Dielectric Materials

→ Insulating and dielectric materials do not have free charges. The charges in dielectrics are bound by the finite forces & hence called bound charges.

→ As the charges are bound and not free the dielectrics cannot contribute to the conduction process.

If subjected to an electric field $\vec{E}$, they shift their relative positions. This shift in the relative positions of bound charges allows dielectric to store energy.

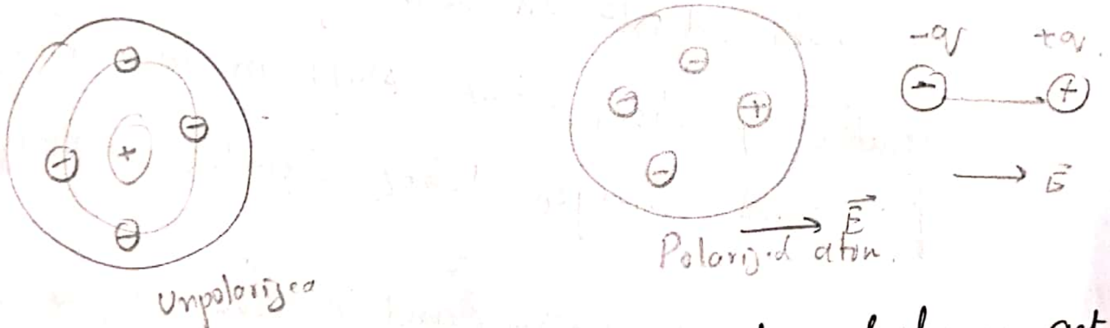
→ The shift in positive and negative charges are in opposite direction under the influence of applied electric field, such charges act as small electric dipoles.

This process due to which separation of bound charges results to produce electric dipoles, under the influence of $\vec{E}$ is called polarization.

→ This displacement against restraining force is analogous to lifting a weight or stretching a string (spring) and represents potential energy.

Polarization

Consider an atom of a dielectric. This consists of nucleus with positive charge & negatively charged electrons revolve around the nucleus in the orbits. In the absence of $\vec{E}$ the number of positive charges is same as number of negative charges. & hence atom is electrically neutral. The is no existence of electric dipole. Hence this is known as unpolarized atom.



When $\vec{E}$ is applied, the symmetric distribution of charges get disturbed. Now there is separation between the nucleus & the centre of the electron cloud. Such atom is known as polarized atom.

There are two types of dielectric

Non Polar : The dipole arrangement is totally absent in absence of $\vec{E}$. The polarization happens only when an external field $\vec{E}$ is applied to it.

Polar : The permanent displacement exists between positive & negative charges and each pair of charges act as dipole. But such dipoles are randomly oriented. $\vec{E}$ makes dipoles to experience torque & they align with the direction of applied electric field $\vec{E}$.

Both the types of dipole may be described by its dipole moment $\vec{p}$: given by

$$\vec{p} = Q\vec{d}$$

Q → magnitude of one of the two charges
$\vec{d}$ → distance vector from negative to positive charge.

If there are n dipoles per unit volume & we deal with volume Δv then there ar $n\Delta v$ dipoles, and total dipole moment is obtained by vector sum

$$\vec{P}_{total} = \sum_{i=1}^{n\Delta v} \vec{p}_i$$

The polarization $\vec{P}$ is defined as the total dipole moment per unit volume

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n\Delta V} Q_i \vec{d}_i}{\Delta V} \quad C/m^2$$

It can be seen that unit of polarization is same as flux density. Thus polarization increases the electric flux density in the dielectric medium. Hence flux density in a dielectric is

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

For isotropic medium $\vec{P}$ & $\vec{E}$ are related as.

$$\vec{P} = \chi_e \varepsilon_0 \vec{E}$$

$\chi_e \rightarrow$ Dimensionless quantity called electric susceptibility of the material.

$$\vec{D} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E}$$

$$\vec{D} = \vec{E} \varepsilon_0 [\chi_e + 1]$$

$$\vec{D} = \varepsilon \vec{E}$$

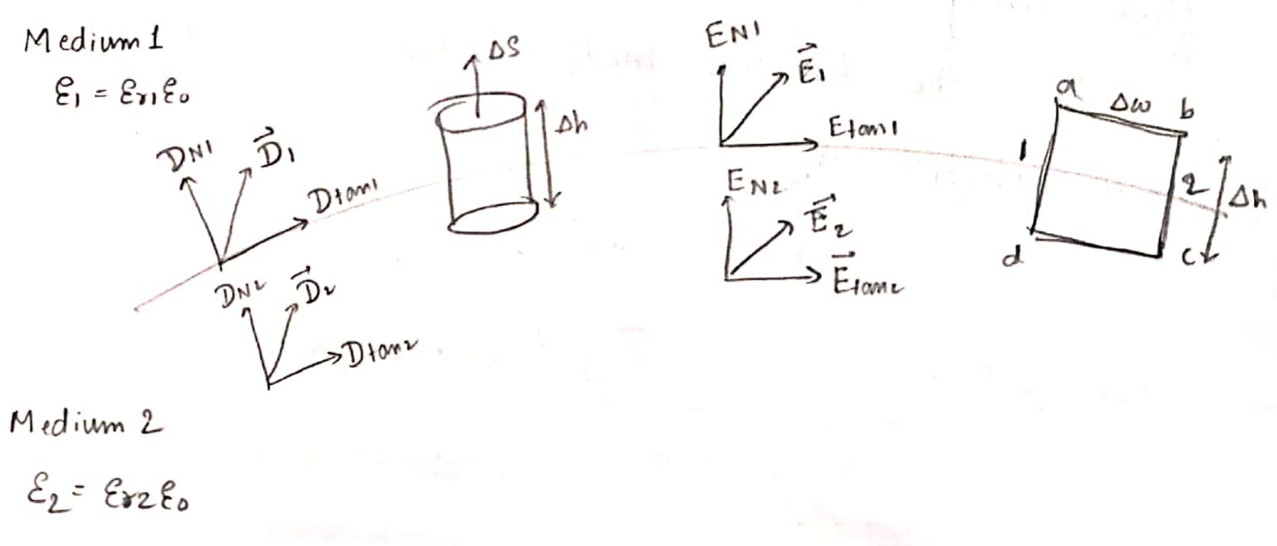
where $\varepsilon = \varepsilon_R \varepsilon_0 = (\chi_e + 1) \varepsilon_0$

$\varepsilon_R = \chi_e + 1 \rightarrow$ Relative permittivity or dielectric Constant

Medium is isotropic if $\vec{D}, \vec{E}$ & $\vec{P}$ are parallel i.e., in the same direction.

<u>Boundary Conditions between two dielectric medium</u>

Let us consider the boundary between two perfect dielectrics. One medium has permittivity ε_1 while other has permittivity ε_2



The $\vec{E}$ & $\vec{D}$ are to be obtained by resolving each into tangential & normal components.

Consider a closed path abcda rectangular in shape having elementary height Δh & elementary width Δw. Placed in such a way that $\Delta h/2$ is in dielectric 1 & $\Delta h/2$ is in dielectric 2.

$$\oint \vec{E}\cdot d\vec{L} = 0$$

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L} = 0$$

Now $\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1N}$ Let $|\vec{E}_{1t}| = E_{tan1}$ $|\vec{E}_{2t}| = E_{tan2}$

$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2N}$ $|\vec{E}_{1N}| = E_{1N}$ $|\vec{E}_{2N}| = E_{2N}$

From above figure

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^2 \vec{E}\cdot d\vec{L} + \int_2^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^1 \vec{E}\cdot d\vec{L} + \int_1^a \vec{E}\cdot d\vec{L} = 0$$

$$E_{tan1}(\Delta w) - \cancel{E_{1N}\left(\frac{\Delta h}{2}\right)} - \cancel{E_{2N}\left(\frac{\Delta h}{2}\right)} + \left(-E_{tan2}\,\Delta w\right) + \cancel{E_{2N}\left(\frac{\Delta h}{2}\right)} + \cancel{E_{1N}\left(\frac{\Delta h}{2}\right)} = 0$$

$$E_{tan1}\,\Delta w - E_{tan2}\,\Delta w = 0$$

$$\boxed{E_{tan1} = E_{tan2}}$$

Thus the tangential components of field intensity at the boundary in both dielectrics remain same.

W.K.T $\vec{D} = \varepsilon \vec{E}$

$$D_{tan1} = \varepsilon E_{tan1} \qquad D_{tan2} = \varepsilon E_{tan2}$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

Tangential Components of $\vec{D}$ undergoes some change across the interface hence tangential $\vec{D}$ is discontinous across the boundary.

To find the normal component let us apply Gauss's law to right circular cylinder (Gaussian surface) which is having Δh height.

$$\oint \vec{D} \cdot d\vec{s} = Q$$

$$\int_{Top} \vec{D} \cdot d\vec{s} + \int_{Bottom} \vec{D} \cdot d\vec{s} + \int_{Lateral} \vec{D} \cdot d\vec{s} = Q$$

$\int_{Lateral} \vec{D} \cdot d\vec{s} = 0$ as $\vec{D}_N$ and $\vec{s}$ are normal to each other.

Let $|\vec{D}| = D_{N1}$ & D_{N2} for dielectric 2 as flux leaving from top & bottom surfaces of cylinder is normal to surface

$$\int_{top} \vec{D} \cdot d\vec{s} = D_{N1} \Delta S$$

$$\int_{Bottom} \vec{D} \cdot d\vec{s} = -D_{N2} \Delta S$$

$$D_{N1}\Delta S - D_{N2}\Delta S = Q$$

$$Q = \rho_s \Delta S$$

$$D_{N1} - D_{N2} = \rho_s$$

There is no free charge available in perfect dielectric

Hence no free charge can exist on the surface

Hence at the ideal dielectric medium boundary

ρ_s can be assumed zero

$$\rho_s = 0$$

hence $\boxed{D_{N1} = D_{N2}}$

$$\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}$$

$$\boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

Normal component of flux density $\vec{D}$ is continous at the boundary between the two perfect dielectrics. &

Normal component of $\vec{E}$ are inversely proportional to the relative permittivities of two media

Poisson's & Laplace's Equations

Poisson's & laplace's equations are mathematical tools to find the potential & field in a given region with known values of charge density on the boundary surfaces.

From the Gauss's law in the point form Poisson's equation can be derived

$$\nabla \cdot \vec{D} = \rho_v \rightarrow \text{Gauss's law in point form}$$

where $\vec{D}$ is flux density & ρ_v is volume charge density.

For homogenous, isotropic and linear medium, flux density and electric field intensity are directly proportional Thus

$$\vec{D} = \varepsilon \vec{E}$$

$$\nabla \cdot \varepsilon \vec{E} = \rho_v \text{, from gradient relationship } \vec{E} = -\nabla V$$

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v$$

$$-\varepsilon \left[\nabla \cdot \nabla V\right] = \rho_v$$

$$\nabla \cdot \nabla V = \frac{-\rho_v}{\varepsilon}$$

Now $\nabla \cdot \nabla$ operation is called 'del squared' operation ∇^2

$$\boxed{\nabla^2 V = \frac{-\rho_v}{\varepsilon}}$$

This is called Poisson's Equation

If in certain region, volume charge density is zero ($\rho_v = 0$) which is true for dielectric medium. Then Poisson's equation takes the form

$$\boxed{\nabla^2 V = 0}$$

This is special case of Poisson's equation and is known as Laplace's equation.

∇^2 operation in different coordinate systems.

Cartesian Coordinate System

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Cylindrical Coordinate System

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left[\rho \frac{\partial V}{\partial \rho}\right] + \frac{1}{\rho^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] + \frac{\partial^2 V}{\partial z^2} = 0$$

Spherical Coordinate System

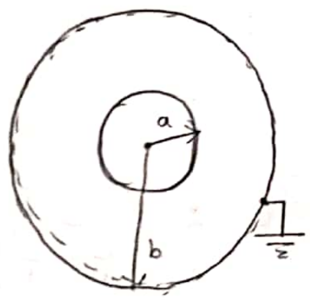
$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Calculating Capacitance using Laplace's Equation

The Laplace's equation can be used to find the capacitance under various conditions.

Example 1

Solve the Laplace's equation for the potential field in the homogenous region between the two concentric spheres with radii a & b, such that b>a if potential $V=0$ at $r=b$ & $V=V_0$ at $r=a$. Find the capacitance between the two concentric spheres.



At $r=b$, $V=0$ hence the outer sphere is shown at zero potential.

The field intensity $\vec{E}$ will be only in radial direction hence V is changing only in radial direction.

$$\nabla^2 V = 0 \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = 0$$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = 0$$

Integrating

$$r^2 \frac{\partial V}{\partial r} = C_1$$

$$\frac{\partial V}{\partial r} = \frac{C_1}{r^2}$$

$$\frac{\partial V}{\partial r} = C_1 r^{-2}$$

Integrating $V = \int C_1 r^{-2} dr$

$$V = -\frac{C_1 r^{-1}}{1} + C_2$$

$$V = \frac{-C_1}{r} + C_2$$

Using boundary Conditions

$V=0$ at $r=b$ & $V=V_0$ at $r=a$.

$$0 = -\frac{C_1}{b} + C_2 \quad \text{and} \quad V_0 = \frac{-C_1}{a} + C_2$$

Subtracting above 2 equations

$$-V_0 = \frac{-C_1}{b} - \left(\frac{-C_1}{a}\right)$$

$$-V_0 = C_1\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C_1 = \frac{+V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)} \quad \& \quad C_2 = \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]}$$

$$V = \frac{-V_0}{r\left[\frac{1}{b} - \frac{1}{a}\right]} + \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]}$$

This is the potential field in the region between the two spheres.

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = \frac{-\partial}{\partial r}\left[\frac{-V_0}{r\left(\frac{1}{b}-\frac{1}{a}\right)}\right]\hat{a}_r$$

$$\vec{E} = \frac{+V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)}\left(\frac{-1}{r^2}\right) = \frac{-V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)r^2}\hat{a}_r \ V/m$$

$$\vec{D} = \varepsilon\vec{E} = \frac{-\varepsilon V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)r^2}\hat{a}_r \ C/m^2$$

As per the boundary conditions between the dielectric and conductor $\vec{D}$ is always normal to the surface

$$\rho_s = |\vec{D}_N| = |\vec{D}| = \frac{\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)r^2} \ C/m^2$$

$C = \frac{Q}{V}$ → charge on surface of sphere of radius r → Potential between two spheres.

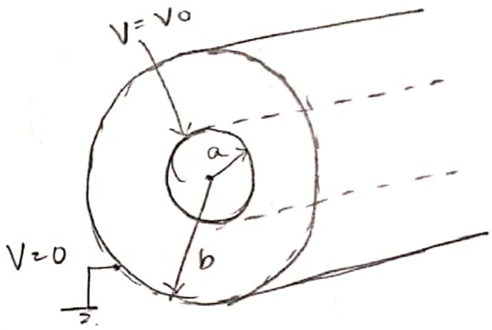
$$C = \frac{(4\pi \cancel{r^2}) \times \frac{\varepsilon \cancel{V_0}}{\left(\frac{1}{a}-\frac{1}{b}\right)\cancel{r^2}}}{\cancel{V_0}}$$

$$\boxed{C = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \ F}$$

This is the capacitance of spherical capacitor.

Example 2

Use Laplace's equation to find the capacitance per unit length of a co-axial cable of inner radius 'a' m and outer radius 'b' m. Assume $V = V_0$ at $r = a$ & $V = 0$ at $r = b$



Consider cylindrical co-ordinate system. The field intensity $\vec{E}$ is in radial direction from inner to outer cylinder hence V is a function of ρ only & not function of ϕ & z.

Using Laplace's equation

$$\nabla^2 V = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Integrating $\rho \frac{\partial V}{\partial \rho} = C_1$

$$\frac{\partial V}{\partial \rho} = \frac{C_1}{\rho}$$

Integrating $V = C_1 \ln(\rho) + C_2$

Using boundary conditions $V = 0$ at $\rho = b$ & $V = V_0$ at $\rho = a$

$0 = C_1 \ln(b) + C_2$ $\qquad$ $V_0 = C_1 \ln(a) + C_2$

Subtracting $$-V_0 = C_1 \{\ln(b) - \ln(a)\} = C_1[\ln(b/a)]$$

$$C_1 = \frac{-V_0}{\ln(b/a)} = \frac{V_0}{\ln(a/b)} \quad \& \quad C_2 = -C_1 \ln(b) = \frac{-V_0 \ln(b)}{\ln(a/b)}$$

$$V = \frac{V_0}{\ln(a/b)} \ln(\rho) - \frac{V_0 \ln(b)}{\ln(a/b)} \text{ V}$$

$$\vec{E} = -\nabla V = \frac{-\partial V}{\partial \rho} \hat{a}_\rho = -\frac{\partial}{\partial \rho}\left[\frac{V_0 \ln(\rho)}{\ln(a/b)}\right] \hat{a}_\rho$$

$$\boxed{\vec{E} = \frac{-V_0}{\ln(a/b)}\left[\frac{\partial}{\partial \rho} \ln(\rho)\right] \hat{a}_\rho = \frac{-V_0}{\rho \ln(a/b)} \hat{a}_\rho \quad V/m}$$

$$\boxed{\vec{D} = \varepsilon \vec{E} = \frac{V_0 \varepsilon}{\rho \ln(b/a)} \hat{a}_\rho \quad C/m}$$

$\vec{D}$ is existing normal to the surface as per boundary conditions

$$\vec{D} = \vec{D}_N = \frac{V_0 \varepsilon}{\rho \ln(b/a)} \hat{a}_\rho \quad C/m^2$$

$$\boxed{\rho_s = |\vec{D}_N| = \frac{V_0 \varepsilon}{\rho \ln(b/a)} \quad C/m^2}$$

$$C = \frac{Q}{V} = \frac{\rho_s (2\pi \rho) L}{V_0} = \frac{2\pi \cancel{\rho} L \left[\frac{\cancel{V_0} \varepsilon}{\cancel{\rho} \ln(b/a)}\right]}{\cancel{V_0}}$$

for $L = 1$ m

$$\boxed{C = \frac{2\pi \varepsilon}{\ln(b/a)} \quad F/m}$$

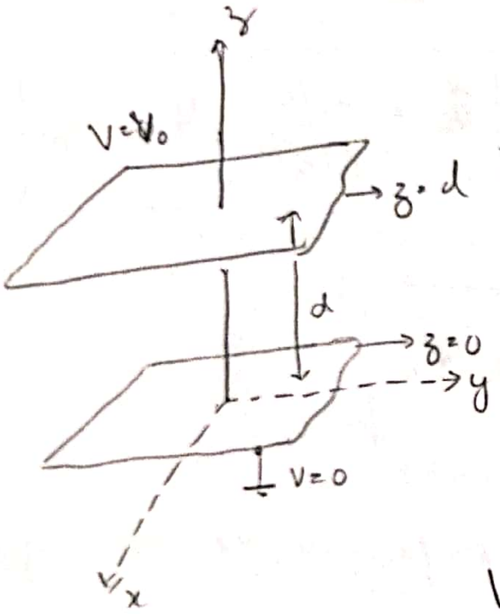
Example 3

Two plates of parallel plate capacitors are separated by distance 'd' and maintained at potential zero and V_0 respectively. Assuming negligible fringing effect determine

a) Potential at any position between the plates

b) Surface charge density on the plates

c) Capacitance between the plates.



Assume that the plates are placed parallel to xy plane shown in figure using Laplaces equation in cartesian system

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

V is function of 'z' only

$$\nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0$$

Integrating $\frac{\partial V}{\partial z} = C_1$

Integrating $V = C_1 z + C_2$

At $z=0,\ V=0$ & at $z=d,\ V=V_0$

$C_2 = 0$ & $C_1 = V_0/d$

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$$\boxed{V = \frac{V_0 z}{d} \ \text{V}}$$

b) $\vec{E} = -\nabla V = \frac{-\partial V}{\partial z} \hat{a}_z = -\frac{V_0}{d}$ V/m

$$\boxed{\vec{D} = \varepsilon \vec{E} = \frac{-\varepsilon V_0}{d} \ \text{C/m}^2}$$

$\vec{D}$ is normal to the surface as per boundary conditions $\boxed{\vec{D} = \vec{D}_N = \rho_s = \frac{V_0 \varepsilon}{d} \ \text{C/m}^2}$

c) $Q = \rho_s A$

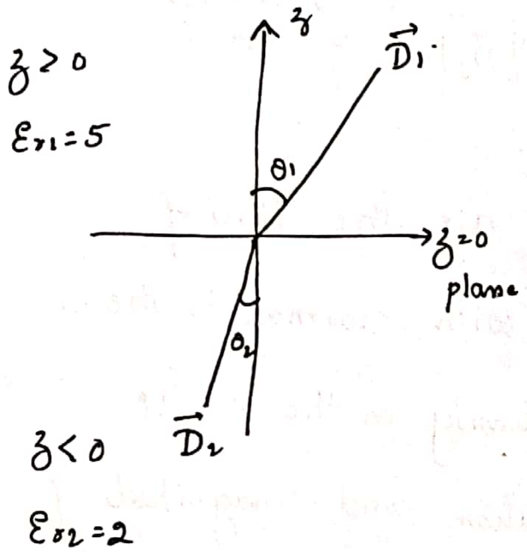
$$C = \frac{Q}{V} = \frac{\frac{\cancel{V_0} \varepsilon}{d} \times A}{\cancel{V_0}} = \frac{\varepsilon A}{d}$$

$$\boxed{C = \frac{\varepsilon A}{d}} \quad \text{F}$$

1) The region with $z<0$ is characterized by $\varepsilon_{r2}=2$ & $z>0$ by $\varepsilon_{r1}=5$. If $\vec{D}_1 = 2\hat{a}_x + 5\hat{a}_y - 3\hat{a}_z \; nc/m^2$ find

a) $\vec{D}_2$ b) $\vec{D}_{N2}$ c) $\vec{D}_{tan2}$ d) Energy density in each region

e) The angle that $\vec{D}_2$ makes with z axis f) $\frac{|D_2|}{|D_1|}$ g) $\frac{|P_2|}{|P_1|}$



$$\vec{D}_1 = 2\hat{a}_x + 5\hat{a}_y - 3\hat{a}_z \; nc/m^2$$

$$\vec{D}_1 = \vec{D}_{N1} + \vec{D}_{tan1}$$

Normal direction to the surface is $\pm\hat{a}_z$

$$\vec{D}_{N1} = -3\hat{a}_z \; nc/m^2$$

$$\vec{D}_{tan1} = \vec{D} - \vec{D}_{N1} = 2\hat{a}_x + 5\hat{a}_y \; nc/m^2$$

b) $\vec{D}_{N2} = \vec{D}_{N1} = -3\hat{a}_z \; nc/m^2$

$$\frac{\vec{D}_{tan1}}{\vec{D}_{tan2}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

c) $\vec{D}_{tan2} = \frac{2}{5}(2\hat{a}_x + 5\hat{a}_y) = 0.8\hat{a}_x + 2\hat{a}_y \; nc/m^2$

a) $\vec{D}_2 = \vec{D}_{N2} + \vec{D}_{tan2} = 0.8\hat{a}_x + 2\hat{a}_y - 3\hat{a}_z \; nc/m^2$

d) Energy density $W_{E1} = \frac{1}{2}\frac{|D_1|^2}{\varepsilon_1} = \frac{1}{2}\frac{|D_1|^2}{\varepsilon_0 \varepsilon_{r1}} = 0.4291 \; \mu J/m^3$

$$W_{E2} = \frac{1}{2}\frac{|D_2|^2}{\varepsilon_0 \varepsilon_{r2}} = 0.3851 \; \mu J/m^3$$

e) To find angle of $\vec{D}_2$ with z-axis

$$(\vec{D}_2)\cdot(-\hat{a}_z) = |\vec{D}_2||\hat{a}_z| \cos\theta_2$$

$$\theta_2 = \cos^{-1}\frac{(\vec{D}_2)\cdot(-\hat{a}_z)}{|\vec{D}_2|} = 35.678^\circ$$

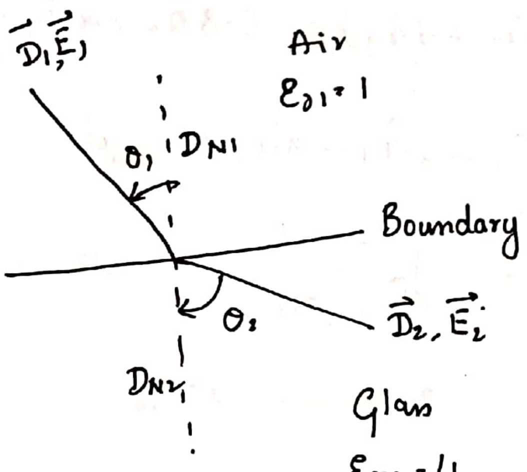
f) $$\frac{|\vec{D}_2|}{|\vec{D}_1|} = \frac{\sqrt{(0.8)^2+(2^2)+(-3)^2}}{\sqrt{2^2+5^2+(-3)^2}} = \frac{3.6932}{6.1644} = 0.599$$

g) $$|\vec{P}| = \chi_e \varepsilon_0 |\vec{E}| = \chi_e \varepsilon_0 \frac{|\vec{D}|}{\varepsilon_0 \varepsilon_r} = \frac{(\varepsilon_r - 1)}{\varepsilon_r} |\vec{D}|$$ Since $\chi_e + 1 = \varepsilon_r$

$$\frac{|\vec{P}_2|}{|\vec{P}_1|} = \frac{(\varepsilon_{r2}-1)}{\varepsilon_{r2}} |\vec{D}_2| \times \frac{\varepsilon_{r1}}{(\varepsilon_{r1}-1)|\vec{D}_1|} = \underline{\underline{0.3743}}$$

2) At the boundary between class and air, the lines of electric field make an angle 40° with normal to the boundary. If the electric flux density in the air is 0.25 μC/m², determine the orientation and magnitude of electric flux density in the glass.

W.K.T $\vec{D}_1 = 0.25\,\mu C/m^2$



$$D_{N1} = D_1 \cos\theta_1$$
$$= 0.25 \times \cos 40° \times 10^{-6}$$
$$D_{N1} = 0.1915\,\mu C/m^2$$
$$D_{N2} = D_{N1} = 0.1915\,\mu C/m^2$$

$$D_1 = \sqrt{(D_{N1})^2 + (D_{tan1})^2}$$
$$0.25 = \sqrt{(0.1915)^2 + (D_{tan1})^2}$$
$$D_{tan1} = 0.1607\,\mu C/m^2$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

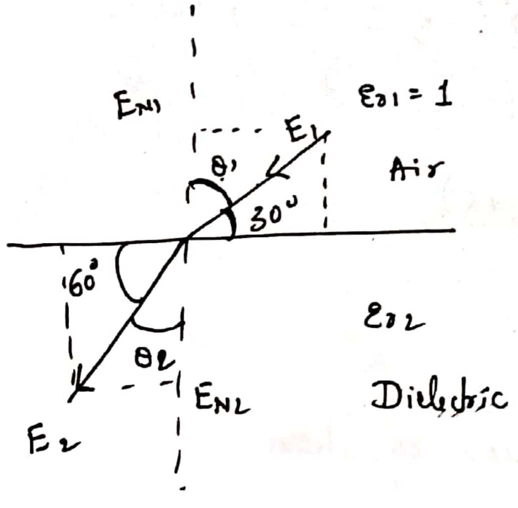
$$D_{tan2} = \frac{4}{1}(0.1607) = 0.6428\ \mu C/m^2$$

$$D_2 = \sqrt{(D_{N2})^2 + (D_{tan2})^2} = \underline{\underline{0.6707\ \mu C/m^2}}$$

$$D_{N2} = D_2 \cos\theta_2$$

$$\theta_2 = \cos^{-1}\left(\frac{D_{N2}}{D_2}\right) = \underline{\underline{73.41^\circ}}$$

3) An Electric field strength 3V/m in air enters a dielectric. The orientations of electric fields with respect to boundary in air and dielectric are 30° & 60° respectively. Find the relative permittivity of the dielectric. Also find electric field strength in dielectric



From arrangement shown

$$\theta_1 = 90^\circ - 30^\circ = 60^\circ$$

$$\theta_2 = 90^\circ - 60^\circ = 30^\circ$$

$$\tan\theta_1 = \frac{E_{tan1}}{E_{N1}} \qquad \tan\theta_2 = \frac{E_{tan2}}{E_{N2}}$$

But W.K.T $E_{tan1} = E_{tan2}$.

Consider ratio

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{E_{N2}}{E_{N1}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

$$E_{N2} \cdot \frac{\tan 60^\circ}{\tan 30^\circ} = \frac{1}{\varepsilon_{r2}}$$

$\varepsilon_{r_2} = 0.333$

From the arrangement shown

$$\cos\theta_1 = \frac{E_{N1}}{E_1} \qquad \cos\theta_2 = \frac{E_{N2}}{E_2}$$

$$\frac{\cos\theta_1}{\cos\theta_2} = \frac{E_{N1}}{E_1} \times \frac{E_2}{E_{N2}} = \frac{E_2}{E_1}\,\frac{\varepsilon_1}{\varepsilon_2}$$

$$\frac{\cos 60^\circ}{\cos 30^\circ} = \frac{\cancel{2}\,1}{0.333} \times \frac{E_2}{3}$$

$$E_2 = 0.5767\ V/m$$

4) Determine whether the following potential fields satisfy the Laplace's equation

a) $V = x^2 - y^2 + z^2$ b) $V = \rho\cos\phi + z$ c) $V = r\cos\theta + \phi$

a) $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 z}{\partial z^2}$$

$$= \frac{\partial^2 (x^2-y^2+z^2)}{\partial x^2} + \frac{\partial^2 (x^2-y^2+z^2)}{\partial y^2} + \frac{\partial^2 (x^2-y^2+z^2)}{\partial z^2}$$

$$\nabla^2 V = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(2z)$$

$$\nabla^2 V = 2 - 2 + 2 = 2 \neq 0$$

Field V does not satisfy Laplace's equation.

b) $V = \rho\cos\phi + z$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial(\rho\cos\phi + z)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2(\rho\cos\phi + z)}{\partial \phi^2} + \frac{\partial^2(\rho\cos\phi + z)}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho(\cos\phi) + \frac{1}{\rho^2}\frac{\partial(-\rho\sin\phi)}{\partial \phi} + \frac{\partial}{\partial z}(1)$$

$$= \frac{1}{\rho}\cos\phi - \frac{1}{\rho^2}(\rho\cos\phi) + 0$$

$$\nabla^2 V = \frac{\cos\phi}{\rho} - \frac{\cos\phi}{\rho} + 0 = 0$$

This field satisfies Laplace's equation.

c) $V = r\cos\theta + \phi$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial(r\cos\theta+\phi)}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial(r\cos\theta+\phi)}{\partial \theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2(r\cos\theta+\phi)}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\cos\theta\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left[-\sin^2\theta\, r\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}(1)$$

$$= \frac{1}{r^2}(2r\cos\theta) + \frac{(-r)}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin^2\theta) + 0$$

$$\nabla^2 V = \frac{2\cos\theta}{r} - \frac{1}{r\sin\theta}2\sin\theta\cos\theta = \frac{2\cos\theta}{r} - \frac{2\cos\theta}{r} = 0$$

This field satisfies Laplace's eqn

5. In a free space $\rho_v = \dfrac{200\,\varepsilon_0}{r^{2.4}}$

i) Use Poisson's equation to find V as a function of r, if it is assumed that $r^2 E_r \to 0$ as $r \to 0$ and $V \to 0$ as $r \to \infty$

Use spherical coordinate system.

ii) Find potential V as a function of r using Gauss's law and line integral.

i) From Poisson's equation

$$\nabla^2 V = \frac{-\rho_v}{\varepsilon_0} \quad \text{as it is free space}$$

$$\nabla^2 V = -\frac{200\,\varepsilon_0}{\varepsilon_0\, r^{2.4}} = \frac{-200}{r^{2.4}}$$

V is function of 'r' not function of θ & ϕ hence $\dfrac{\partial V}{\partial \theta} = \dfrac{\partial V}{\partial \phi} = 0$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = \frac{-200}{r^{2.4}} \quad \text{integrating both sides}$$

$$\int \frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = -\int r^{-0.4}\, 200\, dr$$

$$r^2 \frac{\partial V}{\partial r} = \frac{-200\,(r^{0.6})}{0.6} + C_1 = -333.33\, r^{0.4} + C_1 \quad \rightarrow (1)$$

As $\vec{E}$ is function of r only

$$\vec{E} = -\nabla V = \frac{-\partial V}{\partial r}\hat{a}_r = E_r \hat{a}_r$$

From eq^n (1) $\quad -r^2 E_r = -333.33\, r^{0.6} + C_1$

Bus as $r \to 0$, $r^2 E_r \to 0$

$0 = 0 + C_1 \Rightarrow C_1 = 0$ hence eq^n 1 reduces to

$$r^2 \frac{\partial v}{\partial r} = -333.33\, r^{0.6} + C_1 \to 0$$

$$\frac{\partial v}{\partial r} = -333.33\, r^{-1.4}$$

Integrating b.s with respect to r.

$$V = -333.33 \int r^{-1.4}\, dr = (-333.33) \frac{r^{-0.4}}{-0.4} + C_2$$

$$V = \frac{833.325}{r^{0.4}} + C_2$$

Using boundary conditions $V \to 0$ as $r \to \infty$

$$0 = \frac{833.325}{(\infty)^{0.4}} + C_2 \Rightarrow C_2 = 0.$$

$$\boxed{V = \frac{833.325}{r^{0.4}}}$$

ii) Let us verify this using Gauss's Law

$$\nabla \cdot \vec{D} = \rho_v.$$

$$\nabla \cdot \varepsilon_0 \vec{E} = \rho_v$$

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0} = \frac{200}{r^{2.4}}$$

Divergence of $\vec{E}$ (considering radial component of $\vec{E}$)

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{200}{r^{2.4}}$$

$$\frac{\partial}{\partial r}(r^2 E_r) = \frac{200}{r^{0.4}}$$

Integrating $\quad r^2 E_r = \frac{200\, r^{0.6}}{0.6} + C_1 = 333.33\, r^{0.6} + C_1$

Applying boundary conditions $\quad r^2 E_r \to 0$ as $r \to 0$ hence $C_1 = 0$

$$r^2 E_r = 333.33\, r^{0.6}$$

$$\vec{E} = E_r \hat{a}_r = 333.33\, r^{-1.4} \hat{a}_r \text{ V/m}$$

$$V = -\int \vec{E}\cdot d\vec{l} = -\int (333.33\, r^{-1.5}\, \hat{a}_r)\cdot(dr\, \hat{a}_r) = -333.33 \int r^{-1.5}\, dr$$

$$= -333.33 \frac{r^{-0.4}}{-0.4} + C_2 = \frac{833.33}{r^{0.4}} + C_2$$

But $V = 0$ as $r \to \infty$ hence $C_2 = 0$

$$\boxed{V = \frac{833.33}{r^{0.4}} \text{ V}}$$

6. If $V = 2V$ at $x = 1\,mm$ and $V = 0$ at $x = 0$ and charge density ρ_v is $-10^6 \varepsilon_0\ C/m^3$ constant throughout the region between $x = 0$ to $x = 1\ mm$, calculate V at $x = 0.5\,mm$ and E_x at $x = 1\ mm$ in free space.

From Poisson's equation

$$\nabla^2 V = \frac{-\rho_v}{\varepsilon} = \frac{-[-10^6 \varepsilon_0]}{\varepsilon_0} = 10^6$$

As V is the function of x only

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 10^6$$

Integrating b.s $\quad \frac{\partial V}{\partial x} = 10^6 x + C_1$

Integrating b.s $\quad V = \frac{10^6 x^2}{2} + C_1 x + C_2$

at $x = 0$, $V = 0$ hence $\quad 0 = 0 + 0 + C_2 \Rightarrow C_2 = 0$

at $x = 1mm$, $V = 2V$ hence $\quad 2 = \frac{10^6 (10^{-3})^2}{2} + C_1 (1 \times 10^{-3}) + 0$

$$2 = 0.5 + C_1 (1 \times 10^{-3})$$

$$C_1 = 1.5 \times 10^3 = 1500$$

$$V = \frac{10^6 x^2}{2} + 1500\,x$$

$$\boxed{V = 0.5 \times 10^6 x^2 + 1500\,x}$$

V at $x = 0.5\ mm$ $\quad \boxed{V = 0.875\,V}$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial x}\hat{a}_x = -\left[\frac{1 \times 10^6}{2}(2x) + 1500\right] = -2500\ V/m$$

7) If the field of a region in space is given by

$\vec{E} = 5\cos(z)\,\hat{a}_z$ V/m Is the region free of charge?

We need to verify that potential in the region satisfies Laplace's equation or not

$$\vec{E} = -\nabla V$$

$$\nabla V = -5\cos(z)\,\hat{a}_z$$

$$\nabla \cdot \nabla V = \nabla \cdot \left[-5\cos(z)\,\hat{a}_z\right]$$

$$= \left[\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right] \cdot \left[-5\cos z\,\hat{a}_z\right]$$

$$\nabla^2 V = -\frac{\partial(5\cos z)}{\partial z} = +5\sin z$$

$\nabla^2 V \neq 0$ Hence the region is not free of charge.

8) Given the potential field $V = \left[A\rho^4 + B\rho^{-4}\right]\sin(4\phi)$

i) Show that $\nabla^2 V = 0$

ii) Select A & B such that $V = 100$ V and

$|\vec{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5°, z = 2)$

i) $$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\left(4A\rho^3 - 4B\rho^{-5}\right)\sin 4\phi + \frac{1}{\rho^2}\frac{\partial 4}{\partial \phi}\left(\left(A\rho^4 + B\rho^{-4}\right)\cos(4\phi)\right) + \frac{\partial}{\partial z}(0)$$

$$= \frac{1}{\rho}\left(16A\rho^3 - 16B\rho^{-5}\right)\sin 4\phi - \frac{16}{\rho^2}\left(A\rho^4 + B\rho^{-4}\right)\cdot\sin(4\phi)$$

$$\nabla^2 V = \sin 4\phi\left[16A\rho^2 - 16B\rho^{-6}\right] - \sin 4\phi\left[16A\rho^2 - 16B\rho^{-6}\right] = 0$$

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ii) At $P(\rho = 1, \phi = 22.5^\circ, z = 2)$ and $V = 100V$

$$100 = [A+B] \sin(22.5^\circ \times 4) = A+B \rightarrow (1)$$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right]$$

$$= -\left[(4A\rho^3 - 4B\rho^{-5})\sin(4\phi)\,\hat{a}_\rho + \frac{4}{\rho}(A\rho^4 + B\rho^{-4})\cos 4\phi\right]$$

at P

$$\vec{E} = -(4A - 4B)\,\hat{a}_\rho \text{ V/m}$$

$$|\vec{E}| = (4A - 4B) = 500 \rightarrow (2)$$

Solving 1 & 2 $\quad A = 112.5$ & $B = -12.5$

Q) There exists a potential of $V = -2.5V$ on a conductor at $0.02m$ and $V = 15V$ at $r = 0.35m$, determine E & D by solving Laplace's equation in spherical coordinates.

As the V is a function of r only,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

Integrating $\quad r^2\frac{\partial V}{\partial r} = C_1$

$$\frac{\partial V}{\partial r} = C_1 r^{-2}$$

Integrating. $\quad V = \frac{-C_1}{r} + C_2$

Using boundary Conditions.

$r = 0.02\ m$, $V = -2.5V$ & $r = 0.35\ m$, $V = 15V$

$-2.5 = -50C_1 + C_2$

$15 = -2.85714C_1 + C_2$

Solving above equations $C_1 = 0.3712$ & $C_2 = 16.0606$

$$\boxed{V = -\frac{0.3712}{r} + 16.0606}$$

$$\boxed{\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = -\frac{0.3712}{r^2}\hat{a}_r \ V/m}$$

$$\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = -\frac{0.3712}{r^2}\varepsilon_0 \varepsilon_r \hat{a}_r \ C/m^2}$$

10) Conducting spherical shells with radii $a = 10cm$ and $b = 30cm$ are maintained at a potential difference of 100V such that $V(r=b) = 0$ and $V(r=a) = 100V$. Determine V & E in the region between the shells. If $\varepsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance

Using Laplace's equation

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = 0$$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = 0$$

Integrating $r^2 \frac{\partial V}{\partial r} = C_1 \Rightarrow \frac{\partial V}{\partial r} = C_1 r^{-2}$

Integrating $V = \frac{-C_1}{r} + C_2$

Using boundary conditions

$r = b = 0.3m$, $V = 0$ & $r = a = 0.1m$, $V = 100V$

$0 = -3.33C_1 + C_2$ $100 = -10C_1 + C_2$

Solving $C_1 = -15$ $C_2 = -50$

$$\boxed{V = \frac{+15}{r} - 50}$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = \frac{-\partial[15/r - 50]}{\partial r}\hat{a}_r$$

$$\boxed{\vec{E} = +\frac{15}{r^2}\hat{a}_r \ V/m}$$

$$\vec{D} = \varepsilon\vec{E} = \frac{\varepsilon_0 \varepsilon_r 15}{r^2}\hat{a}_r \ C/m^2$$

$\rho_s = |\vec{D}_N|$

$\rho_s = \frac{15\varepsilon_0\varepsilon_r}{r^2} \ C/m^2$

$Q = \iint \rho_s \, ds$ where $ds = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$

$$Q = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{15\varepsilon_0\varepsilon_r}{r^2} \times r^2 \sin\theta \, d\theta \, d\phi = 15\varepsilon_0\varepsilon_r(4\pi)$$

$Q = 60\pi\varepsilon_0\varepsilon_r \ C$

$$C = \frac{Q}{V} = \frac{60\varepsilon_0\varepsilon_r\pi}{100} = 41.72pF$$

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11) Find the potential function and the electric field intensity for the region between two concentric right circular cylinders where $V=0$ at $r=1\,mm$ & $V=150\,V$ at $r=20\,mm$, if $\varepsilon_r = 3.6$. Find the surface charge density on each cylinder. Determine the capacitance between the conducting cylinders per meter length.

The potential is constant with ϕ & z hence from Laplace's equation

$$\nabla^2 V = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Integrating $\rho \frac{\partial V}{\partial \rho} = C_1$

$$\frac{\partial V}{\partial \rho} = \frac{C_1}{\rho}$$

Integrating $V = C_1 \ln \rho + B$

at $\rho = 1\,mm$ $V=0$ & at $\rho = 20\,mm$ $V = 150\,V$

$0 = A \ln(1\times10^{-3}) + B$ $\quad$ $150 = A \ln(20\times10^{-3}) + B$

Solving both $A = 50.1$ & $B = 345.9$

$$\boxed{V = 50.1 \ln \rho + 345.9\,V}$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\hat{a}_\rho = -\frac{\partial}{\partial \rho}(50.1 \ln \rho + 345.9)\,\hat{a}_\rho$$

$$\vec{E} = -\frac{50.1}{\rho}\,\hat{a}_\rho \ V/m$$

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = -\frac{1.5969}{\rho}\,\hat{a}_\rho \ nC/m^2$$

On Cylinder: $|D_N| = \rho_s = |\vec{D}|$

at $r = 1mm$ $\quad \rho_s = \left| \frac{-1.5969 \times 10^{-9}}{1 \times 10^{-3}} \right| = 1.5969 \, \mu C/m^2$

at $r = 20mm$ $\quad \rho_s = \left| \frac{-1.5969 \times 10^{-9}}{1 \times 10^{-3}} \right| = 79.845 \, nC/m^2$

$$Q = \iint \rho_s \, ds = \int_{z=0}^{z} \int_{\phi=0}^{2\pi} \frac{1.5969 \times 10^{-9}}{\rho} \rho \, d\phi \, dz$$

$$Q = 1.5969 \times 10^{-9} (2\pi) \times z$$

$$\boxed{Q = 1.00336 \times 10^{-8} z \ C}$$

for 1 meter length $\quad Q = 1.00336 \times 10^{-8} C$

$C = Q/V \rightarrow (V \text{ at } r = 20mm) - (V \text{ at } r = 1mm)$

$$C = \frac{1.00336 \times 10^{-8}}{(50.1 \ln(1 \times 10^{-3}) + 345.9) - (50.1 \ln(20 \times 10^{-3}) + 345.9)}$$

$$\boxed{C = 66.8522 \, pF/m}$$