UNIT-4

PART-D

TOPIC-1 :- Identities on analytic function

D-1 If $f(z) = u+iv$ is a regular (analytic) function then ST $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

or $\nabla^2[|f(z)|^2] = 4|f'(z)|^2$

Solution:

$f'(z) = u_x + iv_x \; ; \; |f'(z)| = \sqrt{u_x^2+v_x^2}$

$\therefore |f'(z)|^2 = u_x^2+v_x^2$

RHS $= 4|f'(z)|^2 = 4(u_x^2+v_x^2)$ — ①

To find LHS:- Let $\phi = |f(z)|^2$

Given $f(z) = u+iv \; ; \; |f(z)| = \sqrt{u^2+v^2}$

$\therefore \phi = |f(z)|^2 = u^2+v^2$

$\phi_x = 2uu_x + 2vv_x$

$\phi_{xx} = 2[uu_{xx}+u_x^2+vv_{xx}+v_x^2]$ — ②

||ly $\phi_{yy} = 2[uu_{yy}+u_y^2+vv_{yy}+v_y^2]$ — ③

$\because \phi$ is a symmetric function

From ② and ③

$\phi_{xx}+\phi_{yy} = 2\left[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2+v_x^2+u_y^2+v_y^2\right]$

$= 2[u(0)+v(0)+u_x^2+v_x^2+u_y^2+v_y^2]$

$\because u_{xx}+u_{yy}=0$ and $v_{xx}+v_{yy}=0$

$\therefore \phi_{xx} + \phi_{yy} = 2\left[u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2 \right]$

$\because u_x = v_y \; \& \; v_x = -u_y$

$= 2\left[2(u_x^2 + v_x^2) \right] = 4(u_x^2 + v_x^2)$

$= RHS$ using (1)

$\boxed{D\ 2}$ If $f(z) = u(x,y) + i\,v(x,y)$ is analytic function then prove that $\left[\frac{\partial}{\partial x} |f(z)|\right]^2 + \left[\frac{\partial}{\partial y} |f(z)|\right]^2$

$= |f'(z)|^2$

Proof: - Given $f(z) = u + iv$ is analytic $\therefore$ CR equations are satisfied by u and v

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$f(z) = u + iv$, $|f(z)| = \sqrt{u^2 + v^2}$

Let $\phi = |f(z)| = \sqrt{u^2 + v^2}$

$LHS = \phi_x^2 + \phi_y^2$

$\phi_x = \frac{1}{2\sqrt{u^2+v^2}} (2u u_x + 2v v_x)$

$|||^{ly}$ $\phi_y = \frac{u u_y + v v_y}{\sqrt{u^2+v^2}}$

$$\phi_x^2 = \frac{1}{u^2+v^2}\left(u^2 u_x^2 + v^2 v_x^2 + 2uv\,u_x v_x\right)$$

and $$\phi_y^2 = \frac{1}{u^2+v^2}\left[u^2 u_y^2 + v^2 v_y^2 + 2uv\,u_y v_y\right]$$

$$LHS = \phi_x^2 + \phi_y^2 = \frac{1}{u^2+v^2}\left[u^2\left(u_x^2+u_y^2\right) + v^2\left(v_x^2+v_y^2\right) + 2uv\left(u_x v_x + u_y v_y\right)\right]$$

$$LHS = \phi_x^2 + \phi_y^2 = \frac{1}{u^2+v^2}\left[u^2\left\{u_x^2 + (-v_x)^2\right\} + v^2\left\{v_x^2 + (u_x)^2\right\} + 2uv(0)\right]$$

$\because v_x = -u_y$ & $u_x = v_y$

$$\therefore LHS = \frac{1}{u^2+v^2}\left[\left(u_x^2+v_x^2\right)\left(u^2+v^2\right)\right]$$

$$= u_x^2 + v_x^2$$

$$= |f'(z)|^2. \qquad \because f'(z) = u_x + i v_x$$

$= RHS.$

$\boxed{D\text{-}3}$ If $f(z)$ is a holomorphic (or analytic function) then PT $\nabla^2\left[Re\{f(z)\}\right]^2 = 2\,|f'(z)|^2$

Proof:- Given $f(z) = u(x,y) + iv(x,y)$ is analytic function, CR equations are satisfied

$$u_x = v_y \text{ and } v_x = -u_y$$

$f(z) = u + iv$, $Re\{f(z)\} = u$

and $[Re\{f(z)\}]^2 = u^2$

Let $\phi = u^2$, then LHS $= \nabla^2\phi$ or $\phi_{xx} + \phi_{yy}$

$\phi_x = 2u\,u_x \Rightarrow \phi_{xx} = 2[u\,u_{xx} + u_x^2]$

$\phi_y = 2u\,u_y \Rightarrow \phi_{yy} = 2[u\,u_{yy} + u_y^2]$

LHS $= \nabla^2\phi = 2[u(u_{xx} + u_{yy}) + u_x^2 + u_y^2]$

$= 2[u(0) + u_x^2 + (-v_x)^2]$

$\because u$ is harmonic and $v_x = -u_y$

LHS $= 2(u_x^2 + v_x^2) =$ RHS $= 2|f'(z)|^2$

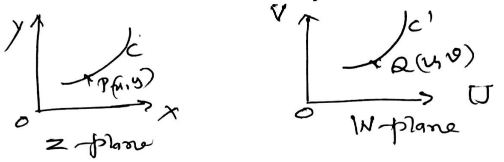
$\because f'(z) = u_x + iv_x$

$\boxed{\text{TOPIC} - 2}$ Conformal Mapping or (transformation)

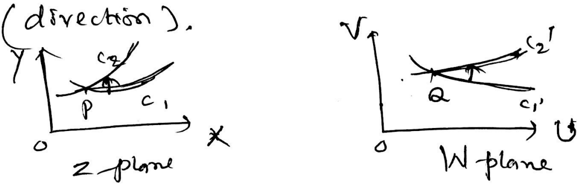
Transformation (or mapping)

A single valued function $w = f(z)$ represents geometrically a transformation from z-plane onto w-plane, because the point $P(x, y)$ corresponding to the complex number $z = x + iy$ is mapped onto the point $Q(u, v)$ in the

W - plane, such that for each value of z, there exists a unique value of w.



<u>Conformal mapping</u> :- The mapping or transformation $w = f(z)$ is said to be conformal mapping if the angle between two curves intersecting at a point is preserved both in magnitude and in sense ~~sese~~ (direction).



<u>Result</u> :- The mapping $w = f(z)$ is conformal mapping if and only if $f(z)$ is analytic function and $f'(z) \neq 0$.

D1 To discuss the transformation $w = z^2$

Proof: Given $w = z^2$

i.e. $u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy$

$\Rightarrow u = x^2 - y^2$ — ① and $v = 2xy$ — ②

Case (i) Let $x = a$ (straight line $\parallel$ to y-axis)

From ① and ② $u = a^2 - y^2$ and $v = 2ay$

$\therefore v^2 = 4a^2y^2 = 4a^2(a^2 - u) = -4a^2(u - a^2)$

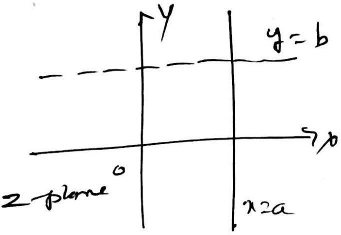
This represents a parabola in the W-plane having vertex at $(a^2, 0)$ and the negative U-axis as its axis as indicated in figure.

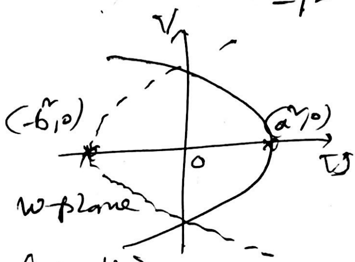
Case (ii) Let $y = b$ (a straight line parallel to x-axis)

From ① and ②, $u = x^2 - b^2$ and $v = 2bx$

$\therefore v^2 = 4b^2x^2 = 4b^2(x^2 + b^2)$

which represents a parabola in the W-plane, having vertex at $(-b^2, 0)$ and positive U-axis as the axis, as shown in figure





Note: $w = z^2$ is a conformal mapping

$\boxed{Q2}$ To discuss the mapping $w = z + \frac{a^2}{z}$ $(z \neq 0)$

Proof:- Consider the transformation $w = z + \frac{a^2}{z}$ ⟶ (1)

[This is of the form $w = f(z)$

$f'(z) = 1 - \frac{a^2}{z^2}$, $f'(z)$ exists and

$f'(z) \neq 0$ when $z \neq 0$ and $z^2 \neq a^2$.

The given transformation is conformal at all points except at $z = 0$ and $z = \pm a$.]

Let $z = re^{i\theta}$

(1) ⟹ $u + iv = re^{i\theta} + \frac{a^2}{re^{i\theta}} = re^{i\theta} + \frac{a^2}{r} e^{-i\theta}$

$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)$

$= \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta$

$\Rightarrow u = \left(r + \frac{a^2}{r}\right)\cos\theta$ ⟶ (2) and $v = \left(r - \frac{a^2}{r}\right)\sin\theta$ ⟶ 3

Case(i) Eliminating θ between (2) and (3)

using $\cos^2\theta + \sin^2\theta = 1$, we get

$$\frac{u^2}{\left(r+\frac{a^2}{r}\right)^2} + \frac{v^2}{\left(r-\frac{a^2}{r}\right)^2} = 1 \quad \text{--- (4)}$$

consider $r = b$ (a constant), which represents a circle centred at the origin in the z-plane. Then (4) represents an ellipse having centre at the origin of the w-plane with u- and v- axes as its axes as shown in figure.

Case(ii) Eliminating r from the relations (2) and (3)

using $(A+B)^2 - (A-B)^2 = 4AB$, to get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4a^2 \quad \& \quad \frac{u^2}{(2a\cos\theta)^2} - \frac{v^2}{(2a\sin\theta)^2} = 1 \quad \text{--- (5)}$$

For $\theta = c$ (a constant) a radial line in z-plane, (5) represents a hyperbola having centre at the origin of w-plane and u- and v-axes as its axes as shown in figure