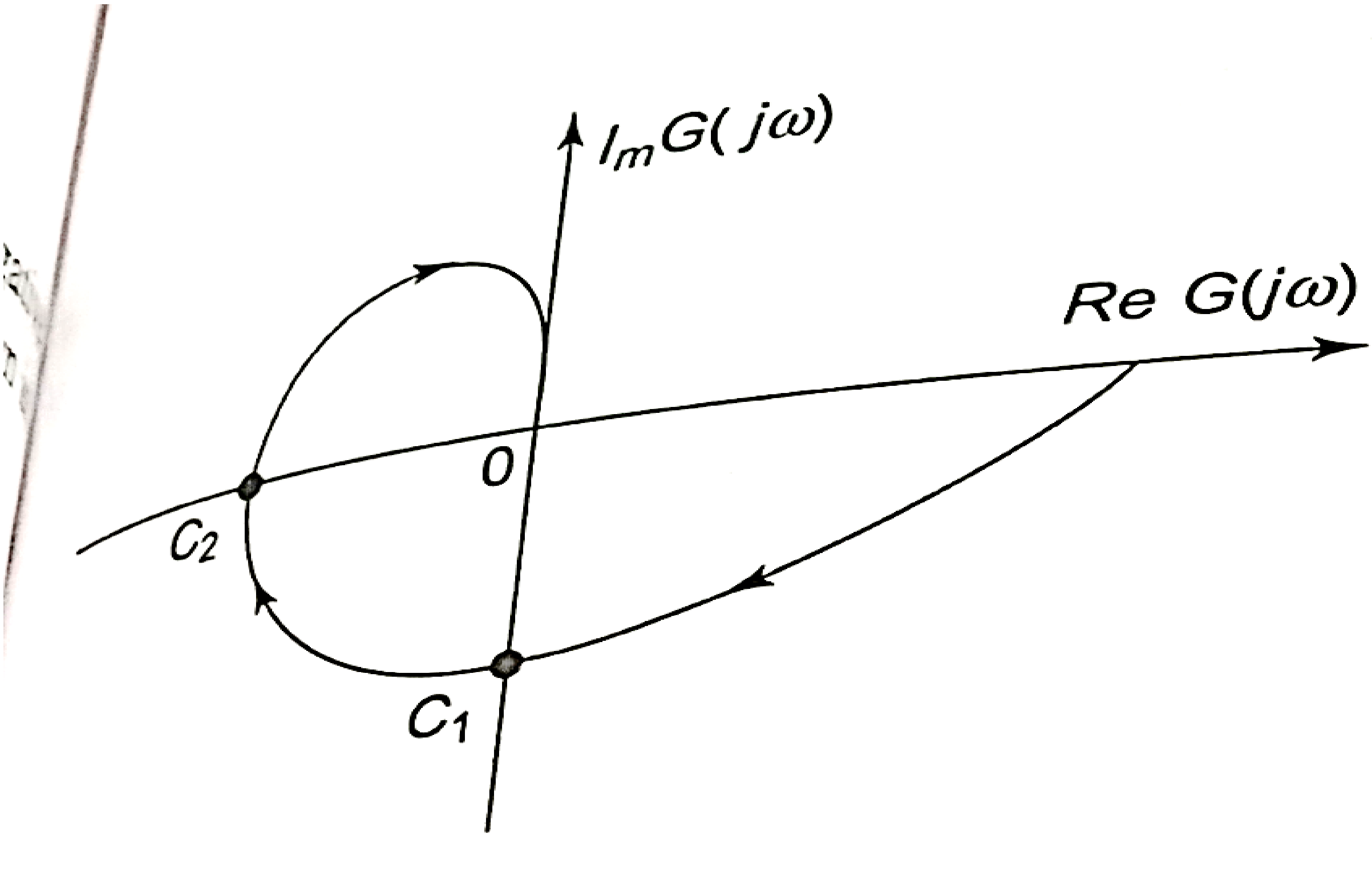
Point of intersection with Re Axis:

Equating Im{G(jω)}=0,

Im{(4-jω)(8-ω^2 - 4jω)}=0

We get ω2 =√24 (C2)

Polar plot



PM

$$G(j\omega) = \frac{40}{(j\omega+4)(-\omega^2+4j\omega+8)}$$

$$|G(j\omega)| = 1 \Rightarrow \sqrt{16+\omega^2}\,\sqrt{(8-\omega^2)^2+16\omega^2} = 40 = \sqrt{1600}$$

$$\sqrt{16+\omega^2}\,\sqrt{(8-\omega^2)^2+16\omega^2} = \sqrt{16+\omega^2}\,\sqrt{64+\omega^4-16\omega^2+16\omega^2}$$

$$\therefore \sqrt{16+\omega^2}\,\sqrt{64+\omega^4} = 40 = \sqrt{1600} = \sqrt{20}\,\sqrt{80} = \sqrt{16+2^2}\,\sqrt{64+2^4}$$

$$\therefore \omega = 2$$

At ω_1, the magnitude of $G(j\omega_1)$ is unity. Hence, this is the gain crossover frequency.

$\phi(\omega_1) = -90°$.

$\therefore PM = -90° + 180° = 90°$.

GM

$$G(j\omega) = \frac{40}{(j\omega+4)(-\omega^2+4j\omega+8)}$$

$$\left|G(j\omega)\right| = \frac{40}{\sqrt{16+\omega^2}\,\sqrt{(8-\omega^2)^2+16\omega^2}}$$

$$At\ \omega = \sqrt{24,} \quad \left|G(j\omega)\right| = 0.25$$

$$GM = \frac{1}{0.25} = 4 = 12.04dB$$