Example 10. *In an examination taken by 500 candidates, the average and S.D. of marks obtained are 40% and 10% respectively. Assuming normal distribution, find (i) how many have scored above 60%., (ii) how many will pass if 50% is fixed as the minimum for passing, and (iii) what should be the minimum for 350 candidates to pass.*

» Here $\mu = 40$ and $\sigma = 10$ in percentages. Therefore, if x is the percentage of marks scored by a student, the corresponding standard normal variate is

$$z = \frac{x - \mu}{\sigma} = \frac{x - 40}{10}$$

(i) The probability that a candidate scores above 60% of marks is

$$P(x > 60) = P\left(z > \frac{60 - 40}{10}\right) = P(z > 2)$$

$$= 0.5 - P(0 < z < 2) = 0.5 - A(2)$$

$$= 0.5 - 0.4772 = 0.0228.$$

Therefore, the number of candidates (out of 500) who have scored above 60% is

$$0.0228 \times 500 = 11.4 \approx 11.$$

(ii) For $x = 50$, we have $z = 1$. Therefore, the probability that a candidate passes if 50% is fixed as the minimum for passing is

$$P(x \geq 50) = P(z \geq 1) = P(z \geq 0) - P(0 \leq z < 1)$$

$$= 0.5 - A(1) = 0.5 - 0.3413 = 0.1587.$$

Therefore, among 500 candidates, the number of candidates who pass with 50% as the minimum for passing is

$$0.1587 \times 500 = 79.35 \approx 79.$$

(iii) Let M % of marks be the minimum for passing if 350 of the 500 candidates are to pass. Then, we should have

$$500 \times P(x \geq M) = 350.$$

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or $$P(x \geq M) = \frac{350}{500} = 0.7.$$

When $x = M$, we have $z = \dfrac{M-40}{10} = -z_1$, say, where $z_1 > 0$.

Then

$$0.7 = P(z \geq -z_1) = P(-z_1 \leq z < 0) + P(z \geq 0)$$
$$= P(0 < z \leq z_1) + 0.5 = A(z_1) + 0.5$$

or $$A(z_1) = 0.2.$$

The normal probability table shows that $A(z_1) = 0.2$ if $z_1 = 0.55$. Thus,

$$0.55 = \frac{40-M}{10}, \text{ or } M = 34.5 \approx 35.$$

Thus, if 35% is fixed as the minimum marks for passing, then 350 candidates out of 500 will pass. ∎

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Since $x = 45$ corresponds to $z = z_1$, and $x =$ –
$z = z_2$, we have

$$z_1 = \frac{45 - \mu}{\sigma}, \quad z_2 = \frac{64 - \mu}{\sigma}$$

Substituting for z_1 and z_2 from expressions (i) and (ii) in these, we get

$$\frac{45 - \mu}{\sigma} = -0.5 \quad \text{and} \quad \frac{64 - \mu}{\sigma} = 1.4$$

i.e. $\quad \mu - (0.5)\,\sigma = 45 \quad$ and $\quad \mu + (1.4)\sigma = 64.$

Solving these equations, we get $\mu = 50$ and $\sigma = 10$. These are respectively the mean and standard deviation of the given normal distribution. ∎

Example 12. *In a normal distribution, 7% are under 35 and 89% are under 63. Find the mean and the standard deviation, given that $A(1.23) = 0.39$ and $A(1.48) = 0.43$ in the usual notation.*

» Let x denote the normal variate and z the standard normal variate for the distribution being considered. Also, let $z = z_1$ correspond to $x = 35$ and $z = z_2$ correspond to $x = 63$. Then, we have, from what is given

$$\int_{-\infty}^{z_1} \varphi(z)\, dz = \frac{7}{100} = 0.07 \qquad \ldots \text{(i)}$$

and
$$\int_{-\infty}^{z_2} \varphi(z)\, dz = \frac{89}{100} = 0.89 \qquad \ldots \text{(ii)}$$

Since 7% < 50% and 89% > 50%, the ordinate $z = z_1$ lies to the left of the line $z = 0$ and the ordinate $z = z_2$ lies to the right. That is, $z_1 < 0$ and $z_2 > 0$ (see Figure 7.4). Accordingly, using the symmetry of the normal curve and expressions (i) and (ii) we find that

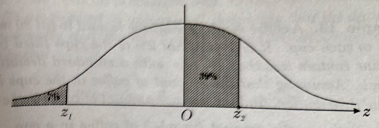

Figure 7.4

$$A(-z_1) = \int_0^{-z_1} \varphi(z)\, dz = \int_{z_1}^{0} \varphi(z)\, dz = \int_{-\infty}^{0} \varphi(z)\, dz - \int_{-\infty}^{z_1} \varphi(z)\, dz$$

$$= 0.5 - 0.07 = 0.43 \qquad \ldots \text{(iii)}$$

and
$$A(z_2) = \int_0^{z_2} \varphi(z)\, dz = \int_{-\infty}^{z_2} \varphi(z)\, dz - \int_{-\infty}^{0} \varphi(z)\, dz$$

$$= 0.89 - 0.5 = 0.39 \qquad \ldots \text{(iv)}$$

It is given that

$$0.43 = A(1.48) \quad \text{and} \quad 0.39 = A(1.23).$$

Therefore, (iii) and (iv) give

$$z_1 = -1.48, \quad z_2 = 1.23$$

Since $z = (x - \mu)/\sigma$, and $z = z_1$ correspond to $x = 35$ and $z = z_2$ to $x = 63$, we have

$$-1.48 = z_1 = \frac{35 - \mu}{\sigma} \quad \text{and} \quad 1.23 = z_2 = \frac{63 - \mu}{\sigma}$$

These give
$$\mu - (1.48)\,\sigma = 35$$
and
$$\mu + (1.23)\,\sigma = 63.$$

Solving these equations, we get $\mu \approx 50.29$ and $\sigma \approx 10.33$. These are the required mean and standard deviation respectively.

Example 13. *A coffee vending machine is used to fill 80 mls. of coffee to each cup. It is found that 2% of the cups filled by the machine contain less than 80 mls. with a standard deviation of 0.2 ml. Assuming that the amount of coffee in the cups has a normal distribution, find the average amount of coffee in the cups.*

» Let x denote the amount (in mls.) of coffee in a cup. From what is given, the probability that a cup contains less than 80 mls is 2%; that is $P(x < 80) = 0.02$.

If z is the corresponding standard normal variate and $z = z_1$ for $x = 80$, then $P(z < z_1) = 0.02$. This gives

$$\int_{-\infty}^{z_1} \varphi(z)\, dz = 0.02 \text{ with } z_1 < 0$$

Therefore,

$$A(-z_1) = \int_0^{-z_1} \varphi(z)\, dz = \int_{z_1}^{0} \varphi(z)\, dz = \int_{-\infty}^{0} \varphi(z)\, dz - \int_{-\infty}^{z_1} \varphi(z)\, dz$$

$$= 0.5 - 0.02 = 0.48$$

Using the Normal probability table, we find that $0.48 = A(2.05)$. Hence $-z_1 = 2.05$.

Since $z = (x - \mu)/\sigma$ and $z = z_1$ corresponds to $x = 80$, and σ is given to be equal to 0.2, we have

$$-2.05 = z_1 = \frac{80 - \mu}{0.2}$$

This gives

$$\mu = 80 + (2.05) \times (0.2) = 80.41$$

Accordingly, the average amount of coffee in the cups is about 80.41 mls.

Example 14. *Steel rods are manufactured to be 3 cms in diameter but they are acceptable if they are inside the limits 2.99 cms and 3.01 cms. It is observed that 5% are rejected as oversized and 5% are rejected as undersized. Assuming that the diameters are normally distributed, find the standard deviation of the distribution.*

$$\left[\textit{Use: } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1.65} e^{-z^2/2}\, dz = 0.05\right]$$

» Here $\mu = 3$. Let x denote the diameter of a rod, and z be the corresponding standard normal variate. Then

$$z = \frac{x - \mu}{\sigma} = \frac{x - 3}{\sigma}$$

Let $z = z_1$ for $x = 2.99$ and $z = z_2$ for $x = 3.01$. Then

$$z_1 = \frac{2.99 - 3}{\sigma} = \frac{-0.01}{\sigma}, \quad z_2 = \frac{3.01 - 3}{\sigma} = \frac{0.01}{\sigma}$$

Thus,

$$z_2 = -z_1 = \frac{0.01}{\sigma} \qquad \ldots \text{(i)}$$

Since $\sigma > 0$, we have $z_2 > 0$ and $z_1 < 0$.

From what is given, we have $P(x < 2.99) = 5\%$ and $P(x > 3.01) = 5\%$, or equivalently, $P(z < z_1) = 5\%$ and $P(z > z_2) = 5\%$. Thus,

$$\int_{-\infty}^{z_1} \varphi(z)\, dz = 5\% = \frac{5}{100} = 0.05$$

and

$$\int_{z_2}^{\infty} \varphi(z)\, dz = 5\% = 0.05$$

Using these expressions and the symmetry of the normal curve as well as the fact that $z_1 < 0$ and $z_2 > 0$, we find that

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$$A(-z_1) = \int_0^{-z_1} \varphi(z)\,dz = \int_{z_1}^{0} \varphi(z)\,dz = \int_{-\infty}^{0} \varphi(z)\,dz - \int_{-\infty}^{z_1} \varphi(z)\,dz$$

$$= 0.5 - 0.05 = 0.45$$

and $$A(z_2) = \int_0^{z_2} \varphi(z)\,dz = \int_0^{\infty} \varphi(z)\,dz - \int_{z_2}^{\infty} \varphi(z)\,dz$$

$$= 0.5 - 0.05 = 0.45.$$

Thus,

$$A(-z_1) = A(z_2) = 0.45 \qquad \ldots$$

It is given that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1.65} e^{-z^2/2}\,dz = 0.05;$$

that is, $$\int_{-\infty}^{-1.65} \varphi(z)\,dz = \int_{1.65}^{\infty} \varphi(z)\,dz = 0.05.$$

Therefore,

$$A(1.65) = \int_0^{1.65} \varphi(z)\,dz = \int_0^{\infty} \varphi(z)\,dz - \int_{1.65}^{\infty} \varphi(z)\,dz$$

$$= 0.5 - 0.05 = 0.45 \qquad \ldots$$

From (ii) and (iii), we find that $z_2 = -z_1 = 1.65$. According[ly]
(i) gives

$$1.65 = z_2 = \frac{0.01}{\sigma}, \quad \text{or} \quad \sigma = \frac{0.01}{1.65} = 0.0061.$$

Thus, the standard deviation of the given distribution [is]
$\sigma = 0.0061$.

Example 15. *A certain number of articles manufactued in a batch were classified into three categories, being less than 50, between 50 and 60 and greater than 60. If this characteristic is known to be normally distributed, determine the mean and standard deviation for this batch if 60%, 35% and 5% were found in these categories.*

» Let z be the standard normal variate associated with the given characteristic, say x. Let z_1 correspond to $x = 50$ and z_2 to $x = 60$. Then, from what is given, we have

$$\int_{-\infty}^{z_1} \varphi(z)\,dz = 60\% = 0.6, \quad z_1 > 0^*$$

and $$\int_{z_2}^{\infty} \varphi(z)\,dz = 5\% = 0.05, \quad z_2 > 0$$

These and the symmetry of the normal curve yield

$$A(z_1) = \int_0^{z_1} \varphi(z)\,dz = \int_{-\infty}^{z_1} \varphi(z)\,dz - \int_{-\infty}^{0} \varphi(z)\,dz$$

$$= 0.6 - 0.5 = 0.1$$

$$A(z_2) = \int_0^{z_2} \varphi(z)\,dz = \int_0^{\infty} \varphi(z)\,dz - \int_{z_2}^{\infty} \varphi(z)\,dz$$

$$= 0.5 - 0.05 = 0.45$$

Thus, $A(z_1) = 0.1$ and $A(z_2) = 0.45$. From the normal probability table, we find that $z_1 \approx 0.25$ and $z_2 \approx 1.65$. Accordingly,

$$0.25 = z_1 = \frac{50 - \mu}{\sigma} \quad \text{and} \quad 1.65 = z_2 = \frac{60 - \mu}{\sigma}$$

(*) Since $\int_{-\infty}^{z_1} \varphi(z)\,dz > 50\%$, we have $z_1 > 0$.

These give

$$\mu = 50 - (0.25)\,\sigma \qquad \text{and} \qquad \mu = 60 - (1.65)\,\sigma.$$

Solving these, we find $\mu \approx 48.21$ and $\sigma = 7.14$. These are the mean and the S.D. respectively of the given normal distribution.