

Fourth Semester B.E. Course-(AS/ME//ECE/ET//EIE)
Course Title: Complex Analysis, Probability and Statistical Methods
Course Code: 23MA4BSCPS

UNIT1: COMPLEX ANALYSIS

1. Is the function $u(x, y) = 2xy + 3xy^2 - 2y^3$ harmonic. **Ans:** u is not harmonic
2. If $f'(z) = 0$ then show that $f(z)$ is constant.
3. If $f(z)$ is an analytic function with constant modulus show that $f(z)$ is constant.
4. If $f(z)$ is a holomorphic function of z , show that $\left\{ \frac{\partial}{\partial x} |f(z)| \right\}^2 + \left\{ \frac{\partial}{\partial y} |f(z)| \right\}^2 = |f'(z)|^2$
5. If $f(z)$ is a regular function of z , prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2$
6. If $f(z)$ is an analytic function of z , prove that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] |Rf(z)|^2 = 2 |f'(z)|^2$
7. Determine the analytic function (or imaginary part of the function) whose real part is
 - a) $\log \sqrt{x^2 + y^2}$ **Ans:** $\log z$
 - b) $\frac{y}{(x^2 + y^2)}$ **Ans:** $\frac{i}{z}$
 - c) $u = e^{2x} (x \cos 2y - y \sin 2y)$ **Ans:** $ze^{2z} + ic$
 - d) $y + e^x \cos y$ **Ans:** $e^2 - iz$
 - e) $x \sin x \cos h y - y \cos x \sin h y$ **Ans:** $z \sin z$
8. Find the regular function (or the real part of the function) whose imaginary part is
 - a) $\frac{(x - y)}{(x^2 + y^2)}$ **Ans:** $\frac{(1+i)}{z} + c$
 - b) $e^x \sin y$ **Ans:** e^z
 - c) $-\sin x \sinh y$ **Ans:** $\cos z + c$
 - d) $e^{-x} (x \sin y - y \cos y)$ **Ans:** $\bar{z} e^{-\bar{z}} + c$
 - e) $e^{-x} (x \cos y + y \sin y)$ **Ans:** $1 + iz e^{-2}$
9. If $f(z) = u + iv$ is an analytic function of z , find $f(z)$ if
 - a) $u + v = \sin x \cosh y + \cos x \sinh y$ **Ans:** $\sin z + c / (1+i)$
 - b) $2u + v = e^x (\cos y - \sin y)$ **Ans:** $\frac{(1+3i)}{5} e^z + c$
 - c) $u - v = 2xy + x^2 - y^2 + x - y$ **Ans:** $z - iz^2 + c$



10. If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = (x^2 - y^2) + \frac{x}{x^2 + y^2}$, find the complex potential as a function of the complex variable z and hence determine ϕ .

$$\text{Ans: } -2xy + \frac{y}{x^2 + y^2} + c$$

11. Find analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$.

$$\text{Ans: } u = -r^2 \sin 2\theta + r \sin \theta + c, \quad f(z) = i(r^2 e^{2i\theta} - r e^{i\theta}) + c + 2i$$

12. Find the analytic function $f(z) = u + iv$, given $v = (r - 1/r) \sin \theta, r \neq 0$.

$$\text{Ans: } \left(r + \frac{1}{r}\right) \cos \theta + i \left(r - \frac{1}{r}\right) \sin \theta + c.$$

13. Verify that the given function is harmonic and find its harmonic conjugate. Express $u + iv$ as analytic function $f(z)$:

a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Ans: $v = 3x^2 y - y^3 + 6xy$.

b) $u = x^2 - y^2 - y$

Ans: $v = 2xy + x + y, f(z) = z^2 + iz + c$

c) $v = x^2 - y^2 + \frac{x}{x^2 + y^2}$

Ans: $u = -2xy + \frac{y}{x^2 + y^2} + c, w = i \left(z^2 + \frac{1}{2}\right) + c$

d) $u = 3xy^2 - x^3$

Ans: $v = y^3 - 3x^2 y + c, f(z) = -z^3 + ic$

e) $v = y^2 - x^2$

Ans: $u = 2xy + c, f(z) = iz^2 + c$

f) $u = e^{-x} (x \sin y - y \cos y)$

Ans: $v = e^{-x} (y \sin y + x \cos y) + c, f(z) = iz e^{-z}$

g) $u = \frac{x^2 - y^2}{(x^2 + y^2)^2}$

Ans: $v = \frac{-2xy}{(x^2 + y^2)^2}, f(z) = \frac{1}{z^2} + c$

h) $u = 3x^3 y + 2x^2 - y^3 - 2y^2$

Ans: not harmonic

i) $u = -e^{-2xy} \sin(x^2 - y^2)$

Ans: $v = -e^{-2xy} \cos(x^2 - y^2) + c, f(z) = -i e^{iz^2} + ci$.

j) $u(r, \theta) = -r^3 \sin 3\theta$.

Ans: $v = r^3 \cos 3\theta + c$

14. Show that $U(x, y) = e^u \cos v, V(x, y) = e^u \sin v$ are harmonic conjugate of each other if $f(z) = u + iv$ is analytic.

15. Find the orthogonal trajectories of the family of curves $x^3 y - x y^3 = c = \text{constant}$.

Ans: $x^4 + y^4 - 6x^2 y^2 = \text{constant}$

16. Find the orthogonal trajectories of the family of curves $r^2 \cos 2\theta = c_1$. **Ans:** $v = r^2 \sin 2\theta$

17. Discuss the transformation $w = z^2$.

18. Discuss the transformation

$$w = z + \frac{k^2}{z}, \quad (z \neq 0).$$

**Complex Integration:**

1. Evaluate $\int_0^{1+i} (x^2 + iy) dz$ along the paths $y = x$ and $y = x^2$.
2. Evaluate $\int_0^{2+i} \left(\frac{-}{z}\right)^2 dz$ along
 - i) the line $y = \frac{x}{2}$.
 - ii) the real axis to 2 and then vertically to $2+i$.
3. Evaluate $\int_C |z|^2 dz$ over the curve made up of the vertices (0, 0), (1,0), (1,1) and (0,1).
4. Show that $\int_C (z-a)^n dz = \begin{cases} 0 : n \neq -1 \\ 2\pi i : \text{if } n = -1 \end{cases}$ C is the circle $|z-a| = r$.

Cauchy's Theorem:

1. Verify the Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle having vertices $-1, 1, 1+i, -1+i$.
2. Verify the Cauchy's theorem for the integral of $\frac{1}{z}$ taken over the boundary of the triangle having vertices (1, 2), (1, 4) & (3, 2).
3. Verify the Cauchy's theorem for the function $f(z) = z^2$ over the boundary of square having vertices (0,0), (1,0), (1,1) and (0,1).
4. Show that $\int_C |z|^2 dz = i-1$ where C the square having vertices (0, 0), (1, 0), (1, 1) & (0, 1). Give the reason for Cauchy's theorem not being satisfied.
5. Verify the Cauchy's for the function $f(z) = ze^{-z}$ over the unit circle with origin as the center.

Cauchy's Integral formula and Generalized Cauchy's Integral formula:

1. Evaluate $\int_C \frac{z^2 - z + 1}{z-1} dz$ where C is the circle (i) $|z|=1$ (ii) $|z| = \frac{1}{2}$
2. Evaluate $\int_C \frac{\cos(\pi z)}{z^2 - 1} dz$ around a rectangle with vertices $2 \pm i, -2 \pm i$
3. Evaluate
 - a. $\int_C \frac{e^{2z} dz}{(z-1)(z-2)}$ where $C: |z|=3$
 - b. $\int_C \frac{\sin^2 z}{(z - \pi/6)^3} dz$ where $C: |z|=1$
 - c. $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where $C: |z|=2$.
 - d. $\int_C \frac{e^z}{(z^2 + \pi^2)^2} dz$ where $C: |z|=4$.
 - e. $\oint_C \frac{z^3 - 2z + 1}{(z-i)^2} dz$ where $C: |z|=2$.
 - f. $\oint_C \frac{z}{z^2 - 3z + 2} dz$ where $C: |z-2| = \frac{1}{2}$.
 - g. $\oint_C \frac{e^z dz}{(z+1)^2}$ where $C: |z-1|=3$.
 - h. $\oint_C \frac{\log z}{(z-1)^3} dz$ where $C: |z-1| = \frac{1}{2}$
 - i. $\oint_C \frac{z+4}{z^2 + 2z + 5} dz$ where $C: |z+1-i|=2$.
 - j. $\int_C \frac{e^{3z}}{z^2} dz$ where $C: |z|=1$.



- f. $\oint_C \frac{e^{-z}}{(z-1)(z-2)^2} dz$ where $C: |z|=3$.
- g. $\int_C \frac{3z^2+7z+1}{z+1} dz$ where $C: |z|=\frac{1}{2}$.
- h. $\int_C \frac{2z^2+1}{z^2+z} dz$ where $C: |z|=\frac{1}{2}$.
- i. $\int_C \frac{z^2+1}{z^2(2z+1)} dz$ where $C: |z|=1$.
- o. $\int_C \frac{z^2+z+1}{(z-2)^3} dz$ where $C: |z|=3$.
- p. $\int_C \frac{e^{\pi z}}{(2z-i)^3} dz$ where $C: |z|=1$.
- q. $\int_C \frac{dz}{(z^2+4)^2}$ where $C: |z-i|=2$.
- r. $\int_C \frac{e^{2z}}{(z+1)^2(z-2)} dz$ where $C: |z|=3$.
4. Evaluate $\int_C \frac{dz}{z^2-4}$ over the circle i) $|z|=1$ ii) $|z|=3$ iii) $|z+2|=1$.
5. Evaluate $\int_C \frac{e^z}{z+i\pi} dz$ over the circle i) $|z|=2\pi$ ii) $|z|=\pi/2$.
6. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ over the circle i) $|z|=3$ ii) $|z|=1/2$ iii) $|z+2|=3$.
7. Evaluate $\int_C \frac{z^2+1}{z^2-1} dz$ over the Circle i) $|z+1|=1$ ii) $|z-1|=1$.