Vector Analysis

Scalar : The term scalar refers to a quantity whose value may be represented by a single number (positive or negative)

Eg: * Temperature on any point in a geographical area
* Speed of a vehicle on road
* Pressure at any point in a liquid in vessel etc

Vector : A vector quantity has both magnitude and direction in space. There may be n-dimensional space.

Eg: * Velocity (car's velocity is 70 km/hr, south)
* If a person steps 1 step forward and 1 step backward then his velocity is zero.
* Accelaration due to gravity
* Force applied on door to push

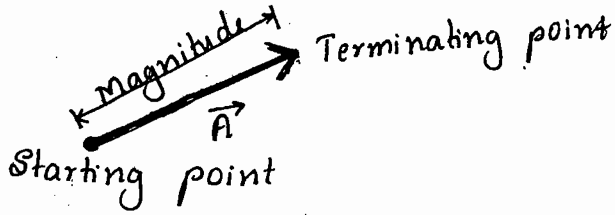
Fields : Field is a physical quantity that takes on different values at different locations.

Scalar Fields : Eg: The temperature throughout the bowl of soup

Vector Fields : Eg: Voltage grdient in a cable

The value of a field varies in general with both position & time.

In two-dimension vector can be represented by a straight line with an arrow in a plane. The length of the segment is magnitude, and arrow indicates direction



<u>Vector Algebra</u> : It includes addition, subtraction, scaling multiplication of vectors.

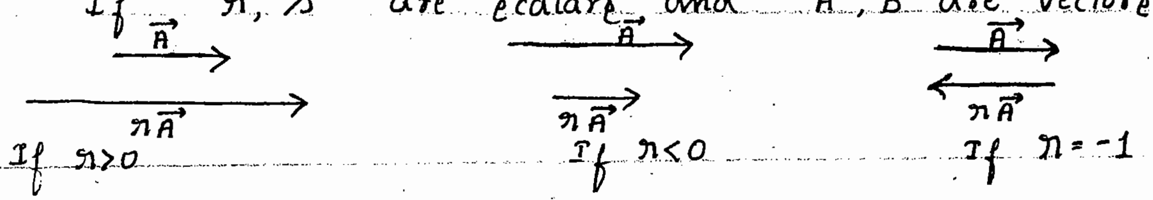
<u>Scaling of Vectors</u>

→ It is multiplication of a vector by scalar.

→ Scaling changes magnitude but the direction remains same.

→ When the scalar is negative it reverses the direction.

If r, s are scalars and $\vec{A}, \vec{B}$ are vectors

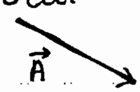


→ Scaling of a vector obeys associative law and distributive law

$$(r+s)(\vec{A}+\vec{B}) = r(\vec{A}+\vec{B}) + s(\vec{A}+\vec{B})$$

$$(r+s)(\vec{A}+\vec{B}) = r\vec{A} + r\vec{B} + s\vec{A} + s\vec{B}$$

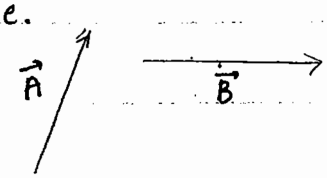
→ Division of a vector by a scalar is just multiplication by the reciprocal of the scalar.



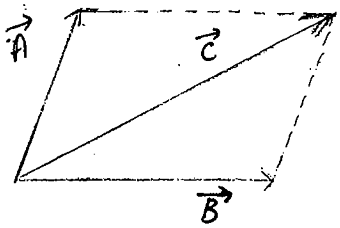
$\vec{A}(\frac{1}{2}) = \vec{A}/2$

<u>Addition of Vectors</u>

Vector addition can be done using parallelogram rule.

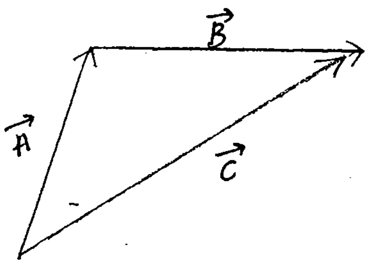


If vector $\vec{A}$ and $\vec{B}$ are to be added then, move one of two vectors parallel to itself without changing its direction to origin of other vector Construct the parallelogram.



Vector $\vec{C}$ is the result of addition of 2 vectors i.e, $\vec{C} = \vec{A} + \vec{B}$.

Two vectors can be added by beginning the second vector from head of first and completing triangle. i.e, Head to tail rule of addition of vectors.

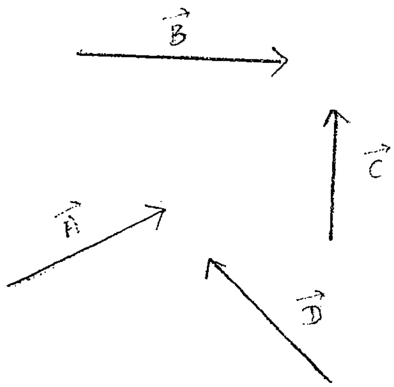


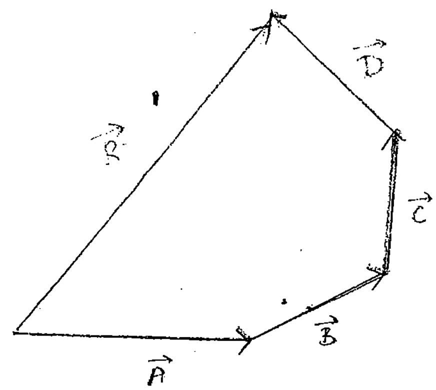
$\vec{A} + \vec{B}$ $\quad$ $\vec{B} + \vec{A}$

Vector addition also obeys associative law

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

If $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ are vectors



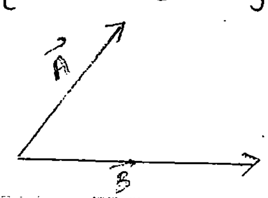


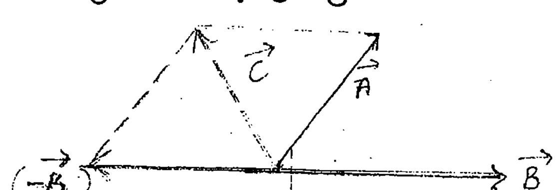
The resultant of all vectors $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$

Subtraction of Vectors

This can be obtained from vector addition. If $\vec{B}$ is to be subtracted from $\vec{A}$ then $\vec{C} = \vec{A} + (-\vec{B})$

where $(-\vec{B})$ is reverse of $\vec{B}$ by multiplying with -1.





Coordinate Systems

The coordinate systems provide specific lengths, directions, angles, projections to describe a vector accurately.

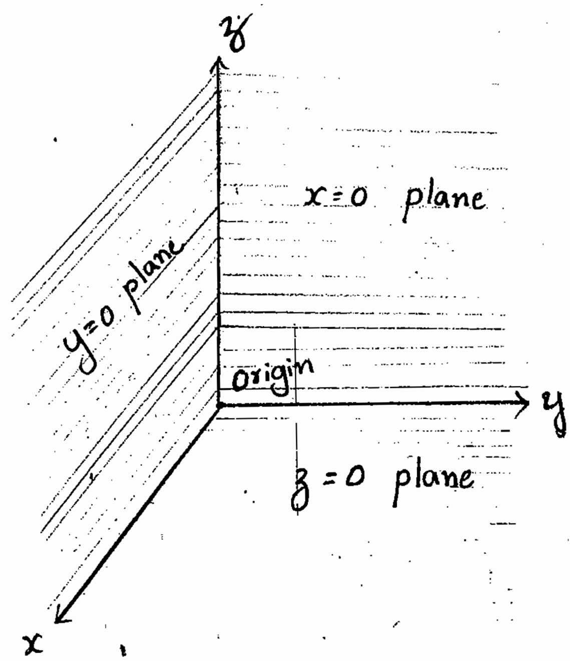
The three simple coordinate systems are

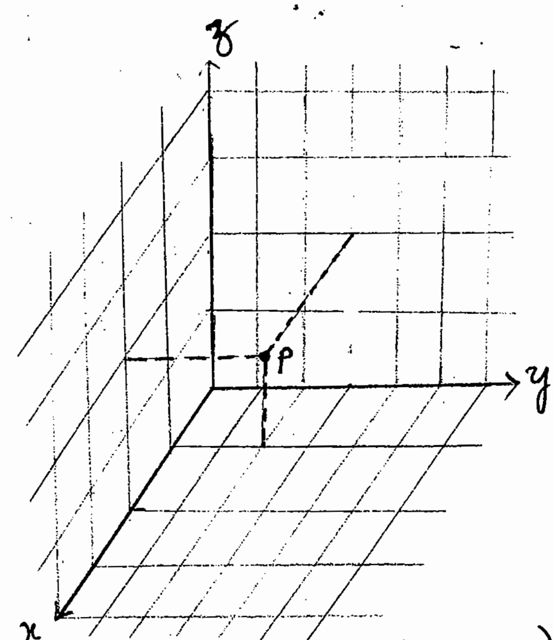
1. Cartesian rectangular coordinate system
2. Circular cylindrical coordinate system
3. Spherical coordinate system.

① Rectangular Coordinate System

→ There are three coordinate axes mutually at right angles to each other and call them as x, y and z.

→ A pt is located by giving its x, y and z coordinates. These are respectively the distances from the origin to the intersection of a perpendicular dropped from the point to x, y and z planes where x=0, y=0 and z=0 planes





Point P has coordinates (2, 2, 2)

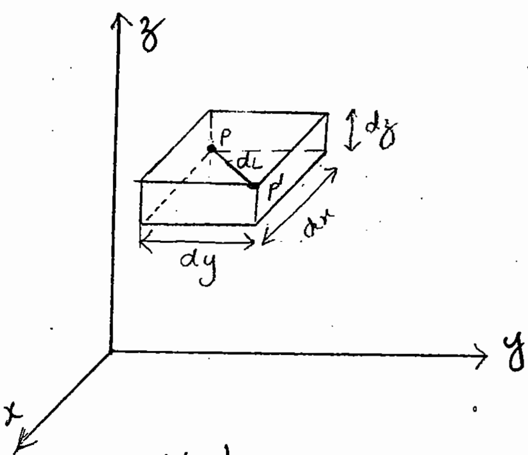
If we visualize three planes intersecting at general point P whose coordinates are x, y and z, we may increase each coordinate value by a differential amount

It will form three slightly displaced planes at P' whose coordinates are $(x+dx, y+dy, z+dz)$

The six planes define parallelpiped whose volume is
$dv = (dx\ dy\ dz)$

The surfaces have differential areas ds of $dxdy$, $dydz$, $dzdx$

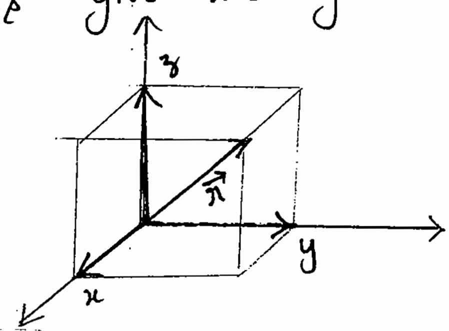
The distance dL from P to P' is the diagonal of parallelpiped and has length of $\sqrt{(dx)^2+(dy)^2+(dz)^2}$. P is located at invisible corner



Vector Components and Unit Vectors

Let us consider a vector $\vec{r}$ extending outward from the origin

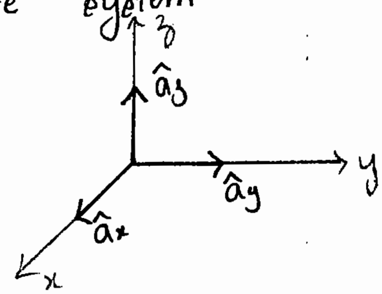
Logically this vector can be identified by giving component vectors along three coordinate axes. The vector sum of component vectors give the given vector $\vec{r}$



i.e, the component vectors have magnitudes which depend on given vector R but they each have a constant direction.

Unit vectors have unit magnitude and directed along the coordinate axes in the direction of increasing coordinate values.

Unit vectors can be denoted as $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ in rectangular coordinate system

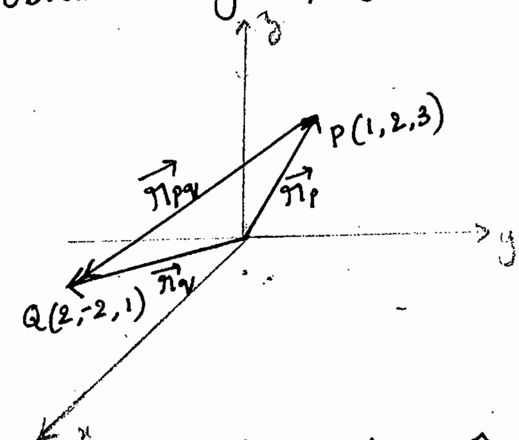


A vector $\vec{r}_p$ pointing from origin to point $P(1,2,3)$ is written $\vec{r}_p = \hat{a}_x + 2\hat{a}_y + 3\hat{a}_z$ and $\vec{r}_q$ pointing from origin to point $Q(2,-2,1)$ is written as $\vec{r}_q = 2\hat{a}_x - 2\hat{a}_y + \hat{a}_z$

The vector from P to Q can be obtained by applying the rule of addition of vectors

i.e,
$$\vec{r}_p + \vec{r}_{pq} = \vec{r}_q$$
$$\vec{r}_{pq} = \vec{r}_q - \vec{r}_p$$

III'ly
$$\vec{r}_{qp} = \vec{r}_p - \vec{r}_q$$



Any vector $\vec{B}$ then may be described by $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

The magnitude of B is $|B| = \sqrt{B_x^2 + B_y^2 + B_z^2}$

III'ly unit vector in given direction of vector B is

$$\vec{a}_B = \frac{B}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{B}{|B|}$$

The Dot Product

The dot product or scalar product is defined as the product of the magnitude of $\vec{A}$, the magnitude of $\vec{B}$ and the cosine of smaller angle between them

$$\vec{A} \cdot \vec{B} = |A||B| \cos \theta_{AB}$$

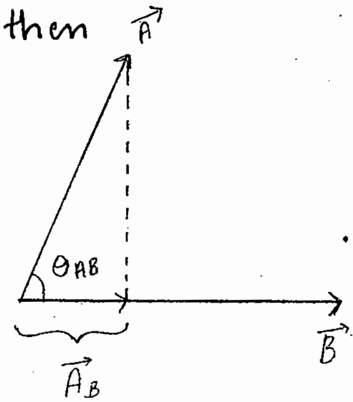
The geometrical term projection is also used in dot product

If $\vec{A_B}$ is projection of $\vec{A}$ on $\vec{B}$ then

$$|\vec{A_B}| = \cos \theta_{AB} |\vec{A}|$$

$$\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos \theta_{AB}$$

$$\vec{A} \cdot \vec{B} = |\vec{B}| \, |\vec{A_B}|$$



Properties of Dot Product

ⓐ If 2 vectors are parallel to each other, $\theta = 0^\circ$ then

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$$

ⓑ If 2 vectors are perpendicular then $\theta = 90^\circ$

$$\vec{A} \cdot \vec{B} = 0$$

ⓒ Dot product obeys commutative law

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

ⓓ Dot product obeys distributive law

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

ⓔ The dot product of a vector with itself is the square of magnitude of that vector

$$\vec{A} \cdot \vec{A} = |\vec{A}| \, |\vec{A}| \cos(0) = |\vec{A}|^2$$

(f) If $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ are unit vectors in cartesian coordinate system. All these vectors are mutually perpendicular to each other then

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$

(g) Any unit vector dotted with itself is unity

$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$$

(h) If $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$\vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z$

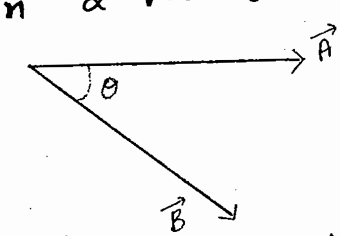
Then $\vec{A} \cdot \vec{B}$ has 9 scalar terms among them 6 terms equal to zero because of property (f)

$$\vec{A} \cdot \vec{B} = A_x B_x (\hat{a}_x \cdot \hat{a}_x) + \cancelto{0}{A_x B_y (\hat{a}_x \cdot \hat{a}_y)} + \cancelto{0}{A_x B_z (\hat{a}_x \cdot \hat{a}_z)}$$
$$+ \cancelto{0}{A_y B_x (\hat{a}_y \cdot \hat{a}_x)} + A_y B_y (\hat{a}_y \cdot \hat{a}_y) + \cancelto{0}{A_y B_z (\hat{a}_y \cdot \hat{a}_z)}$$
$$+ \cancelto{0}{A_z B_x (\hat{a}_z \cdot \hat{a}_x)} + \cancelto{0}{A_z B_y (\hat{a}_z \cdot \hat{a}_y)} + A_z B_z (\hat{a}_z \cdot \hat{a}_z)$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

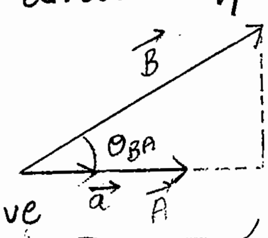
Applications of Dot product

① Finding angle between 2 vectors $\theta = \cos^{-1} \left\{ \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|} \right\}$



② To find component of a vector in a given direction

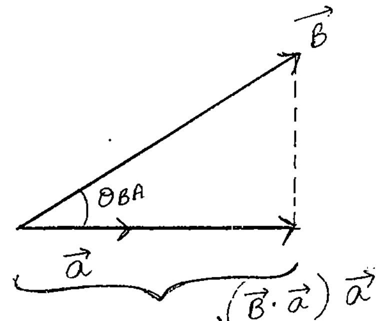
If we need component of $\vec{B}$ in the direction specified by unit vector $\hat{a}$



• If $0 \leq \theta_{BA} \leq 90^\circ$ Sign of component is positive

$90^\circ < \theta_{BA} \leq 180^\circ$ Sign of component is negative

③ Similarly component vector of vector B in the direction of unit vector $\vec{a}$ by just multiplying the component by $\vec{a}$ i.e,



Hence component of a vector in any direction becomes the problem of finding unit vector in that direction.

Note: ① Component of a vector in given direction → scalar

② Component vector in a given direction → Vector

④ Physical work done by a constant force can be expressed as dot product of 2 vectors

$$W = |\vec{F}|\, d \cos\theta = \vec{F}\cdot\vec{d}$$

If force varies along with path then total work done is

$$W = \int \vec{F}\cdot\vec{dL}$$

Vector Field

Vector field is vector function of a position vector. The magnitude and direction of the function will change as we move throughout the region.

In rectangular coordinate system vector field should be a function of x, y and z.

If position vector is $\vec{r}$ vector field $\vec{G}$ can be written as $\vec{G}(r)$

Eg: Velocity vector $\vec{V} = V_x(r)\,\hat{a}_x + V_y(r)\,\hat{a}_y + V_z(r)\,\hat{a}_z$

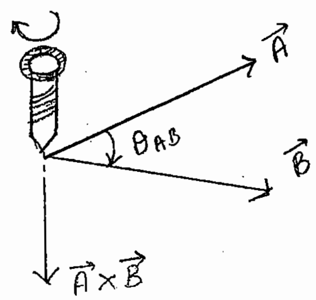
Each component V_x, V_y and V_z may be function of three variables

Cross Product

The cross product $\vec{A} \times \vec{B}$ (A cross B) is a vector, the magnitude of $\vec{A} \times \vec{B}$ is equal to the product of the magnitudes of $\vec{A}, \vec{B}$ and the sine of the smaller angle between $\vec{A}$ & $\vec{B}$. The direction of $\vec{A} \times \vec{B}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$

The direction of resultant $\vec{A} \times \vec{B}$ is along one of 2 possible perpendiculars which is in the direction of advance of a right handed screw as $\vec{A}$ is turned into $\vec{B}$.

$$\vec{A} \times \vec{B} = \vec{a}_N \, |\vec{A}| \, |\vec{B}| \, \sin(\theta_{AB})$$



$a_N \rightarrow$ Unit vector
$N \rightarrow$ Stands for Normal

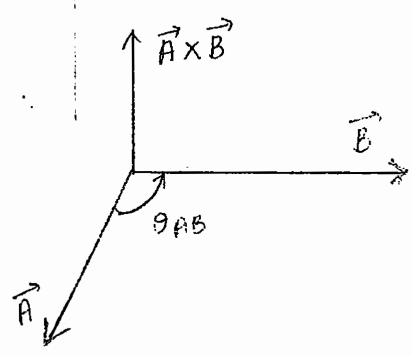
Properties of Cross Product

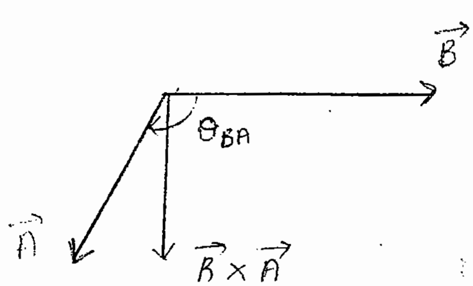
(1) The commutative law is not applicable for cross product

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

Reversing the order of vector reverses the resultant vector

i.e, $$[\vec{A} \times \vec{B}] = -[\vec{B} \times \vec{A}]$$





(2) The cross product is not associative

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

(3) Cross product is distributive

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(4) If two vectors are in same direction i.e, $\theta = 0°$ then the cross product is zero

(5) The cross product of a vector with itself is 0

$$\vec{A} \times \vec{A} = 0$$

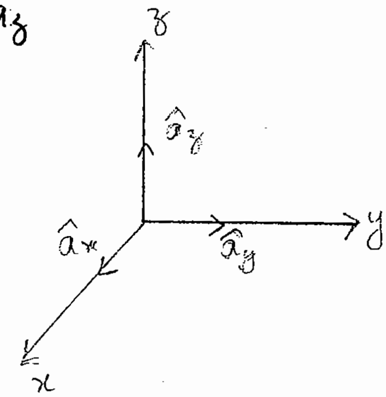
(6) Cross product of unit vectors

$$\hat{a}_x \times \hat{a}_y = |a_x|\,|a_y|\,\sin(90°)\,\vec{a}_N$$

$$\sin(90°) = |a_x| = |a_y| = 1 \quad \& \quad \vec{a}_N = \hat{a}_z$$

Hence

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$
$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$
$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$



(7) Cross product in determinant form

Consider two vectors $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$\vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z$

$$\vec{A} \times \vec{B} = \cancel{A_x B_x (\hat{a}_x \times \hat{a}_x)}^{0} + A_x B_y (\hat{a}_x \times \hat{a}_y) + A_x B_z (\hat{a}_x \times \hat{a}_z) +$$
$$A_y B_x (\hat{a}_y \times \hat{a}_x) + \cancel{A_y B_y (\hat{a}_y \times \hat{a}_y)}^{0} + A_y B_z (\hat{a}_y \times \hat{a}_z) +$$
$$A_z B_x (\hat{a}_z \times \hat{a}_x) + A_z B_y (\hat{a}_z \times \hat{a}_y) + \cancel{A_z B_z (\hat{a}_z \times \hat{a}_z)}^{0}$$

$$= A_x B_y (\hat{a}_z) + A_x B_z (-\hat{a}_y) + A_y B_x (-\hat{a}_z) + A_y B_z (\hat{a}_x) +$$
$$A_z B_x (\hat{a}_y) + A_z B_y (-\hat{a}_x)$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\,\hat{a}_x + (A_z B_x - A_x B_z)\,\hat{a}_y + (A_x B_y - A_y B_x)\,\hat{a}_z$$

In determinant form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Circular Cylindrical Coordinates

→ It is a three dimensional coordinate system.

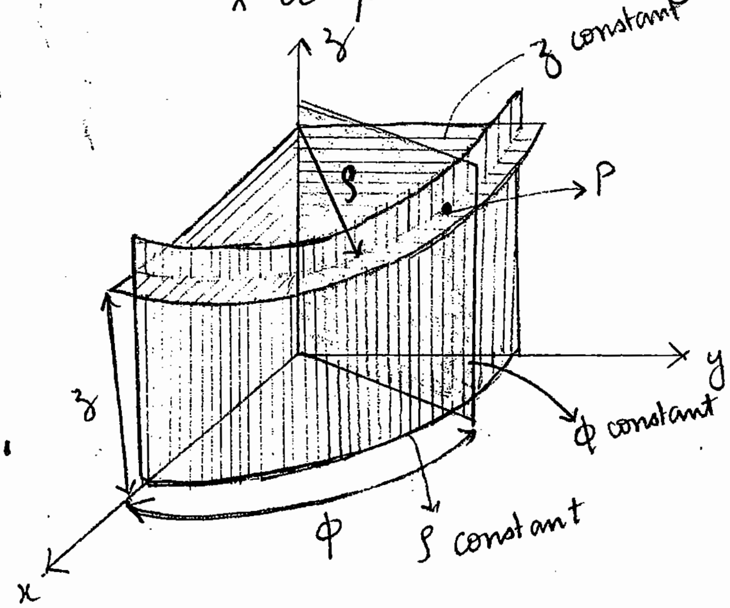
→ It consists of a point which is located in a plane by giving its distance ρ from the origin.

→ An angle ϕ between the line from the point to the origin and an arbitrary radial line taken as $\phi = 0$

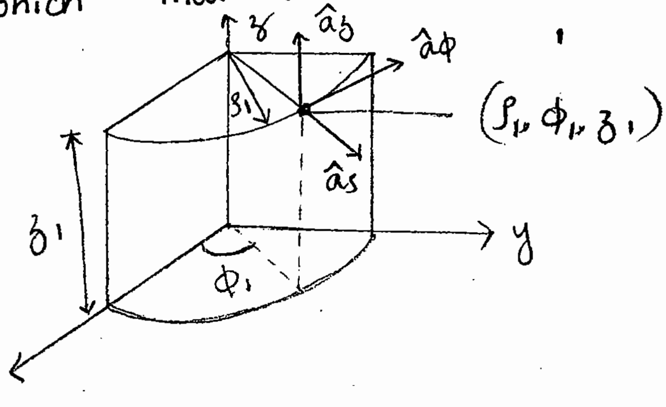
→ The distance z of the point from an arbitrary $z=0$ reference plane which is perpendicular to the line $\rho = 0$.

→ Any point on cylindrical coordinate system can be considere as intersection of three mutually perpendicular planes.

These planes are
* a circular Cylinder ρ = constant
* a plane where ϕ = constant
* a plane where z = constant

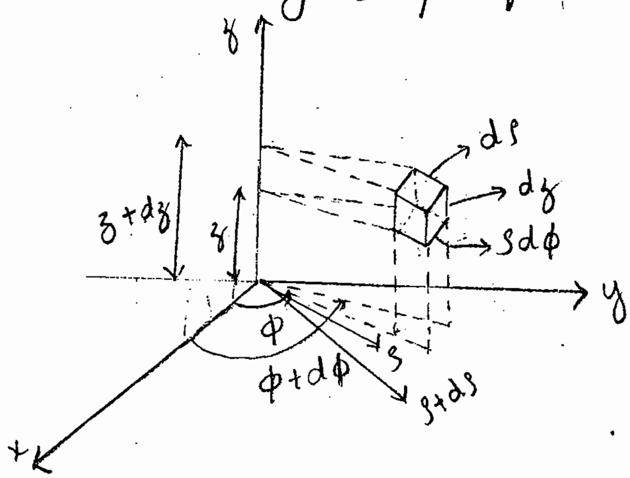


Unit vector can be considered directed towards the increasing coordinate values and are perpendicular to the surface on which that coordinate value is constant.



A differential volume in cylindrical coordinates may be obtained by increasing ρ, ϕ and z by differential increments $d\rho$, $d\phi$ and dz.

The two cylinders of radius ρ and $\rho + d\rho$
The radial planes at angles ϕ and $\phi + d\phi$
Two horizontal planes at elevation z and $z + dz$
give a small volume having shape of truncated wedge



This forms rectangular parallelpiped with sides of length $d\rho$, dz, $\rho d\phi$
The surfaces have areas $\rho d\phi \, d\rho$, $d\rho dz$, $\rho d\phi \, dz$
Volume becomes $\rho d\phi \, d\rho \, dz$.

The relationship between rectangular and cylindrical systems.

$$x = \rho \cos\phi \qquad\qquad \rho = \sqrt{x^2+y^2}$$
$$y = \rho \sin\phi \qquad \& \qquad \phi = \tan^{-1}(y/x)$$
$$z = z \qquad\qquad z = z$$

Using above equations scalar functions in one coordinate system can be easily converted to other

If $\vec{A} = A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z$ and we need vector in cylindrical coordinates as $\vec{A} = A_\rho\,\hat{a}_\rho + A_\phi\,\hat{a}_\phi + A_z\,\hat{a}_z$

To find desired component of a vector we find dot product of the vector and unit vector in desired direction.

i.e, If A_ρ is scalar component of vector $\vec{A}$ in ρ direction then

$$A_\rho = \vec{A}\cdot\hat{a}_\rho = (A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z)\cdot\hat{a}_\rho = A_x\,\hat{a}_x\cdot\hat{a}_\rho + A_y\,\hat{a}_y\cdot\hat{a}_\rho + \cancel{A_z\,\hat{a}_z\cdot\hat{a}_\rho}$$

$$\boxed{A_\rho = A_x\,\hat{a}_x\cdot\hat{a}_\rho + A_y\,\hat{a}_y\cdot\hat{a}_\rho}$$

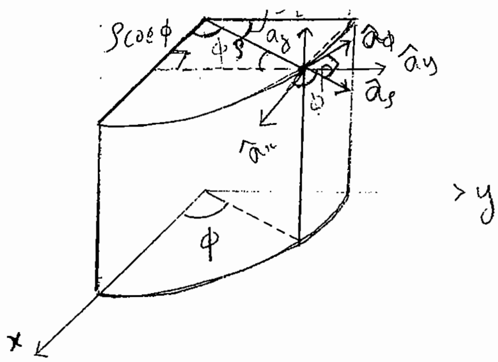
$$A_\phi = \vec{A}\cdot\hat{a}_\phi = (A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z)\cdot\hat{a}_\phi = A_x\,\hat{a}_x\cdot\hat{a}_\phi + A_y\,\hat{a}_y\cdot\hat{a}_\phi + \cancel{A_z\,\hat{a}_z\cdot\hat{a}_\phi}$$

$$\boxed{A_\phi = A_x\,\hat{a}_x\cdot\hat{a}_\phi + A_y\,\hat{a}_y\cdot\hat{a}_\phi}$$

$$A_z = \vec{A}\cdot\hat{a}_z = (A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z)\cdot\hat{a}_z = A_z$$

$$\boxed{A_z = A_z}$$

In above equations the dot product can be solved by applying definition of dot product. Since these are unit vectors the dot product will be cosine of angle between two vectors



	a_ρ	a_ϕ	a_z
$a_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$a_y \cdot$	$\sin\phi$	$\cos\phi$	0
$a_z \cdot$	0	0	1

The Spherical Coordinate System

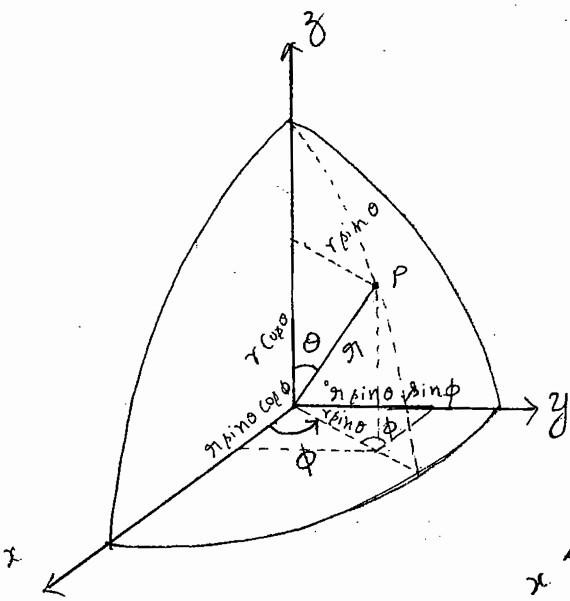
→ The first coordinate in spherical coordinate system is the distance from the origin to any point as r.

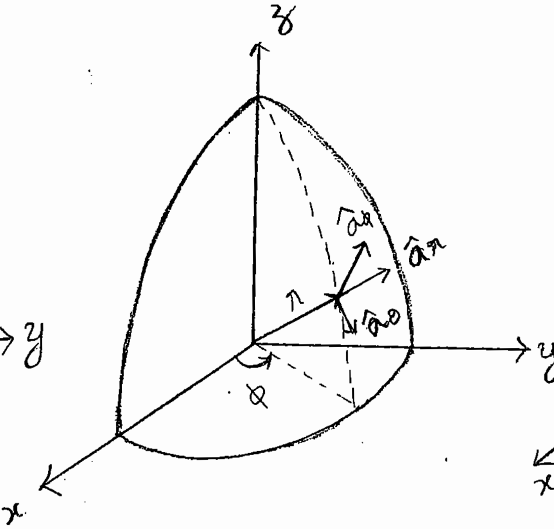
→ The second coordinate is the angle θ between the z axis and the line drawn from the origin to the point P

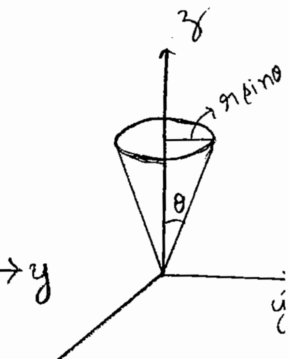
→ The surface θ = constant is a cone. The surface of sphere and surface of cone are everywhere perpendicular to each other. The intersection of θ = Constant and surface of sphere forms a circle of radius $r\sin\theta$.

→ The third coordinate ϕ is also an angle between the x axis and the projection of point P on $z=0$ plane

→ Unit vectors may be defined at any point. Each unit Vector is $\perp^{lar}$ to one another and oriented in the increasing direction of Coordinates

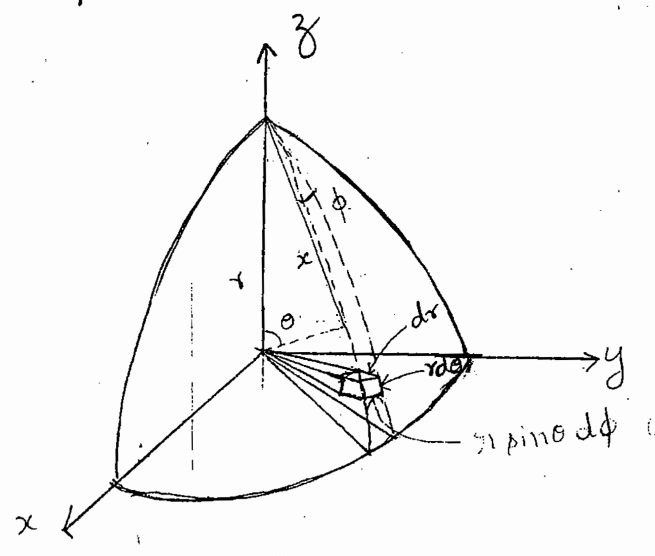


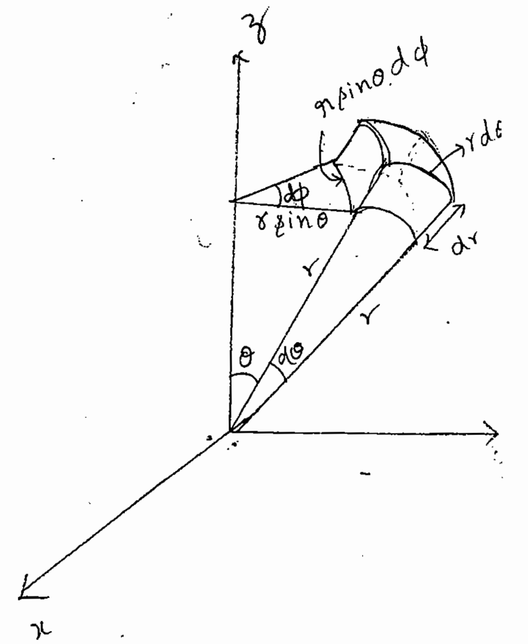




θ = Constant

→ A differential volume element may be constructed in spherical coordinate by increasing r, θ and φ by dr, dθ and dφ as shown in figure.





→ The volume i.e $r^2 \sin\theta \, dr \, d\theta \, d\phi$

→ The transformations of scalars from rectangular to spherical coordinate system i.e

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

The transformation in the reverse direction is achived

$$r = \sqrt{x^2+y^2+z^2} \qquad (r \geq 0)$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

$$\phi = \tan^{-1}(y/x)$$

The transformation of vectors requires the determination of the products of the unit vectors in rectangular & spherical coordinates

	$\hat{a}_r$	$\hat{a}_\theta$	$\hat{a}_\phi$
$\hat{a}_x \cdot$	$\sin\theta \cos\phi$	$\cos\theta \sin\phi$	$-\sin\phi$
$\hat{a}_y \cdot$	$\sin\theta \sin\phi$	$\cos\theta \sin\phi$	$\cos\phi$
$\hat{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Unit - 1

Introduction to Electrostatics

Coulomb's Law and Electric field Intensity

Coulomb stated that the force between two very small objects separated in vacuum or free space by distance R which is large compared to their size is proportional to the charge on each and inversely proportional to the square of distance between them.

$$F = k\frac{Q_1 Q_2}{R^2} \quad (\text{measured in } N)$$

Q_1 & Q_2 → Positive or negative quantity of charges

R → Separation measured in meters

k → Proportionality constant

This will be achieved if $k = \frac{1}{4\pi\epsilon_0}$

ϵ_0 permittivity of free space (measured in F/m^2)

$$\epsilon_0 = 8.854 \times 10^{-12} = \frac{1}{36\pi}10^{-9} \; F/m$$

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

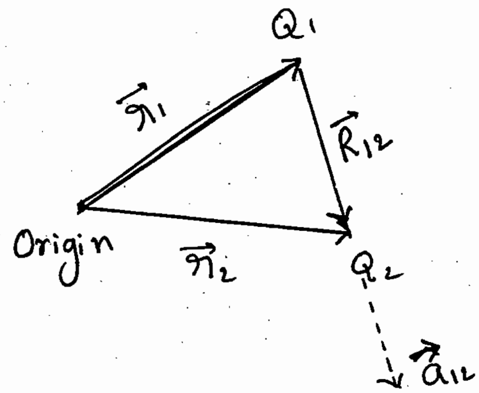
Vector form of above equation can be written by considering the force acting along the line joining the 2 charges. Force is repulsive if they have same sign and attractive if opposite sign.

Let the vector $\vec{r}_1$ locates Q_1 and $\vec{r}_2$ locates Q_2 then $\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$ directed line segment from Q_1 to Q_2

i.e, $$\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\, \vec{a}_{12}$$

$\vec{F}_2$ is the force on Q_2 and $\vec{a}_{12}$ is the unit vector in the direction of $\vec{R}_{12}$

$$\vec{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$



Electric Field Intensity.

If Q_1 is the charge fixed in a position and Q_t is test charge moving around Q_1. Then Q_1 will exert force on Q_t

This force is given by $\vec{F}_t = \dfrac{Q_1 Q_t}{4\pi\epsilon_0 |\vec{R}_{1t}|^2} \vec{a}_{1t}$

$$\text{Force/unit charge is } \frac{\vec{F}_t}{\vec{Q}_t} = \frac{Q_1}{4\pi\epsilon_0 |R_{1t}|^2} \vec{a}_{1t}$$

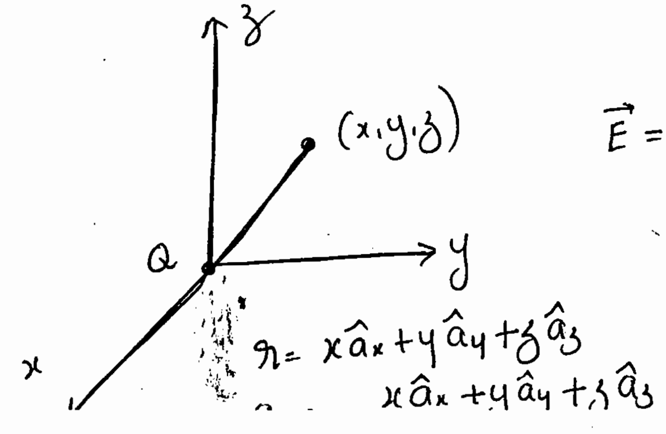
The quantity $\dfrac{Q_1}{4\pi\epsilon_0 |R_{1t}|^2} \vec{a}_{1t}$ is function of only Q_1 and directed line segment from Q_1 to Q_t. This describes a vector field known as Electric Field Intensity.

The unit of electric field intensity is N/c i.e, Force/unit charge is also measured in V/m.

Electric field intensity $\vec{E} = \vec{F}_t / Q_t$

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2} \hat{a}_{1t}$$

If charge Q is at the origin and $\vec{E}$ at (x,y,z) is



$$\vec{E} = \frac{Q}{4\pi\epsilon_0 [x^2+y^2+z^2]} \frac{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}{[x^2+y^2+z^2]^{3/2}}$$

If charge is not at the origin of coordinate system :,
if a charge Q located at $r' = x'\hat{a}_x + y'\hat{a}_y + z'\hat{a}_z$
need its field intensity at point $r = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$
by expressing R as $r - r'$

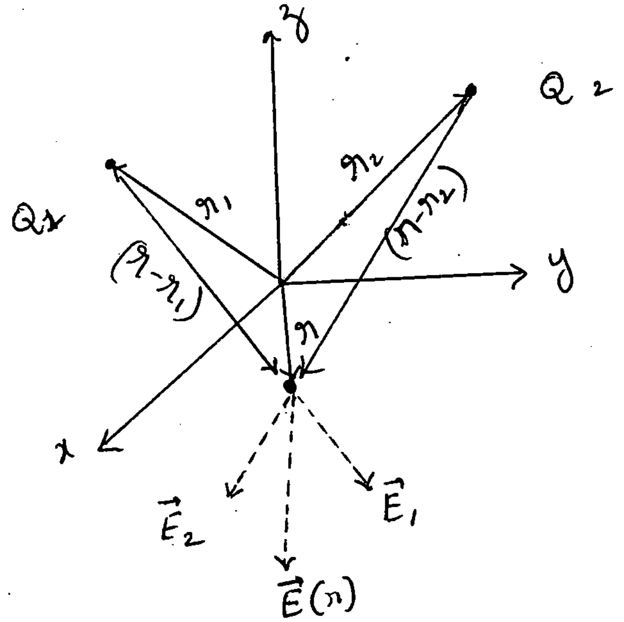
$$E(r) = \frac{Q}{4\pi\epsilon_0 (|r-r'|)^2} \frac{(r-r')}{|r-r'|} = \frac{Q(r-r')}{4\pi\epsilon_0 (r-r')^3}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{(x-x')\hat{a}_x + (y-y')\hat{a}_y + (z-z')\hat{a}_z}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}$$

→ Coulomb forces are linear i.e, electric field intensity due to two point charges Q_1 at r_1 and Q_2 at r_2 is sum of forces on Q_t caused by Q_1 and Q_2 acting alone

$$\vec{E}(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|^2} \vec{a}_1 + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|^2} \vec{a}_2$$

where a_1 and a_2 are unit vectors in the direction $(r-r_1)$ and $(r-r_2)$



The field

$$\vec{E}(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|^2}\vec{a}_1 + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|^2}\vec{a}_2 + \cdots + \frac{Q_n}{4\pi\epsilon_0 |r-r_n|^2}\vec{a}_n$$

$$\vec{E}(r) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |r-r_m|^2}\vec{a}_m \quad V/m$$

Field due to a continous volume charge distribution

The space filled with large number of charges with smooth continous distribution described by a volume charge density

Volume charge density is denoted by ρ_v (C/m^3)

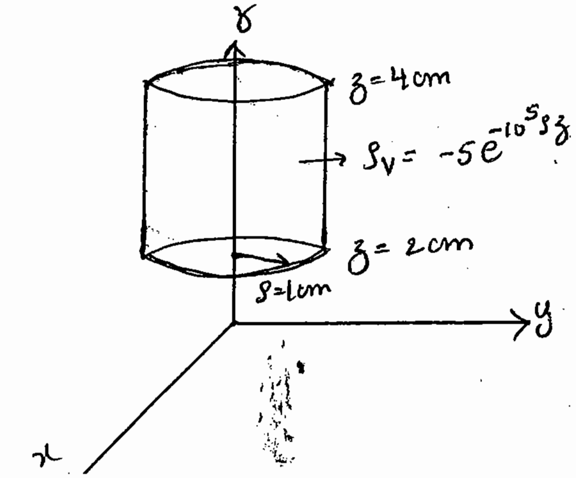
The small amount of charge ΔQ in small volume ΔV is

$$\Delta Q = \rho_v \Delta V$$

The total charge within some finite volume is

$$Q = \int_{vol} \rho_v \, dv$$

Eg: The total charge contained in 2cm length electron beam as shown in the figure is



The total charge

$$Q = \int_{vol} \rho_v \, dv$$

$$= \int_{z=0.02}^{0.04} \int_{\rho=0}^{0.01} \int_{\phi=0}^{2\pi} \rho_v \, \rho \, d\rho \, d\phi \, dz$$

$$Q = \int_{z=0.02}\int_{\rho=0}\int_{\phi=0} -5\times10^{-6}\, e^{-10^5 \rho z}\, \rho\, d\rho\, d\phi\, dz$$

$$= \int_{z=0.02}^{0.04}\int_{\rho=0}^{0.01} -5\times10^{-6}\, e^{-10^5\rho z}\, \rho\, d\rho\, (2\pi)$$

$$= -10\pi\times10^{-6}\int_{z=0.02}^{0.04}\int_{\rho=0}^{0.01} e^{-10^5\rho z}\, \rho\, d\rho\, dz$$

$$= -\pi\times10^{-5}\int_{\rho=0}^{0.01} \frac{-1}{10^5\rho}\, e^{-10^5\rho z}\Big]_{0.02}^{0.04} \rho\, d\rho$$

$$= \frac{\pi\times10^{-5}}{10^5}\int_{\rho=0}^{0.01}\left(e^{-10^3\rho\cdot 4} - e^{-10^3 2\rho}\right) d\rho$$

$$= 10^{-10}\pi\left[\int_{\rho=0}^{0.01} e^{-4000\rho}\, d\rho - \int_{\rho=0}^{0.01} e^{-2000\rho}\, d\rho\right]$$

$$= 10^{-10}\pi\left\{\left(\frac{-1}{4000}\right) e^{-4000\rho}\Big]_0^{0.01} - \left(\frac{-1}{2000}\right) e^{-2000\rho}\Big]_0^{0.01}\right\}$$

$$= 10^{-10}\pi\left\{\frac{-1}{4000}\left[e^{-40} - e^{0}\right] + \frac{1}{2000}\left[e^{-20} - e^{0}\right]\right\}$$

$$10^{-10}\pi\left[\frac{1}{[illegible]} - \frac{1}{[illegible]}\right] = 0.0785\ pC \quad \text{electron}$$

Field of a Line Charge

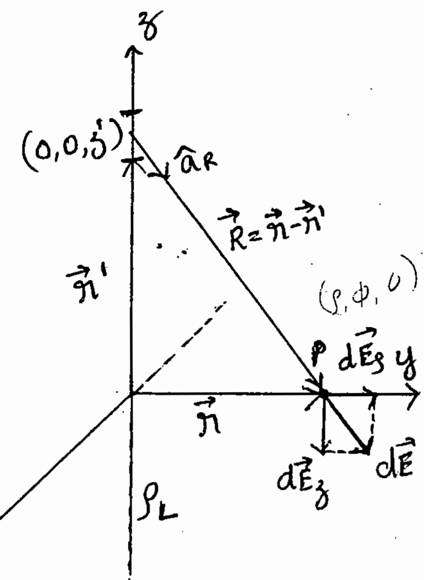
→ The charge distribution in a charged conductor with small radius can be treated as line charge density ρ_L C/m

→ If electron motion is steady and uniform and if we ignore magnetic field for a moment these electron beam may be considered as stationary electrons.

· Consider a straight line charge extending along the z-axis in a cylindrical coordinate system from $-\infty$ to ∞ we desire the electric field intensity $\vec{E}$ at any point from a uniform line charge density

· Symmetry should be considered first in order to determine two specific factors

1) with which coordinates the field does not vary
2) which component of the field are not present.



Lets analyze first factor

From figure, as we move around the line charge varying ϕ while keeping ρ and z constant the line charge appears the same from every angle i.e, azimuthal symmetry & no field component may vary with ϕ

Again if ρ and ϕ are maintained constant while moving z along the line charge the line charge will not vary i.e, axial symmetry leads to fields which are not functions of z.

If ϕ and ρ are ... as ρ increases. Hence to coulomb's law field become weaker as ρ increases. Hence field varies only with ρ.

→ Each incremental length of line charge acts as point charge and produces an incremental contribution to the electric field.

→ No incremental length produces electric field in ϕ direction hence $\vec{E}_\phi = 0$

→ E_z component is also zero because of presence of charges which are of equal distances above and below the point, this will cancel the field.

→ Therefore there is only $\vec{E}_\rho$ that varies along with ρ coordina

Choose a point $P(0,y,0)$ on y-axis, the incremental charge $dQ = \rho_L dz'$ we h field at P because of incremental charge

$$d\vec{E} = \frac{\rho_L \, dz' (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}$$

$$\vec{r} = y\hat{a}_y = \rho\hat{a}_\rho$$

$$\vec{r}' = z'\hat{a}_z$$

$$\vec{r} - \vec{r}' = \rho\hat{a}_\rho - z'\hat{a}_z$$

$$d\vec{E} = \frac{\rho_L \, dz' \, (\rho\hat{a}_\rho - z'\hat{a}_z)}{4\pi\epsilon_0 \left[\rho^2 + z'^2\right]^{3/2}}$$

Only $\vec{E}_\rho$ component is present we may simplify

$$dE_\rho = \frac{\rho_L \, dz' \, \rho}{4\pi\epsilon_0 \left[\rho^2 + z'^2\right]^{3/2}}$$

$$E_\rho = \int_{-\infty} \frac{\rho_L \, \rho \, dz'}{4\pi\epsilon_0 \left[\rho^2 + z'^2\right]^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho \, dz'}{\left[\rho^2 + z'^2\right]^{3/2}}$$

Let $z' = \rho \tan\theta$

$dz' = \rho \sec^2\theta \, d\theta$

if $z' = -\infty \Rightarrow \theta = -\pi/2$

$z' = +\infty \Rightarrow \theta = \pi/2$

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\rho^2 \sec^2\theta \, d\theta}{\left(\rho^2 + \rho^2 \tan^2\theta\right)^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\rho^2 \sec^2\theta}{\rho^3 \sec^3\theta} \, d\theta$$

$$= \frac{\rho_L}{4\pi\epsilon_0 \rho} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{\rho_L}{4\pi\epsilon_0 \rho} \sin\theta \Big]_{-\pi/2}^{\pi/2} = \frac{\rho_L}{4\pi\epsilon_0 \rho} \left[\sin(\pi/2) - \sin\left(\frac{-\pi}{2}\right)\right]$$

$$\boxed{E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}}$$

$$d\vec{E} = \frac{\rho_s\, dy'}{2\pi\epsilon_0 \sqrt{x^2+y'^2}} \frac{(x\hat{a}_x - y'\hat{a}_y)}{\sqrt{x^2+y'^2}}$$

$$d\vec{E} = \frac{\rho_s\, dy' (x\hat{a}_x - y'\hat{a}_y)}{2\pi\epsilon_0 (x^2+y'^2)}$$

$y'\hat{a}_y = 0$ as Fields will cancel because of +ve and negative y-axis. Field along x-direction

$$dE_x = \frac{\rho_s\, dy'}{2\pi\epsilon_0} \frac{x}{(x^2+y'^2)}$$

$$E_x = \int_{-\infty}^{\infty} \frac{\rho_s\, dy'}{2\pi\epsilon_0} \frac{x}{(x^2+y'^2)}$$

$$E_x = \frac{\rho_s}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x}{(x^2+y'^2)}\, dy'$$

$y' = x \tan\theta$

$dy' = x \sec^2\theta\, d\theta$

$$E_x = \frac{\rho_s}{2\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{x^2 \sec^2\theta\, d\theta}{(x^2 + x^2\tan^2\theta)}$$

$$= \frac{\rho_s}{2\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} d\theta$$

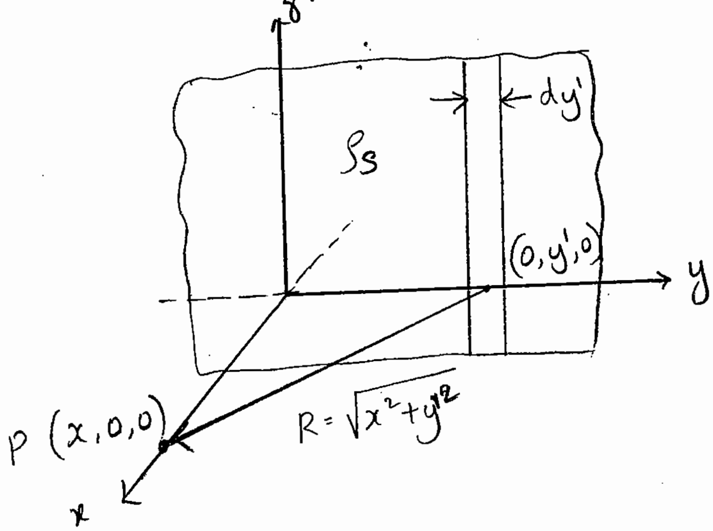
$$E_x = \frac{\rho_s}{2\epsilon_0}$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0} \hat{a}_x}$$

Field of a Sheet of Charge

→ This is charge configuration in the infinite sheet of charge having uniform density of ρ C/m^2, is known as surface charge density. Eg: Strip transmission line parallel plate capacitor.

→ Let us place a sheet of charge along yz plane. Considering symmetry the field will not vary with y & z then the field $\vec{E_y}$ and $\vec{E_z}$ will cancel. Hence only $\vec{E_x}$ is present which is function of x alone



Let us use the field of infinite line charge by dividing the infinite sheet charge into differential width stripe

$\rho_L = \rho_S \, dy'$

The distance from point $P(x,0,0)$ on x-axis is $R = \sqrt{x^2+y'^2}$ differential width stripe to

dE_x is the contribution of differential width stripe to E_x at P.

Electric flux density.

→ Electric flux is line of force around the charge. This line of force starts from positive charge and terminate on the negative charge.

→ Michael Faraday had a pair of concentric metallic spheres. The outer sphere consists of 2 hemispheres which can be clamped together

→ Faraday dismantled outer sphere and he charged inner sphere with some positive charges
Then the outer sphere joined together with dielectric about 2cm between inner and outer sphere.

→ He discharged outer sphere momentarily by connecting it to ground

→ Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed in the inner sphere.

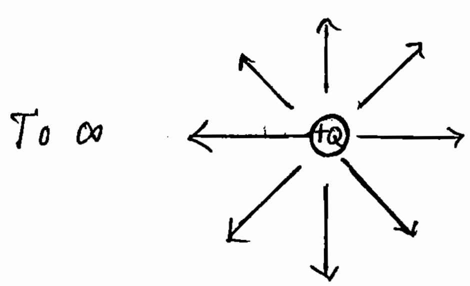
→. Faraday concluded that there was some sort of displacement from inner sphere to the outer sphere which was independent of the medium.
This is refered as displacement flux or simply electric flux.
Thus total number of lines of force in any perticular electric field is called flux.

→ If electric flux is denoted as Ψ ... on the inner sphere by Q then

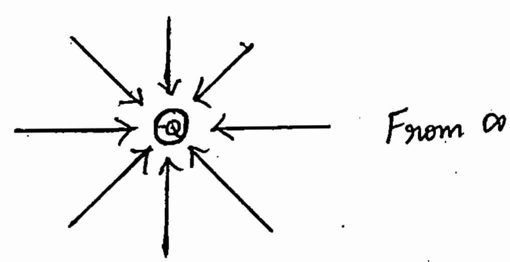
$$\Psi = Q \quad (\text{unit } C)$$

Properties of flux Lines

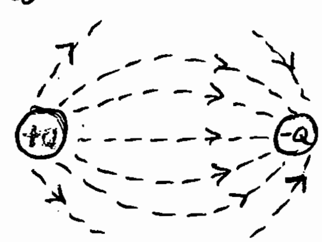
1. Electric flux lines start from positive charge and terminate on negative charge
2. If negative charge is absent then flux lines terminate at
3. There will be more number of lines if electric field is strong.
4. The flux lines will be parallel & never cross other flux line
5. Flux lines are independent of medium in which charges are placed.
6. The lines always enter and leave the charge surface normally.



Flux lines +Q



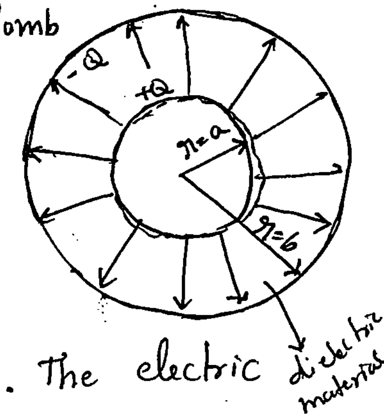
Flux lines -Q



Flux lines +Q to -Q

∴ If inner sphere of radius 'a' and outer sphere of radius 'b' with charges +Q and -Q respectively is considered

At the surface of inner sphere Ψ Coulomb of electric flux are produced by charge Q Coulomb



The density of the flux at this surface is

$$\frac{\Psi}{4\pi a^2} \quad \text{or} \quad \frac{Q}{4\pi a^2} \quad C/m^2$$

The flux density is denoted by letter $\vec{D}$. The electric flux density is a vector field

→ The direction of $\vec{D}$ is direction of flux lines and magnitude is given by the number of flux lines crossing a surface normal to the lines divided by surface area

$$\vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\hat{a}_r$$

$$\vec{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\hat{a}_r$$

at radial distance r where $a \leq r \leq b$

$$\left\langle \vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r \right\rangle$$

W.K.T

$$\left\langle \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r \right\rangle$$

Therefore $\boxed{\vec{D} = \epsilon_0 \vec{E}}$ for free space

For general volume charge distribution

$$\vec{E} = \int_{Vol} \frac{\rho_v dv}{4\pi\epsilon_0 R^2} \hat{a}_R$$

$$\vec{D} = \int_{Vol} \frac{\rho_v dv}{4\pi r^2} \hat{a}_r$$

Gauss's Law.

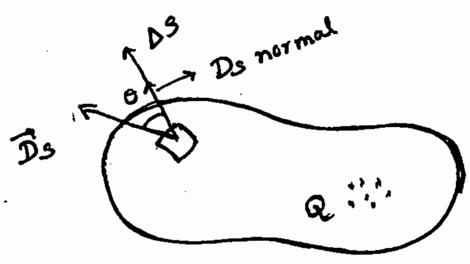
The generalization of Faraday's experiment leads to Gauss's law.

+Q Coulomb of any inner conductor would produce an induced charge of -Q Coulomb on the surrounding conducting surface can be defined as.

i.e, Gauss's law

"The electric flux passing through any closed surface is equal to the total charge enclosed by that surface."

Let us imagine distribution of charges as shown, surrounded by a closed surface of any shape.



If the total charge is Q, then Q Coulomb of electric flux will pass through the enclosing surface

At every point on the surface the electric flux density vector $\vec{D}$ will have some value $\vec{D}_s$. Subscript s indicates that $\vec{D}$ must be evaluated at surface. $\vec{D}_s$ will vary in magnitude and direction from one point to other

→ ΔS is nearly a portion of plane surface. ΔS is a vector quantity as it includes both magnitude & direction in space i.e, $\Delta \vec{S}$

→ Let $D_{s,norm}$ is the normal to surface $\Delta\vec{S}$ at point P.

The total flux crossing ΔS is

$$\Delta\Psi = \text{flux crossing } \Delta\vec{S} = D_{S,norm}\Delta S = D_S \cos\theta \Delta S$$

$$= \vec{D}_S \cdot \Delta\vec{S}$$

The total flux passing through closed surface is

$$\Psi = \int d\psi = \oint_{\substack{\text{Closed}\\ \text{Surface}}} \vec{D}_S \cdot d\vec{S} = \text{charge enclosed} = Q$$

The charge enclosed might be

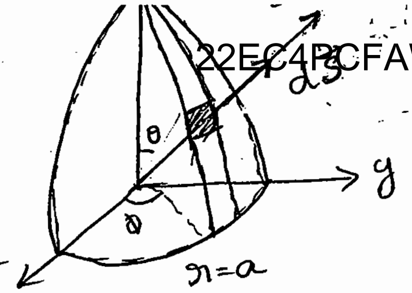
Several point charges $Q = \sum Q_n$

Line charge $Q = \int \rho_L \, dL$

Surface charge $Q = \int \rho_S \, ds$

Volume charge $Q = \int \rho_v \, dv$

Let's find the charge enclosed in sphere with radius a with $\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\vec{a}_r$



Proof: WKT $E = \dfrac{Q}{4\pi\epsilon_0 r^2}\, a_r$

$$\vec{D} = \epsilon_0 \vec{E}$$

$$\vec{D} = \frac{Q}{4\pi r^2}\,\vec{a_r}$$

At surface $d\vec{s}$ of sphere $\vec{D}_s = \dfrac{Q}{4\pi a^2}\,\hat{a}_r$

$$ds = a^2 \sin\theta \, d\theta \, d\phi$$

$$d\vec{s} = a^2 \sin\theta \, d\theta \, d\phi \, \vec{a_r}$$

$$\vec{D_s} \cdot d\vec{s} = \frac{Q}{4\pi a^2}\,\vec{a_r} \cdot a^2 \sin\theta \, d\theta \, d\phi \, \vec{a_r}$$

$$d\psi = \vec{D_s} \cdot d\vec{s} = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\psi = \oint_s \vec{D_s} \cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{Q}{4\pi}(2\pi) \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$= \frac{Q}{4\pi}(2\pi) \left. -\mathrm{Cos}\,\theta \right]_0^{\pi}$$

$$= \frac{Q}{\cancel{4\pi}} \cancel{(2\pi)}\cancel{(2)}$$

$$\underline{\underline{\psi = Q}}$$

Applications of Gauss's Law (Symmetrical Charge distributions)

Gauss's law is another way of stating Coulomb's law

Gauss's law can be used to find $\vec{E}$ & $\vec{D}$ for symmetrical charge distributions like point charge, infinite line charge, an infinite sheet charge and spherical distribution of charges

- It also help us to find total charge present inside the closed surface

- Gauss's law holds good for all closed surfaces symmetrical or non symmetrical distribution of charges, but Gauss's law we can find $\vec{E}$ & $\vec{D}$ only for symmetrical charge distributions

→ Evaluating $\vec{E}$ & $\vec{D}$ still can be made easy if the surface we select satisfies two conditions.

1. $\vec{D}_s$ is everywhere either normal or tangential to the closed surface, so that $\vec{D}_s \cdot d\vec{s}$ is either 0 or $D\,ds$ respectively

2. If $\vec{D}_s \cdot d\vec{s}$ is not zero then D_s is constant over a portion of closed surface. These assumptions allows us to replace the dot product with the product of scalars D_s and ds.

If we select symmetrical surface then both conditions can be easily satisfied.

① Field Intensity because of point charge

→ Let us consider a point charge Q at origin of spherical coordinate system

→ The surface is spherical surface. Ds is everywhere normal to the surface. Ds has same value at all points in the surface

$$Q = \oint_S \vec{D_s} \cdot d\vec{s} = \oint_{Sphere} D_s \, ds$$

$$= D_s \oint_{Sphere} ds = D_s \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta$$

$$Q = D_s \left(4\pi r^2\right)$$

$$D_s = \frac{Q}{4\pi r^2}$$

Ds is directed radially outwards

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2} \vec{a_r}} \qquad \boxed{\vec{E} = \frac{Q}{4\pi \epsilon_0 r^2} \vec{a_r}}$$

② Field Intensity of a Infinite line charge

Consider an infinite line charge of density ρ_L c/m lying along z-axis from $-\infty$ to ∞

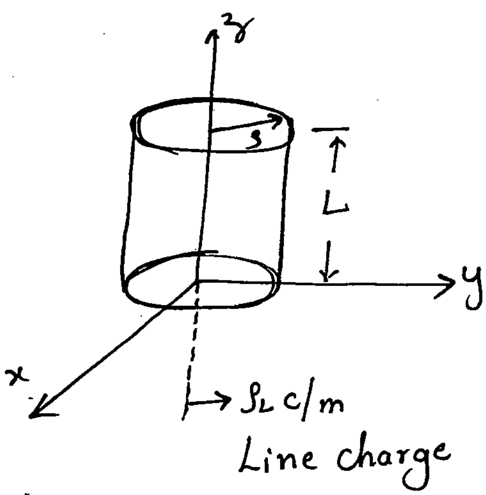
Consider Guassian surface as the right circular cylinder with z-axis with radius ρ. The length of cylinder is L

11

→ The flux density at any point on the surface of cylinder is directed radially outwards. $\vec{a}_r$

→ Only radial component of D is present which varies only with ρ.

→ If we consider Gaussian surface as cylindrical surface then D_ρ will be everywhere normal.



Applying Gauss's law

$$Q = \oint_{cyl} \vec{D_s} \cdot d\vec{S} = D_s \int_{sides} ds + 0 \int_{top} ds + 0 \int_{bottom} ds$$

Since $\vec{D}$ has only radial component and no component along $\vec{a}_z$ & $-\vec{a}_z$

$$\oint_{top} \vec{D_s} \cdot d\vec{s} = \oint_{bottom} \vec{D_s} \cdot d\vec{s} = 0$$

then $$Q = D_s \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = 2\pi \rho D_s L$$

$$D_\rho = \frac{Q}{2\pi \rho L}$$

The total charge enclose in L length line charge

$$Q = \rho_L (L)$$

then $$\boxed{\vec{D} = \frac{\rho_L \vec{a}_\rho}{2\pi \rho}}$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi \epsilon_0 \rho} \vec{a}_\rho \ \ V/m}$$

③ Field Intensity of Coaxial Cable

→ To find electric field intensity of coaxial cable is difficult from standpoint of coulomb's law.

→ Consider coaxial cylindrical conductors with inner conductor radius a and outer conductor radius b each infinite in length.

→ Let ρ_s be charge distribution on outer surface of inner conductor

→ Considering symmetry, only D_ρ component is present and it is function of only ρ

Consider a right circular cylinder of radius ρ and length L. where $a < \rho < b$.

From discussion of line charge

$$Q = D_\rho 2\pi \rho L \longrightarrow ①$$

Total charge on a length L of inner conductor is

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_s \, a \, d\phi \, dz$$

$$Q = 2\pi \rho_s a L \longrightarrow ②$$

Equating ① & ②

$$D_\rho \cancel{2\pi} \rho \cancel{L} = \cancel{2\pi} \rho_s a \cancel{L}$$

$$D_\rho = \frac{\rho_s a}{\rho}$$

$$\boxed{\vec{D} = \frac{\rho_s a}{\rho} \hat{a}_\rho} \qquad a < \rho < b$$

ρ_s can be expressed in terms of ρ_L

$$\rho_L = \frac{\rho_s \times \text{surface area}}{\text{Total length}} = \frac{\rho_s \times 2\pi a \cancel{L}}{\cancel{L}}$$

$$\rho_L = (2\pi a)\rho_s$$

$$\vec{D} = \frac{\cancel{a}\rho_L}{(2\pi\cancel{a})\rho}\hat{a}_\rho = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

$$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho}$$

Every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of outer cylinder the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{s(\text{inner cyl})}$$

$$2\pi b L \rho_{s(\text{outer cyl})} = -2\pi a L \rho_{s(\text{inner cyl})}$$

$$\rho_{s(\text{outer cyl})} = \frac{a}{b}\rho_{s(\text{inner cyl})}$$

If $\rho > b$ then total charge is zero i.e,

$$D_\rho = 0 \qquad \rho > b$$

$$D_\rho = 0 \qquad \rho < a$$

Application of Gauss's law : Differential volume element.

→ Gauss law can be applied to the non symmetrical surface.

→ For non symmetric surface simple Gaussian surface cannot be chosen such that normal component of $\vec{D}$ is either 0 or constant.

→ This problem can be solved by choosing very small surface such that D is almost constant over the surface and the small change in $\vec{D}$ may be adequately represented by using the first two terms of Taylor's series expansion for D.

→ Let us consider any point P located in a rectangular cordinate system

Let the value of $\vec{D}$ at the point P may be expressed as

$$\vec{D}_0 = \vec{D}_{x0}\,\hat{a}_x + \vec{D}_{y0}\,\hat{a}_y + \vec{D}_{z0}\,\hat{a}_z$$

→ We choose as our closed surface the small rectangular box centered at P having sides of length $\Delta x, \Delta y$ and Δz and apply Gauss's law

$$\oint_S \vec{D}\cdot d\vec{s} = Q$$

Integral can be divided on six faces one over each face.

$$\oint_S \vec{D}\cdot d\vec{s} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

Consider the first of these in detail.
Since surface element is very small, $\vec{D}$ is essentially constant

$$\int_{front} = \vec{D}_{front} \cdot \Delta \vec{S}_{front}$$

$$= \vec{D}_{front} \cdot \Delta y \, \Delta z \, \hat{a}_x$$

$$= D_{x,front} \, \Delta y \, \Delta z$$

We have to approximate only value of D_x at this face. Front face is at distance $\frac{\Delta x}{2}$ from P.

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

D_{x0} is value of D_x at P. Since D_x in general also varies with y and z, partial derivative must be used to represent rate of change of D_x with x

$$\int_{front} = \left[D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right] \Delta y \, \Delta z$$

Considering integral over back surface

$$\int_{back} = \vec{D}_{back} \cdot \Delta \vec{S}_{back}$$

$$= \vec{D}_{back} \cdot (-\Delta y \, \Delta z \, \hat{a}_x)$$

$$= -D_{x,back} \, \Delta y \, \Delta z$$

$$D_{x,back} = D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{back} = \left[-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right] \Delta y \, \Delta z$$

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly same process

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

These results are collected to yield

$$\oint_S \vec{D} \cdot d\vec{s} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] \Delta x \, \Delta y \, \Delta z$$

$$\oint \vec{D} \cdot d\vec{s} = Q = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] \Delta V$$

Divergence

The expression for charge enclosed by volume ΔV becomes exact by allowing the volume element ΔV to shrink to zero

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] = \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta V} = \frac{Q}{\Delta V}$$

as a limit $$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] = \lim_{\Delta V \to 0} \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta V} = \lim_{\Delta V \to 0} \frac{Q}{\Delta}$$

The last term is the volume charge density ρ_v

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta V \to 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta V} = \rho_v$$

We can write it as two separate equations

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \lim_{\Delta v \to 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta v} \rightarrow (a)$$

and

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \rho_v \rightarrow (b)$$

Equation (a) involves no charge density. This operation is known as "divergence", divergence of any vector is defined as the outflow of flux from for small closed surface per unit volume as the volume shrinks to zero.

$$\text{div } D = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] \rightarrow (c)$$

The positive divergence for any vector quantity indicates a source of that vector quantity and negative divergence indicates a sink.

Equation (b) is result of applying definition of divergence to differential volume in rectangular coordinate system. Similary divergence of cylindrical coordinate system is

$$\text{div } D = \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}\right]$$

in Spherical system

$$\text{div } D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Maxwell's first equation

With the concept of divergence for given Gauss's law

$$\oint_s \vec{D} \cdot d\vec{s} = Q$$

per unit volume

$$\frac{\oint \vec{D} \cdot d\vec{s}}{\Delta V} = \frac{Q}{\Delta V}$$

As the volume shrinks to zero

$$\lim_{\Delta V \to 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta V} = \lim_{\Delta V \to 0} \frac{Q}{\Delta V}$$

i.e,

$$\boxed{\text{div } \vec{D} = \rho_v}$$

This is first of Maxwell's four equations, it states that electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density. This equation is also known as point form of Gauss's law.

The vector operator ∇ and the divergence theorem

Divergence is an operation on vector yielding scalar, same as dot product. It is possible to find something which may be dotted formally with $\vec{D}$ to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

This can be accomplished by dot product i.e, we define operator ∇ as a vector operator

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

Considering $\nabla \cdot \vec{D}$

$$\nabla \cdot \vec{D} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z\right)$$

$$\nabla \cdot \vec{D} = \left[\frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}D_z\right]$$

This divergence of $\vec{D}$ i.e,

$$\text{div } \vec{D} = \nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

From Gauss's law

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

$$Q = \int_{vol} \rho_v \, dv$$

$$\int_{vol} \nabla \cdot D \, dv = \int_{vol} \rho_v \, dv$$

$$\boxed{\oint_S \vec{D} \cdot d\vec{s} = \int_{vol} \nabla \cdot \vec{D} \, dv}$$

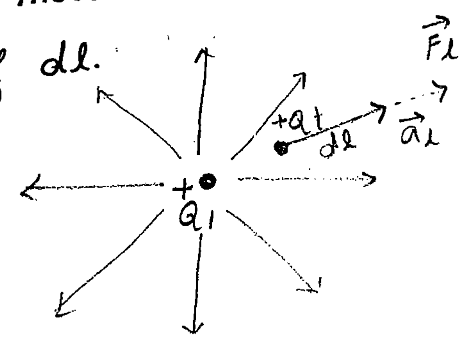
Energy & Potential.

Electric scalar Potential can be used to obtain electric field intensity $\vec{E}$. This is another method of obtaining vector field $\vec{E}$ from electric scalar potential.

Energy Expended in moving a point charge in an Electric Field

Electric field intensity is defined as the force on a unit test charge at that point at which we want to find value of $\vec{E}$.

Consider a positive charge Q_1 and its electric field $\vec{E}$. If a positive test charge Q_t is placed in this field, it will move due to force of repulsion. Let movement of charge Q_t is dl. The direction in which the movement has taken is denoted by $\hat{a}_L$ in the direction of dl.



The force exerted by field $\vec{E}$ on Q_t is

$$\vec{F} = Q_t \vec{E} \quad N$$

But the component of a vector in the direction of the unit vector is the dot product of the vector with that unit vector

$$F_l = \vec{F} \cdot \hat{a}_L = Q_t \vec{E} \cdot \hat{a}_L \quad N$$

If we wish to move charge in the direction of field by same distance dl, our energy expenditure turns to be -ve. i.e,

$$F_{applied} = -F_l = -Q_t \vec{E} \cdot \hat{a}_L \; N$$

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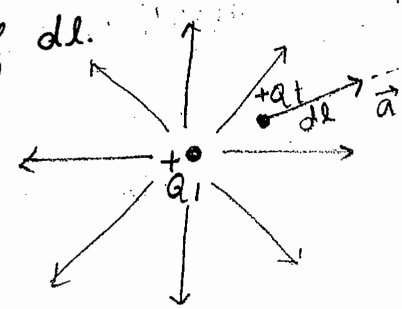

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Mathematically the differential work done by external source moving Q is

$$dW = (F_{applied}) \times (dl) = (-Q_t \vec{E} \cdot \hat{a}_L)(dl)$$

$$\boxed{dW = -Q_t \vec{E} \cdot d\vec{L}}$$

Total work done if a charge is moved from initial position to the final position against direction of electric field is

$$W = \int_{\text{Initial position}}^{\text{Final Position}} dW =$$

$$\boxed{W = -Q \int_{\text{Initial}}^{\text{Final}} \vec{E} \cdot d\vec{L} \quad J} \text{ units}$$

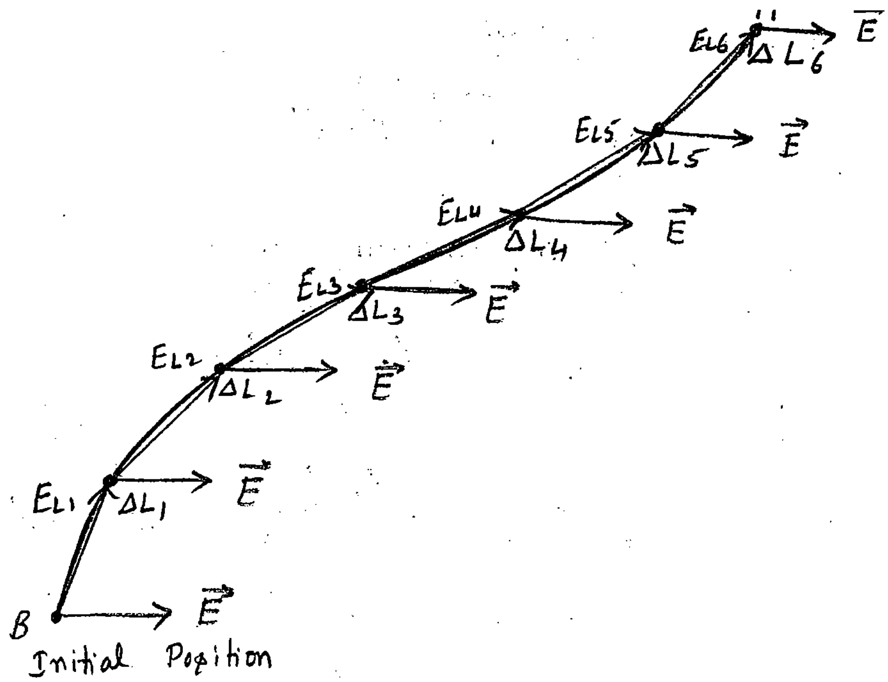
Line Integral

Consider a charge Q is moved from initial position B to final position A against the electric field $\vec{E}$ then

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L}$$

$\vec{E} \cdot d\vec{L}$ gives component of $\vec{E}$ along $d\vec{L}$ direction.

Mathematically this involves choosing any arbitrary path B to A, break up into a large number of very small segments, multiply the component of the field along each segment by the length of the segment, then add the result of all the segments



A path has been chosen from B to A, and a uniform electric field is chosen.

The path is divided into 6 line segments $\Delta L_1, \Delta L_2 \ldots \Delta L_6$ and components of $\vec{E}$ along each segment are denoted by $E_{L1}, E_{L2} \ldots E_{L6}$. The work involved in moving a charge Q from B to A is then

$$W = -Q\left[E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6\right]$$

Using vector notation

$$W = -Q\left[\vec{E_1}\cdot\vec{\Delta L_1} + \vec{E_2}\cdot\vec{\Delta L_2} + \cdots + \vec{E_6}\cdot\vec{\Delta L_6}\right]$$

Since we have assumed uniform electric field $\vec{E} = \vec{E_1} = \vec{E_2} \cdots =$

$$W = -Q\,\vec{E}\cdot\left[\vec{\Delta L_1} + \vec{\Delta L_2} + \cdots + \vec{\Delta L_6}\right]$$

The sum $\left[\vec{\Delta L_1} + \vec{\Delta L_2} + \cdots + \vec{\Delta L_6}\right]$ is result obtained by parallelogram law of addition. the sum is vector directed fro B to A i.e., $\vec{L_{BA}}$ therefore

$$\boxed{W = -Q\,\vec{E}\cdot\vec{L_{BA}}}$$

From the integral expression

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L}$$

The electric field is uniform hence

$$W = -Q \vec{E} \cdot \int_B^A d\vec{L}$$

Last integral is $\vec{L}_{BA}$

$$\boxed{W = -Q \vec{E} \cdot \vec{L}_{BA}}$$ uniform $\vec{E}$.

While solving the problem it is necessary to select $d\vec{L}$. The expression according to coordinate systems selected. The expression for $d\vec{L}$ in three coordinate systems are

Cartesian $\quad d\vec{L} = dx\, \hat{a}_x + dy\, \hat{a}_y + dz\, \hat{a}_z$

Cylindrical $\quad d\vec{L} = dr\, \hat{a}_r + r\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z$

Spherical $\quad d\vec{L} = dr\, \hat{a}_r + r\, d\theta\, \hat{a}_\theta + r \sin\theta\, d\phi\, \hat{a}_\phi$

Potential Difference and Potential

→ The work done in moving a point charge Q from point B to A in the electric field $\vec{E}$ is given by

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L}$$

If Q is selected as unit charge then from the above equation we get the work done in moving unit charge from B to A.

This work done in moving unit charge from point B to A in the field $\vec{E}$ is called potential difference between the points B and A. i.e,

Potential difference $= V_{AB} = -\int_B^A \vec{E} \cdot d\vec{L}$ Volts or J/C

From the integral expression

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L}$$

The electric field is uniform hence

$$W = -Q\, \vec{E} \cdot \int_B^A d\vec{L}$$

Last integral is $\vec{L}_{BA}$

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$$\text{Potential difference} = V_{AB} = -\int_B^A \vec{E} \cdot d\vec{L} \quad \text{Volts or J/}$$

→ If B is initial position and A is the final point then the potential difference is denoted as V_{AB} which indicates the potential difference between the points A and B & charge is moved from B to A.

→ If V_{AB} is positive then work done by external source in moving the unit charge from B to A is against direction of $\vec{E}$

→ One volt potential difference is one joule of work done in moving unit charge from one point to other in the field $\vec{E}$

$$1\,\text{volt} = \frac{1\,\text{joule}}{1\,\text{coulomb}}$$

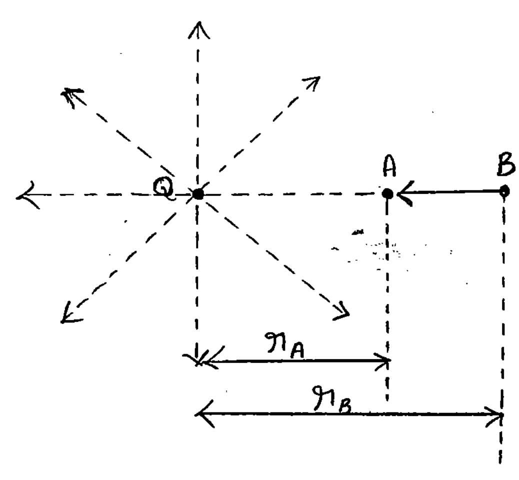
→ It is convenient to express absolute potentials than potential difference. Absolute potentials are measured with respect to the specified reference position. Such reference position is assumed to be at zero potential. For practical circuits such zero potential is selected as ground.

The potential field of a point charge

Consider a point charge located at the origin of a spherical coordinate system, producing $\vec{E}$ radially in all the directions

The field $\vec{E}$ due to a point charge Q at distance r from origin is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r$$



Consider a unit charge which is placed at a point at a radial distance r_B from the origin. It is moved against the direction of $\vec{E}$ from point B to point A. The point A is at radial distance r_A from the origin

The differential length in spherical coordinate system is

$$d\vec{L} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$

The potential difference between A & B is V_{AB} where

$$V_{AB} = -\int_B^A \vec{E}\cdot d\vec{L}$$

$B = r_B$ & $A = r_A$

$$V_{AB} = -\int_{r_B}^{r_A}\left(\frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r\right)\cdot\left(dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi\right)$$

$$V_{AB} = -\int_{r_B}^{r_A}\frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$= \frac{-Q}{4\pi\epsilon_0}\int_{r_B}^{r_A}\frac{1}{r^2}\,dr$$

$$V_{AB} = \frac{-Q}{4\pi\epsilon_0}\left(\frac{-1}{r}\right)_{r_B}^{r_A}$$

$$V_{AB} = \frac{+Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \text{ V}$$

→ Absolute potential can be defined by considering a reference point. The simplest possibility is to let $V=0$ at infinity. If $r = r_B$ recede to infinity the potential at r_A becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

→ Thus the expression for potential at any distance r from point charge Q at the origin is

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}}$$

→ If charge Q_1 is at $\vec{r}_1$ and potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0 |\vec{r}-\vec{r}_1|}$$

→ If charges Q_1 is at $\vec{r}_1$ and Q_2 is at $\vec{r}_2$ then potential at r is

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\vec{r}-\vec{r}_2|}$$

In general potential arising from n point charges is

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\vec{r}-\vec{r}_2|} + \cdots + \frac{Q_n}{4\pi\epsilon_0|\vec{r}-\vec{r}_n|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\vec{r}-\vec{r}_m|}$$

→ If number of charges are infinite then

$$V(\vec{r}) = \int_{vol} \frac{\rho_v(r')\, dv'}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

In above expression $\rho_v(r')\,dv'$ represents differential amount of charge located at r'. The distance $|r-r'|$ is the distance from source point to the field point.

→ If charge distribution is in form of line charge ~~then~~

$$V(\vec{r}) = \int \frac{\rho_L(r')\, dL'}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

→ If charge distribution is over surface

$$V(\vec{r}) = \int \frac{\rho_s(r')\, ds'}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

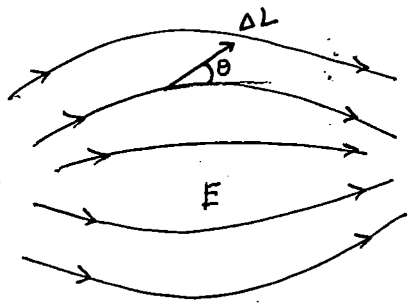
Potential Gradient

we know the line integral relationship

$$V = -\int \vec{E}\cdot d\vec{L}$$

If the above expression is applied to short element length $\Delta\vec{L}$ leading to incremental potential difference ΔV.

$$\Delta V = -\vec{E}\cdot\Delta\vec{L}$$



Consider a general region of space in which $\vec{E}$ and V both changes as we move from point to point

$\Delta V = -\vec{E}\cdot\Delta\vec{L}$ tells us to choose an incremental vector element of length $\Delta\vec{L} = \Delta L\,\hat{a}_L$ and multiply its magnitude by the component of $\vec{E}$ in the direction of $\hat{a}_L$ to obtain small potential difference between the final and initial points of $\Delta\vec{L}$

If θ is angle between $\Delta \vec{L}$ and $\vec{E}$ then

$$\Delta V = -E \, \Delta L \cos\theta$$

$$\frac{\Delta V}{\Delta L} = -E\cos\theta$$

Applying limit and Considering derivative $\lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \frac{dv}{dL}$

$$\frac{dv}{dL} = -E\cos\theta$$

Maximum positive increment of potential ΔV_{max} will occur when $\Delta \vec{L}$ points in the direction opposite to $\vec{E}$ i.e, $\cos\theta = -1$ or

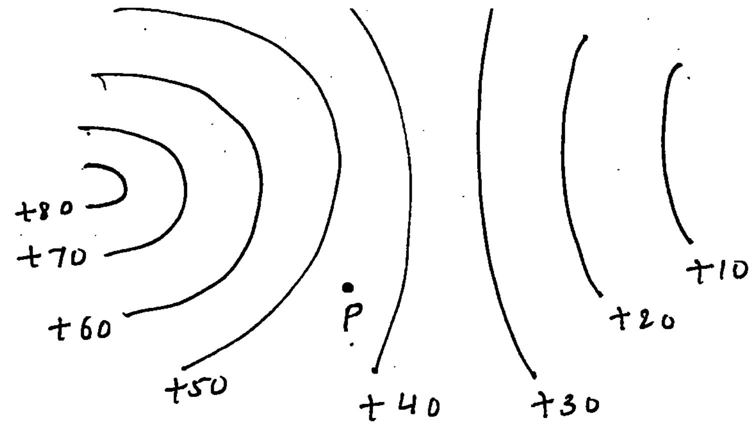
$$\left.\frac{dV}{dL}\right|_{max} = E$$

Two characteristics of the relationship between E and V at any point can be defined.

ⓐ Magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance

ⓑ This maximum value is obtained when the direction of the distance increment is opposite to $\vec{E}$.

The rate of change of potential with respect to the distance is called the potential gradient.



A potential field by equipotential surfaces.

If we consider a point P in equipotential surface. We desire information about electric field intensity $\vec{E}$.

The magnitude of $\vec{E}$ is given by maximum rate of change of V with distance. From above potential field towards left field is varying (increasing) rapidly, therefore the electric field will be oppositely directed.

Mathematically let $\hat{a}_N$ be the unit vector normal to the equipotential surface directed towards higher potential.

$$\vec{E} = -\left.\frac{dv}{dL}\right|_{max} \hat{a}_N$$

The above equation is physical interpretation of the process of finding the electric field intensity from the potential

The operation on V by which $\vec{E}$ is obtained is known as gradient i.e,

$$\boxed{\vec{E} = -\text{grad } V}$$

V is a unique function of x, y & z then

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \longrightarrow (a)$$

But we also have

$$dV = -\vec{E}\cdot d\vec{L} = -E_x dx - E_y dy - E_z dz \longrightarrow (b)$$

Comparing above two expressions

$$\vec{E}_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

These results may be combined vectorially

$$\boxed{\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right]}$$

we know that $\boxed{\vec{E} = -\text{grad } V}$ thus

$$\boxed{\text{grad } V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z}$$

The vector operator $\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$ can be used

$$\boxed{\vec{E} = -\nabla V}$$

Energy Density in the Electrostatic field.

We know that when a unit positive charge is moved from infinity to a point in a field the work is done by the external source and energy is expended.

To hold the charge at a point in an electrostatic field an external source has to to work. This energy gets stored in the form of potential energy.

when the external source is removed potential energy gets converted to kinetic energy.

→ Consider an empty space where there is no electric field at all.

→ Charge Q_1 is moved from infinity to a point in the space say P_1. This requires no work as there is no electric field

→ The charge Q_2 is to be placed at P_2, but now there is a field due to charge Q_1 and Q_2 is required to move against the field of Q_1.

Potential = Work done per unit charge

$$V = \frac{W}{Q}$$

work done to position Q_2 at P_2 = $V_{2,1} Q_2$

where $V_{2,1}$ = potential at P_2 due to Q_1

If charge Q_3 is to be moved from ∞ to P_3 then

work done to position Q_3 at P_3 = $V_{3,1} Q_3 + V_{3,2} Q_3$

similarly work done to position Q_4 at P_4 = $V_{4,1} Q_4 + V_{4,2} Q_4 + V_{4,3} Q_4$

The total work done to position all the charges

$$W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \cdots \longrightarrow (c)$$

Total work done is nothing but potential energy in the system of charges.

Consider $Q_3 V_{3,1} = Q_3 \dfrac{Q_1}{4\pi \epsilon_0 R_{13}} = Q_1 \dfrac{Q_3}{4\pi \epsilon_0 R_{31}} = Q_1 V_{1,3}$

where R_{13} and R_{31} are scalar distance between Q_1 & Q_3

$Q_1 V_{1,3}$ is equivalent to $Q_3 V_{3,1}$

Hence by replacing each term in the expression of W_E

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \cdots \longrightarrow (d)$$

Adding equations ⓒ & ⓓ

$$2W_E = Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \dots) + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \dots) + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \dots) + \dots \longrightarrow (e)$$

Each sum of potentials in parantheses is the combined potential due to all the charges except for the charge at the point where this combined potential is being found.

i.e., $V_{1,2} + V_{1,3} + V_{1,4} + \dots = V_1$

where V_1 is potential at location of Q_1 due to charges Q_2, Q_3 -

hence from above expressions

$$2W_E = Q_1 V_1 + Q_2 V_2 + \dots$$

$$2W_E = \sum_{m=1}^{N} Q_m V_m$$

$$\boxed{W_E = \frac{1}{2}\sum_{m=1}^{N} Q_m V_m} \longrightarrow (f)$$

The expression for energy stored in a region of continous charge distribution is obtained by replacing each charge by $\rho_v \, dv$ in equation (f) the summation becomes integral

$$W_E = \frac{1}{2}\int_{vol} (\rho_v \, dv) V \longrightarrow (g)$$

By Maxwell's first equation $\nabla \cdot \vec{D} = \rho_v$

By vectore identity

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot \nabla V$$

$$\nabla\cdot(V\vec{D}) = V\rho_v + \vec{D}\cdot\nabla V$$

But $\vec{E} = -\nabla V$

So, $\nabla\cdot(V\vec{D}) = V\rho_v - \vec{D}\cdot\vec{E}$

$$\rho_v V = \nabla\cdot(V\vec{D}) + \vec{D}\cdot\vec{E} \longrightarrow (h)$$

From equations (g) and (h) we have

$$W_E = \frac{1}{2}\int_{vol}\left(\nabla\cdot(V\vec{D}) + \vec{D}\cdot\vec{E}\right) dv$$

$$W_E = \frac{1}{2}\int_{vol}\nabla\cdot(V\vec{D})dv + \frac{1}{2}\int_{vol}\vec{D}\cdot\vec{E}\,dv$$

Using divergence theorem i.e, $\oint_S \vec{D}\cdot d\vec{s} = \int_{vol}\nabla\cdot\vec{D}\,dv$ the first volume integral can be changed to closed surface integral. i.e.,

$$W_E = \frac{1}{2}\oint_S (V\vec{D}\cdot d\vec{s}) + \frac{1}{2}\int_{vol}\vec{D}\cdot\vec{E}\,dv$$

In the above expression surface integral is zero, surrounding the universe V is approaching zero at the rate $\frac{1}{r}$ and $\frac{1}{r^2}$ but the surface area is increasing at the rate as r^2 consequently in the limit $r\to\infty$ the increasing integration becomes zero hence

$$\boxed{W_E = \frac{1}{2}\int_{vol}\vec{D}\cdot\vec{E}\,dv = \frac{1}{2}\int_{vol}\epsilon_0 E^2\,dv}$$