equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \qquad (1)$$

known as the Legendre's differential equation.

The parameter n is given integer, (although it could be a real number).

The solution of Legendre's Equation (1) is known as Legendre's function of order n.

Assume a power series solution of (1) as

$$y(x)=\sum_{m=0}^{\infty} a_m x^m \qquad (2)$$

Substitute (2) and its derivatives in (1), then

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}-2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

where $k=n(n+1)$. Rewriting

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty} m(m-1)a_m x^m -2\sum_{m=1}^{\infty} m a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0 \qquad (3)$$

(3) is an identity since (2) is a solution of (1). So equate the sum of the coefficients of each power of x to zero.

Coefficient of x^0 arise from 1st and fourth series in (3). Thus

$$2a_2+n(n+1)a_0=0 \qquad (4)$$

coefficient of x^1 arise from 1st, 3rd and 4th series in (3). So

$$6a_3+[-2+n(n+1)]a_1=0 \qquad (5)$$

All the four series in (3) contribute coefficients of x^s for $s \geq 2$. Thus

$$(s+2)(s+1)a_{s+2}+[-s(s-1)-2s+n(n+1)]a_s=0 \qquad (6)$$

Solving (6)

$$a_{s+2}=-\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \qquad (7)$$

for $\quad s=0,1,2,\ldots$

since $-s(s-1)-2s+n(n+1)$

$$= -s^2+s-2s+n^2+n$$
$$= (n^2-s^2+n-s)$$
$$= (n-s)(n+s+1).$$

(7) is known as a recurrence relation or recursion formula, which determines all coefficients in terms of a_1 or a_0. Here a_0 and a_1 are arbitrary constants, to be chosen appropriately. Thus

$$a_2 = \frac{-(n)(n+1)}{2!}a_0, \quad a_3 = \frac{-(n-1)(n+2)}{3!}a_1$$
$$a_4 = \frac{-(n-2)(n+3)}{4\cdot 3}a_2, \quad a_5 = \frac{-(n-3)(n+4)}{5\cdot 4}a_3$$
$$a_4 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0,$$
$$a_5 = \frac{-(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

In general, the coefficients with even subscripts are

$$a_{2m} = \frac{-(n-2m+2)(n+2m-1)}{(2m)(2m-1)}a_{2m-2}$$

and the coefficients with odd subscripts are

$$a_{2m+1} = \frac{-(n-2m+1)(n+2m)}{(2m)(2m+1)}a_{2m-1}$$

Substituting these coefficients in (2), we get

$$y(x) = a_0y_1(x) + a_1y_2(x) \tag{8}$$

where $y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots \quad (9)$

and $y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots \quad (10)$

Both the series (9) and (10) converge for $|x| < 1$. y_1 and y_2 are linearly independent (i.e., y_1/y_2 is not constant) because y_1 contains only even powers of x while y_2 contains only odd powers of x. Thus $y(x)$ given by (8) is the general solution of Legendre's equation (1) and is valid for $-1 < x < 1$.

Legendre Polynomials

Assume that the parameter n is non-negative integer i.e., $n \geq 0$, n integer. Since $(n-s)$ appears in the recurrence relation (7), with $s = n$, the coefficients $a_{n+2}, a_{n+4}, a_{n+6}$, etc., are all zero i.e., $a_{m+2} = 0$ when $m \geq n$. If n is even, then $y_1(x)$ reduces to a polynomial of degree n (while $y_2(x)$ remains an infinite series).

Similarly if n is odd, then $y_2(x)$ becomes a polynomial of degree n (while $y_1(x)$ remains an infinite series).

In either of these cases, the series which reduces to a finite sum (a polynomial), multiplied by some constant, is known as the Legendre polynomial or zonal harmonic of order n denoted by $P_n(x)$. The series which remain infinite is known as the Legendre's function of the second kind denoted by $Q_n(x)$. Thus for a non-negative integer n, the general solution (2) of Legendre's Equation (1) is the sum a polynomial solution and an infinite series solution i.e.,

$$y(x) = AP_n(x) + BQ_n(x) \tag{11}$$

Note: $Q_n(x)$ is unbounded at $x = \pm 1$.

Derivation of Legendre Polynomial $P_n(x)$

Rewriting (7),

$$a_m = \frac{-(m+2)(m+1)}{(n-m)(n+m+1)}a_{m+2} \quad \text{for } m \leq n-2 \tag{12}$$

(12) expresses all non-vanishing coefficients in terms of a_n, which is coefficient of the highest power of x i.e., x^n in the polynomial. The arbitrary coefficient a_n may be chosen as

$$a_n = 1 \quad \text{for } n = 0$$

and
$$a_n = \frac{(2n)!}{2^n(n!)^2} \quad \text{for } n = 1, 2, \ldots \tag{13}$$

For this choice of a_n, $P_n(x=1) = 1$.

The non-vanishing coefficients are obtained from (12) and (13)

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)}a_n \quad (\text{for } m = n-2)$$
$$= \frac{-n(n-1)}{2(2n-1)}\cdot\frac{(2n)!}{2^n(n!)^2}$$