21-05-2021 CS IV SEM 'D'

UNIT 2:

Controllers and time response analysis
Step response of first order and second order systems, Response specifications, steady state error and error constants. Effect of PI, PD and PID controllers on the time response of the system

-x-

Time domain response: In the analysis of control systems, when variables such as reference input and controlled output are expressed as functions of time i.e. as $r(t)$ and $c(t)$ → it is called time domain analysis

Time domain response $C(t)$ has two parts

1) Transient response $C_t(t)$
2) steady state response $C_{ss}(t)$

$$C(t) = C_t(t) + C_{ss}(t)$$

A physical control system with mechanical components like Mass, inertia & friction and electrical components like Inductors and capacitors cannot follow instantaneously, any sudden change in the input.

i.e. whenever the i/p to the system changes suddenly the system takes some time to give steady output.

Scanned by TapScanner

Transient response is the immediate response when the i/p is changed suddenly. After a transient period, system reaches a steady state

Order of a system: The highest power of 'S' present in the characteristic equation, i.e. in the denominator polynomial of a closed loop T.F of a system is called 'order' of a system.

eg: $T.F = \frac{1}{s(s+a)}$ is a 2nd order system

Type of a System: Refers to the number of poles at origin of open loop T.F, $G(s)H(s)$

eg $TF = \frac{1}{s(s+a)}$ has only one pole at origin

$\therefore$ it is a type '1' system.

<u>Analysis of First order System</u>:

Total response is obtained as $C(t) = C_t(t) + C_{ss}(t)$

First we consider a first order system with step input and obtain its transient response.

Mathematically step i/p is expressed as

$$r(t) = A \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$

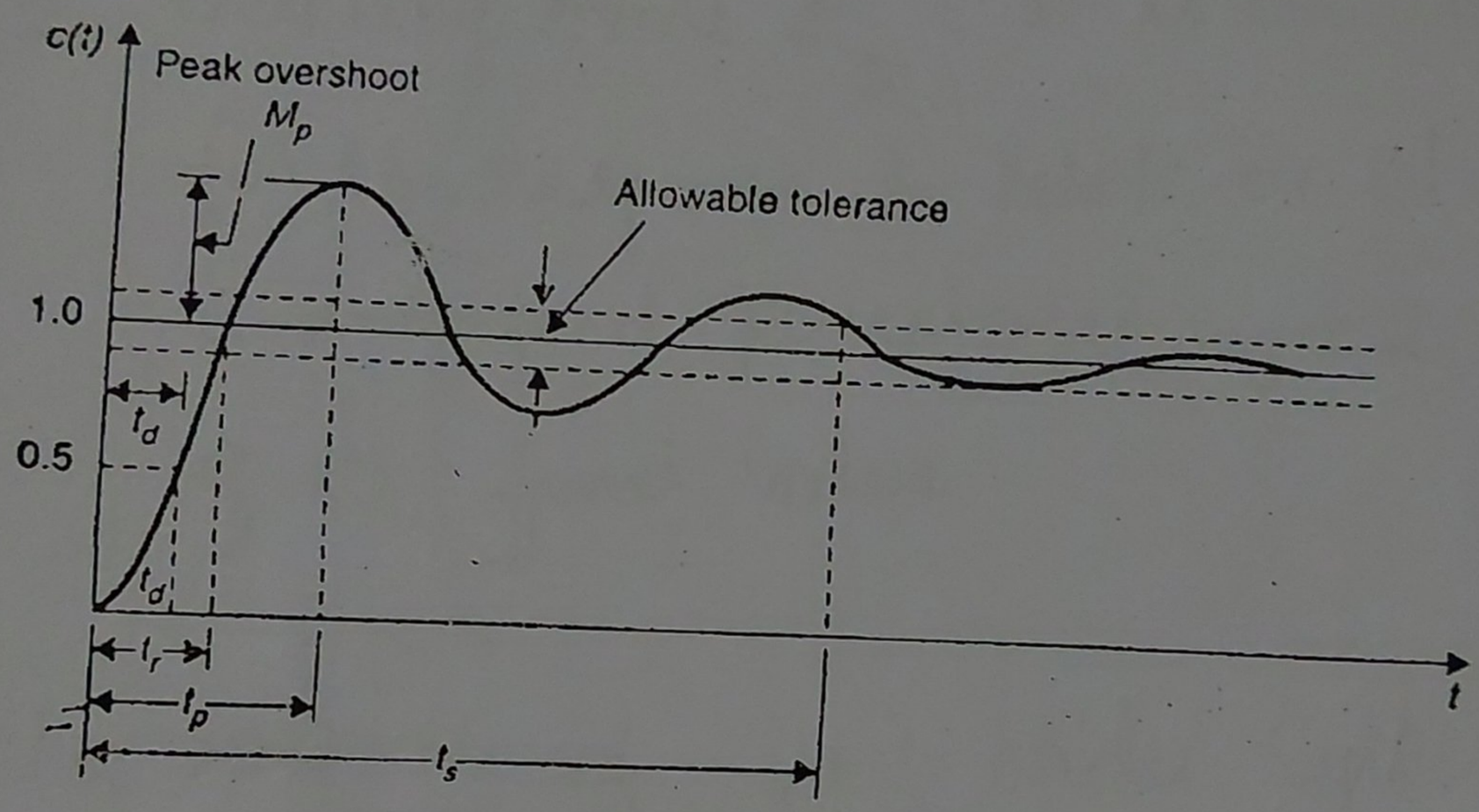
Transient response of a system does not depend on the type of input $\therefore$ we analyze transient response using step i/p.

Scanned by TapScanner

24.05.2021 CS IV Sem 'D'

Step response of first order system:

Response Specifications for step input




1. Delay time td : It is the time required for the response to reach 50% of its final value in first attempt.

2. Rise time tr : It is the time required for the o/p to rise from 10% to 90% of its final value

Scanned by TapScanner

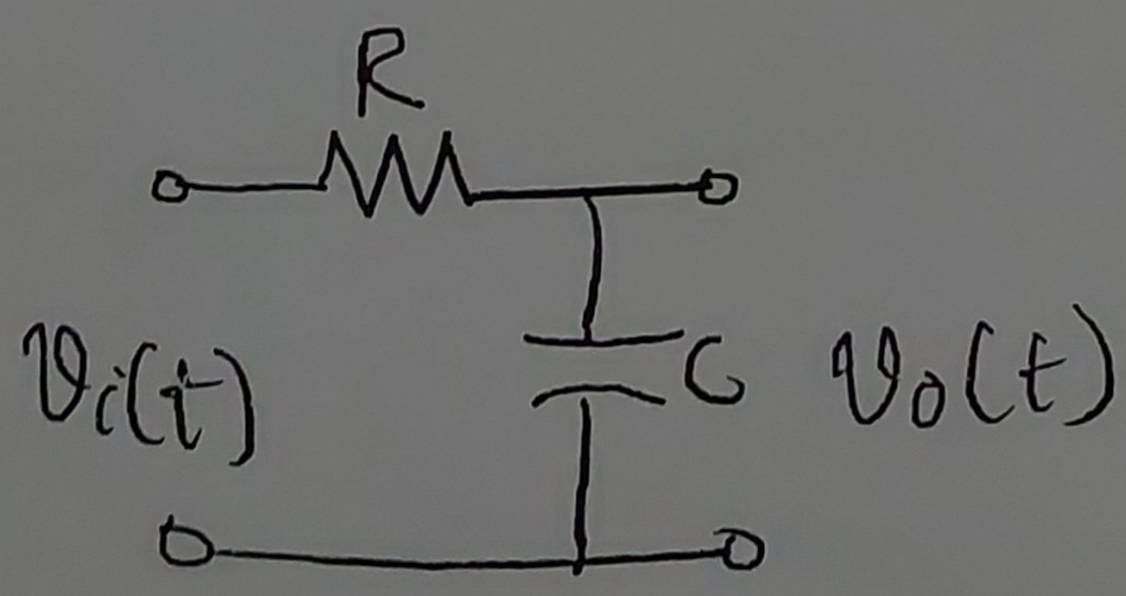
3 Peak time $t_p$: It is the time required for the response to reach its peak value.

4. peak overshoot $M_p$: It is the largest error b/n reference input and the output during the transient response

5. Settling time $t_s$: It is the time required for the response to settle or stay within a specified percentage (say ±2% or ±5%) of its final value

## Step response of first order System:

Consider the circuit shown




Let $v_i(t)$ be step i/p

$v_i(t) = 1; t \geq 0$

$v_i(t) = 0; t < 0$

$$v_o(t) = \frac{1}{C}\int i \cdot dt \Rightarrow V_o(s) = \frac{I(s)}{Cs}$$

$$v_i(t) = iR + v_o(t) \Rightarrow V_i(s) = I(s)R + \frac{I(s)}{Cs}$$

Scanned by TapScanner

$$T.F = \frac{V_o(S)}{V_i(S)} = \frac{\frac{I(S)}{CS}}{I(S)\left(R+\frac{1}{CS}\right)} = \frac{1}{RCS+1}$$

$\downarrow$

Pole at $S = -\frac{1}{RC}$

$v_i(t) \longrightarrow$ step i/p

$\therefore V_i(S) = \frac{1}{S}$

$$V_o(S) = \frac{1}{RCS+1} \times \frac{1}{S} = \frac{1}{S(RCS+1)}$$

$$V_o(S) = \frac{A}{S} + \frac{B}{1+RCS} \Rightarrow A = 1$$

$B = -RC$

$A(1+RCS) + BS = 0$ i.e, $A + S(ARCS + B) = 0$.

$A = 1$

$ARC + B = 0$

$B = -RC$

$$\frac{1}{S} - \frac{RC}{1+RCS} = \frac{1}{S} - \frac{1}{\frac{1}{RC}+S}$$

Taking Inverse Laplace Transform,

$$v_o(t) = 1 - e^{-t/RC}$$

$\rightarrow$ transient term is totally dependant on RC & Rate of exponential decay is controlled by $-1/RC$, which is pole of the system

Scanned by TapScanner

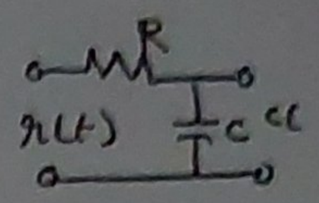
26.05.2021

CS

IV Sem 'D'

Step response of First order system (contd)

Step response of First order system considered, i.e.,




$$c(t) = 1 - e^{-t/RC}$$

let $RC = T$ (time constant)

Transient term is totally dependant on $T$ and rate of exponential decay is controlled by $-\frac{1}{T}$, which is the pole of the system.

Value of R&C i.e., T will not affect the steady state part

| $t$ | $c(t)$ |
|---|---|
| 0 | 0 |
| T | 0.632 |
| 2T | 0.86 |
| 3T | 0.95 |
| 4T | 0.982 |
| ⋮ | |
| $\infty$ | 1 |

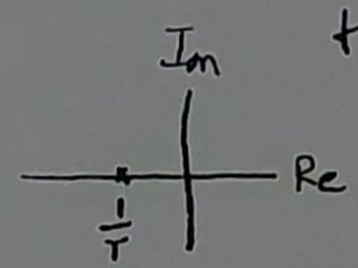
$S = -\frac{1}{T}$ represents the p

when T is small, $\frac{1}{T}$ i

large & the pole move

away from Im axis

towards Left Hand si




& the system is

more stable since

transients die out

quickly

(i) First order system with a zero:

$$T.F = \frac{C(S)}{R(S)} = \frac{K(1+\alpha S)}{1+TS} \longrightarrow$$

a pole at $S = -\frac{1}{T}$

a zero at $S = -\frac{1}{\alpha}$

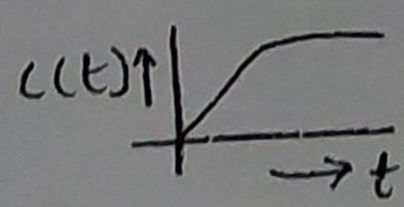
For step input, $R(S) = \frac{1}{S}$

$$C(S) = \frac{K(1+\alpha S)}{S(1+TS)} = \frac{K}{S} + \frac{K(\alpha - T)}{1+TS}$$

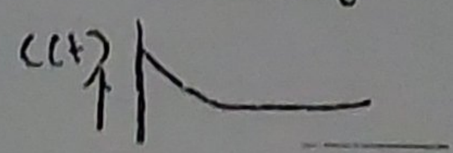
To get the response in time domain, take Laplace inverse

$$c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}$$

when $T > \alpha$




when $T < \alpha$



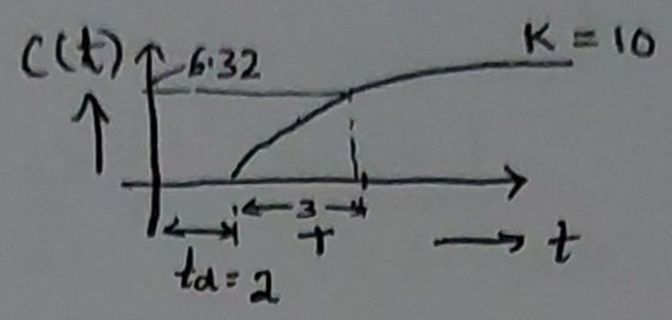

Scanned by TapScanner

(ii) First order system with delays.

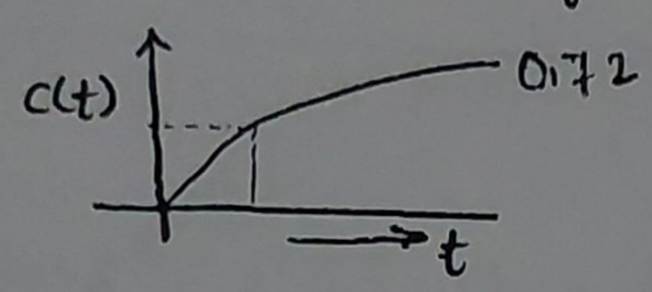
$$T \cdot F = \frac{C(S)}{R(S)} = \frac{K}{1+TS} e^{-t_d s}$$ where $t_d$ = delay

eg:

if $T \cdot F = \frac{10e^{-2S}}{3S+1}$




(iii) Determination of T·F of $1^{st}$ order system when step response is given




T·F is of the form

$$TF = \frac{K}{1+TS}$$

$K = 0.72$

T = time required to reach

$K \times 0.63 = 0.72 \times 0.63 = 0.45$

$\Rightarrow T = 0.135$

$$T = \frac{0.72}{1 + 0.13S}$$

Example: A system has following T·F

$$\frac{C(S)}{R(S)} = \frac{20}{S+10}$$, determine the step response assuming zero initial conditions

For step i/p, $R(S) = \frac{1}{S}$.

$$\therefore C(S) = \frac{1}{S} \cdot \frac{20}{S+10} = \frac{A}{S} + \frac{B}{S+10}$$

$A(S+10) + BS =$

$AS + 10A + BS =$

$A = \frac{2}{1}$

$= 2$

$B = -2$

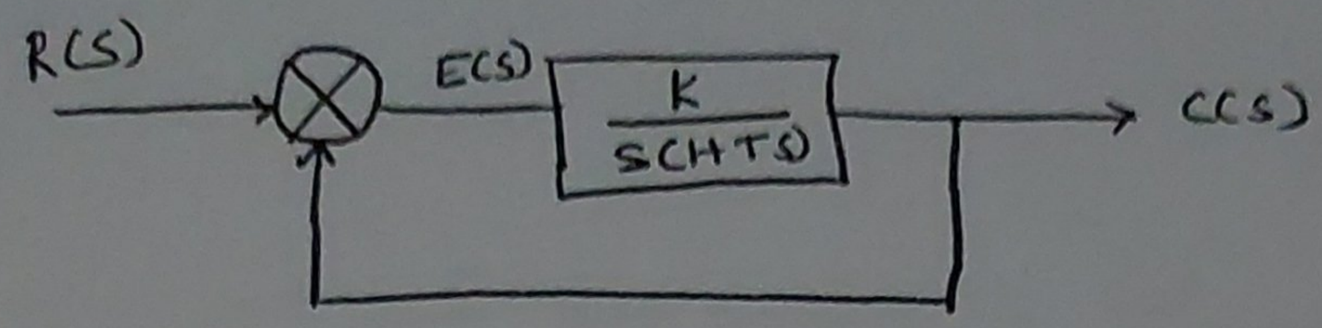
$$C(S) = \frac{2}{S} - \frac{2}{S+10}$$

To obtain time response, take Laplace inverse.

$$c(t) = 2 - 2e^{-10t}$$

Scanned by TapScanner

## Second order system:




$G(S) = \frac{K}{S(1+TS)}$

$H(S) = 1$

$$\text{closed loop T.F} = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{K/S(1+TS)}{1 + K/S(1+TS)}$$

$$\frac{K}{S(1+TS)+K} = \frac{K}{TS^2+S+K} = \frac{K/T}{S^2+\frac{S}{T}+\frac{K}{T}}$$

The standard C.L.T.F of a second order system is of the form

$$C.L.T.F = \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

$\therefore K/T = \omega_n^2$ and $\frac{1}{T} = 2\xi\omega_n$

$\omega_n$ = undamped natural frequency

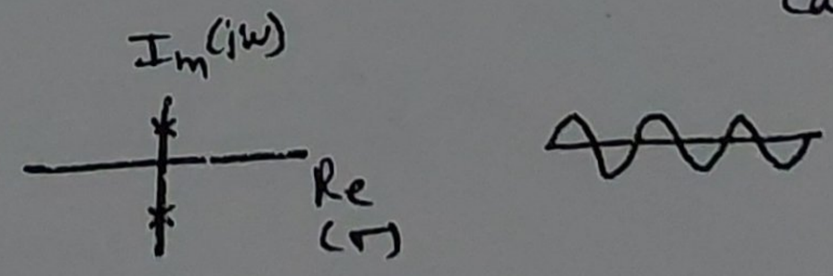
$\xi$ = damping factor $= \frac{\text{damping coefficient}}{\text{Critical damping coefficient}}$

$$= \frac{1}{2T\omega_n}$$

2nd order system can display damped or pure oscillations for its transient response

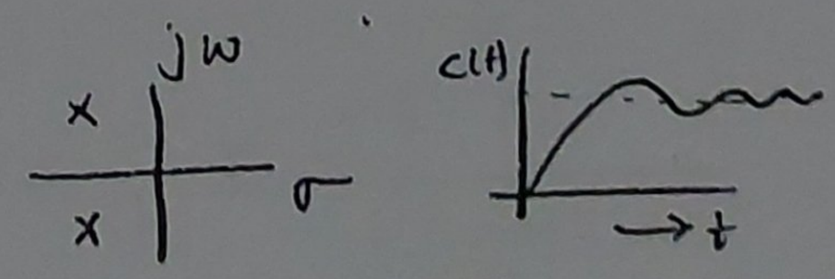
Depending on the value of $\xi$, the system can be put into 4 categories

1) $\xi = 0 \rightarrow$ undamped



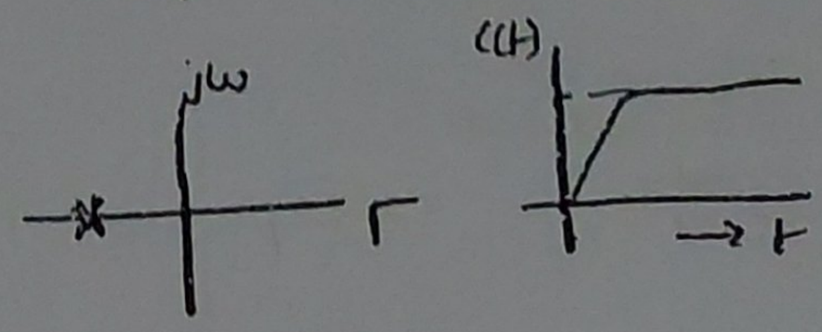

There are two Im poles

2) $0 < \xi < 1 \rightarrow$ under damped




There are two complex poles

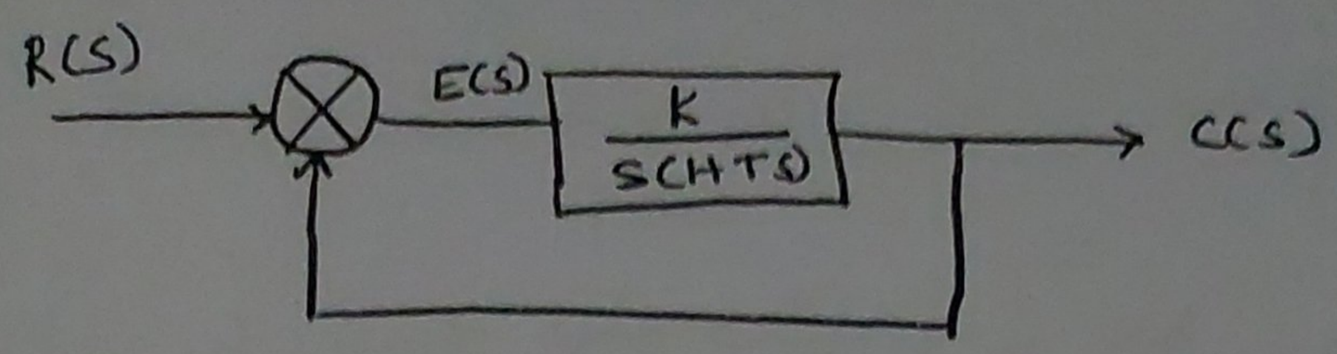
3) $\xi = 1 \rightarrow$ critically damped




There are two real equal poles

Scanned by TapScanner

## Second order system:




$G(S) = \frac{K}{S(1+TS)}$

$H(S) = 1$

$$\text{closed loop T.F} = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{K/S(1+TS)}{1 + K/S(1+TS)}$$

$$\frac{K}{S(1+TS)+K} = \frac{K}{TS^2+S+K} = \frac{K/T}{S^2+\frac{S}{T}+\frac{K}{T}}$$

The standard C.L.T.F of a second order system is of the form

$$C.L.T.F = \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

$\therefore K/T = \omega_n^2$ and $\frac{1}{T} = 2\xi\omega_n$

$\omega_n$ = undamped natural frequency

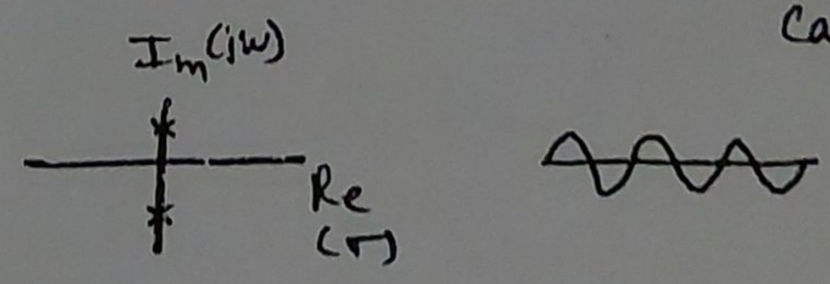
$\xi$ = damping factor $= \frac{\text{damping coefficient}}{\text{Critical damping coefficient}}$

$$= \frac{1}{2T\omega_n}$$

$2^{nd}$ order system can display damped or pure oscillations for its transient response

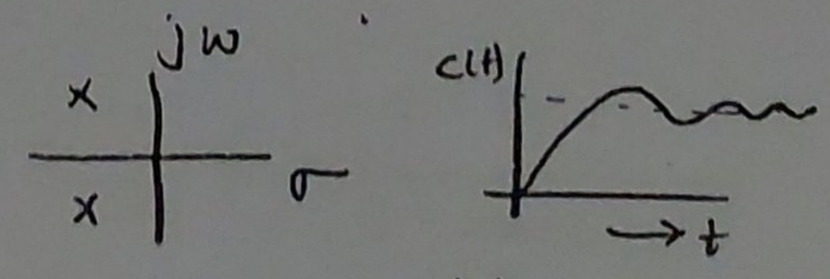
Depending on the value of $\xi$, the system can be put into 4 categories

1) $\xi = 0 \rightarrow$ undamped



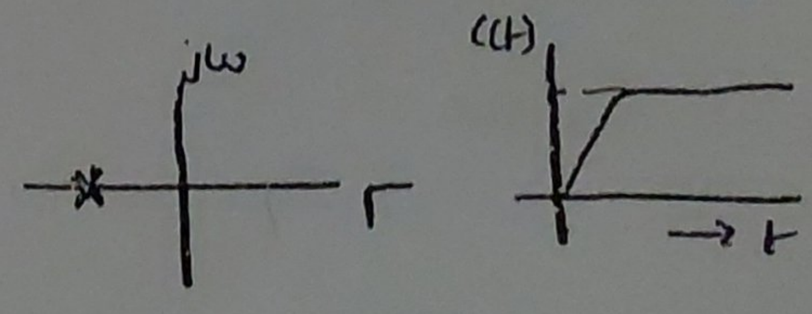

There are two Im poles

2) $0 < \xi < 1 \rightarrow$ under damped




There are two complex poles

3) $\xi = 1 \rightarrow$ critically damped

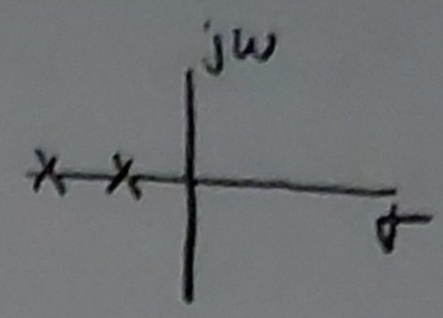


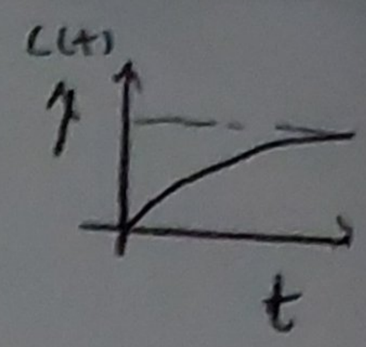

There are two real equal poles

Scanned by TapScanner

4) $\xi > 1 \rightarrow$ Overdamped

There are two distinct real poles




## Derivation of unit step response of 2nd order system (under damped case $0 < \xi < 1$)

The standard C.L.T.F of 2nd order system is of the form

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\frac{A(s^2 + 2\xi\omega_n s + \omega_n^2) + Bs^2 + Cs}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

equating constants

$A\omega_n^2 = \omega_n^2$

$\Rightarrow A = 1$

equating coefficients of $s^2$,

$0 = A + B \Rightarrow B = -1$

equating coefficients of $s$,

$2A\xi\omega_n + C = 0$

$2\xi\omega_n + C = 0 \Rightarrow C = -2\xi\omega_n$

$$C(s) = \frac{1}{s} + \frac{(-1)s + (-2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

let $\xi\omega_n = \alpha$, $C(s) = \frac{1}{s} + \frac{-s - 2\alpha}{s^2 + 2\alpha s + \omega_n^2}$

add and subtract $\alpha^2$

$$C(s) = \frac{1}{s} - \frac{(s + 2\alpha)}{\underbrace{s^2 + 2\alpha s + \alpha^2}_{(s+\alpha)^2} + \omega_n^2 - \alpha^2}$$

Scanned by TapScanner

$$C(s) = \frac{1}{s} - \frac{s+2\alpha}{(s+\alpha)^2+\omega_d^2}$$

where $\omega_n^2 - \alpha^2$

$\omega_n^2 - (\xi\omega_n)^2$

$\omega_n^2(1-\xi^2) = \omega_d^2$

$\omega_d = \omega_n\sqrt{1-\xi^2}$

$$C(s) = \frac{1}{s} - \left\{\frac{s+\alpha}{(s+\alpha)^2+\omega_d^2} + \frac{\alpha}{(s+\alpha)^2+\omega_d^2}\right\}$$

W.K.T

$$L^{-1}\left[\frac{s+a}{(s+a)^2+\omega^2}\right] = e^{-at}\cos\omega t \quad \text{and} \quad L^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t$$

$$C(s) = \frac{1}{s} - \left\{\frac{s+\alpha}{(s+\alpha)^2+\omega_d^2} + \frac{\alpha}{\omega_d}\frac{\omega_d}{(s+\alpha)^2+\omega_d^2}\right\}$$

Taking Laplace inverse

$$c(t) = 1 - e^{-\alpha t}\cos\omega_d t - \frac{\alpha}{\omega_d}e^{-\alpha t}\sin\omega_d t$$

But, $\alpha = \xi\omega_n$, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

let $\xi = \cos\theta \Rightarrow \sqrt{1-\xi^2} = \sin\theta$

then

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

$\sin(\omega_d t+\theta) = \sin\omega_d t\cos\theta + \cos\omega_d t\sin\theta$

Scanned by TapScanner

$$T_d = \frac{1+0.7\xi}{\omega_n}$$

$$0.5 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$0.5 = e^{-\xi\omega_n t_d}$$

$$\xi\omega_n t_d = \ln 0.5$$

$$-0.697$$

$$t_d = \frac{-.7}{\xi\omega_n}$$

$$\frac{1}{\xi\omega_n} \rightarrow T_s$$

Scanned by TapScanner

Time domain response specifications of 2nd order system:
(under damped)

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad ; \quad \cos\theta = \xi, \quad \omega_d = \omega_n\sqrt{1-\xi^2} \quad \text{— (1)}$$

1) Rise time ($t_r$): The time required for the response to rise from 0 to 100%. For under damped (10% – 90% for over damped)

$c(t) = 1$ at $t = t_r$

$$1 = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta)$$

$$\Rightarrow \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

→ not zero when $0 < \xi < 1$.

$\therefore \sin(\omega_d t_r + \theta) = 0$

$\Rightarrow \omega_d t_r + \theta = n\pi$ where $n = 0, 1, 2 \ldots$

$\omega_d t_r = n\pi - \theta$

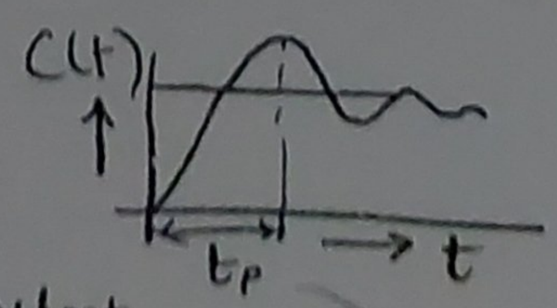
$t_r = \frac{n\pi - \theta}{\omega_d}$

if $n = 1$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_n\sqrt{1-\xi^2}}}$$

2) Peak time ($t_p$):

It is the time taken by the response to reach peak of the overshoot




$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Scanned by TapScanner

Maximum value is obtained when $\frac{d\,c(t)}{dt} = 0$ $\quad \xi = \cos\theta$, $\sqrt{1-\xi^2} = \sin\theta$

$$\frac{d\,c(t)}{dt} = 0 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\cos\omega_d t_p + \theta\right]\omega_d - \frac{\xi\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t_p} \sin(\omega_d t + \theta)$$

$$= \frac{-e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\left[\left(\cos\omega_d t_p + \theta\right)\omega_n\sqrt{1-\xi^2} - \xi\omega_n \sin(\omega_d t_p + \theta)\right]$$

$$\frac{d\,c(t)}{dt} = -\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\,\omega_n\left[\underbrace{(\cos\omega_d t_p + \theta)\sin\theta - \sin(\omega_d t_p + \theta)\cos\theta}\right] = 0$$

not zero

For $0 < \xi < 1$

$\sin(A-B) = \sin A\cos B - \cos A \sin$

$$\Rightarrow \left[\sin(\omega_d t_p + \theta - \theta)\right] = 0$$

i.e., $\sin(\omega_d t_p) = 0 \Rightarrow \omega_d t_p = n\pi$ where $n = 0, 1, 2.$

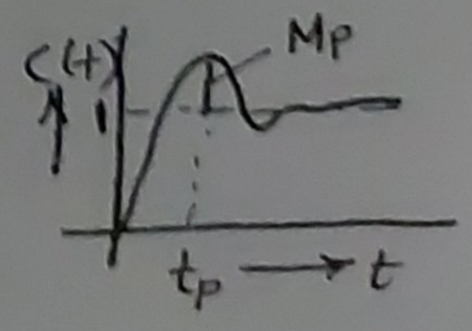
$$t_p = \frac{n\pi}{\omega_d}$$

For $n = 1$, $\boxed{t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$

3) The Maximum over shoot $M_p$:

Maximum over-shoot occurs at $t = t_p$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$




overshoot:

$$M_p = c(t_p) - 1$$

$$M_p = -\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$$

$t_p = \frac{\pi}{\omega_d}$.

Scanned by TapScanner

$$M_p = -\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\pi+\theta)$$

$\sin(\pi+\theta) = -\sin\theta$

$$= \frac{-e^{-\xi\omega_n t_p}}{\sin\theta} \times (-\sin\theta)$$

$\sqrt{1-\xi^2} = \sin\theta$

$$M_p = e^{-\xi\omega_n t_p} \quad \text{—} \quad \omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}}, \quad t_p = \frac{\pi}{\omega_d}.$$

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\frac{\xi\omega_d}{\sqrt{1-\xi^2}} \cdot \frac{\pi}{\omega_d}$$

$$\boxed{M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}}$$

4) Settling time ($t_s$): Time required by the response to reach and stay within a specified tolerance band of its final value

$\downarrow$ $\pm 2\%$ or $\pm 5\%$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Let the tolerance band be $\pm 2\%$

$$\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} = 0.02.$$

For underdamped case $\xi < 1$. & $\xi^2 << 1$

$$e^{-\xi\omega_n t_s} \approx 0.02$$

$$-\xi\omega_n t_s \approx \ln(0.02)$$

Scanned by TapScanner

$$\boxed{t_s = \frac{3.912}{\xi \omega_n}} \rightarrow \pm 2\%$$

$$\approx 4T$$

$$\frac{1}{\xi \omega_n} = T.$$

|||ly

$$\left.\begin{aligned} t_s &\approx 3T \\ &\approx \frac{3}{\xi \omega_n} \end{aligned}\right\} \rightarrow \pm 5\%$$

Scanned by TapScanner

28-05-2021 CS IV Sem 'D'

1) The closed loop TFs of a 2nd order unity feedback systems are given below. Determine the type of damping in them

(a) $\frac{25}{S^2+4S+25}$ (b) $\frac{2}{S^2+3S+2}$ (c) $\frac{4}{S^2+4S+4}$ (d) $\frac{8}{S^2+16}$

Soln: Standard form of C.L.T.F of 2nd order system is

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

(a) comparing with the standard form

$2\xi\omega_n = 4$ & $\omega_n = 5$

$\xi = \frac{4}{2\times 5} = \frac{4}{10} = 0.4$ ; i.e. $0<\xi<1$

Type of damping is underdamping

(b) comparing with std form

$2\xi\omega_n = 3$ and $\omega_n = \sqrt{2} \Rightarrow \xi = \frac{3}{2\sqrt{2}} = 1.060$

i.e, $\xi > 1$.

Type of damping is overdamping

(c) comparing with std. form

$\omega_n = 2$ and $2\xi\omega_n = 4$

$\Rightarrow \xi = \frac{4}{2\times 2} = 1. \rightarrow$ critical damping

(d) comparing with std form $\xi = 0 \rightarrow$ undamped.

2) A 2nd order system is excited by unit step input. if $\xi = 0.5$ and $\omega_n = 6$ rad/sec, determine $t_r$, $t_p$, $t_s$ and $M_p$

Scanned by TapScanner

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - \cos^{-1}(\xi)}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi - \pi/3}{6\sqrt{1-0.25}}$$

$$= \frac{2\pi/3}{6\times 0.75} = \frac{2\pi}{3}\times\frac{1}{6\times\sqrt{0.75}} = 0.403 \text{ secs}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{3.14}{6\sqrt{1-0.25}} = \frac{3.14}{6\times\sqrt{0.75}} = 0.604 \text{ sec.}$$

$$t_s = \frac{4}{\xi\omega_n} \text{ for } (\pm 2\%) = \frac{4}{0.5\times 6} = \frac{4}{3} = 1.33 \text{ secs.}$$

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = e^{-3.14\times 0.5/\sqrt{0.75}} = 0.163 = 16.3\%$$

3) A 2nd order c.s is governed by following differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 16y = 16x \; ; \; y \text{ is o/p } \& \; x \text{ is i/p}$$

if $x$ is unit step function, find $t_r$, $M_p$ and $t_s$ ($\pm 5\%$) 1

The T.F is $\frac{Y(s)}{X(s)}$ ; Taking L.T of the given differential eq

$$s^2Y(s) + 4sY(s) + 16Y(s) = 16X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{16}{s^2+4s+16}$$

1.

Comparing this with standard form

$$\omega_n = 4 \text{ and } 2\xi\omega_n = 4 \Rightarrow \xi = \frac{4}{8} = \frac{1}{2}$$

(i) $$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - \cos^{-1}(\frac{1}{2})}{4\sqrt{1-(0.5)^2}} = \frac{\pi - \pi/3}{4\sqrt{0.75}}$$

$$\frac{2\pi}{3}\times\frac{1}{4\sqrt{0.75}} = 0.605 \text{ sec.}$$

(ii) $$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.163 \simeq 16.3\%$$

(iii) $$t_s \simeq \frac{3}{\xi\omega_n} \simeq \frac{3}{0.5\times 4} = \frac{3}{2} = 1.5 \text{ sec.}$$

Scanned by TapScanner

28-05-2021

CS

IV Sem 'D'

1) The closed loop TFs of a 2nd order unity feedback systems are given below Determine the type of damping in them

(a) $\frac{25}{S^2+4S+25}$ (b) $\frac{2}{S^2+3S+2}$ (c) $\frac{4}{S^2+4S+4}$ (d) $\frac{8}{S^2+16}$

Soln: Standard form of C.L.T.F of 2nd order system is

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

(a) comparing with the standard form

$$2\xi\omega_n = 4 \quad \& \quad \omega_n = 5$$

$$\xi = \frac{4}{2\times 5} = \frac{4}{10} = 0.4 \text{ ; i.e. } 0<\xi<1$$

Type of damping is under damping

(b) comparing with std form

$2\xi\omega_n = 3$ and $\omega_n = \sqrt{2} \Rightarrow \xi = \frac{3}{2\sqrt{2}} = 1.060$

i.e, $\xi > 1$.

Type of damping is overdamping.

(c) comparing with std. form

$\omega_n = 2$ and $2\xi\omega_n = 4$

$\Rightarrow \xi = \frac{4}{2\times 2} = 1. \rightarrow$ critical damping

(d) comparing with std form $\xi = 0 \rightarrow$ undamped.

7) A 2nd order system is excited by unit-step input. if

Scanned by TapScanner

5) if System response $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ for a step i/p of 1V, determine $\omega_n$, $\xi$ and obtain unit ramp response.

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$ , for unit step i/p $R(s) = \frac{1}{s}$

~~TF = C(s)/R(s)~~ $\therefore$ TF = ~~R(s).~~ $= \frac{1}{s}\left[\frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}\right]$

$$= \frac{600}{s^2+70s+600}$$

$\therefore$ $2\xi\omega_n = 70$ and $\omega_n^2 = 600$

$\Rightarrow \omega_n = \sqrt{600} = 24.5$ rad/sec.

$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+10} + \frac{D}{s+60}$

$\xi = \frac{70}{2 \times 24.5} = 1.4286 \Rightarrow$ Overdamped System.

For unit ramp $R(s) = 1/s^2 \Rightarrow$ TF $= \frac{600}{s^2(s+10)(s+60)}$

Response for unit ramp, using partial fraction $A = -0.11$, $B = 1$, $C = 0.12$ and $D = 0.003$

$\therefore c(t) = -0.11 + t + 0.12e^{-10t} - 0.003e^{-60t}$

Scanned by TapScanner

31-05-2021 CS IV Sem 'D'

Time domain response - 2nd order system (contd).

4) The response of a system subjected to unit-step is

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

obtain the expression for closed loop T.F. Determine $\omega_n$ & $\xi$

Soln:

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

Taking L.T, $C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$

For step input $R(s) = \frac{1}{s}$

$$C.L.T.F = \frac{C(s)}{R(s)} = s\left[\frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}\right]$$

$$C.L.T.F \quad \frac{C(s)}{R(s)} = 1 + \frac{0.2s}{s+60} - \frac{1.2s}{s+10}$$

$$= \frac{(s+60)(s+10) + 0.2s(s+10) - 1.2s(s+60)}{(s+10)(s+60)}$$

$$= \frac{s^2 + 70s + 600 + 0.2s^2 + 2s - 1.2s^2 - 72s}{s^2 + 70s + 600}$$

$$= \frac{600}{s^2 + 70s + 600}$$

comparing this with the standard form,

$\omega_n^2 = 600$ $\quad 2\xi\omega_n = 70$

$\omega_n = \sqrt{600}$ $\quad \xi = 35/\sqrt{600} = 1.4288$

$= 24.49$ rad/sec

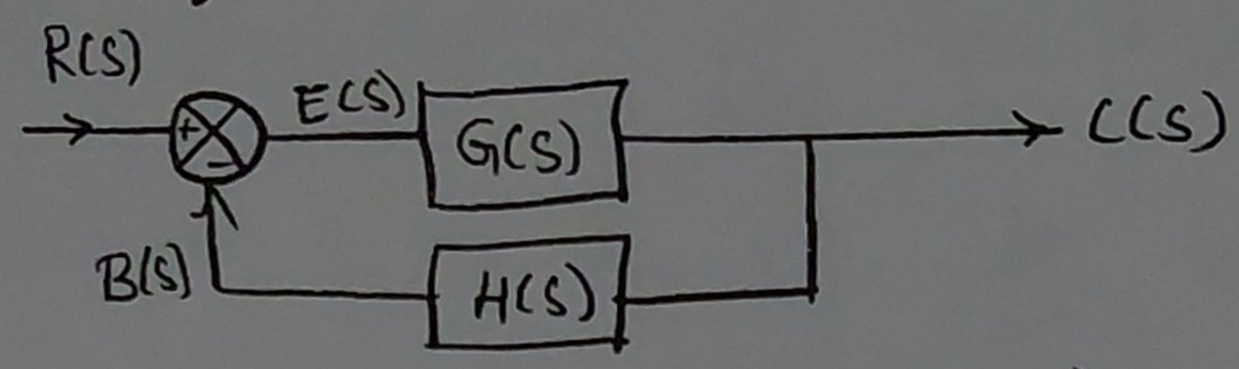
Scanned by TapScanner

31-05-2021.

# Steady State Analysis

Steady state error ($e_{ss}$) → is the difference b/n the actual o/p and desired o/p

## Derivation of steady state error




$$E(S) = R(S) - B(S) = R(S) - C(S)H(S)$$

$$= R(S) - E(S)G(S)H(S)$$

$$E(S)[1 + G(S)H(S)] = R(S)$$

$$E(S) = \frac{R(S)}{1 + G(S)H(S)} \quad \text{—— (1)}$$

In time domain,

$$e_{ss} = \lim_{t \to \infty} e(t)$$

using final value theorem

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{S \to 0} S\,E(S)$$ → substitute from (1)

$$\therefore \quad e_{ss} = \lim_{S \to 0} S \frac{R(S)}{1 + G(S)H(S)}$$

The steady state error

→ depends on R(S), reference input – its type and magnitude

→ depends on G(S)H(S) → open loop T.F

→ dominant non-linearity, if any

Scanned by TapScanner

I. Effect of Input on $e_{ss}$ :

let us consider step input of amplitude 'A'

$$\therefore R(s) = \frac{A}{s}$$

$$e_{ss} = \lim_{s \to 0} \frac{s \cdot A/s}{1+G(s)H(s)} = \frac{A}{1+G(s)H(s)}$$

For a given system $\lim_{s \to 0} G(s)H(s)$ is constant

Denoting it by $K_p$ i.e., $\lim_{s \to 0} G(s)H(s) = K_p$

$$\therefore e_{ss} = \frac{A}{1+K_p}$$ ; $K_p$ is called position control error coefficient

|||ly For ramp i/p $[R(s) = A/s^2]$

$$e_{ss} = \lim_{s \to 0} \frac{A}{s[1+G(s)H(s)]} = \lim_{s \to 0} \frac{A}{s+sG(s)H(s)}$$

$$= \frac{A}{\lim_{s \to 0} sG(s)H(s)}$$

let $K_v = \lim_{s \to 0} sG(s)H(s) \longrightarrow$ Velocity error coefficient

|||ly for parabolic i/p $\left[R(s) = \frac{1}{s^3} \cdot A\right]$

$K_a = \lim_{s \to 0} s^2 G(s)H(s) \longrightarrow$ Acceleration error coefficient

$$e_{ss} = A/K_a.$$

Effect of 'type' of system on $e_{ss}$ :

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)+\cdots}{s^j(1+T_a s)(1+T_b s)+\cdots}$$

K = gain and j = type of system

Scanned by TapScanner

'Type' is property of O.L.T.F $\rightarrow G(s)H(s)$

'Order' is property of C.L.T.F $\rightarrow \dfrac{G(s)}{1+G(s)H(s)}$

To find error constants, the system must be stable, since for unstable system, there is no steady state

Type '0' system :

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots}{(1+T_a s)(1+T_b s)\cdots}$$

1) $$K_p = \lim_{s \to 0} G(s)H(s) = K.$$

i.e., $K_p = K$

$$e_{ss} = \frac{A}{1+K_p} = \frac{A}{1+K}$$

$A \rightarrow$ amplitude

$K \rightarrow$ gain

Type '0' system follows step input with finite error

$$e_{ss} = \frac{A}{1+K}$$

By adjusting gain K of open loop or Amplitude of i/p steady state error is minimized.

2) $$K_v = \lim_{s \to 0} s\,G(s)H(s) = 0$$

$$e_{ss} = \frac{A}{K_v} = \infty$$

$\rightarrow$ Gives very large error. ramp i/p not to be given to type '0' system

3) $K_a = \infty$

Scanned by TapScanner

02-06-2021

CS

IV Sem 'D'

II. Effect of 'type' of System on steady state error (contd)

Type '1' system:

$$G(S)\,H(S) = \frac{K(1+T_1S)(1+T_2S)\cdots}{S(1+T_aS)(1+T_bS)\cdots} \quad \text{—(1)}$$

Steady state error $\rightarrow e_{SS} = \lim_{t\to\infty} e(t)$

using Final value theorem,

$$e_{SS} = \lim_{S\to 0} S\,\frac{R(S)}{1+G(S)H(S)}$$

For step input

$$e_{SS} = \frac{A}{1+K_p} \quad \text{and } K_p = \lim_{S\to 0} G(S)H(S)$$

$$K_p = \lim_{S\to 0} G(S)H(S) = \lim_{S\to 0} \frac{K(1+T_1S)(1+T_2S)\cdots}{S(1+T_aS)(1+T_bS)\cdots}$$

when $S\to 0$, $K_p = \infty$

$$e_{SS} = \frac{A}{1+\infty} = 0$$

For ramp i/p

$$K_v = \lim_{S\to 0} S\,G(S)H(S)$$

$$K_v = K.$$

Type '2' system:

$$G(S)\,H(S) = \frac{K(1+T_1S)(1+T_2S)\cdots}{S^2(1+T_aS)(1+T_bS)\cdots}$$

For step input,

$$K_p = \lim_{S\to 0} G(S)H(S)$$

$$K_p = \lim_{S\to 0} \frac{K(1+T_1S)(1+T_2S)\cdots}{S^2(1+T_aS)(1+T_bS)\cdots}$$

when $S\to 0$, $K_p = \infty$.

$$e_{SS} = \frac{A}{1+K_p} = 0$$

Scanned by TapScanner

⇒ Type '1' and above systems follow step input with negligible error.

1) A -ve unity feedback system has $K_p = 5$. Find steady state error $e_{ss}$ if i/p is

(i) $6\,u(t)$
(ii) $6t\,u(t)$

Soln: (i) This is step input with amplitude $A = 6$

$$e_{ss} = \frac{A}{1+K_p} = \frac{6}{1+5} = 1$$

∵ error is for the system type '0'

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} G(s)$$

∵ unity feedback

$$R(s) = \frac{6}{s}$$

$$C.L.T.F = \frac{G(s)}{1+G(s)} = \frac{C(s)}{R(s)}$$

$$E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{R(s)}{1+G(s)}$$

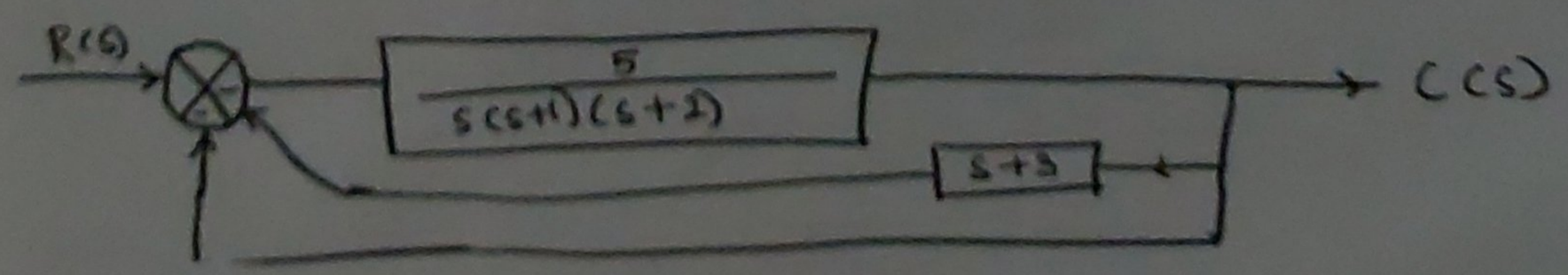
$$\frac{1}{s} = \frac{6/s}{1+G(s)}$$

$$1+G(s) = 6 \Rightarrow G(s) = 6-1 = 5$$

(ii) $$K_v = \lim_{s\to 0} s\,G(s) = 0$$

$$\therefore e_{ss} = \frac{6}{K_v} = \frac{6}{0} = \infty$$

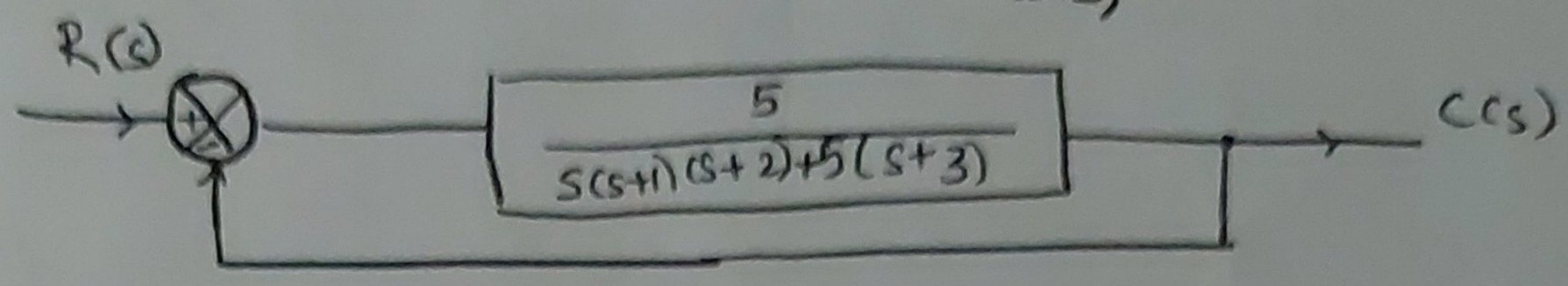
2. For the system shown, find system type, state error constants $K_p, K_v, K_a$ and $e_{ss}$ for an i/p, $r(t) = 5\,u(t)$




Scanned by TapScanner

$$= \frac{5}{s(s+1)(s+2)} \Big/ 1 + \frac{(s+3)5}{s(s+1)(s+2)}$$

$$= \frac{5}{s(s+1)(s+2) + 5(s+3)}$$




Here $G(s) = \dfrac{5}{s^3+3s^2+7s+15}$

$H(s) = 1.$

$\therefore$ Open loop T.F : $G(s)H(s) = \dfrac{5}{s^3+3s^2+7s+15}$

O.L.T.F is NOT having any pole at origin (i.e. at $s=0$)

it is a 'Type 0' system

Static error constants :

$$K_p = \lim_{s\to 0} G(s)H(s) = \frac{5}{15} = \frac{1}{3}$$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = 0$$

$$K_a = \lim_{s\to 0} s^2\,G(s)H(s) = 0$$

Steady state error $e_{ss}$ for $r(t) = 5u(t)$

$A = 5$

$$e_{ss} = \frac{A}{1+K_p} = \frac{5}{1+\frac{1}{3}} = \frac{5\times 3}{4} = \frac{15}{4} = 3.75$$

3. open loop T.F of a unity feedback system is given by

$$G(s) = \frac{K}{(1+Ts)s}$$ where K and T are constants

Determine the factor by which K should be multiplied so that overshoot of step response is reduced from 75% to 25%

Scanned by TapScanner

Soln:

$$C.L.T.F = \frac{G(s)}{1+G(s)} = \frac{K/s(1+Ts)}{1+\frac{K}{s(1+Ts)}}$$

$$= \frac{K/T}{s^2+\frac{1}{T}s+K/T}$$

The std. form of closed loop T-F of 2nd order system is

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

comparing,

$$\omega_n = \sqrt{K/T}, \quad \xi = \frac{1}{2\sqrt{KT}}$$

$$2\xi\omega_n = \frac{1}{T}$$

$$\xi = \frac{1}{2T\sqrt{K/T}} = \frac{1}{2\sqrt{KT}}$$

To calculate the factor by which K should be multiplied to reduce $M_p$ from 0.75 to 0.25 :

For 75% overshoot:

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$0.75 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\Rightarrow \ln(0.75) = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

$$0.2876 = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

Squaring both sides

$$0.0827(1-\xi^2) = \pi^2\xi^2$$

$$0.0827 = (\pi^2+0.0827)\xi^2$$

$$\xi^2 = 8.31\times10^{-3}$$

$$\xi_{old} = 0.091 \rightarrow \xi_{old} = \frac{1}{2\sqrt{K_{old}T}}$$

For 25% overshoot.

$$\ln(0.25) = -\frac{\pi\xi}{\sqrt{1-\xi^2}}$$

$$\xi_{new} = 0.4039$$

$\therefore$ $\xi$ must be changed from 0.091 to 0.403.

$$\xi_{new} = \frac{1}{2\sqrt{KT}} = 0.403 = \frac{1}{2\sqrt{K_{new}T}}$$

Scanned by TapScanner

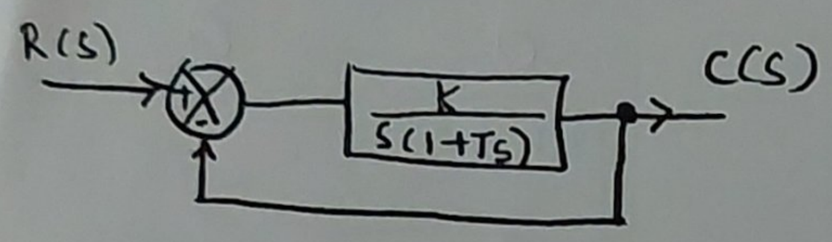
$$\frac{\xi_{new}}{\xi_{old}} = \frac{0.403}{0.0911} = \frac{1/2\sqrt{K_{new}T}}{1/2\sqrt{K_{old}T}}$$

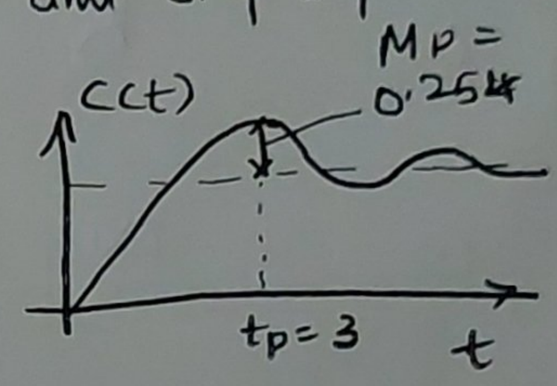
$$\frac{(0.403)^2}{(0.0911)^2} = \frac{K_{old}}{K_{new}} \Rightarrow K_{new} = K_{old}\left[\frac{0.0911}{0.403}\right]^2$$

$K_{old}$ should be multiplied by 0.0511

K should be multiplied by 0.0511 to reduce peak overshoot from 75% to 25%.

4. The system shown below is subjected to unit step input & the response is also shown. Determine K and T




Soln;

$$C.L.T.F = \frac{K/s(1+Ts)}{1+K/s(1+Ts)}$$

$$= \frac{K/T}{s^2 + \frac{1}{T}s + \frac{K}{T}} \Rightarrow \omega_n = \sqrt{\frac{K}{T}}, \quad \xi = 1/2\sqrt{KT}$$

$$M_p = 0.254 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

Solving, $\xi = 0.4$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \Rightarrow \omega_n = 1.1425 \text{ rad/sec.}$$

$$2\xi\omega_n = \frac{1}{T} \Rightarrow T = 1.094.$$

$$\frac{K}{T} = \omega_n^2 \Rightarrow K = \omega_n^2 T = 1.428$$

Scanned by TapScanner

$e_{ss}$ for

| Type | $K_p$ | $K_v$ | $K_a$ | Step i/p | Ramp i/p | parabolic |
|---|---|---|---|---|---|---|
| 0 | $k$ | 0 | 0 | $\frac{A}{1+K}$ | $\infty$ | $\infty$ |
| 1 | $\infty$ | $K$ | 0 | 0 | $\frac{A}{K}$ | $\infty$ |
| 2 | $\infty$ | $\infty$ | $K$ | 0 | 0 | $A/K$ |

Scanned by TapScanner