Current & Current Density

→ Electric charges in motion constitute electric current. The unit of current is Ampere (A). 1 ampere current is said to be flowing across a surface when 1 Coulomb of charge is passing across surface in one second.

Current is symbolized as I

$$I = \frac{dQ}{dt}$$

Current density is a vector represented by $\vec{J}$, measured in ampere / square meter (A/m^2)

The increment of current ΔI crossing an incremental surface ΔS normal to the current density is

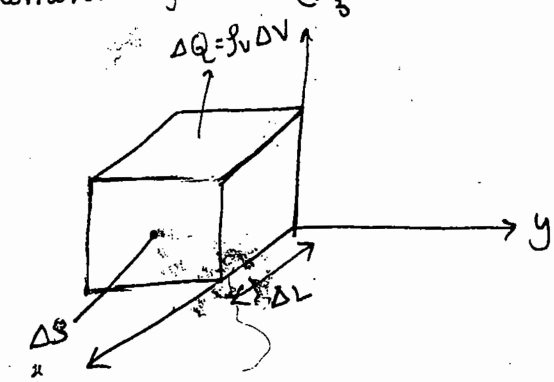
$$\Delta I = J_N \Delta S$$

If current density is not normal to ΔS

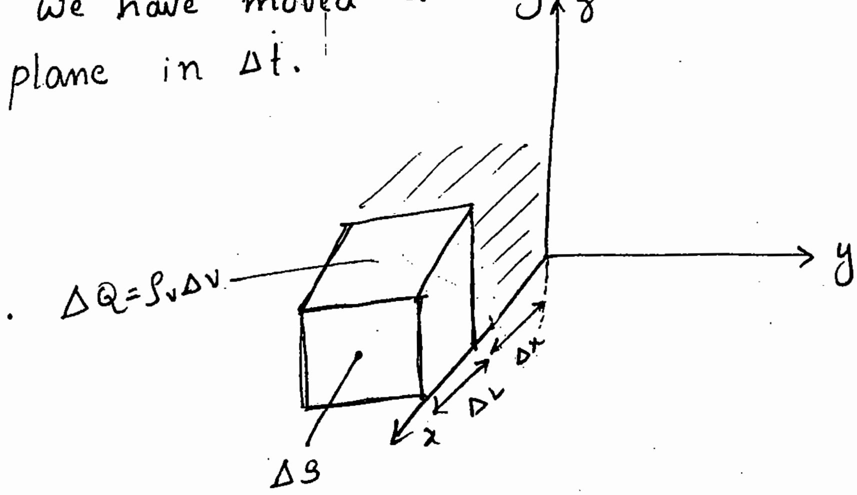
$$\Delta I = \vec{J} \cdot \Delta \vec{S}$$

Total current $\quad I = \int_S \vec{J} \cdot d\vec{S}$

Consider element of charge $\Delta Q = \rho_v \Delta V = \rho_v \Delta S \, \Delta L$ as shown below



Let the charge is moving in x-direction with velocity $\vec{V}$ and thus velocity has only x component V_x i.e., we have moved a charge $\Delta Q = \rho_v \Delta S \Delta x$ through a reference plane in Δt.



In the time interval Δt the element of charge moved through distance Δx in direction of x-axis as shown in above figure. The resultant current is

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

Now $\Delta Q = \rho_v \Delta S \Delta x$ and

$$\Delta I = \frac{\rho_v \Delta S \Delta x}{\Delta t}$$

$\frac{\Delta x}{\Delta t} = V_x$ velocity in x direction

$$\Delta I = \rho_v \Delta S V_x$$

In terms of current density

$$J = \rho_v V_x$$

In general $\vec{J} = \rho_v \vec{V}$

$\vec{V} \rightarrow$ Velocity vector

The principle of conservation of charges states that charges can neither be created nor destroyed although equal amount of positive & negative charges may be simultaneously created, obtained by separation, destroyed, or lost by recombination.

The continuity equation follows this principle when we consider any region bounded by a closed surface

The current through the closed surface is

$$I = \oint_S \vec{J} \cdot d\vec{s}$$

This outward flow of charge must be balanced by a decrease of positive charge within the closed surface

If the charge inside the closed surface is denoted by Q_i, then rate of decrease is $-dQ_i/dt$

$$I = \oint \vec{J} \cdot d\vec{s} = -\frac{dQ_i}{dt}$$

Above equation is integral form of continuity equation, the point form is obtained by following steps.

using divergence theorem $\oint_S \vec{J} \cdot d\vec{s} = \int_{vol} (\nabla \cdot \vec{J})\, dv$

Enclosed charge Q_i is volume integral of the charge density

$$\int_{vol} (\nabla \cdot \vec{J})\, dv = -\frac{d}{dt} \int_{vol} \rho_v \, dv$$

If we keep surface constant, the derivative becomes partial derivative i.e.,

$$\int_{vol} (\nabla \cdot \vec{J})\, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t} \, dv$$

This expression is true for any volume however small,

$$(\nabla \cdot \vec{J})\, \Delta v = -\frac{\partial \rho_v}{\partial t} \Delta v$$

$$\boxed{(\nabla \cdot \vec{J}) = -\frac{\partial \rho_v}{\partial t}}$$

This equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

Metallic Conductors

→ The range of energies that an electron may possess in an atom is known as the energy band.

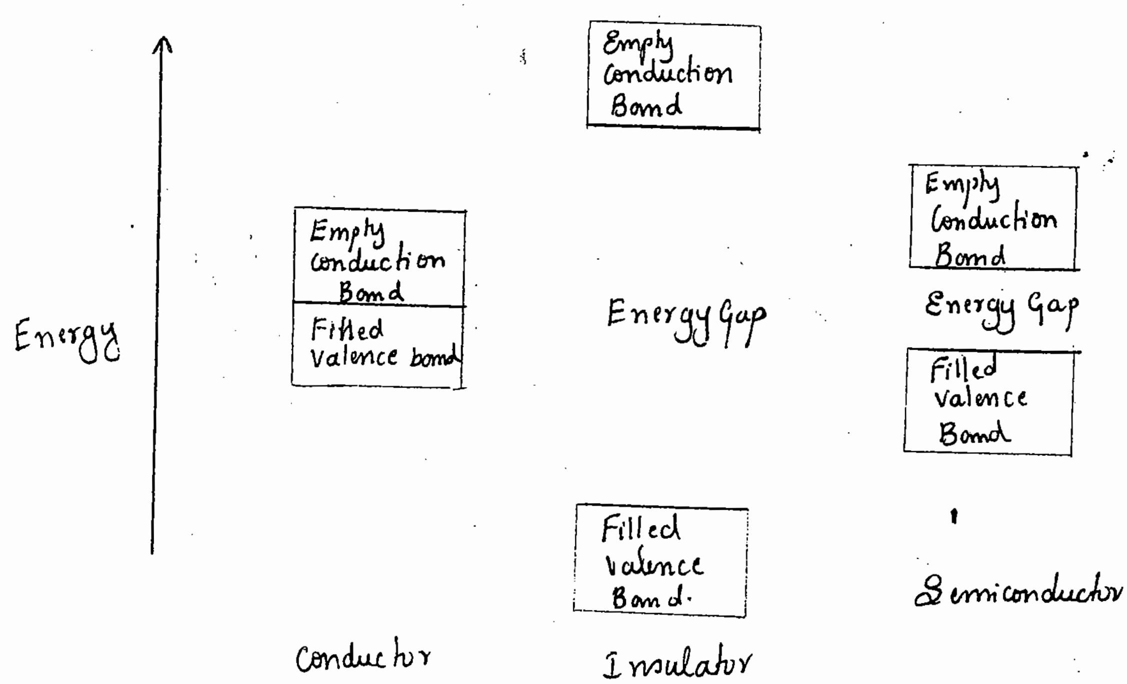
→ According to the quantum theory, only certain discrete energy levels or energy states are permissible in a given atom.

∴ electrons are located in valence band.

→ In conductors the conduction and valence bonds are overlapped, additional kinetic energy may be given to the valence electron by an external source resulting in electron flow.

→ In case of insulators gap exists between the valence band and the conduction band, the electron cannot accept energy in smaller amount. if it accepts the insulator breaks down

→ In semiconductors an intermediate condition occurs when only a small "forbidden region" separates 2 bonds. Small amount of energy in the form of heat, light or an electric field may raise the energy of the electrons at the top of the filled band and drag to conduction bond



Energy bond Structure in three different types of materials

Conductors

→ In conductors the free electrons move under the influence of electric field

If electric field is $\vec{E}$ and an electron having charge $Q = -e$ will experience a force $\vec{F} = -e\vec{E} \longrightarrow (1)$

→ In free space the electron continously increases its velocity In crystalline material because of collisions with thermally excited crystalline lattice structure constant velocity is attained

This velocity v_d is known as drift velocity. The drift velocity is linearly related with field intensity and mobility of electron in the given material.

$$\vec{v}_d = -\mu_e \vec{E} \longrightarrow (2)$$

mobility is measured in terms of m^2/Vs and $\vec{E}$ as V/m

we know that $\vec{J} = \rho_v \vec{v}$ substituting for $\vec{v}$

$$\vec{J} = -\rho_e \mu_e \vec{E} \longrightarrow (3)$$

→ The relation between $\vec{E}$ and $\vec{J}$ is also given as.

$$\vec{J} = \sigma \vec{E} \longrightarrow (4)$$

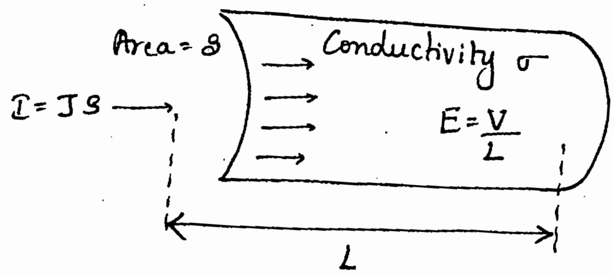
where σ is conductivity measured in Siemens/meter.

1 Siemen is the basic unit of conductance in the SI system and is defined as 1 ampere/volt.

Later the unit of conductance called mho i.e, ℧

From equation (3) & (4) $\sigma = -\rho_e \mu_e$

Let us assume J and E are uniform,



$$I = \int_S \vec{J} \cdot d\vec{S} = JS$$

$$V_{ab} = -\int_b^a \vec{E} \cdot d\vec{L}$$

$$V_{ab} = -\vec{E} \cdot \vec{L}_{ba}$$

or

$$V = EL$$

Thus $J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$

$$V = \frac{L}{\sigma S} I$$

$$\boxed{V = RI}$$

where R is resistance of the cylinder above equation is known as Ohm's law.

Conductor Properties & Boundary Conditions

Property 1: If number of electrons are placed interior of a conductor, as there is no positive charge to neutralize the electrons begin to accelarate away from each other. This continues until the electrons reach the surface of the conductor. Outward progress of electrons stops as material surrounding the conductor is insulator. No charge may remain within the conductor.

Property 2: No current may flow during the static condition, the electric field intensity within the conductor is zero.

Summarizing for electrostatics no charge and no field exists within a conducting material.

The charge may appear on the surface as surface charge density. The external fields are related to the charge on the surface of the conductor.

Boundary Conditions between conductor & free space

When an electric field passes from one medium to other medium it is important to study the conditions at the boundary between the two media.

The conditions existing at boundary of the two media when field passes from one medium to other are called boundary conditions.

There are 2 cases of boundary conditions.

1) Boundary between conductor and free space
2) Boundary between two dielectrics with different properties

We know from Maxwell's equations that

$$\oint \vec{E}\cdot d\vec{L} = 0 \quad \text{and} \quad \oint \vec{D}\cdot d\vec{S} = Q$$

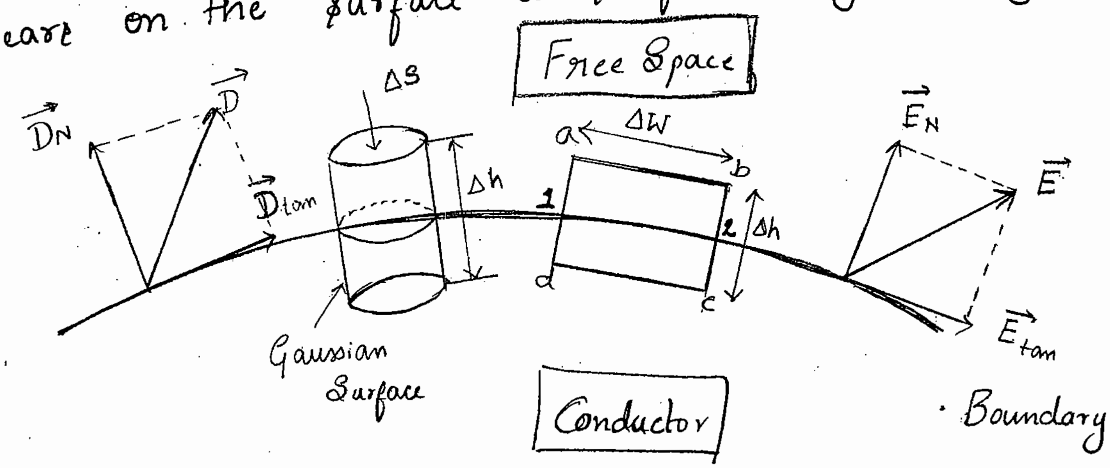
The field intensity and flux density can be [illegible] into two components namely tangential to boundary and normal to boundary hence at any point on boundary

$$\vec{E} = \vec{E}_{tan} + \vec{E}_N$$

$$\vec{D} = \vec{D}_{tan} + \vec{D}_N.$$

For ideal conductors it is known that

1. The field intensity and flux density inside the conductor is zero
2. No charge exists inside the conductor, the charge appears on the surface as surface charge density.



Consider the conductor free space boundary as shown in the above figure.

We know that $\oint \vec{E} \cdot d\vec{L} = 0$

i.e, workdone in carrying a unit positive charge around a closed path is zero.

Consider a rectangular closed path a-b-c-d-a traced in clockwise direction. $\oint \vec{E}\cdot d\vec{L}$ can be divided into 4 parts

$$\oint \vec{E}\cdot d\vec{L} = \int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L}$$

The closed contour is placed such that two sides a-b and c-d are parallel to tangential direction and b-c and d-a are parallel to normal direction of the surface

Let height and width of elementary rectangle is Δh and Δw respectively. Thus $\Delta h/2$ is in the conductor and $\Delta h/2$ is in free space.

The portion c-d is in the conductor where $\vec{E}=0$ hence

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L} = 0$$

as Δw is small $\vec{E}$ over it can be assumed constant

hence above integral can be written as

$$\int_a^b \vec{E}\cdot d\vec{L} = \vec{E}\int_a^b d\vec{L} = \vec{E}\,(\Delta W)$$

$\vec{E}$ is along tangential direction to the boundary

$\vec{E} = \vec{E}_{tan}$ $\quad E_{tan} = |\vec{E}_{tan}|$

$$\int_a^b \vec{E}\cdot d\vec{L} = E_{tan}\,(\Delta W)$$

b-c is parallel to the normal component so we have E_N along this direction. Let $E_N = |\vec{E}_N|$

Over the small height Δh, E_N can be assumed constant and can be taken out of integration

$$\int_b^c \vec{E} \cdot d\vec{L} = \vec{E} \int_b^c d\vec{L} = E_N \int_b^c d\vec{L}$$

Out of b-c, b-2 is in free space and 2-c is in the conductor where $\vec{E} = 0$

$$\int_b^c d\vec{L} = \int_b^2 d\vec{L} + \int_2^c d\vec{L} = \frac{\Delta h}{2} + 0 = \frac{\Delta h}{2}$$

$$\int_b^c \vec{E} \cdot d\vec{L} = E_N \left(\frac{\Delta h}{2}\right) \qquad \text{—— (8)}$$

Similarly for path d-a the condition is same as for the path b-c, only direction is opposite.

$$\int_d^a \vec{E} \cdot d\vec{L} = -E_N \left(\frac{\Delta h}{2}\right) \qquad \text{—— (9)}$$

Substituting in

$$\int_a^b \vec{E} \cdot d\vec{L} + \int_b^c \vec{E} \cdot d\vec{L} + \int_d^a \vec{E} \cdot d\vec{L} = 0$$

$$E_{tan} \Delta W + E_N \left(\frac{\Delta h}{2}\right) - E_N \left(\frac{\Delta h}{2}\right) = 0$$

$$E_{tan} \Delta W = 0$$

but $\Delta W \neq 0$ as finite

$$\boxed{E_{tan} = 0}$$

Thus tangential component of electric field intensity is zero at the boundary between conductor and free space.

$$\vec{D} = \epsilon_0 \vec{E} \quad \text{for free space}$$

$$D_{tan} = \epsilon_0 E_{tan} = 0$$

$$\boxed{D_{tan} = 0}$$

The tangential component of electric field intensity is zero at the boundary between conductor and free space.

To find normal component of $\vec{D}$ and $\vec{E}$ select a closed Gasian surface in the form of right circular cylinder. Its height is Δh and it is placed such that $\Delta h/2$ is in the conductor and $\Delta h/2$ is in the free space

According to Gauss's law $\oint_S \vec{D} \cdot d\vec{s} = Q$

The surface integral must be evaluated over three surfaces

$$\int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} + \int_{lateral} \vec{D} \cdot d\vec{s} = Q$$

The bottom surface is inside the conductor where $\vec{D} = 0$ hence above equation reduces to

$$\int_{top} \vec{D} \cdot d\vec{s} + \int_{lateral} \vec{D} \cdot d\vec{s} = Q$$

The lateral surface area is $2\pi r \Delta h$ where r is the radius of cylinder. Because D_N is tangential to surface & $\vec{D}_N \cdot d\vec{s} = 0$

The corresponding integral is zero

$$\int_{top} \vec{D}\cdot d\vec{s} + \cancel{\int_{lateral} \vec{D}\cdot d\vec{s}}^{\,0} = Q$$

$$\int_{top} \vec{D}\cdot d\vec{s} = D_N \int_{top} d\vec{s} = D_N \Delta S$$

From Gauss's law

$$D_N \Delta S = Q$$

But at the boundary charges exist as surface charge density $Q = \rho_S \Delta S$ by comparing above two equations

$$D_N \cancel{\Delta S} = \rho_S \cancel{\Delta S}$$

$$\boxed{D_N = \rho_S}$$

$$\boxed{E_N = \frac{\rho_S}{\epsilon_0}}$$

The electric flux leaves the conductor in a direction normal to the surface, and value of electric flux density is numarically equal to the surface charge density.

Zero tangential electric field intensity is the fact that a conductor surface is an equipotential surface

To summarize the concepts which apply to conductors in electrostatic fields

1. The static electric field intensity inside a conductor is zero

2. The static field intensity at the surface of the conductor is everywhere directed normal to that surface.

3. The conductor surface is an equipotential surface.

Dielectric Material

→ Insulating materials or dielectric materials differ from conductor. There will be no free charge that can be transported within them to produce conduction current. All charges are confined to molecular or lattice sites by coulomb forces.

→ By applying electric field has the effect of displacing the charges slightly, leading to the ensemble of electric dipoles.

→ The extent to which this occurs (polarization) is measured by the relative permittivity or dielectric constant.

→ The charge displacement principle constitutes an energy storage mechanism that is used in construction of capacitor.

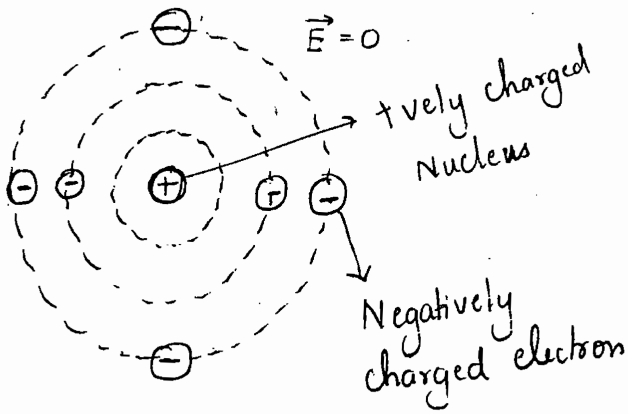
→ The dielectric in an electric field can be viewed ~~as~~ as a free space arrangement of microscopic electric dipoles which are composed of positive and negative charges whose centers do not coincide.

They are not free charges and they cannot contribute to the conduction process.

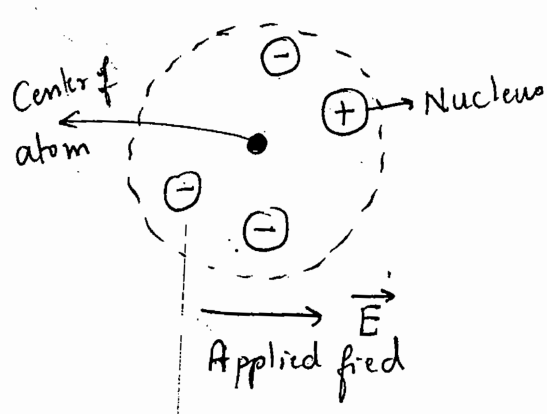
→ The dipoles are known as bound charges, which can be considered as any other sources of the electrostatic field.

→ The common characteristic of dielectric material (whether they are solid, liquid or gas, no crystalline in nature) is their ability to store energy.

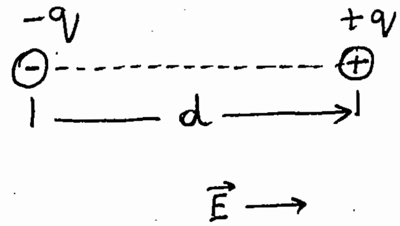
The storage of energy takes place by means of shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces. (Similar to lifting a weight or stretching a spring i.e., potential energy)



Unpolarized atom of dielectric



Polarized atom



Equivalent dipole

Mathematical Expression for Polarization.

When a dipole is formed due to polarization, there exists an electric dipole moment $\vec{p}$.

$$\vec{p} = Q\,\vec{d}$$

Q is magnitude of one of the two charges

$\vec{d}$ distance vector from -ve to positive charge

If there are n dipoles per unit volume, the number of dipoles in volume ΔV is $n\Delta V$, and total dipole moment is

$$\vec{p}_{total} = \sum_{i=1}^{n\Delta V} Q_i \vec{d_i}$$

If dipoles are randomly oriented $\vec{P}_{total}$ is zero but if dipoles are aligned in the direction of applied $\vec{E}$ then $\vec{P}_{total}$ has significant value.

The polarization $\vec{P}$ is defined as the total dipole moment per unit volume

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n\Delta V} Q_i \vec{d_i}}{\Delta V} \quad C/m^2$$

It can be seen that the unit of polarization is the same as that of flux density $\vec{D}$. The increase in polarization leads to increase in flux density in a dielectric medium.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

For isotropic and linear medium, the linear relationship between $\vec{P}$ & $\vec{E}$ is:

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

where χ_e is dimensionless quantity called electric susceptibility of the material. using this relation

$$\vec{D} = \epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E} = (\chi_e + 1) \epsilon_0 \vec{E}$$

The expression within parenthesis defined as

$$\epsilon_r = \chi_e + 1 \quad \text{thus}$$

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E} = \epsilon \vec{E}$$

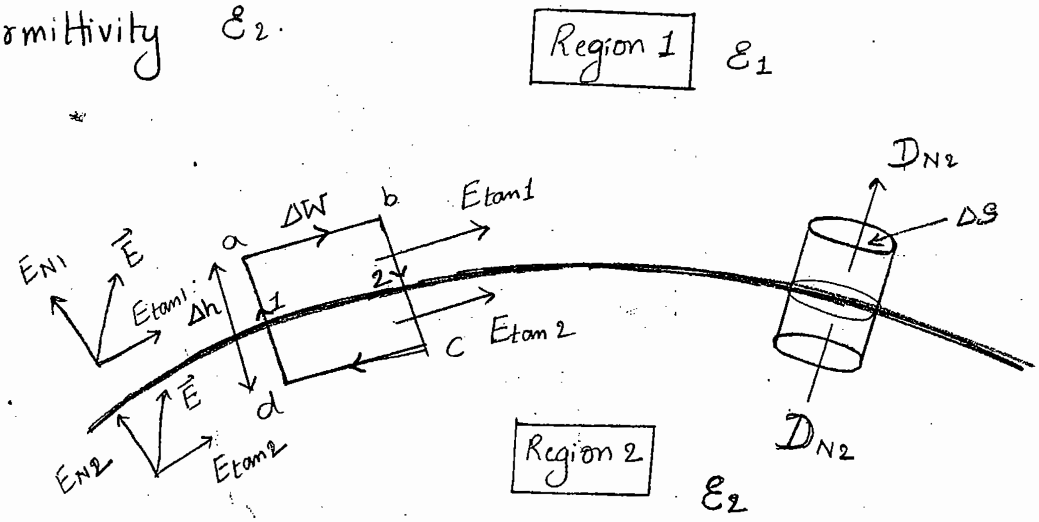
$$\boxed{\epsilon = \epsilon_0 \epsilon_r}$$

ϵ is known as permittivity

ϵ_r is relative permittivity or dielectric constant of the material. (dimensionless)

Boundary condition for perfect dielectric material:

Consider the boundary between two perfect dielectrics. One dielectric has permittivity ε_1 while the other has permittivity ε_2.



The $\vec{E}$ and $\vec{D}$ are to be obtained by resolving each into two components, tangential to the boundary and normal to the surface.

Consider a closed path abcda rectangular in shape having elementary height Δh and elementary width ΔW. It is placed such that $\Delta h/2$ is in dielectric 1 while the remaining is in dielectric 2.

The integral over closed path abcda is

$$\oint \vec{E}\cdot d\vec{L} = 0$$

$$\therefore \int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L} = 0 \quad \longrightarrow (a)$$

and dielectric 2 respectively.

These electric fields have both normal & tangential components i.e.,

$$\vec{E_1} = \vec{E_{1t}} + \vec{E_{1N}}$$
$$\vec{E_2} = \vec{E_{2t}} + \vec{E_{2N}}$$

Let $|\vec{E_{1t}}| = E_{tan1}$, $|\vec{E_{2t}}| = E_{tan2}$

$|\vec{E_{1N}}| = E_{1N}$, $|\vec{E_{2N}}| = E_{2N}$

From equation (a) and above

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L} = 0$$

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^2 \vec{E}\cdot d\vec{L} + \int_2^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^1 \vec{E}\cdot d\vec{L} + \int_1^a \vec{E}\cdot d\vec{L} = 0$$

$$\left[E_{tan1}\,\Delta W\right] + \cancel{\left[-E_{N1}\frac{\Delta h}{2}\right]} + \cancel{\left[-E_{N2}\frac{\Delta h}{2}\right]} + \left[-E_{tan2}\,\Delta W\right] + \cancel{\left[E_{N2}\frac{\Delta h}{2}\right]} + \cancel{\left[E_{N1}\frac{\Delta h}{2}\right]} = 0$$

$$\Delta W\left[E_{tan1} - E_{tan2}\right] = 0$$

ΔW cannot be equal to zero hence

$$\boxed{E_{tan1} = E_{tan2}}$$

Thus the tangential components of field intensity at the boundary in both the dielectrics remain same i.e., electric field intensity is continous across the boundary.

W.K.T $\vec{D} = \varepsilon \vec{E}$

If D_{tan1} and D_{tan2} are tangential components of electric flux density in dielectric 1 and dielectric 2

$$D_{tan1} = \varepsilon_1 E_{tan1} \qquad D_{tan2} = \varepsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}\,\varepsilon_0}{\varepsilon_{r2}\,\varepsilon_0} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

Thus the tangential components of $\vec{D}$ undergoes some changes across the interface hence $\vec{D}$ is discontinous across the boundary.

To find normal components let us use Gauss's law

Consider a Gassian surface in the form of right circular cylinder, placed in such a way that half of its lies in dielectric 1 while the remaining half in dielectric 2

$$\oint \vec{D}\cdot d\vec{S} = Q$$

$$\int_{Top} \vec{D}\cdot d\vec{S} + \int_{bottom} \vec{D}\cdot d\vec{S} + \int_{lateral} \vec{D}\cdot d\vec{S} = Q.$$

$$\int_{top} \vec{D}\cdot d\vec{S} + \int_{bottom} \vec{D}\cdot d\vec{S} + \int_{\substack{lateral\\ Top}} \vec{D}\cdot d\vec{S} + \int_{\substack{lateral\\ bottom}} \vec{D}\cdot d\vec{S} = Q$$

$$D_{N1}\Delta S - D_{N2}\Delta S - \cancel{D_{tan1}\frac{\Delta h}{2}\rho\Delta\phi} + \cancel{D_{tan1}\frac{\Delta h}{2}\rho\Delta\phi}$$

$$-\cancel{D_{tan2}\frac{\Delta h}{2}\rho\Delta\phi} + \cancel{D_{tan2}\frac{\Delta h}{2}\rho\Delta\phi} = Q$$

$$D_{N1}\Delta S - D_{N2}\Delta S = Q$$

$$D_{N1} - D_{N2} = \frac{Q}{\Delta S} = \rho_s$$

$$D_{N1} - D_{N2} = \rho_s$$

There is no charge available in perfect dielectric. As all the charges are bound charges and are not free, ~~hen~~ the surface charge density can be assumed zero

$$\rho_s = 0$$

$$\boxed{D_{N1} = D_{N2}}$$

Hence Normal component of flux density $\vec{D}$ is continous at the boundary between two perfect dielectrics.

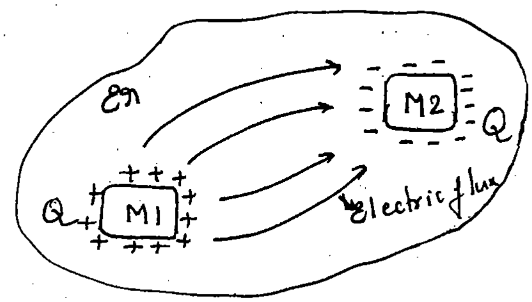
Now $\quad D_{N1} = \varepsilon_1 E_{N1} \qquad D_{N2} = \varepsilon_2 E_{N2}$

$$\boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

The normal component of electric field intensity $\vec{E}$ are inversely proportional to the relative permitivities of two media.

Capacitance

→ Consider M1 & M2 two conducting materials placed in dielectric medium having permittivity ε_r. The material M_1 carries a positive charge and M_2 carries negative charge. The total charge of the system is zero



→ In conductors charges reside on the surface. Such two conducting surfaces carrying equal and opposite charge placed in dielectric medium is called capacitive system giving rise to capacitance.

→ The flux is directed from M_1 to M_2

→ Work must be done to carry a positive unit charge from M2 to M1. i.e, potential difference between M_1 & M_2 is V_{12}.

→ The ratio of magnitude of charge on any one of the conductor & potential difference between two conductor is defined as capacitance

$$C = \frac{Q}{V_{12}}$$

In general $C = \frac{Q}{V}$ Farads

$$1 \text{ Farad} = \frac{1 \text{ coulomb}}{1 \text{ Volt}}$$

$\therefore$ As charges exist only on surface of conductor we can use Gauss's law to find total charges

$$Q = \oint \vec{D} \cdot d\vec{S} = \oint_S \varepsilon_0 \varepsilon_r \vec{E} \cdot d\vec{S} = \oint_S \varepsilon \vec{E} \cdot d\vec{S}$$

→ V is the work done in moving a unit positive charge from negative to positive surface

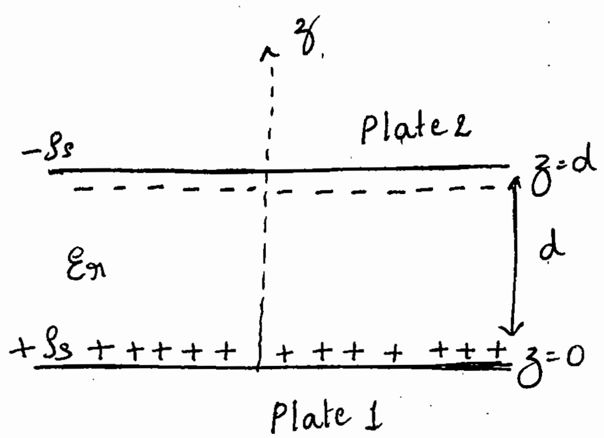
$$V = -\int \vec{E} \cdot d\vec{L} = -\int_{-}^{+} \vec{E} \cdot d\vec{L}$$

Hence $$C = \frac{Q}{V} = \frac{\oint_S \varepsilon \vec{E} \cdot d\vec{S}}{-\int_{-}^{+} \vec{E} \cdot d\vec{L}}$$

→ Capacitance is not dependent on $\vec{E}, \vec{D}$, charge & V

→ Capacitance depends on physical dimensions of the system and properties of dielectric such as permittivity of dielectric.

Parallel plate capacitor



Consider two plates separated by 'd'. Let S be area of cross section of plates

The total charge $Q = \rho_s S$ Coulombs

Magnitude of charge on any one plate

Assuming plate 1 to be infinite sheet charge

$$\vec{E}_1 = \frac{\rho_s}{2\varepsilon} \hat{a}_N = \frac{\rho_s}{2\varepsilon} \hat{a}_z \ V/m$$

$\vec{E_1}$ is normal at the boundary without any tangential component for plate 2

$$\vec{E}_2 = \frac{-\rho_s}{2\varepsilon}\,\hat{a}_N = \frac{-\rho_s}{2\varepsilon}(-\hat{a}_z) = \frac{\rho_s\,\hat{a}_z}{2\varepsilon}$$

In between plates

$$\vec{E} = \vec{E_1} + \vec{E_2} = \frac{\rho_s}{\varepsilon}\,\hat{a}_z \text{ V/m}$$

The potential difference is

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{L} = -\int_{\text{Upper}}^{\text{Lower}} \frac{\rho_s}{\varepsilon}\,\hat{a}_z \cdot d\vec{L}$$

$$V = -\int_{z=d}^{0}\left(\frac{\rho_s}{\varepsilon}\,\hat{a}_z\right)\cdot\left(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z\right) = -\int_{z=d}^{0}\frac{\rho_s}{\varepsilon}\,dz$$

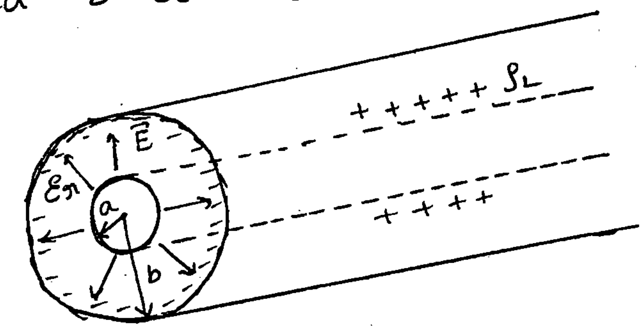
$$V = \frac{-\rho_s}{\varepsilon}(-d) = \frac{\rho_s d}{\varepsilon}$$

$$C = Q/V = \frac{\rho_s S}{\frac{\rho_s d}{\varepsilon}} = \frac{\varepsilon S}{d} \text{ F}$$

$$\boxed{C = \frac{\varepsilon_0\,\varepsilon_r\,S}{d}} \text{ F}$$

Capacitance of coaxial cable

Consider coaxial cable or coaxial capacitor. Let 'a' be inner radius and 'b' be outer radius



The two concentric conductors are separated by dielectric & the length of cable is 'L' m.

The inner conductor carries the charge density ρ_L C/m, on its surface and $-\rho_L$ C/m exists on the outer conductor

$$Q = \rho_L L$$

Assuming cylindrical coordinate system, $\vec{E}$ will be radially outwards from inner to outer conductor. for infinite line charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\hat{a}_\rho$$

$\vec{E}$ is directed from inner conductor to outer conductor.

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{L} = -\int_{\rho=b}^{a} \left(\frac{\rho_L}{2\pi\varepsilon\rho}\hat{a}_\rho\right)\cdot(d\rho\,\hat{a}_\rho)$$

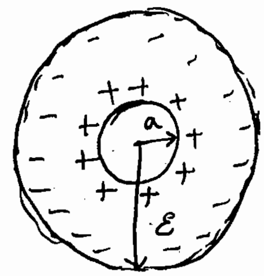
$$V = \frac{-\rho_L}{2\pi\varepsilon}\ln(\rho)\Big]_b^a = \frac{+\rho_L}{2\pi\varepsilon}\ln(b/a)$$

$$\boxed{C = Q/V = \frac{+2\pi\varepsilon L}{\ln(b/a)}} \text{ F}$$

Spherical Capacitor

Consider a spherical capacitor formed by 2 concentric spheres of radius a & b. Let radius of outer sphere is 'b' and inner sphere is 'a' i.e., $a < b$.

The electric field of sphere with charge Q is given by



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r \ \text{V/m}$$

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{L} = -\int_{r=b}^{a} \left(\frac{Q}{4\pi\varepsilon r^2}\,\hat{a}_r\right)\cdot\left(dr\,\hat{a}_r\right)$$

$$= -\int_{r=b}^{a} \frac{Q}{4\pi\varepsilon r^2}\,dr$$

$$= \frac{-Q}{4\pi\varepsilon}\left[\frac{-1}{r}\right]_b^a$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \ \text{F}$$

$$\boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \ \text{F}}$$

Introduction to Magnetostatics

→ Electrostatic field exists due to the static charges, the magnetic field exists due to a permanent magnet, which is a natural magnet.

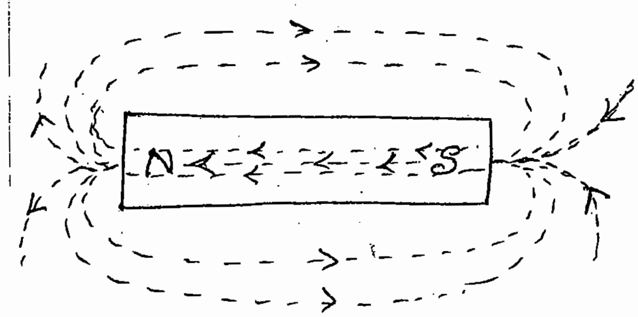
But in electromagnetic engineering a link between electric and magnetic field is required to be studied. That link will be absent with magnetic field due to a natural magnet.

→ The scientist Oersted discovered that when the charges in motion they are surrounded by a magnetic field. Thus flow of charges constitute an electric current. Thus the current carrying conductor is always surrounded by a magnetic field.
If such current is steady (time invarient) then magnetic field produced is also steady magnetic field

→ The study of steady magnetic field existing in a given space produced due to the flow of direct current through a conductor is called magnetostatics.
The various concepts like e.m.f induced, force experienced by a conductor, motoring action, transformer action etc are dependent on the magnetostatics.

Magnetic Field and its properties

A permanent magnet has two poles north (N) & south (S)
The region around a magnet within which the influence of the magnet can be experienced is called magnetic field
Such a field is represented by imaginary lines around the magnet which are called magnetic lines of force.
These lines are introduced by Scientist Michael Faraday.
The direction of lines are always from North pole to South pole. These lines are also called magnetic flux lines

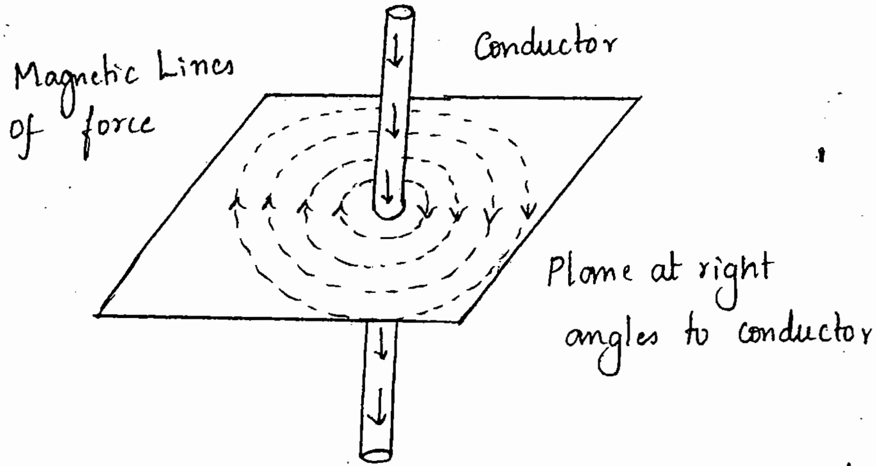


Parmanent magnet and magnetic lines of forces.

Magnetic field due to current carrying conductor

when a straight conductor carries a direct current, it produces a magnetic field around it all along its length. The lines of force in such a case are in the form of concentric circles in the planes at right angles to the conductor.

The direction of concentric circles depends on the direction of current flowing in the conductor.



Magnetic field due to conductor carrying direct current.

A right hand thumb rule is used to determine the direction of magnetic field around a conductor carrying a direct current. It states that hold the current carrying current in the right hand such that the thumb pointing in the direction of current and parallel to the conductors then curled fingers point in the direction of magnetic flux lines around it

Right hand thumb rule.

→ The quantitative measure of strongness or weakness of the magnetic field is given by 'magnetic field intensity.

→ The magnetic field intensity is measured as the force experienced by a unit north pole of one weber strength when placed at that point.

→ The magnetic flux lines are measured in webers (Wb) while magnetic field intensity is measured in N/wb or Amperes/metre. It is denoted as $\vec{H}$

→ The total magnetic lines of force crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density. It is denoted as $\vec{B}$ and is a vector quantity. It is measured in wb/m^2 which is also called Tesla (T)

→ In electrostatics $\vec{E}$ & $\vec{D}$ are related to each other through permittivity ε of the region.

In magnetostatics $\vec{B}$ and $\vec{H}$ are related through the property of the region in which current carrying conductor is placed. It is called permeability denoted as μ.

It is the ability with which the current carrying conductor forces the magnetic flux through the region around it. For the free space permeability is denoted as μ_0 and its value is $4\pi \times 10^{-7}$ Henry/meter.

For any other region relative permeability is specified as μ_r and $\mu = \mu_0 \mu_r$

Thus $\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}$

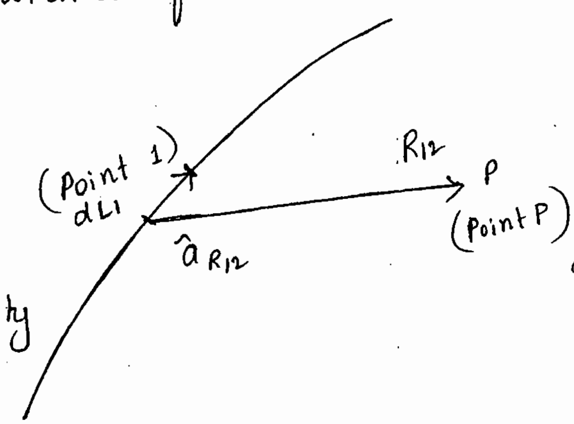
for free space $\vec{B} = \mu_0 \vec{H}$.

Biot Savart Law.

→ Consider a conductor carrying direct current I and a steady magnetic field produced around it.

→ The Biot-Savart law allows us to obtain the differential magnetic field intensity $d\vec{H}$, produced at a point P, due to a differential vector length of the filament $d\vec{L}$

→ The law of Biot-Savart states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of distance from the differential element to the point P.



$$d\vec{H}_2 = \frac{I_1 d\vec{L}_1 \times \hat{a}_{R_{12}}}{4\pi R_{12}^2}$$

The Biot-Savart's law expresses the magnetic field intensity dH_2 produced by a differential current element $I_1 dL_1$

$$d\vec{H} = \frac{I\, d\vec{L} \times \hat{a}_r}{4\pi R^2} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi R^3}$$

Mathematically

$$d\vec{H} \propto \frac{I\,dL\sin\theta}{R^2}$$

$$d\vec{H} = \frac{k\,I\,dl\sin\theta}{R^2}$$

where k = proportionality constant $k = 1/4\pi$

$$d\vec{H} = \frac{I\,dl\sin\theta}{4\pi R^2}$$

If $\hat{a}_r$ is the unit vector in the direction from differential current element to point P. then

$$d\vec{L} \times \hat{a}_r = dL|\hat{a}_r|\sin\theta = dL\sin\theta$$

$$d\vec{H} = \frac{I\,d\vec{L} \times \vec{R}}{4\pi R^3} \quad A/m$$

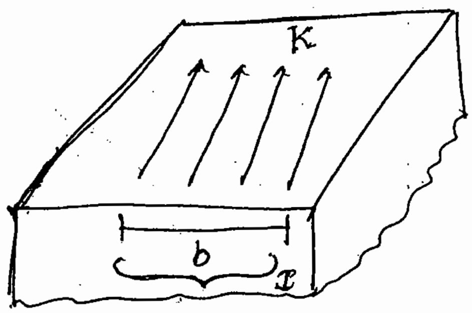
It is impossible to check experimentally the law of Biot Savart law as expressed in above equation. because the differential current element cannot be isolated. Hence integral form of above Biot Savart's law can be verified experimentally

$$\vec{H} = \oint \frac{I\,d\vec{L} \times \hat{a}_r}{4\pi R^2}$$

The Biot-Savart law may also be expressed in terms of distributed sources such as current density J and surface current density K

Biot - Savart Law Interms of Distributed Sources

Consider a surface carrying a uniform current over its surface as shown in the figure..



Then the surface current density is denoted as $\vec{K}$ and measured in A/m. Thus for uniform current density the current I in any width b is given by $I = kb$ where b is $\perp^{lar}$ to direction of current flow.

If dS is the differential surface area considered of a surface having current density $\vec{K}$ then

$$I\,d\vec{L} = \vec{K}\,dS$$

If current density in a volume of given conductor is $\vec{J}$ measured in A/m^2 then for differential volume dv

$$I\,d\vec{L} = \vec{J}\,dv$$

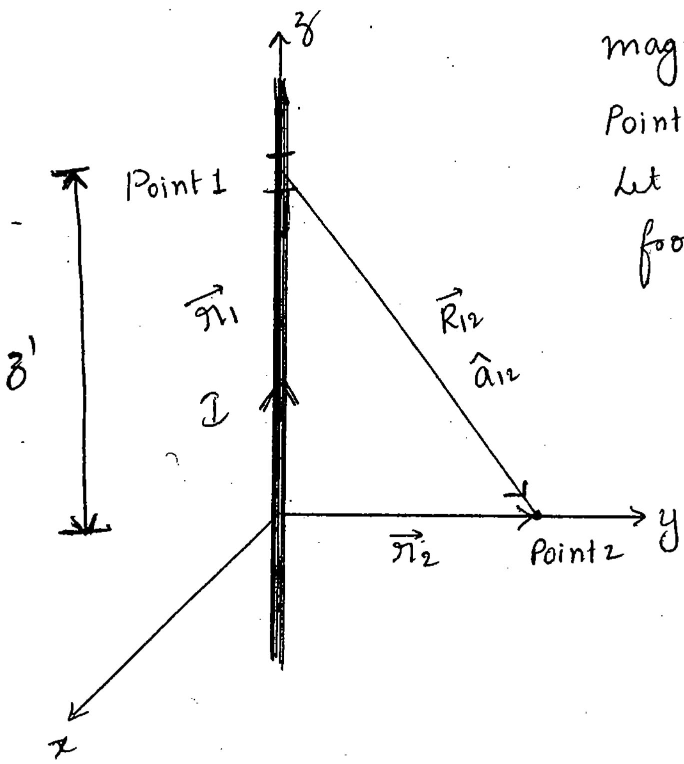
Hence Biot Savart's law can be expressed as:

$$\vec{H} = \int_S \frac{\vec{K} \times \hat{a}_R\, dS}{4\pi R^2} \quad A/m$$

$$\vec{H} = \int_{vol} \frac{\vec{J} \times \hat{a}_R\, dv}{4\pi R^2} \quad A/m$$

$\vec{H}$ due to Infinitely Long Straight Conductor

Consider an infinitely long straight conductor as shown in the figure below. It carries current I Amperes. The conductor is placed in z-axis



At point 2 we need to determine magnetic field intensity.
Point 2 is chosen in $z=0$ plane
Let point 2 is at $\vec{r}_2$ distance from origin. where

$$\vec{r}_2 = \rho\,\hat{a}_\rho$$

Consider a small differential element on conductor at point 1 at distance $\vec{r}_1$ from origin where $\vec{r}_1 = z'\,\hat{a}_z$

So $\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 = \rho\hat{a}_\rho - z'\hat{a}_z$ and $\hat{a}_{12}$ is unit vector along $\vec{R}_{12}$

$$\hat{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\rho\hat{a}_\rho - z'\hat{a}_z}{\sqrt{\rho^2 + z'^2}}$$

and $d\vec{L} = dz'\,\hat{a}_z$

From Biot Savart's Law at point 2 the magnetic field intensity due to dz' element is given by

$$d\vec{H} = \frac{I\, d\vec{L} \times \vec{a}_R}{4\pi R^2} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi\, R^3}$$

$$d\vec{H}_2 = \frac{I\,(dz'\,\hat{a}_z) \times (\rho\,\hat{a}_\rho - z'\,\hat{a}_z)}{4\pi\,(\rho^2 + z'^2)^{3/2}}$$

Since the current is directed toward increasing value of z' the limits are $-\infty$ & ∞ on the integral

$$d\vec{H}_2 = \frac{I\, dz'\, \rho\, \hat{a}_\phi - 0}{4\pi\,(\rho^2 + z'^2)^{3/2}}$$

$$d\vec{H}_2 = \frac{I\, dz'\, \rho\, \hat{a}_\phi}{4\pi\,(\rho^2 + z'^2)^{3/2}}$$

$$\vec{H}_2 = \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{dz'\, \rho\, \hat{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

Here $\hat{a}_\phi$ changes with coordinate ϕ but not with ρ or z. So, $\hat{a}_\phi$ is a constant and can be removed out of integral as integration is w.r.t z'

$$\vec{H}_2 = \frac{I\, \hat{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\, dz'}{(\rho^2 + z'^2)^{3/2}}$$

∴ Since it is even function.

$$\vec{H_2} = \frac{2 I \hat{a}_\phi}{4\pi} \int_0^\infty \frac{\rho \, dz'}{(\rho^2+z'^2)^{3/2}}$$

Substitute $z' = \rho \tan\theta$ $\quad dz' = \rho \sec^2\theta \, d\theta$

$$(\rho^2 + \rho^2 \tan^2\theta)^{3/2} \Rightarrow \rho^3 \sec^3\theta$$

$$z' = 0 \Rightarrow \theta = \tan^{-1}(0) = 0$$

$$z' = \infty \Rightarrow \theta = \tan^{-1}(\infty) = \pi/2$$

$$\vec{H_2} = \frac{2 I \hat{a}_\phi}{4\pi} \int_0^{\pi/2} \frac{\rho^2 \sec^2\theta \cdot d\theta}{\rho^3 \sec^3\theta}$$

$$= \frac{2 I \hat{a}_\phi}{4\pi} \int_0^{\pi/2} \frac{1}{\rho \sec\theta} \, d\theta$$

$$= \frac{I \hat{a}_\phi}{2\pi\rho} \int_0^{\pi/2} \cos\theta \, d\theta$$

$$\boxed{\vec{H_2} = \frac{I \hat{a}_\phi}{2\pi\rho}}$$

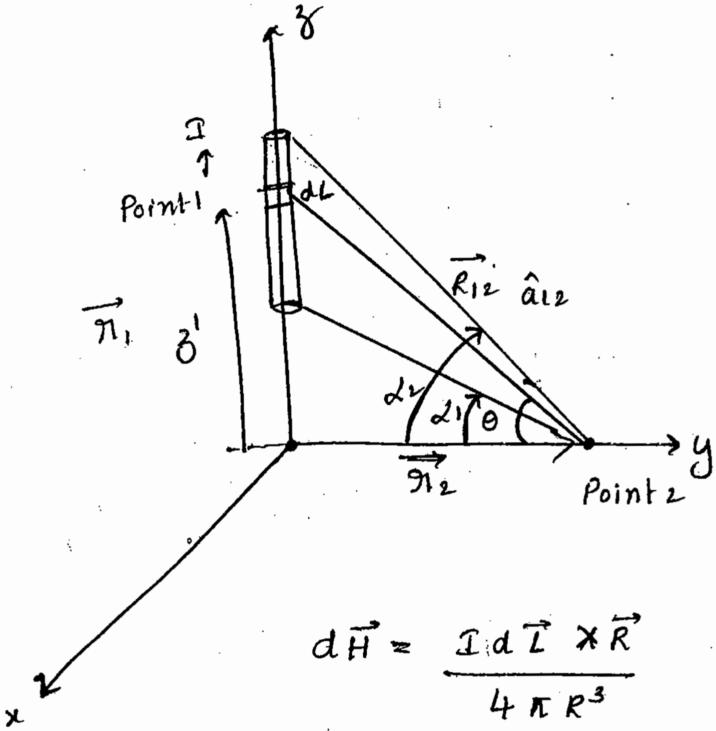
From above expression the magnitude of magnetic field intensity is not function of ϕ or z. It is inversely proportional to ρ which is perpendicular distance from the point to conductor.

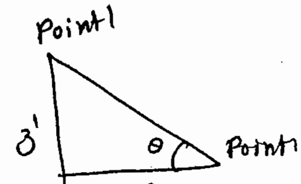
The direction of $\vec{H}$ is tangential i.e, circumferential along $\hat{a}_\phi$

The magnetic flux lines are circuferential

Magnetic field intensity due to finite length current element

Consider a finite-length current element as in figure below





$\tan\theta = z'/\rho$

$z' = \rho\tan\theta$

$$d\vec{H} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi R^3}$$

$$d\vec{H} = \frac{I\, dz'\, \hat{a}_z \times (\rho\hat{a}_\rho - z'\hat{a}_z)}{4\pi\left(\rho^2 + z'^2\right)^{3/2}}$$

$$d\vec{H} = \frac{I\rho\, dz'\, \hat{a}_\phi}{4\pi\left(\rho^2 + z'^2\right)^{3/2}}$$

Substitute $z' = \rho\tan\theta$

$dz' = \rho\sec^2\theta\, d\theta$

$\left(\rho^2 + z'^2\right)^{3/2} = \rho^3\sec^3\theta$

$$d\vec{H} = \frac{I\rho\cdot\rho\sec^2\theta\, d\theta\, \hat{a}_\phi}{4\pi\left(\rho^3\sec^3\theta\right)} = \frac{I\,\hat{a}_\phi}{4\pi\rho}\cos\theta\, d\theta$$

$$\vec{H} = \frac{I\,\hat{a}_\phi}{4\pi\rho} \int_{\theta=\alpha_1} \cos\theta \, d\theta = \frac{I\,\hat{a}_\phi}{4\pi\rho} \sin\theta \Big]_{\alpha_1}$$

$$\vec{H} = \frac{I}{4\pi\rho} \left[\sin\alpha_2 - \sin\alpha_1\right] \hat{a}_\phi$$

Ampere's Circuit Law.

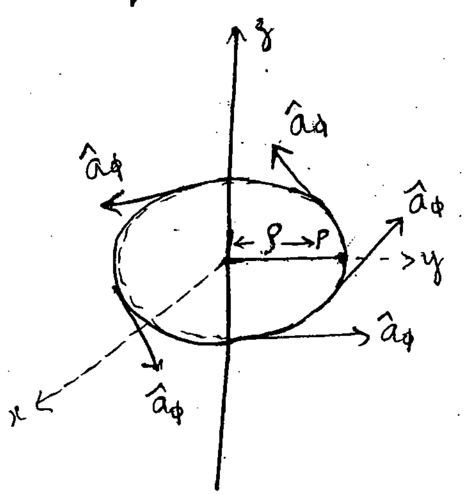
This law is used to solve complex problems in magnetostatics.

Ampere's Circuit law states that

the line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

Mathematically $\oint \vec{H} \cdot d\vec{L} = I$

Proof: Consider a long straight conductor carrying direct current I placed along z-axis as shown in figure.



Point P is at $\perp^{lar}$ distance ρ from conductor. Consider $d\vec{L}$ at point P in $\hat{a}_\phi$ direction

$$d\vec{L} = \rho\, d\phi\, \hat{a}_\phi$$

$\vec{H}$ obtained at point P from Biot-Savart law due to infinitely long conductor is,

$$\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi$$

Then $\vec{H} \cdot d\vec{L} = \frac{I}{2\pi\rho} \hat{a}_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = \frac{I}{2\pi} d\phi$

Integrating over entire closed path

$$\oint \vec{H} \cdot d\vec{L} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi}(2\pi - 0) = I$$

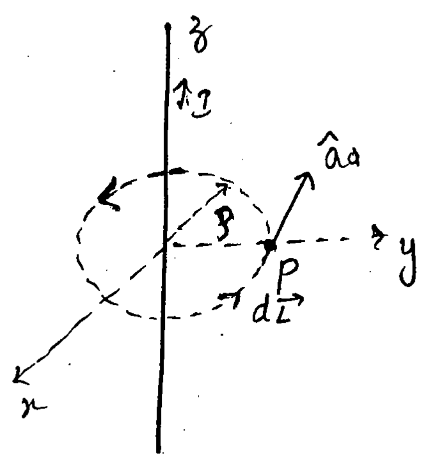
$\therefore \oint \vec{H} \cdot d\vec{L} = I$ Current carried by conductor

→ Ampere's law doesn't depends upon the shape of the path, but the path must be enclosed & is called as Amperian path.

<u>Applications of Ampere's Circuit Law</u>

<u>$\vec{H}$ due to Infinitely long Straight Conductor</u>

Consider an infinite long straight conductor placed along z-axis carrying a direct current I.



From figure at point P

$$\vec{H} = H_\phi \hat{a}_\phi$$

Because the $\vec{H}$ depends on ρ and the direction is always tangential to closed path i.e., $\hat{a}_\phi$

$$d\vec{L} = \rho \, d\phi \, \hat{a}_\phi$$

$$\therefore \vec{H} \cdot d\vec{L} = H_\phi \hat{a}_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = H_\phi \rho \, d\phi$$

From Ampere's ckt law $\oint \vec{H} \cdot d\vec{L} = I$

$$\int_{\phi=0} H_\phi \rho \, d\phi = I$$

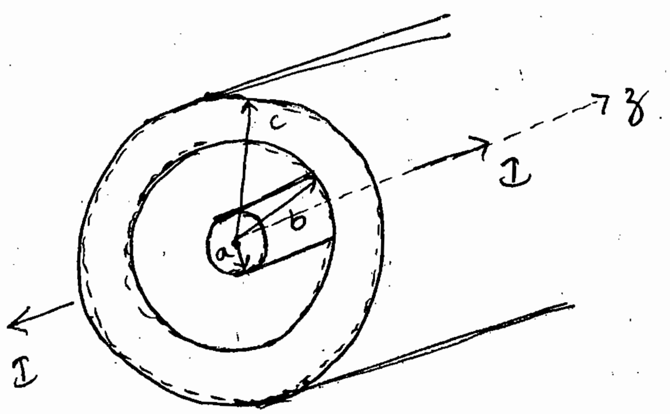
$$H_\phi \rho (2\pi) = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi \quad A/m}$$

$\vec{H}$ due to a Co-axial Cable

Consider a co-axial cable as shown in figure below. The inner conductor radius is 'a' carrying current I. The outer conductor is in the form of concentric cylinder whose inner radius is 'b' and outer radius 'c'. This cable is placed along z-axis



The current I is uniformly distributed in the inner conductor, while $-I$ is uniformly distributed in the outer conductor.

→ The space between inner and outer conductor is filled with dielectric say air.

The calculation of $\vec{H}$ is divided corresponding to various regions of the cable.

Region 1

Within the inner conductor $\rho < a$.

Consider a closed path having radius $\rho < a$

The area of cross section enclosed is $\pi \rho^2 \; m^2$

The total current flowing is I through the area πa^2

Hence the current enclosed by the closed path is

$$I' = \frac{\pi \rho^2}{\pi a^2} I = \frac{\rho^2}{a^2} I$$

$\vec{H}$ is only in $\hat{a}_\phi$ direction and depends on ρ

$$\vec{H} = H_\phi \; \hat{a}_\phi \quad \& \quad d\vec{L} = \rho \, d\phi \; \hat{a}_\phi$$

$$\therefore \; \vec{H} \cdot d\vec{L} = H_\phi \, \rho \, d\phi$$

According to Ampere's circuit law

$$\oint \vec{H} \cdot d\vec{L} = I'$$

$$\oint H_\phi \rho \, d\phi = \frac{\rho^2}{a^2} I$$

$$\int_{\phi=0}^{2\pi} H_\phi \rho \, d\phi = \frac{\rho^2}{a^2} I$$

$$H_\phi \rho (2\pi) = \frac{\rho^2}{a^2} I$$

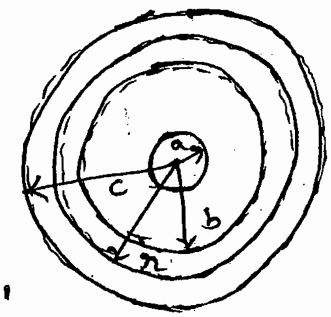
$$\boxed{\vec{H} = \frac{I \rho}{2\pi a^2} \; \hat{a}_\phi} \; A/m$$

Region 2 : within ... which encloses the inner conductor carrying direct current I. This is the case of infinitely long conductor along z-axis. Hence $\vec{H}$ in this region is

$$\boxed{\vec{H} = \frac{I}{2\pi\rho}\,\hat{a}_\phi \ A/m}$$

Region 3 : Within outer conductor $b<\rho<c$ Consider a closed path

The current enclosed by the closed path is only the part of current $-I$ in the outer conductor. The total current $-I$ is flowing through cross section $\pi(c^2-b^2)$ the closed path encloses the cross section $\pi(r^2-b^2)$. Hence the current enclosed by the closed path of outer conductor i.e



$$I' = \frac{\pi(\rho^2-b^2)}{\pi(c^2-b^2)}(-I) = \frac{-(\rho^2-b^2)}{(c^2-b^2)}I$$

$I'' = I$ = Current in inner conductor enclosed

As the closed path also encloses the inner conductor and hence the current I flowing through it

$$I_{enc} = I' + I'' = -\frac{(\rho^2-b^2)}{(c^2-b^2)}I + I$$

$$= I\left[1 - \frac{(\rho^2-b^2)}{c^2-b^2}\right]$$

$$I_{enc} = I\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

According to Ampere's circuit law

$$\oint \vec{H}\cdot d\vec{L} = I_{enc}$$

$$\vec{H} = 0 H_\phi \hat{a}_\phi \quad \& \quad d\vec{L} = \rho\, d\phi\, \hat{a}_\phi$$

$$\vec{H}\cdot d\vec{L} = H_\phi \rho\, d\phi$$

$$\int_{\phi=0}^{2\pi} H_\phi \rho\, d\phi = I_{enc}$$

$$2\pi\rho H_\phi = I\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

$$H_\phi = \frac{I}{2\pi\rho}\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

$$\vec{H} = \frac{I}{2\pi\rho}\left[\frac{c^2-\rho^2}{c^2-b^2}\right]\hat{a}_\phi \quad A/m$$

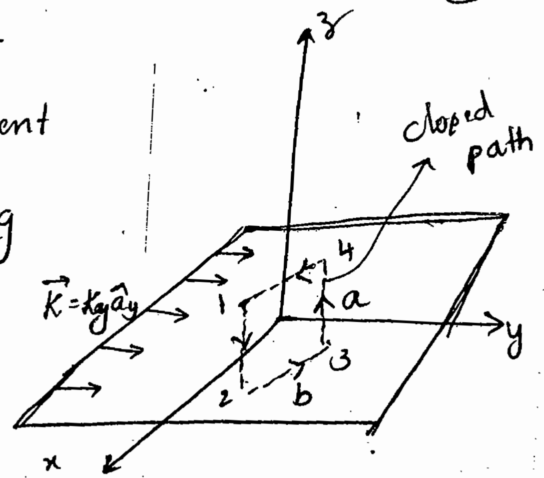
Region 4: Outside the cable $r > c$

$$I_{enc} = +I - I = 0\,A$$

$$\oint \vec{H}\cdot d\vec{L} = 0$$

$$\boxed{\vec{H} = 0 \; A/m.}$$

The magnetic field does not exists outside the cable



Consider an infinite sheet of current in the $z=0$ plane. The surface current density is $\vec{K}$. The current is flowing in positive y direction hence

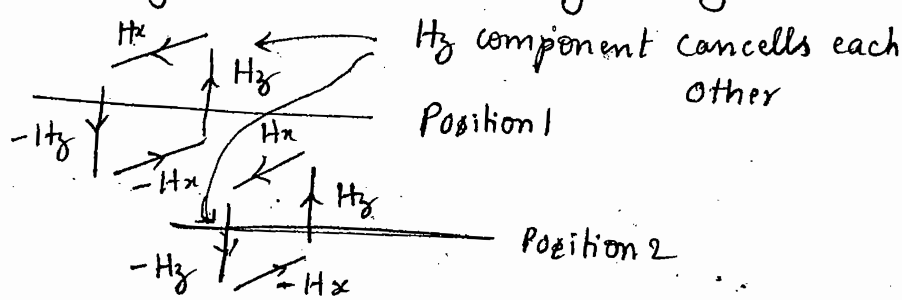
$$\vec{K} = K_y \hat{a}_y.$$

Consider a closed path 1-2-3-4 as shown in the figure. The width of the path is 'b' and the height 'a'.

The current flowing across the distance b is given by

$$I_{enc} = K_y b$$

Consider current in $\hat{a}_y$ direction according to right hand thumb rule.



Between any two very closely spaced conductors, the components of $\vec{H}$ in z direction are oppositely directed. All such components cancel each other and $\vec{H}$ cannot have any component in $\hat{a}_z$ direction.

As current flowing in y direction $\vec{H}$ cannot have any component in y direction So $\vec{H}$ has only component in x direction.

$$\therefore \ \vec{H} = H_x \hat{a}_x \quad \text{for } z>0$$
$$= -H_x \hat{a}_x \quad \text{for } z<0$$

Applying Ampere's ckt law $\oint \vec{H} \cdot d\vec{L} = I_{enc}$

Evaluating integral over the path 1-2-3-4-1

For path 1-2, $d\vec{L} = dz\, \hat{a}_z$

For path 3-4, $d\vec{L} = dz\, \hat{a}_z$ As $\vec{H}$ is in x direction

$\hat{a}_x \cdot \hat{a}_z = 0$

Hence along paths 1-2 and 3-4 $\oint \vec{H} \cdot d\vec{L} = 0$

Consider path 2-3 along which $d\vec{L} = dx\, \hat{a}_x$

$$\therefore \int_2^3 \vec{H} \cdot d\vec{L} = \int_2^3 (-H_x \hat{a}_x) \cdot (dx\, \hat{a}_x) = H_x \int_2^3 dx = b H_x$$

Consider path 4-1 along which $d\vec{L} = dx\, \hat{a}_x$

$$\therefore \int_4^1 \vec{H} \cdot d\vec{L} = \int_4^1 (H_x \hat{a}_x) \cdot (dx\, \hat{a}_x) = H_x \int_4^1 dx = b H_x$$

$$\therefore \oint \vec{H} \cdot d\vec{L} = 2b H_x$$

Equating this to $I_{enc} = K_y b$

$$2b H_x = K_y b$$

$$H_x = \frac{1}{2} K_y$$

Hence,
$$\vec{H} = \frac{1}{2} K_y \hat{a}_x \quad \text{for } z>0$$
$$= -\frac{1}{2} K_y \hat{a}_x \quad \text{for } z<0$$

In general for an infinite sheet of current density K A/m

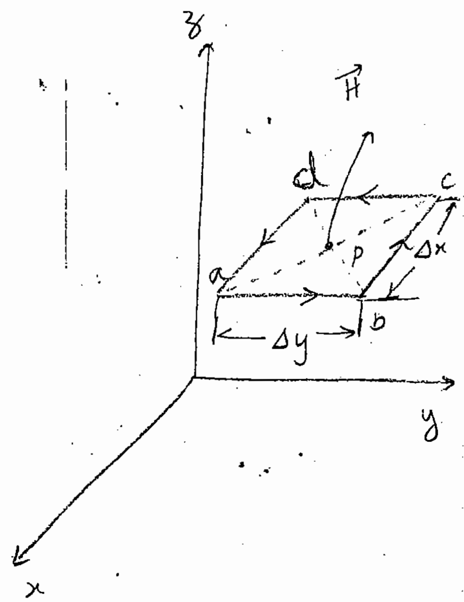
$$\boxed{\vec{H} = \frac{1}{2}\vec{K} \times \hat{a}_N}$$

$\hat{a}_N$ = unit vector normal from the current sheet to the point at which $\vec{H}$ is to be obtained.

Curl

Ampere's circuit law is to be applied to the differential surface element to develop the concept of curl.

Consider a differential surface element having sides Δx and Δy plane. The unknown current has produced $\vec{H}$ at the centre of this incremental closed path



The total magnetic field intensity at the point P which is centre of the small rectangle is

$$\vec{H} = H_{x0}\,\hat{a}_x + H_{y0}\,\hat{a}_y + H_{z0}\,\hat{a}_z$$

To apply Ampere's ckt law to this closed path let us evaluate the closed line integral of $\vec{H}$ about this path in the direction abcda.

According to right hand thumb rule the current is in $\hat{a}_z$ direction

Along path a-b $\vec{H} = H_y \hat{a}_y$ and $d\vec{L} = \Delta y \hat{a}_y$

$\therefore \vec{H} \cdot d\vec{L} = H_y \hat{a}_y \cdot \Delta y \hat{a}_y = H_y \Delta y$

The intensity H_y along a-b can be expressed in terms of H_{y0} existing at P and the rate of change of H_y in x direction with x

$$\therefore \left(\vec{H} \cdot d\vec{L}\right)_{a-b} = \left[H_{y0} + \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}\right] \Delta y$$

Along path b-c $\vec{H} = -H_x \hat{a}_x$ and $d\vec{L} = \Delta x \hat{a}_x$

$\therefore \vec{H} \cdot d\vec{L} = -H_x \Delta x$

$$\left(\vec{H} \cdot d\vec{L}\right)_{b-c} = -\left[H_{x0} + \frac{\Delta y}{2} \frac{\partial H_x}{\partial y}\right] \Delta x$$

Along path c-d $\vec{H} = -H_y \hat{a}_y$ and $d\vec{L} = \Delta y \hat{a}_y$

$\therefore \vec{H} \cdot d\vec{L} = -H_y \Delta y$

$$\left(\vec{H} \cdot d\vec{L}\right)_{c-d} = -\left[H_{y0} - \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}\right] \Delta y$$

Along path d-a $\vec{H} = H_x \hat{a}_x$ and $d\vec{L} = \Delta x \hat{a}_x$

$\therefore \vec{H} \cdot d\vec{L} = H_x \hat{a}_x$

$$\left(\vec{H} \cdot d\vec{L}\right)_{d-a} = \left[H_{x0} - \frac{\Delta y}{2} \frac{\partial H_x}{\partial y}\right] \Delta x$$

The total $\left(\vec{H} \cdot d\vec{L}\right)$ along abcda path ie

$$\vec{H} \cdot d\vec{L} = \cancel{H_{y0} \Delta y} + \frac{\partial H_y}{\partial x} \frac{\Delta x \Delta y}{2} + \frac{\partial H_y}{\partial x} \frac{\Delta x \Delta y}{2} - \cancel{H_{y0} \Delta y} - \cancel{H_{x0} \Delta x} - \frac{\partial H_x}{\partial y} \frac{\Delta x \Delta y}{2} - \frac{\partial H_x}{\partial y} \frac{\Delta x \Delta y}{2} + \cancel{H_{x0} \Delta x}$$

$$\oint \vec{H} \cdot d\vec{L} = \frac{\partial H_y}{\partial x} \Delta x \Delta y - \frac{\partial H_x}{\partial y} \Delta x \Delta y$$

$$\oint \vec{H} \cdot d\vec{L} = \Delta x \Delta y \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right]$$

This integral must be current enclosed by the differential element according to Ampere's ckt law.

Current enclosed = Current Density normal to closed path $\times$ Area of the closed path.

$$I_{enc} = J_z \Delta x \Delta y$$

$J_z \rightarrow$ Current density in $\hat{a}_z$ direction as the current enclosed in $\hat{a}_z$ direction

$$\oint \vec{H} \cdot d\vec{L} = \Delta x \Delta y \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = J_z \Delta x \Delta y$$

$$\frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \Delta y} = \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = J_z$$

This gives accurate result as closed path shrinks to a point i.e., $\Delta x \Delta y$ area ~~d~~ tends to zero

$$\lim_{\Delta x \Delta y \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \Delta y} = \frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial x} = J_z$$

Considering incremental closed path in yz plane we get the current density normal to it i.e., in x directio

i.e., $$\lim_{\Delta y \Delta z \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta y \, \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

Considering incremental closed path in zx plane we get the current density normal to it i.e, in y directio

i.e., $$\lim_{\Delta z \Delta x \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta z \, \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

In general we can write

$$\lim_{\Delta S_N \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S_N} = J_N \quad \text{where}$$

J_N = Current density normal to the surface ΔS

ΔS_N = Area enclosed by closed line integral

The term on left hand side of the equation is called "curl $\vec{H}$".

The total current density is given by

at point P $\quad \vec{J} = J_x \hat{a}_x + J_y \hat{a}_y + J_z \hat{a}_z$

$$\vec{J} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] \hat{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] \hat{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] \hat{a}_z$$

$$\vec{J} = \text{Curl } \vec{H} = \nabla \times \vec{H}$$

The curl $\vec{H}$ is indicated by $\nabla \times \vec{H}$ which is cross product of operator del and $\vec{H}$.

Point form of Ampere's Circuit Law

$$\boxed{\text{Curl } \vec{H} = \nabla \times \vec{H} = \vec{J}}$$

This is Second Maxwell's equations.

The third Maxwell's equation is the point form of

$\oint \vec{E} \cdot d\vec{L} = 0$ $\boxed{\nabla \times \vec{E} = 0}$

Curl in various Coordinate Systems.

1. Cartesian Coordinate System

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\nabla \times \vec{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\hat{a}_z$$

② Cylindrical Coordinate System

$$\nabla \times \vec{H} = \frac{1}{r} \begin{vmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$\nabla \times \vec{H} = \left[\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right]\hat{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\hat{a}_\phi + \left[\frac{\partial r H_\phi}{\partial r} - \frac{\partial H_r}{\partial \phi}\right]\frac{\hat{a}_z}{r}$$

3) Spherical Coordinate System

$$\nabla \times \vec{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\,H_\phi \end{vmatrix}$$

$$\nabla \times \vec{H} = \frac{1}{r\sin\theta}\left[\frac{\partial(H_\phi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right] + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(r.H_\phi)}{\partial r}\right]\hat{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\hat{a}_\phi$$

Significance of a Curl

The curl is a closed line integral per unit area as the area shrinks to a point. It gives circulation density of a vector about a point at which the area is going to shrink. The curl also gives the direction which is along the axis through a point at which curl is defined.

The magnetic field lines produced by the current carrying conductor are rotating in the form of Concentric circles around the conductor. Thus there exists a curl of magnetic field intensity which we have defined as $\nabla \times \vec{H}$

The direction of curl is to be obtained by right hand thumb rule.

If curl of a vector field exists then field is called as rotational. For irrotational vector field curl is zero

Magnetic flux and magnetic flux density

→ In free space magnetic flux density is $\vec{B} = \mu_0 \vec{H}$ where $\vec{B}$ is measured in wb/m^2 or a new ISU Tesla (T) where $1\ T = 1\ wb/m^2$

The constant $\mu_0 = 4\pi \times 10^{-7}$ H/m

where μ_0 is permeability of free space.

→ If magnetic flux is represented by ϕ i-e, flux passing through designated area

$$\left\langle \phi = \int_S \vec{B} \cdot d\vec{S} \quad \text{Webere} \right\rangle$$

Electric flux measured in Coulombs is

$$\left\langle \psi = \oint_S \vec{D} \cdot d\vec{S} = Q \right\rangle$$

→ The magnetic flux lines are closed and do not terminate on a magnetic charge. For a closed surface the number of magnetic flux lines entering must be equal to number of flux lines leaving The single magnetic pole cannot exist. like a single electric charge. Hence the integral $\vec{B} \cdot d\vec{S}$ evaluated over a closed surface is zero

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$

Applying divergence theorem to above equation

$$\oint_S \vec{B}\cdot d\vec{s} = \int_{vol} \nabla\cdot\vec{B}\, dv = 0$$

where dv is the volume enclosed by the closed surfac

$$\nabla\cdot\vec{B} = 0$$

The divergence of magnetic flux density is always zero. This is called Gauss's law in differential form for magnetic fields.

The above expression is last Maxwell's equation.

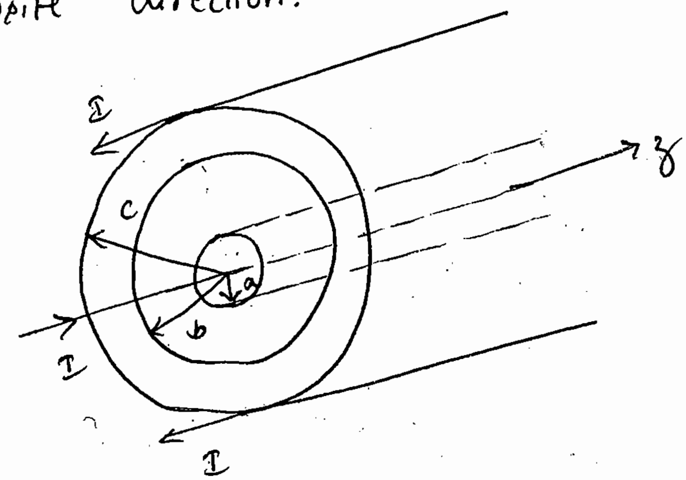
Thus for static electric fields and steady magnetic fields

$$\boxed{\begin{aligned} \nabla\cdot\vec{D} &= \rho_v \\ \nabla\times\vec{E} &= 0 \\ \nabla\times\vec{H} &= \vec{J} \\ \nabla\cdot\vec{B} &= 0 \end{aligned}}$$

The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

$$\boxed{\begin{aligned} &\oint_S \vec{D}\cdot d\vec{s} = Q = \int_{vol} \rho_v\, dv \\ &\oint \vec{E}\cdot d\vec{L} = 0 \\ &\oint \vec{H}\cdot d\vec{L} = I = \int_S \vec{J}\cdot d\vec{s} \\ &\oint_S \vec{B}\cdot d\vec{s} = 0 \end{aligned}}$$

Consider a coaxial cable such that its axis is along the z-axis. It carries a direct current I which is uniformly distributed in the inner conductor. The outer conductor carries the same current I in opposite direction.



$\vec{H}$ in the region $a<\rho<b$ is

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \ A/m$$

We are interested in the flux in the region $a<\rho<b$. The cable is filled with air i.e., $\mu=\mu_0$

$$\vec{B} = \mu_0\vec{H} = \frac{\mu_0 I}{2\pi\rho}\hat{a}_\phi \ wb/m^2$$

Let d be the length of the conductor. The magnetic flux contained between the conductors in a length d is magnetic flux crossing the radial plane from $\rho=a$ to $\rho=b$ and for $z=0$ to $z=d$.

The magnetic flux is given by

$$\phi = \int_S \vec{B} \cdot d\vec{s}$$

$$d\vec{s} = d\rho\, dz$$

$$\phi = \int_S \vec{B} \cdot d\vec{s} = \int_S \frac{\mu_0 I}{2\pi\rho} \hat{a}_\phi \cdot d\rho\, dz\, \hat{a}_\phi$$

$$= \int_{z=0}^{d} \int_{\rho=a}^{b} \frac{\mu_0 I}{2\pi\rho}\, d\rho\, dz = \frac{\mu_0 I}{2\pi} \left[z\right]_0^d \left[\ln \rho\right]_a^b$$

$$\boxed{\phi = \frac{\mu_0 I d}{2\pi} \ln(b/a) \quad wb}$$

The Scalar and Vector Magnetic potentials.

In electrostatics it is seen that there exists a scalar electric potential V which is related to the electric field intensity $\vec{E}$ as $\vec{E} = -\nabla V$

In case of magnetic fields there are two types of potentials can be defined.

1. Scalar magnetic potential denoted as V_m
2. Vector magnetic potential denoted as $\vec{A}$

Two vector identities (Properties of curl) are

$$\nabla \times \nabla V = 0, \quad V = \text{Scalar}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0, \quad \vec{A} = \text{Vector}$$