2.Root Locus-Construction Rules

Properties of the root loci

(that are useful for constructing the root loci manually)

These properties are developed based on the relationship between the poles and zeros of $G(s)H(s)$ and the zeros of $1 + G(s)H(s)$, which are the roots of the characteristic equation.

- $K = 0$ AND $K = \pm\infty$ POINTS
- NUMBER OF BRANCHES ON THE ROOT LOCI
- SYMMETRY OF THE ROOT LOCI
- ANGLES OF ASYMPTOTES OF THE ROOT LOCI AND BEHAVIOR OF THE ROOT LOCI AT $|S| \to \infty$
- INTERSECT OF THE ASYMPTOTES (CENTROID)
- ROOT LOCI ON THE REAL AXIS
- ANGLES OF DEPARTURE AND ANGLES OF ARRIVAL OF THE ROOT LOCI
- INTERSECTION OF THE ROOT LOCI WITH THE IMAGINARY AXIS
- BREAKAWAY POINTS (SADDLE POINTS)

- CALCULATION OF K ON THE ROOT LOCI

1. $K = 0$ AND $K = \pm\infty$ POINTS

The $K = 0$ points on the root loci are at the poles of $G(s)H(s)$.

Let open loop TF be $$G(s)H(s) = \frac{KN(s)}{D(s)}$$

The characterstic equation of closed loop TF is

$$1 + G(s)H(s) = 1 + \frac{KN(s)}{D(s)} \quad \text{or } \mathrm{D(s) + KN(s) = 0}$$

When K=0, the above eqn becomes D(s)=0 and roots of D(s) are poles of open loop TF $G(s)H(s)$.

The $K = \pm \infty$ points on the root loci are at the zeros of $G(s)H(s)$.

The characterstic equation of closed loop TF can be written as

$$1 + G(s)H(s) = \frac{D(s)}{K} + \mathrm{N(s)} = 0$$

When K approaches infinity, it becomes N(s)=0

Roots of N(s)=0 are zeros $G(s)H(s)$

The poles and zeros referred to here include those at infinity, if any.

Poles and zeros at Infinity

Let m=No. of finite open loop zeros
& n= No. of finite open loop poles.

If $m=n$, each branch originates from a finite open loop pole and terminates at a finite open loop zeros

If $m<n$, m branches originates from a finite open loop pole and terminates at m finite open loop zeros . Remaining (n-m) branches originate from remaining open loop poles and approach infinity.

If $m>n$, there are (m-n) poles at infinity(i.e., branches originate from infinity)

2. No. of Branches on the root loci

- The number of branches of root loci is equal to the order of the polynomial

$$1+G(s)H(s)=1+\frac{KN(s)}{D(s)} \text{ or } \mathrm{D(s)+KN(s)=0}$$

- It is important to keep track of the total number of branches of the root loci.
- Each branch of the root locus represents the locus of one CLOSED Loop Pole.

Illustrative example 1

Given the Ch. Equation

$s\ (s + 2)\ (s + 3) + K\ (s + 1) = 0,$

(i)Obtain open loop TF

(ii) Mark K=0& K= ±∞ points on the *S*-plane.

(iii) Comment on the branches

(i)Open loop TF

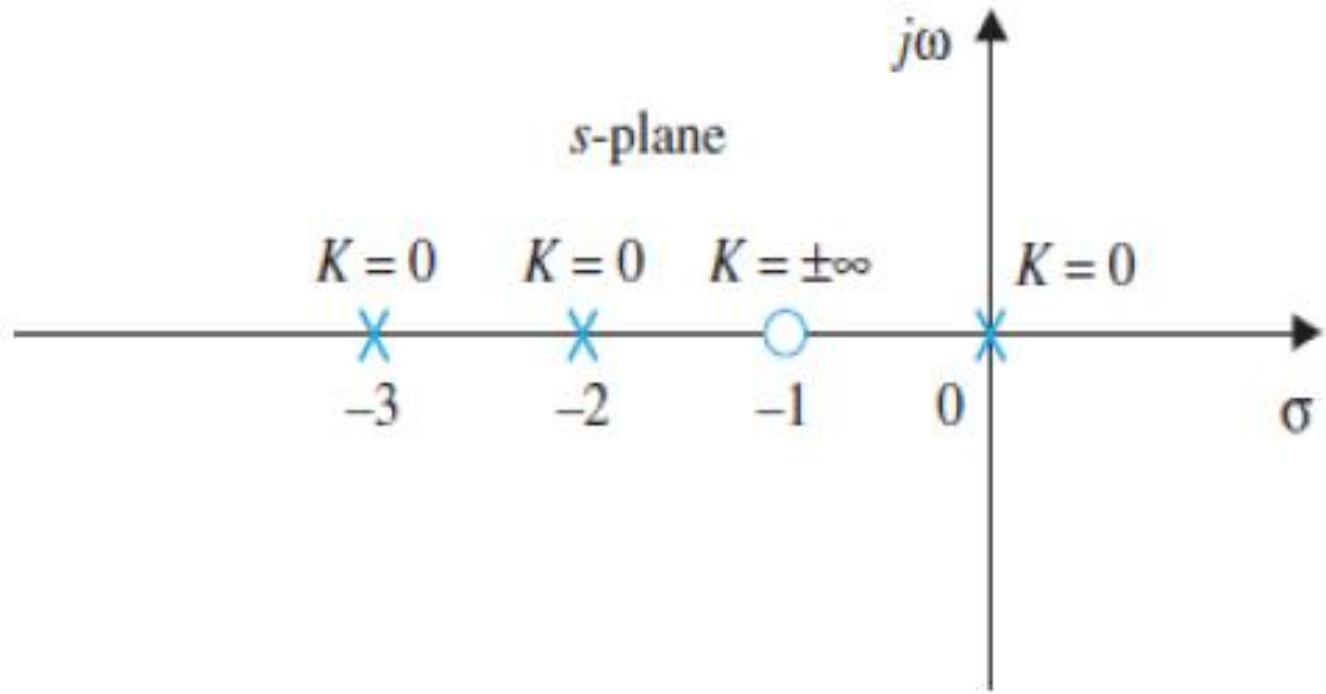
Dividing both sides of given eqn. by the terms that (

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0$$

Which gives pen loop TF as

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

(ii) K=0& K= ±∞ points on the *S*-plane



(iii) Comment on the branches

- There are 3 branches
- There are 3 finite poles, 1 finite zero and 2 zeros at ∞. Hence two branches approach infinity

3. SYMMETRY OF THE ROOT LOCI

- The root loci are symmetrical with respect to the real axis of the *s*-plane.

The locus of closed loop poles , as the gain factor K of the open loop is varied is the Root locus. The closed loop poles, if real lie on the real axis. If they are complex, occur in conjugate pairs which are symmetrical about the real axis. Hence, root loci are symmetrical about the real axis of the *s*-plane.

- In general, the root loci are symmetrical with respect to the axes of symmetry of the pole-zero configuration of $G(s)H(s)$.

Asymptote

Definition of *asymptote*

: a straight line associated with a curve such that as a point moves along an infinite branch of the curve the distance from the point to the line approaches zero and the slope of the curve at the point approaches the slope of the line

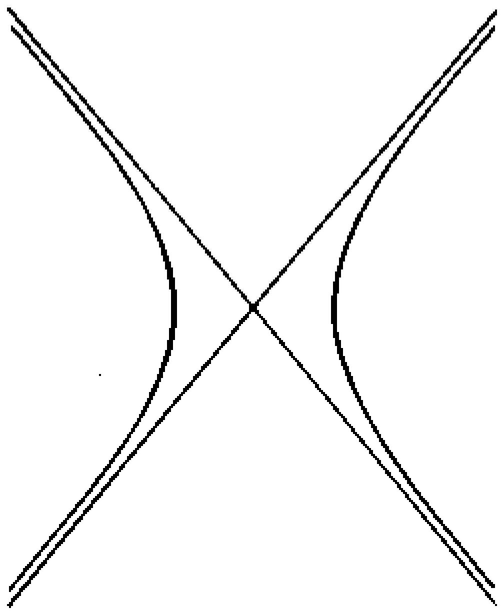


Illustration of *asymptote:*asymptotes to the hyperbola

4. ANGLES OF ASYMPTOTES OF THE ROOT LOCI

(& BEHAVIOR OF THE ROOT LOCI AT $|S| \to \infty$)

- Asymptotes of root loci refers to behavior of root loci at $|s| \to \infty$.
- When m < n, there are (n-m) zeros at ∞.

  (m=No. of finite open loop zeros

  & n= No. of finite open loop poles.)

- i.e., (n-m) branches terminate at infinity. At Infinity, these branches approach st. lines called asymptotes.

Angle of asymptotes

- For large values of *s,* the root loci for $K \geq 0$ (*RL*) are asymptotic to asymptotes with angles given by

$$\theta_i = \frac{(2i+1)}{|n-m|} \times 180^\circ \quad n \neq m$$

Where i = 0, 1, 2,… , |n − m| − 1; and *n* and *m* are the number of finite poles and zeros of $G(s)H(s)$, respectively.

For $K \leq 0$ (*RL*), the angles of the asymptotes are

$$\theta_i = \frac{2i}{|n-m|} \times 180^\circ \quad n \neq m$$

5. INTERSECT OF THE ASYMPTOTES (CENTROID)

The intersection of the asymptotes of the root loci lies on the real axis of the *s*-plane, at

.

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$$

where *n* is the number of finite poles and *m* is the number of finite zeros of *G*(*s*)*H*(*s*), respectively

- The intersection of the asymptotes σ1 represents the center of gravity of the root loci and is always a real number.

Since the poles and zeros of G(s)H(s) are either real or in complex-conjugate pairs, the imaginary parts in the numerator always cancel each other out. Thus, the terms in the summations may be replaced by the real parts of the poles and zeros of G(s)H(s), respectively. That is,

$$\sigma_1 = \frac{\sum \text{real parts of poles } G(s)H(s) - \sum \text{real parts of zeros of } G(s)H(s)}{n-m}$$

Illustrative example 2

Given the transfer function, $G(s)H(s) = \frac{K(s+1)}{s(s+4)(s^2+2s+2)}$

Write the Ch. Eqn. and find

i. $K = 0$ AND $K = \pm\infty$ POINTS

ii. NUMBER OF BRANCHES ON THE ROOT LOCI

iii. SYMMETRY OF THE ROOT LOCI

iv. ANGLES OF ASYMPTOTES OF THE ROOT LOCI AND BEHAVIOR OF THE ROOT LOCI AT $|S| \to \infty$

v. INTERSECT OF THE ASYMPTOTES (CENTROID)

Illustrative example 2:Solution

i. The characteristic equation is

$$s(s+4)(s^2+2s+2)+K(s+1)=0$$

i. $K = 0$: The points at which $K = 0$ on the root loci are at the poles of $G(s)H(s)$: $s = 0, -4, -1 + j$, and $-1 - j$. $K = \pm\infty$: The points at which $K = \pm\infty$ on the root loci are at the zeros of $G(s)H(s)$: $s = -1, \infty, \infty$, and ∞.

ii. There are four root loci branches, since ch.Eqn. And open loop TF are of the fourth order.

iii. The root loci are symmetrical to the real axis.

iv. Since the number of finite poles of $G(s)H(s)$ exceeds the number of finite zeros of $G(s)H(s)$ by three ($n - m = 4 - 1 = 3$), when $K = \pm\infty$, three root loci approach $s = \infty$.

The angles of the asymptotes of the root loci for $K \geq 0$ are given by,

$$j = 0: \quad \theta_0 = \frac{180^\circ}{3} = 60^\circ$$
$$j = 1: \quad \theta_1 = \frac{540^\circ}{3} = 180^\circ$$
$$j = 2: \quad \theta_2 = \frac{900^\circ}{3} = 300^\circ$$

and for $K \leq 0$ are calculated to be 0°, 120°, and 240°.

centroid

v. The intersection of the asymptotes is given by

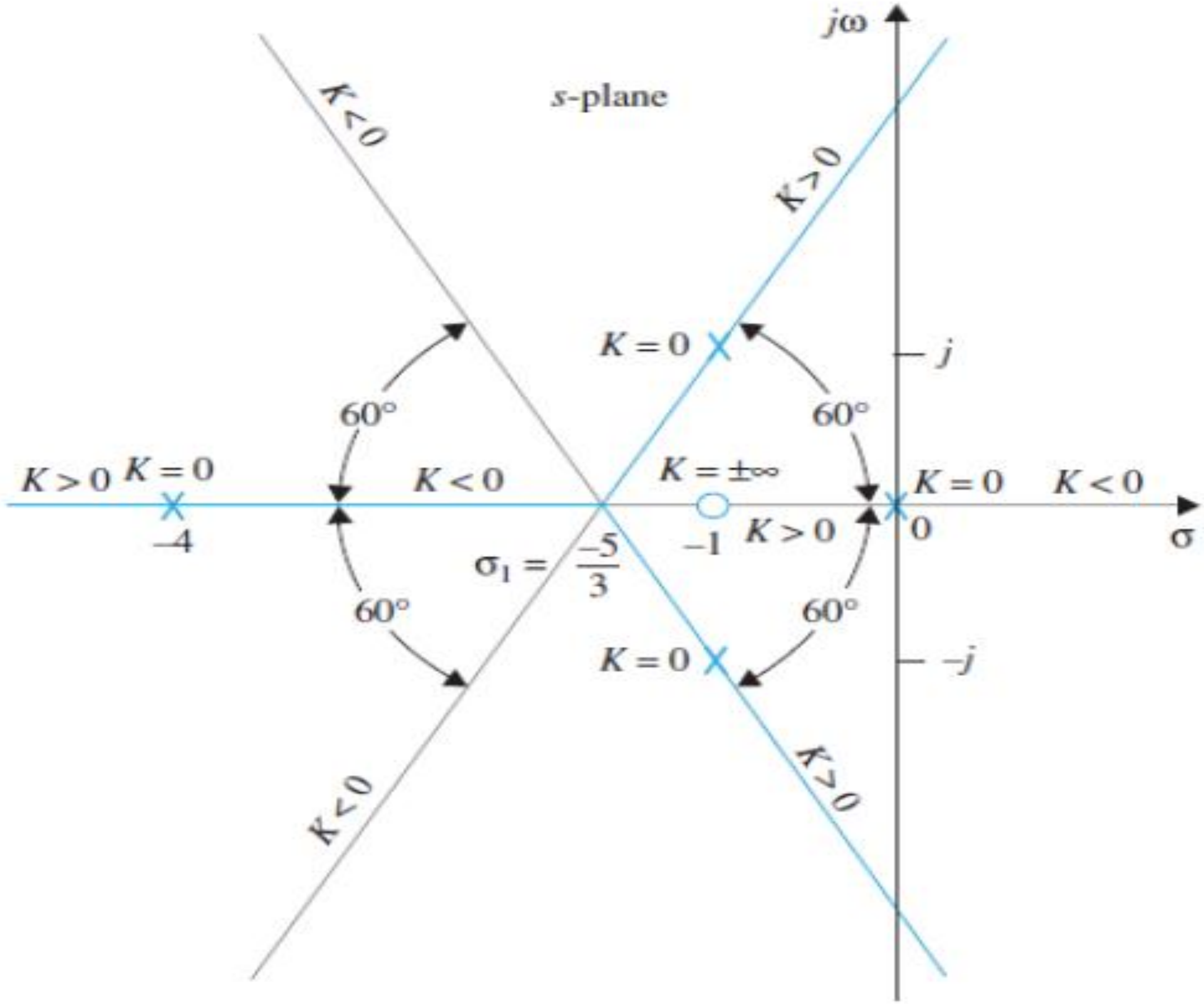
$$\sigma_1 = \frac{\sum \text{real parts of poles } G(s)H(s) - \sum \text{real parts of zeros of } G(s)H(s)}{n - m}$$

poles of $G(s)H(s)$: $s = 0, -4, -1 + j$, and $-1 - j$.

zeros of $G(s)H(s)$: $s = -1, \infty, \infty$, and ∞.

$$\sigma_1 = \frac{(-4-1-1)-(-1)}{4-1} = -\frac{5}{3}$$

Asymptotes of the root loci of s(s + 4)(s² + 2s + 2) + K(s + 1) = 0.



6. ROOT LOCI ON THE REAL AXIS

- $K \geq 0$: On a given section of the real axis, root loci are found in the section only if the total number of poles and zeros of $G(s)H(s)$ to the right of the section is odd.

  ($K \leq 0$: On a given section of the real axis, root loci are found in the section only if the total number of real poles and zeros of $G(s)H(s)$ to the right of the section is even.)

- Complex poles and zeros of $G(s)H(s)$ do not affect the type of root loci found on the real axis.

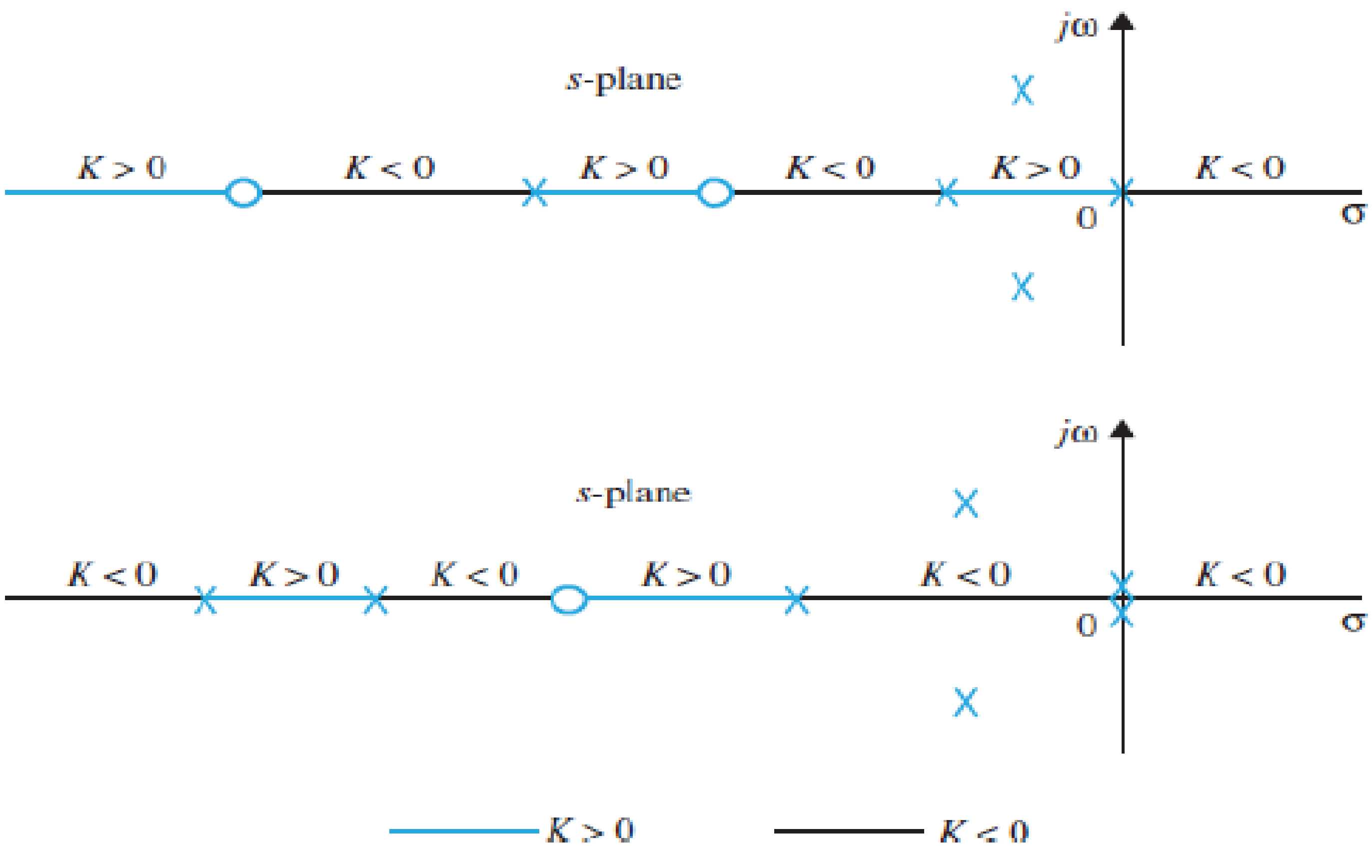
These properties are arrived at based on the following observations:

1. At any point s on the real axis, the angles of the vectors drawn from the complex-conjugate poles and zeros of $G(s)H(s)$ add up to zero. Thus, only the real zeros and poles of $G(s)H(s)$ contribute to the angular relations
2. Only the real poles and zeros of $G(s)H(s)$ that lie to the right of the point s contribute to angular relations because real poles and zeros that lie to the left of the point contribute nothing
3. Each real pole of $G(s)H(s)$ to the right of s contributes $-180°$, and each real zero of $G(s)H(s)$ to the right of s contributes $+180°$

Properties of root loci on the real axis.

The root loci on the real axis for two pole-zero configurations of $G(s)H(s)$ are shown in Fig. Notice that the entire real axis is occupied by the root loci for all values of K.

Properties of root loci on the real axis.



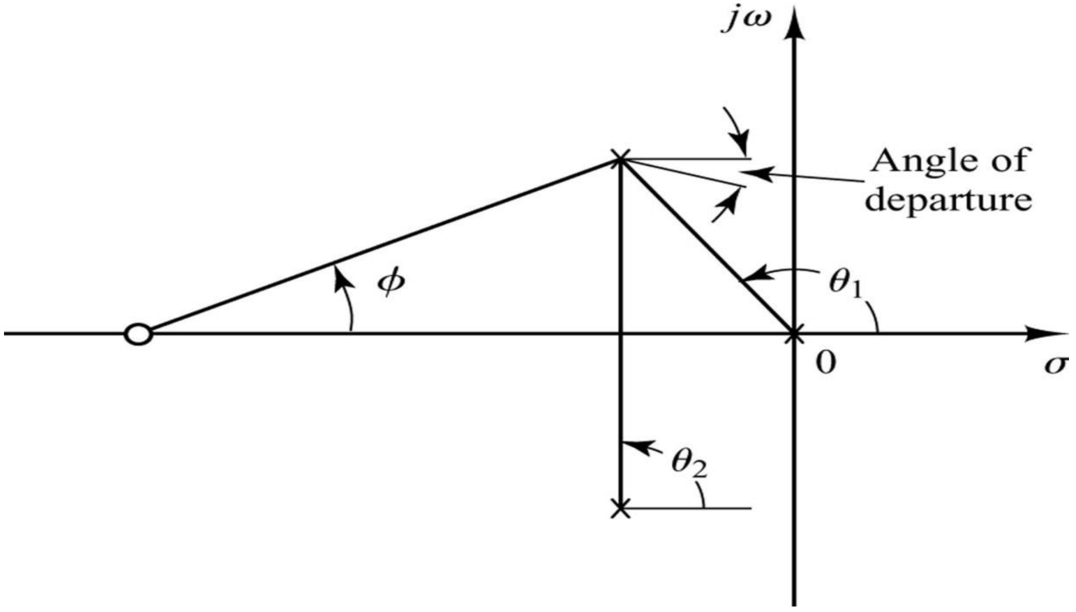
7. ANGLES OF DEPARTURE AND ANGLES OF ARRIVAL OF THE ROOT LOCI

- Root locus starts from a pole of open loop TF. The angle at which it leaves a pole is called

  Angle of Departure (θ_D)
- Root locus terminates on a zero of open loop TF. The angle at which it arrives at a zero is called Angle of Arrival (θ_A)

$$\theta_D = 180 + \{\text{net contribution from the zeros and poles of } G(s)H(s) \text{ evaluated at the pole in question, excluding the contribution from that pole}\}$$

[Angle of departure =180°+{*Φ*- (*Θ*$_1$+*Θ*$_2$)] in degrees



Angle of Arrival

$$\theta_A = 180 - \{\text{net contribution from the zeros and poles of } G(s)H(s) \text{ evaluated at the zero in question, excluding the contribution from that zero}\}$$

8. INTERSECTION OF THE ROOT LOCI WITH THE IMAGINARY AXIS

- Routh-Hurwitz criterion may be used to find the intersection of the root loci on the imaginary axis.
- Routh array is formed from characteristic equation

- If $s=j\omega$ is a closed loop pole on the imaginary axis, then $KG(j\omega)H(j\omega)=-1$

Which we can solve for the two unknowns K and ω(i.e.,the critical gain beyond which the system goes unstable, and the oscillation frequency at the critical gain.)

- An alternative but just as effective method as RH to find intersection of the root loci on the imaginary axis.
- In this example,

$$
\begin{aligned}
KG(s)H(s) &= \frac{K(s+3)}{s(s+1)(s+2)(s+4)} \\
&= \frac{Ks+3K}{s^4+7s^3+14s^2+8s} \Rightarrow \\
KG(j\omega)H(j\omega) &= \frac{jK\omega+3K}{\omega^4-j7\omega^3-14\omega^2+j8\omega}.
\end{aligned}
$$

Setting $KG(j\omega)H(j\omega)=-1$,

$$-\omega^4+j7\omega^3+14\omega^2-j(8+K)\omega-3K=0.$$

Separating real and imaginary parts,

$$\begin{cases} -\omega^4 + 14\omega^2 - 3K &= 0, \\ 7\omega^3 - (8+K)\omega &= 0. \end{cases}$$

In the second equation, we can discard the trivial solution $\omega = 0$. It then yields

$$\omega^2 = \frac{8+K}{7}.$$

Substituting into the first equation,

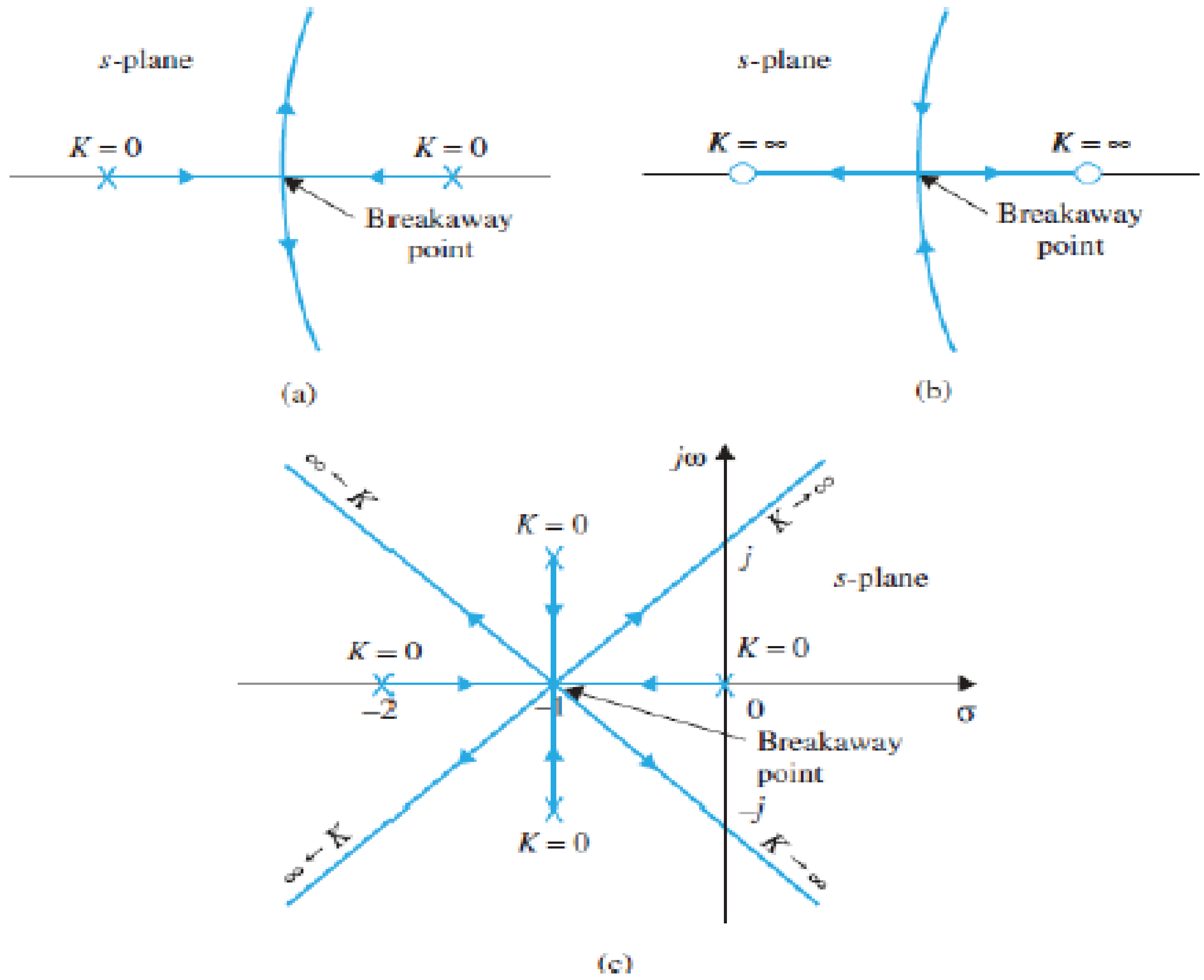
$$\begin{aligned} -\left(\frac{8+K}{7}\right)^2 + 14\left(\frac{8+K}{7}\right) - 3K &= 0 \Rightarrow \\ K^2 + 65K - 720 &= 0. \end{aligned}$$

Of the two solutions $K = -74.65$, $K = 9.65$ we can discard the negative one (negative feedback $\Rightarrow K > 0$). Thus, $K = 9.65$ and $\omega = \sqrt{(8+9.65)/7} = 1.59$.

9. BREAKAWAY POINTS (SADDLE POINTS)

- Breakaway points on the root loci of an equation correspond to multiple-order roots of the equation(Closed loop poles)
- A root-locus plot may have more than one breakaway point.
- Break away points may be real, imaginary or complex.

Examples of breakaway points on the real axis in the s-plane.



- The breakaway points on the root loci of 1 + KG_1 (s)H_1 (s) = 0 must satisfy

$$\frac{dG_1(s)H_1(s)}{ds}=0$$

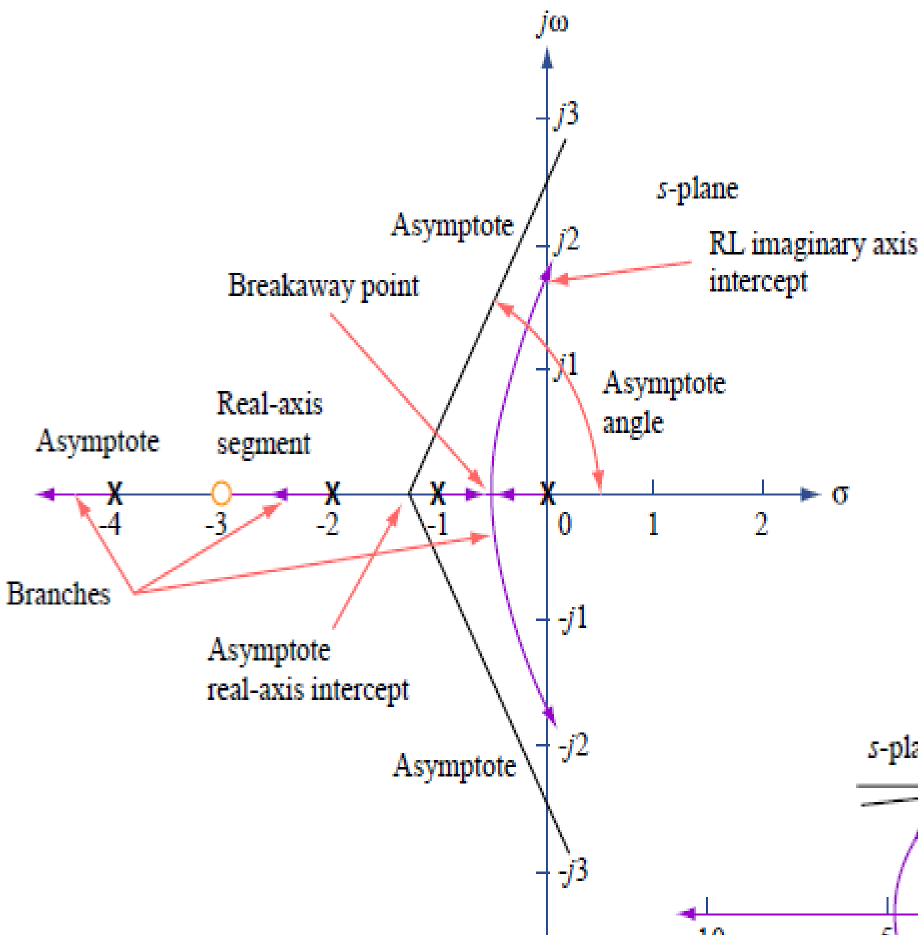
This is necessary but not sufficient. The roots of the above eqn must also lie on Root locus

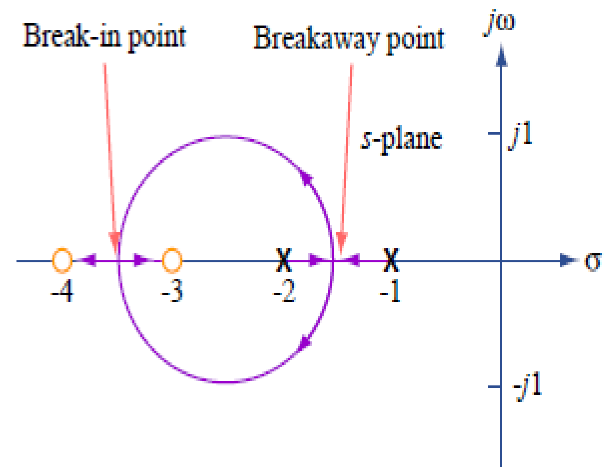
$$K=-\frac{1}{G_1(s)H_1(s)} \qquad \frac{dK}{ds}=\frac{dG_1(s)H_1(s)/ds}{[G_1(s)H_1(s)]^2}$$

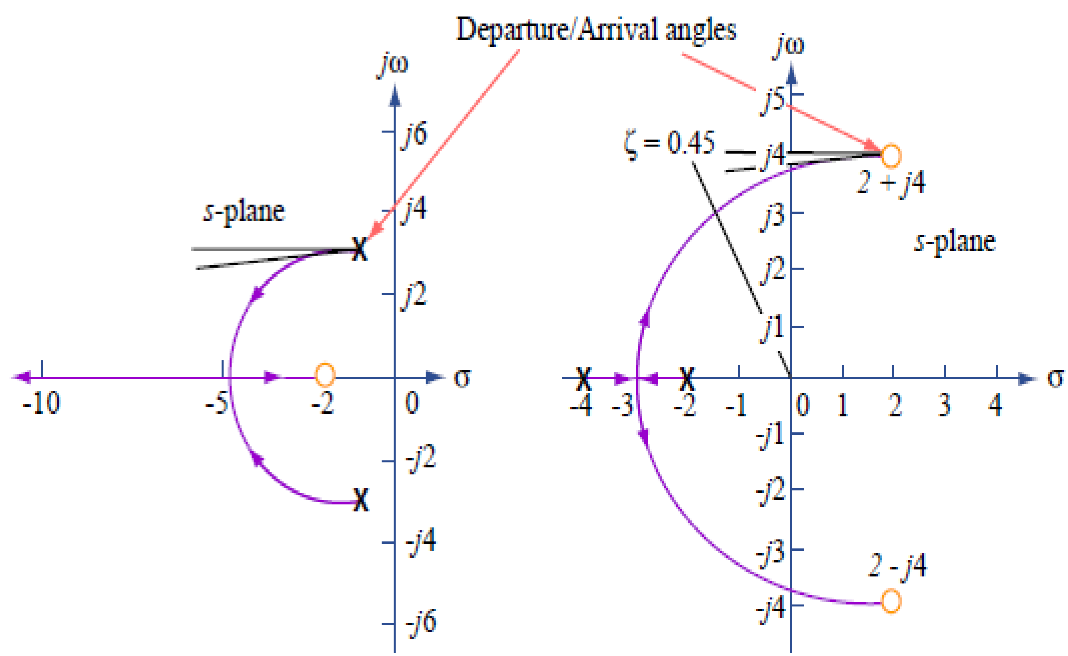
- the breakaway point condition can also be written as

$$\frac{dK}{ds}=0$$

Root locus terminology







Illustrative example

Given

$$s\,(s+2)+K\,(s+4)=0$$

Obtain Break Away points.

Soln:

$$G_1(s)\,H_1(s)=\frac{s+4}{s\,(s+2)}$$

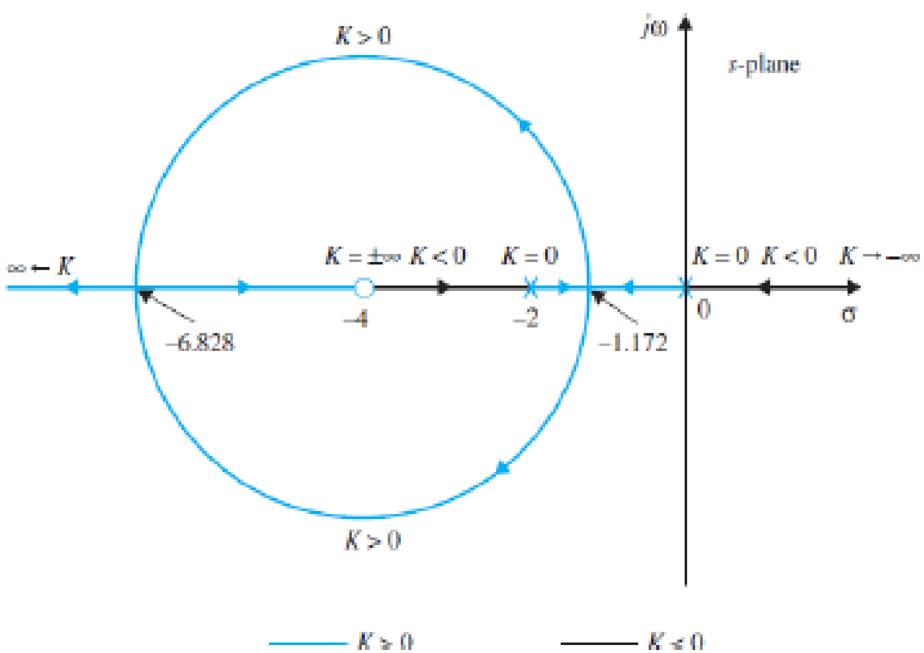
Break away points must satisfy

$$\frac{dG_1(s)\,H_1(s)}{ds}=\frac{s\,(s+2)-2\,(s+1)\,(s+4)}{s^2(s+2)^2}=0$$

Or $$s^2 + 8s + 8 = 0$$

Solving , we find the two breakaway points of the root loci at $s = -1.172$ and -6.828.

Root loci of $s(s + 2) + K(s + 4) = 0$ is as below



10. CALCULATION OF *K* ON THE ROOT LOCI

- There are two Ways.

1. Magnitude Criteria : At any points on the root locus we can apply magnitude criteria as,

$$|G(s)H(s)| = 1$$

Using this formula we can calculate the value of K at any desired point.

2. Using Root Locus Plot : The value of K at any s on the root locus is given by

$$K = \frac{product\ of\ all\ of\ the\ vector\ lengths\ drawn\ from\ the\ poles\ of\ G(s)H(s)\ to\ s}{product\ of\ all\ of\ the\ vector\ lengths\ drawn\ from\ the\ zeros\ of\ G(s)H(s)\ to\ s}$$

Properties of the Root Loci of $F(s) = 1 + KG(s)H(s) = 0$

1. $K = 0$ points	The $K = 0$ points are at the poles of $G(s)H(s)$, including those at $s = \infty$.
2. $K = \pm\infty$ points	The $K = \pm\infty$ points are at the zeros of $G(s)H(s)$, including those at $s = \infty$.
3. Number of separate root loci	The total number of root loci is equal to the order of the characteristic equation $F(s)$.
4. Symmetry of root loci	The root loci are symmetrical about the axes of symmetry of the pole-zero configuration of $G(s)H(s)$.
5. Asymptotes of root loci as $\lvert s\rvert \to \infty$	For large values of s, the root loci for $K > 0$ are asymptotic to asymptotes with angles given by $$\theta_i = \frac{2i+1}{\lvert n-m\rvert} \times 180^\circ$$ For $K < 0$, the root loci are asymptotic to $$\theta_i = \frac{2i}{\lvert n-m\rvert} \times 180^\circ$$ where $i = 0, 1, 2, \ldots, \lvert n - m\rvert - 1$ n = number of finite poles of $G(s)H(s)$, and m = number of finite zeros of $G(s)H(s)$.

6. Intersection of the asymptotes	(a) The intersection of the asymptotes lies only on the real axis in the *s*-plane. (b) The point of intersection of the asymptotes is given by $$\sigma_1 = \frac{\Sigma \text{real parts of poles } G(s)\,H(s) - \Sigma \text{real parts of zeros of } G(s)\,H(s)}{n-m}$$
7. Root loci on the real axis	Root loci for $K > 0$ are found in a section of the real axis only if the total number of real poles and zeros of $G(s)H(s)$ to the **right** of the section is **odd**. If the total number of real poles and zeros to the right of a given section is **even**, root loci for $K < 0$ are found.
8. Angles of departure	The angle of departure or arrival of the root loci from a pole or a zero of $G(s)H(s)$ can be determined by assuming a point s_1 that is very close to the pole, or zero, and applying the equation $$\begin{aligned} \angle G(s_1)H(s_1) &= \sum_{k=1}^{m} \angle(s_1+z_k) - \sum_{j=1}^{n} \angle(s_1+p_j) \\ &= 2(i+1)180^\circ \quad K>0 \\ &= 2i \times 180^\circ \quad K>0 \end{aligned}$$ where $i = 0, \pm 1, \pm 2, \ldots$.
9. Intersection of the root loci	The crossing points of the root loci on the imaginary axis and with the imaginary axis the corresponding values of K may be found by use of the Routh-Hurwitz criterion.

10. Breakaway points	The breakaway points on the root loci are determined by finding the roots of $dK/ds = 0$, or $dG(s)H(s)/ds = 0$. These are necessary conditions only.
11. Calculation of the values of K	The absolute value of K at any point s_1 on the root loci is on the root loci determined from the equation $$\lvert K\rvert = \frac{1}{\lvert G_1(s_1)H_1(s_1)\rvert}$$