NYQUIST PLOTS

Introduction

- Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from $-\infty$ to ∞.
- Nyquist plots are used to draw the complete frequency response of the open loop transfer function.
- The Nyquist stability criterion determines the stability of a closed-loop system from its open-loop frequency response and open-loop poles.

The Nyquist Criterion can be expressed as,

$$Z = P + N$$

where Z = number of zeros of $1 + G(s)H(s)$ on the right-half s-plane

N = net encirclements around the point $(-1+j0)$.

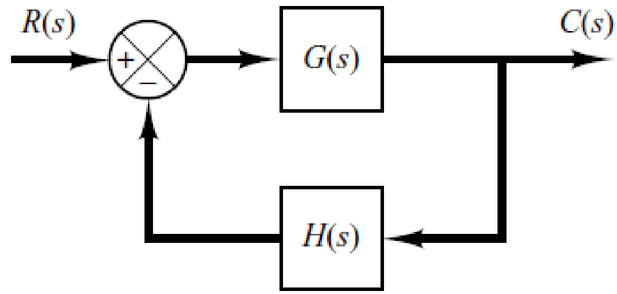
(clockwise encirclements are taken as positive and anticlockwise encirclements are negative)

P = number of poles of $G(s)H(s)$ in the right-half of s-plane

The stability of linear control systems using the Nyquist stability criterion, three possibilities can occur:

1. There is no encirclement of the (–1+j0) point. This implies that the system is stable if there are no poles of G(s)H(s) in the right-half of s- plane; otherwise, the system is unstable.
2. There are one or more counterclockwise encirclements of the (–1+j0) point. In this case the system is stable if the number of counterclockwise encirclements is the same as the number of poles of G(s)H(s) in the right-half of s- plane; otherwise, the system is unstable.
3. There are one or more clockwise encirclements of the (–1+j0) point. In this case the system is unstable.

Consider the closed-loop system shown in Fig.



The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation is $1 + G(s)H(s) = 0$

For stability, all roots of the characteristic equation must lie in the left-half of s- plane.

Example 1

G(s)H(s) = $\frac{(s+1)(s+2)}{s(s+3)}$

Open loop zeros: -1, -2

Open loop poles: 0, -3

The characteristic equation is 1 + G(s)H(s) = 0

$$1+\frac{(s+1)(s+2)}{s(s+3)} = 0$$

$$\frac{(s+0.38)(s+2.62)}{s(s+3)} = 0$$

Roots of the system: - 0.38, - 2.62

Closed-loop system T.F = $\frac{G(s)}{1 + G(s)H(s)} = \frac{(s+1)(s+2)}{(s+0.38)(s+2.62)}$

Closed-loop poles: - 0.38, - 2.62 [zeros of 1 + G(s)H(s)]

Example 2

$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$

Open loop zeros: -2

Open loop poles: 1, -1

The characteristic equation is 1 + G(s)H(s) = 0

$$1 + \frac{(s+2)}{(s+1)(s-1)} = 0$$

$$\frac{(s+0.5 \pm j0.87)}{(s+1)(s-1)} = 0$$

Roots of the system: $-\ 0.5 \pm j0.87$

Closed-loop system T.F = $\frac{G(s)}{1 + G(s)H(s)} = \frac{(s+2)}{(s+0.5 \pm j0.87)}$

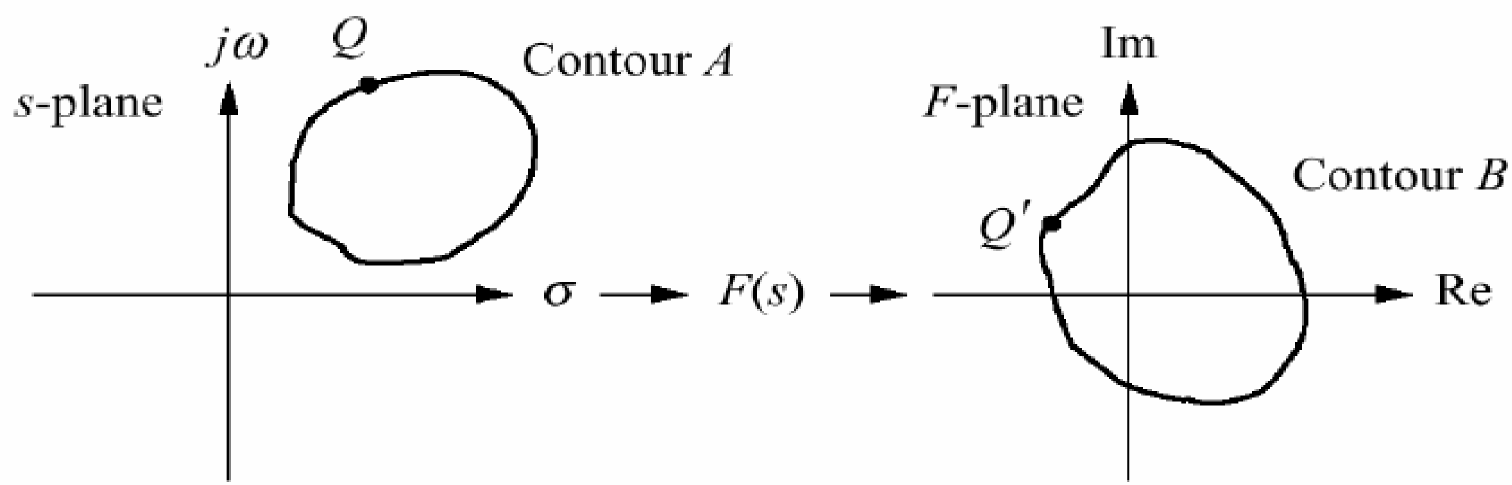
Closed-loop poles: $-\ 0.5 \pm j0.87$ [zeros of 1 + G(s)H(s)]

- The system is stable if all the poles of the closed-loop transfer function are in the left-half of s- plane (Zeros of characteristic function). Although there may be poles and zeros of the open-loop transfer function G(s)H(s) may be in the right- half of s- plane.
- The Nyquist stability criterion relates the open-loop frequency response G(s)H(s) to the number of zeros and poles of 1+G(s)H(s) that lie in the right-half of s- plane.

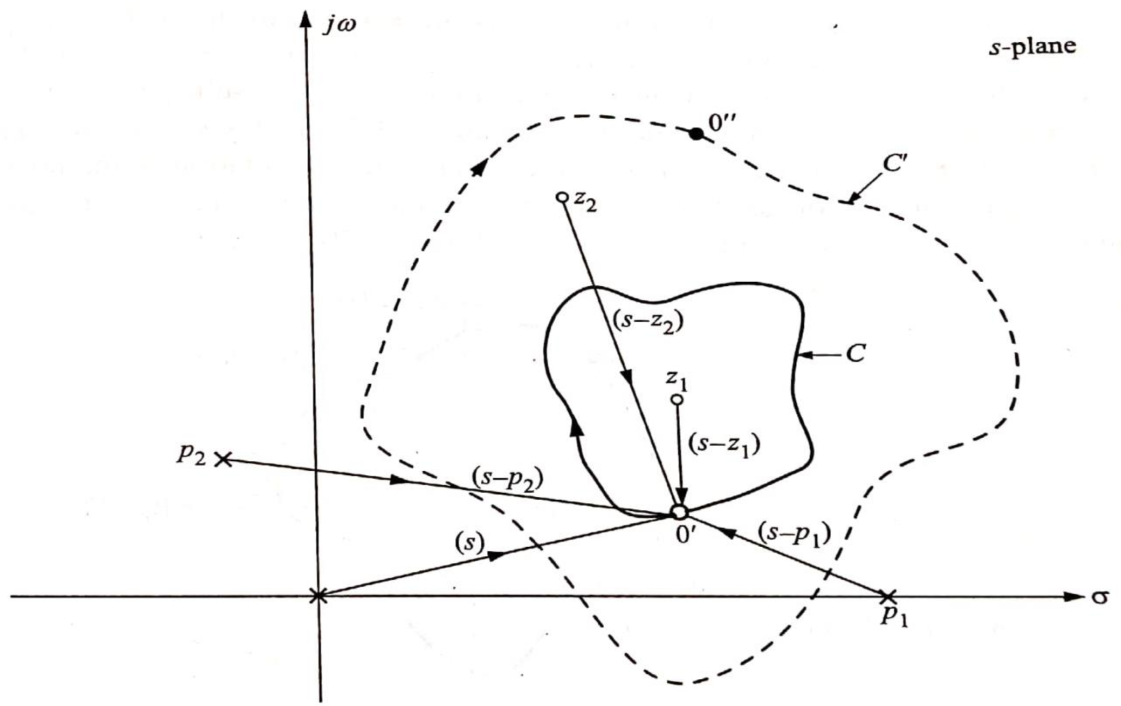
Mapping from *s*-plane to *F* -plane through a function of *F*(*s*)

- **For a point.** Taking a complex number in the *s*-plane and substituting it into a function of *F*(*s*), the result is also a complex number, which is represented in a new complex-plane (called *F*-plane). This process is called mapping, specifically **mapping a point from *s*-plane to *F*-plane through *F*(*s*).**

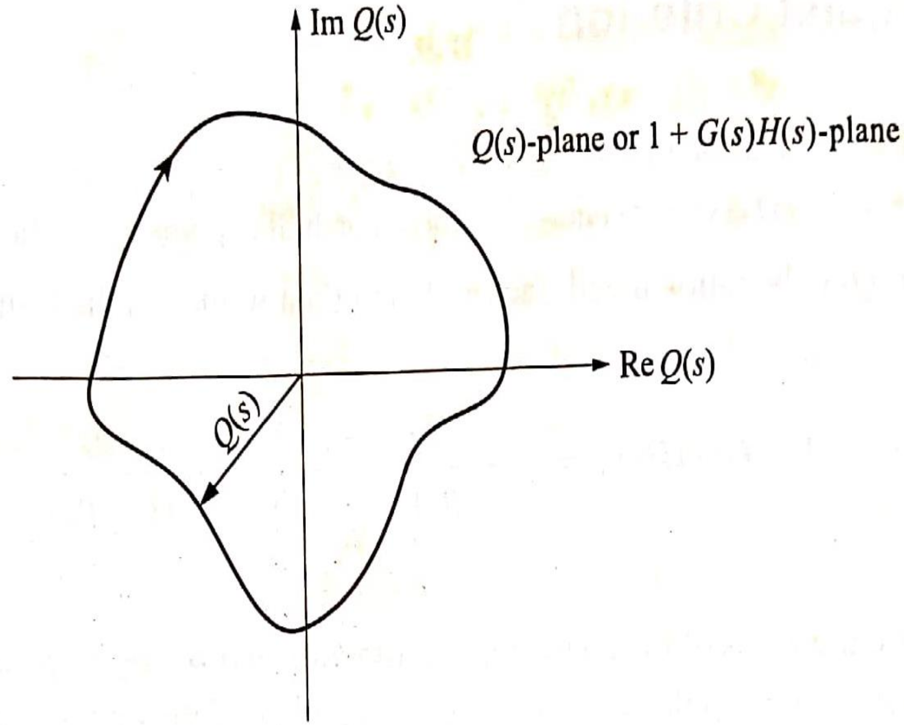
- **For a contour.** Consider the collection of points in the s-plane (called a contour), shown in the following figure as contour A. Using the above point mapping process through *F*(*s*), we can also get a contour in the *F*-plane, shown in the following figure contour B.



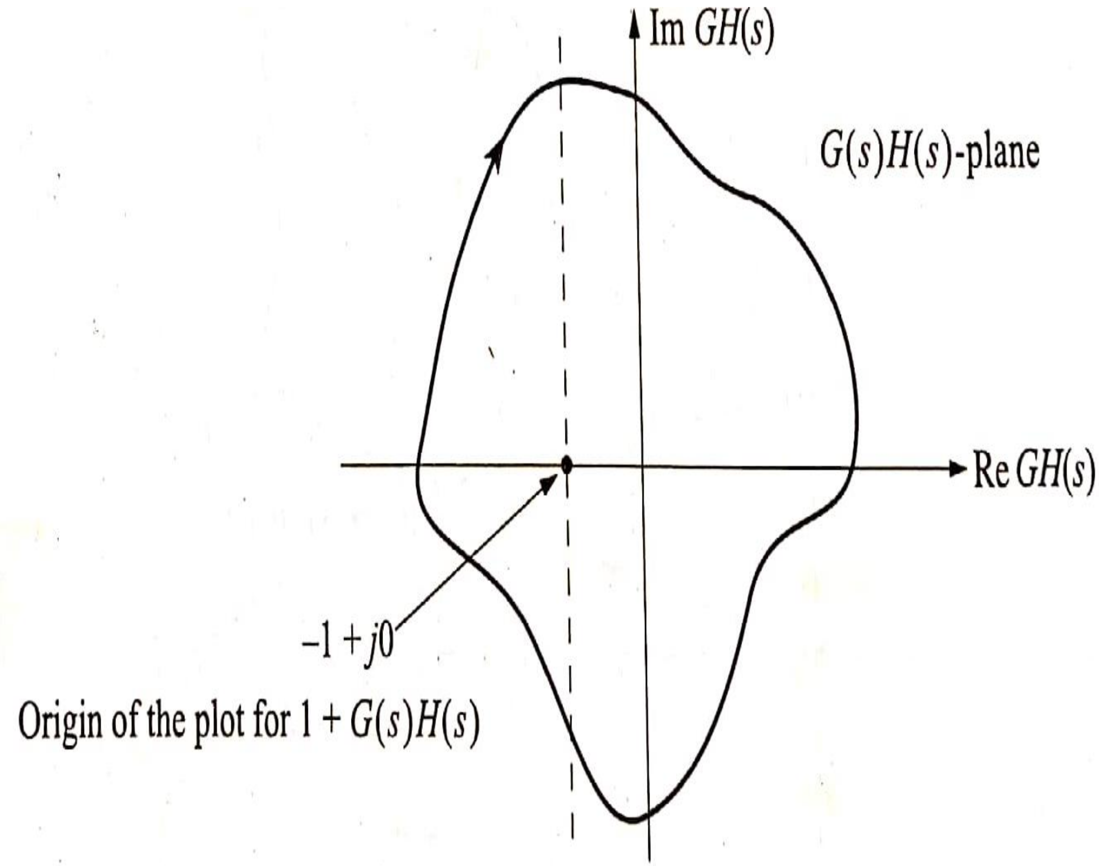
As point o’ is rotated once around the contour in the clockwise direction, the vector $(s-z_1)$ makes one complete clock-wise revolution. Undergoes a net angle of 360^o clockwise



The path traced on the 1+GH(s)-plane corresponding to one rotation of the point o’ on the s-plane also experience a net phase change of 360^o clockwise



- Enclosed
- Encirclement



Procedure for drawing Nyquist plot

Steps

1. Plot the poles of $G(s)H(s)$ on the s-plane. Then find P.
   P = Number of open loop poles on right-half of s-plane
2. Perform the conformal mapping or find the image of the contour 'abcda' enclosing the right-half of s-plane on the $G(s)H(s)$-plane and then determine the number of encirclements (N) of the $-1+j0$ point.
3. Determine Z:

$$Z = P + N$$

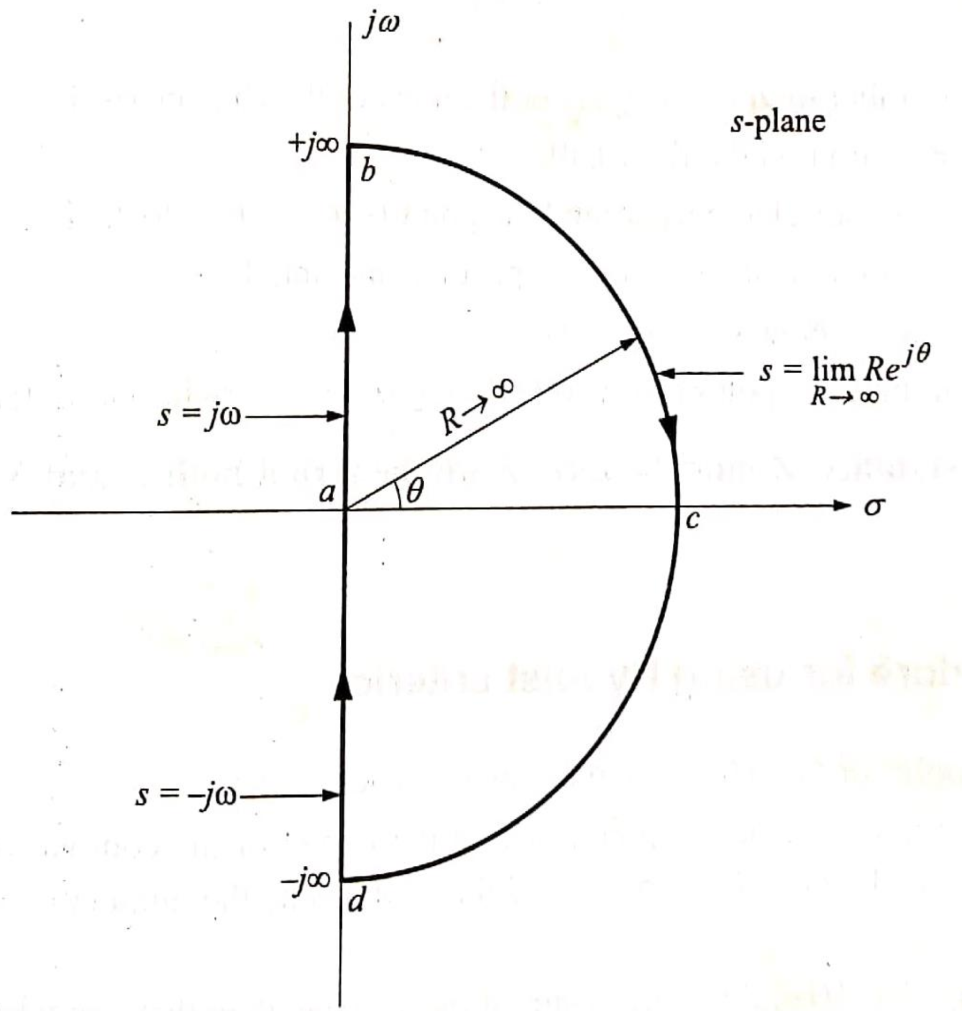
   If Z is zero, the closed-loop system is stable.

The right-half of the s-plane enclosed by the semicircle

Section I : path ab

Section II : path bcd

Section III : path da



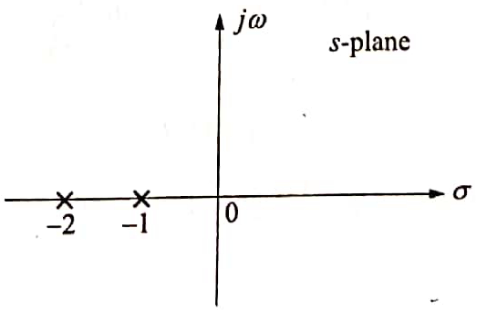
Problem

Using Nyquist stability criterion, Investigate the stability of a closed-loop system whose open-loop transfer function is given by,

$$G(s)H(s) = \frac{10}{(s+1)(s+2)}$$

Solution:

Step 1: Plot the poles of G(s)H(s) on the s-plane



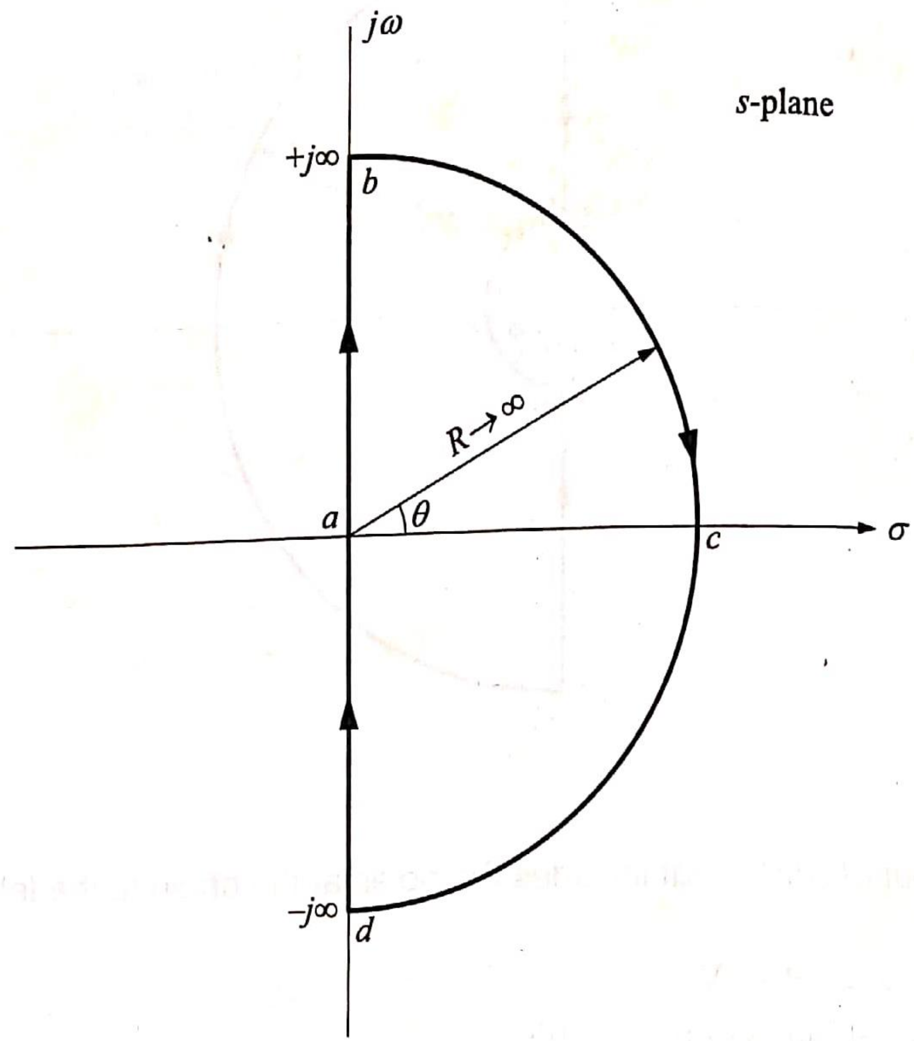
Since both poles lie on the left side of the s-plane, P = 0

Step 2: To find the image of the contour 'abcda' in G(s)H(s) plane and N

Section I: path ab

Section II: path bcd

Section III: path da



Section I : To find the image of path ab (Polar Plot):

$$G(s)H(s) = \frac{10}{(s+1)(s+2)}$$

put s = jω

$$G(j\omega)H(j\omega) = \frac{10}{(j\omega+1)(j\omega+2)}$$

$$= \frac{10}{\{\sqrt{\omega^2+1}\angle\tan -1\omega\}\{\sqrt{\omega^2+4}\angle\tan -1(\frac{\omega}{2})\}}$$

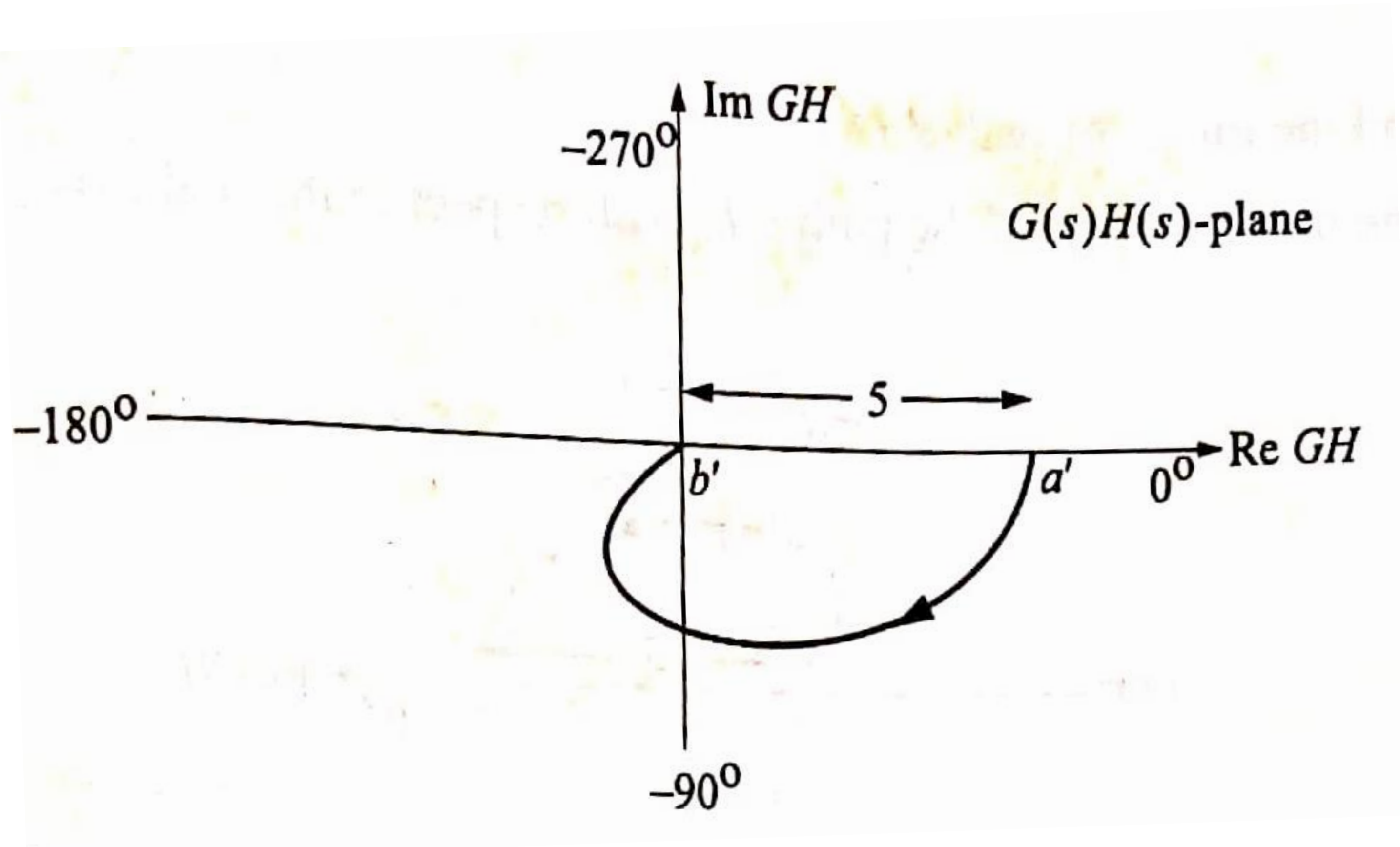
$$= \frac{10}{\{\sqrt{\omega^2+1}\}\{\sqrt{\omega^2+4}\}} - \angle\tan -1\omega - \angle\tan -1\left(\frac{\omega}{2}\right)$$

$$M = \frac{10}{\{\sqrt{\omega^2+1}\}\{\sqrt{\omega^2+4}\}} ;$$

$$\emptyset = -\angle\tan -1\omega - \angle\tan -1\left(\frac{\omega}{2}\right)$$

$$\lim_{\omega \to 0} M\angle\emptyset = 5\angle 0 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle\text{-}180 \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

put s = $\lim_{R \to \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 $\longrightarrow$ 0 $\longrightarrow$ -90

Then, $\lim_{R \to \infty} GH(Re^{j\theta}) = \lim_{R \to \infty} \frac{10}{(Rej^{\theta}+1)(Re^{j\theta}+2)}$

$$= \lim_{R \to \infty} \frac{10}{(Re^{j\theta})(Re^{j\theta})}$$

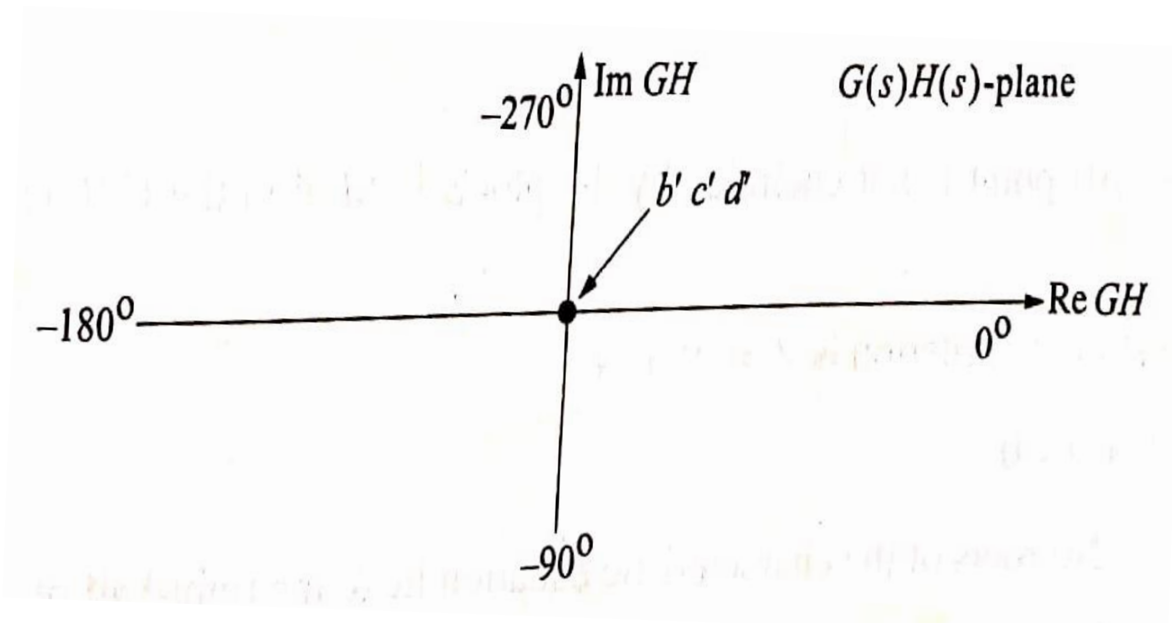
$$= \lim_{R \to \infty} \frac{10}{(R^2 e^{j2\theta})}$$

$$= 0\angle -2\theta$$

$$= 0\angle -180 \longrightarrow 0 \longrightarrow 180$$

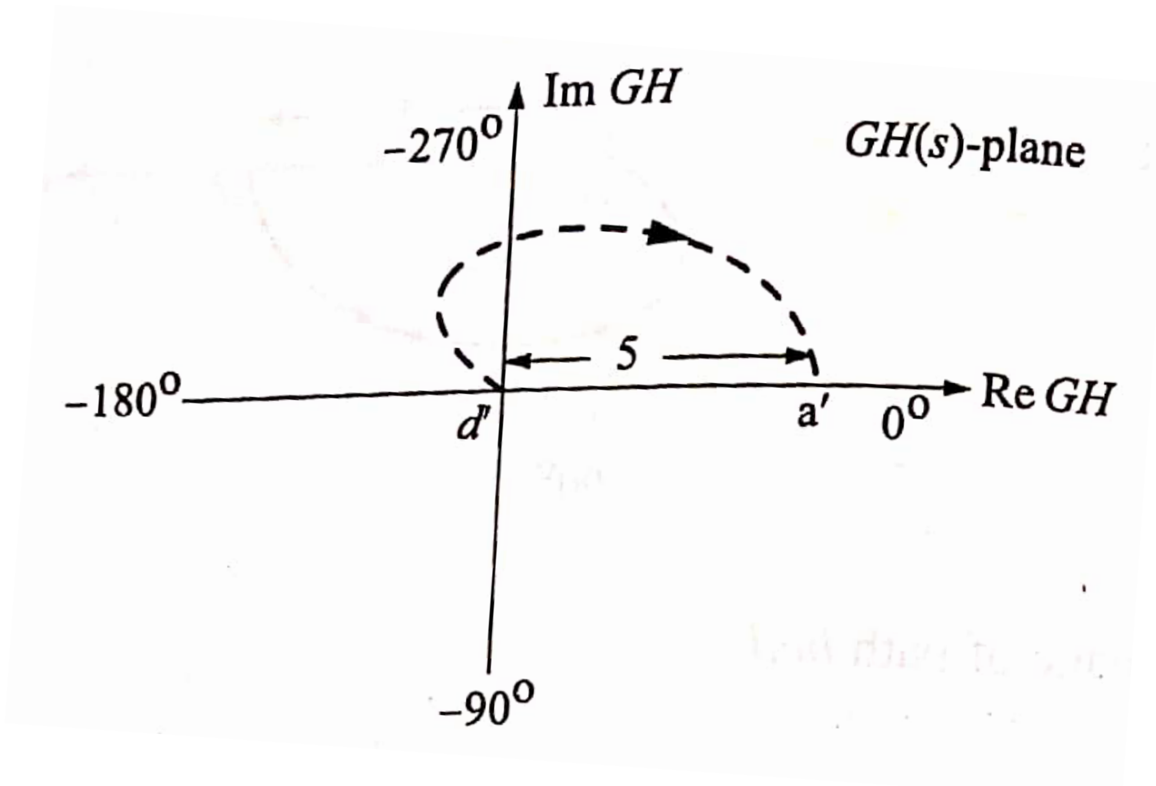
$\uparrow$b' $\uparrow$c' $\uparrow$d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.

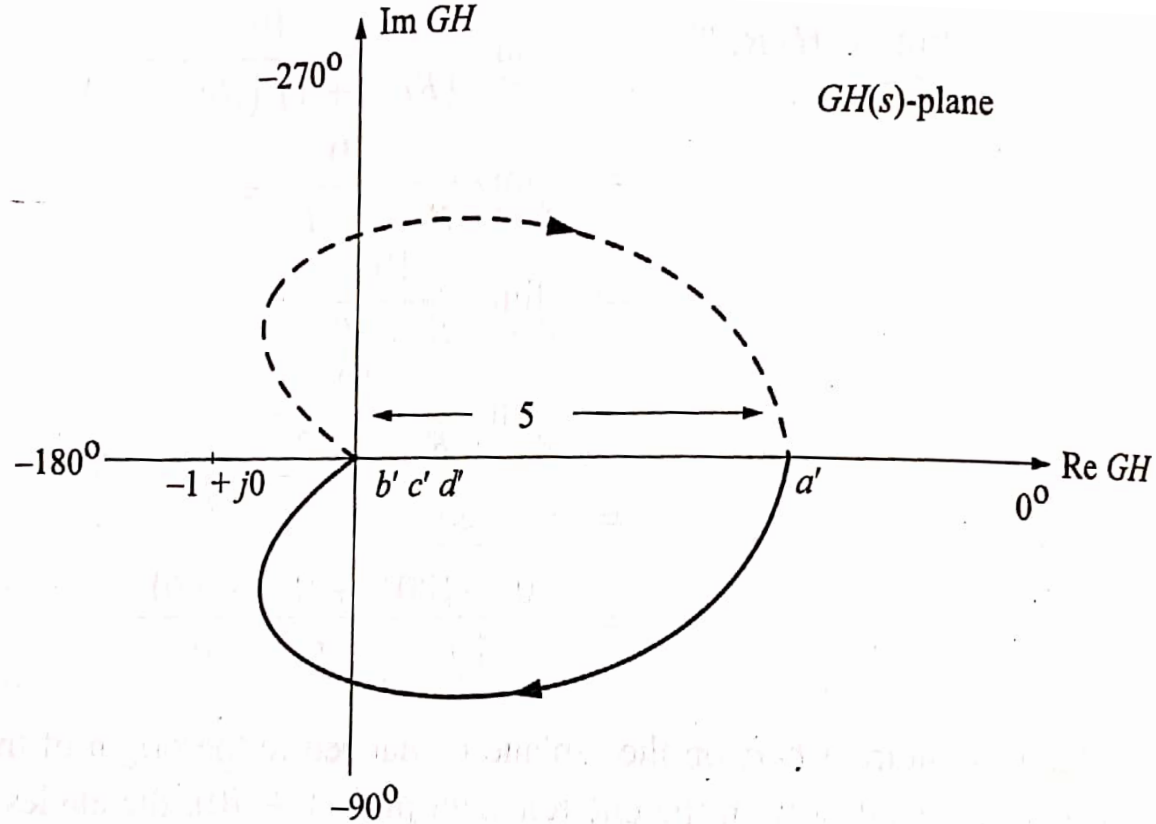


Section III: To find the image of path 'da'

Path d'a' is the mirror image of the path a'b' with respect to real axis.



The complete Nyquist plot is shown below



Since, (-1+j0) point is not encircled by the plot a'b'c'd'a' in the GH(s) plane , N= 0

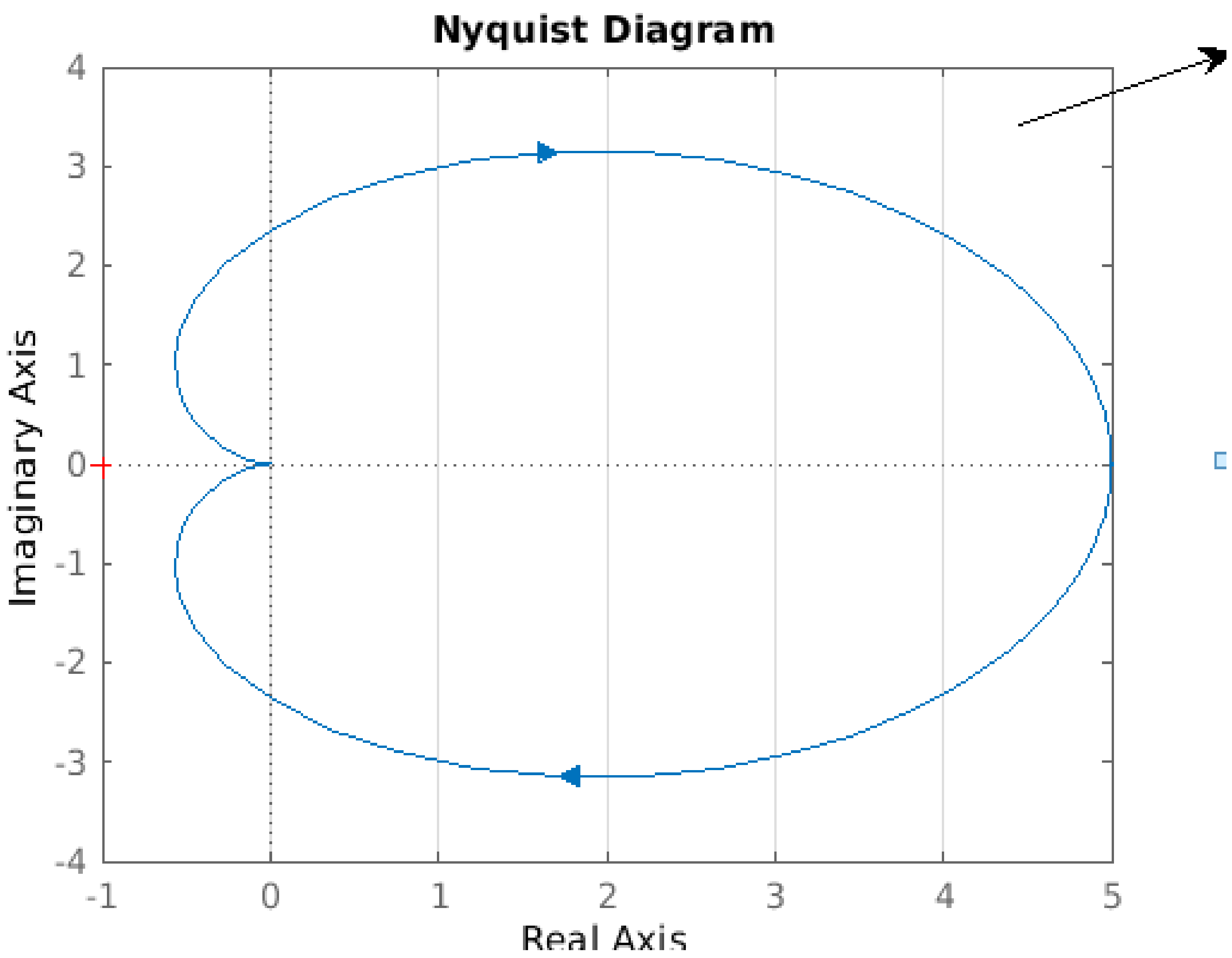
Step 3: The Nyquist stability criterion is Z = P + N

Hence, Z = 0 + 0 = 0

$\Longrightarrow$No roots of the system lie to the right-half of s-plane.

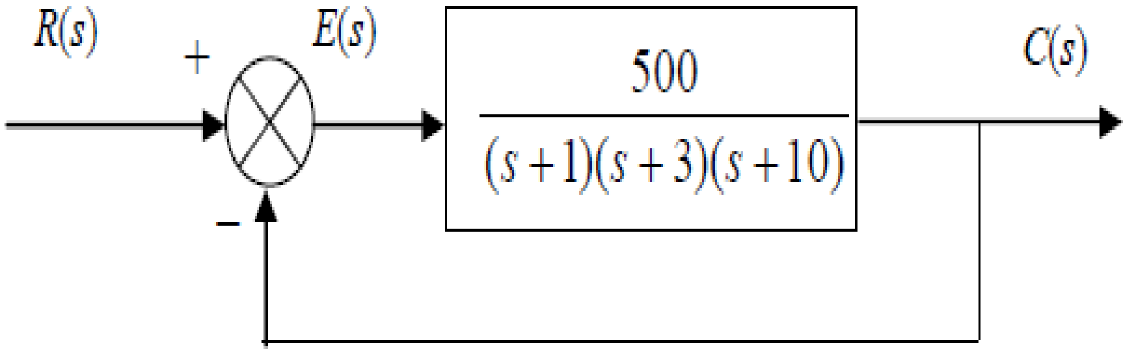
Hence the closed-loop control system is stable.

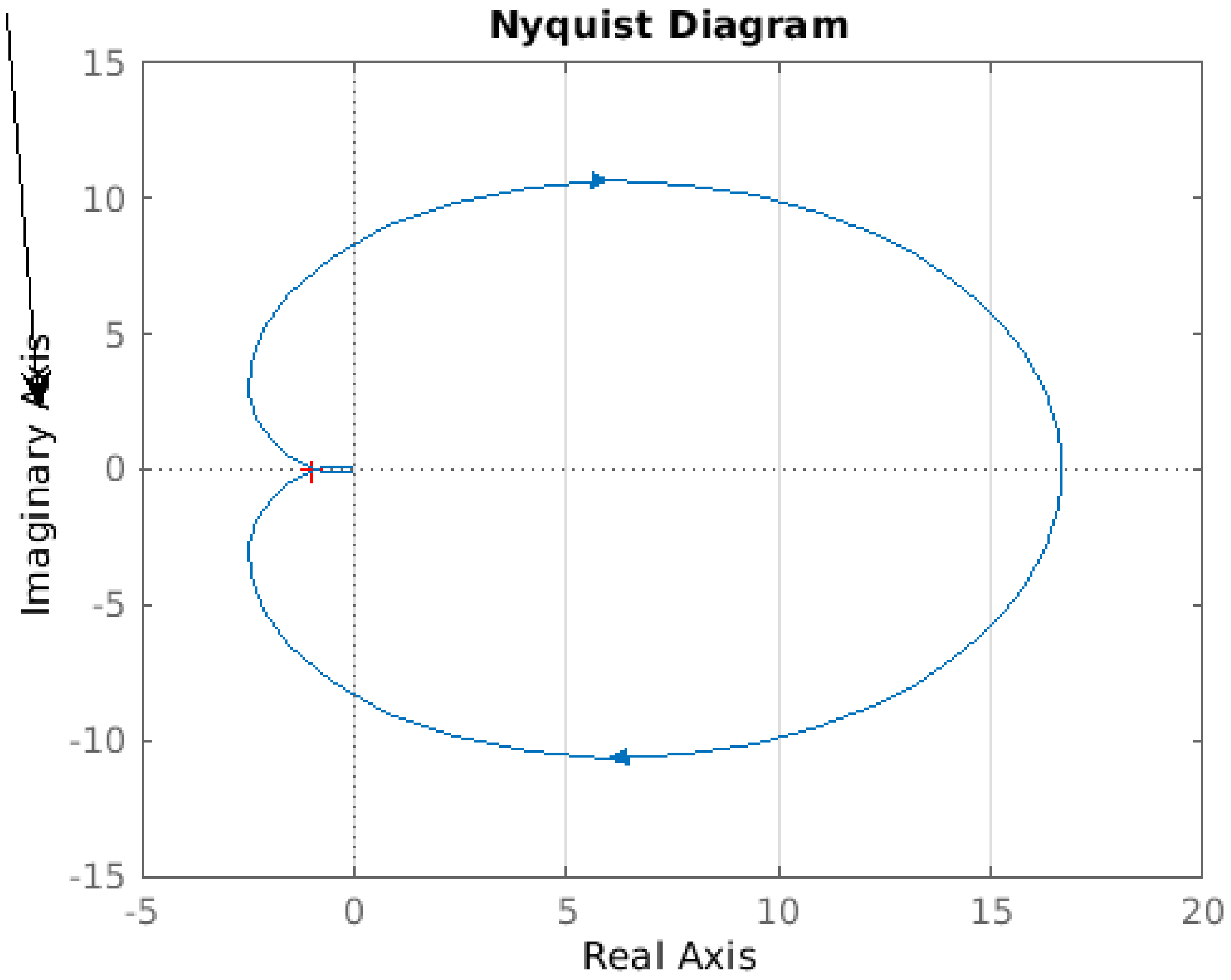
Matlab


Nyquist Diagram
Imaginary Axis
Real Axis
4
3
2
1
0
-1
-2
-3
-4
-1
0
1
2
3
4
5

Problem

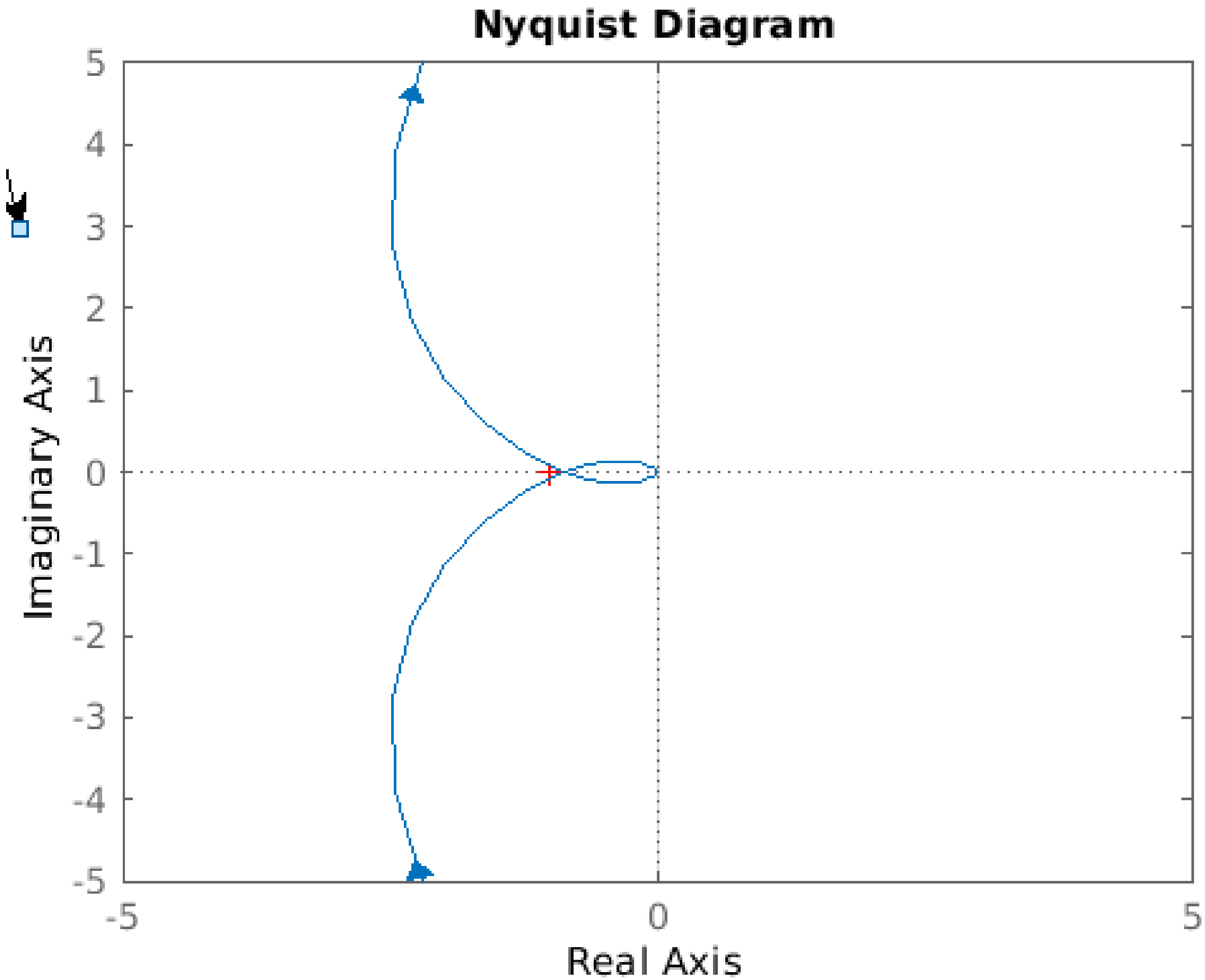
Sketch the Nyquist diagram for the system shown in the following figure, and then determine the system stability using the Nyquist criterion





Nyquist Diagram
15
10
5
0
-5
-10
-15
-5
0
5
10
15
20
Real Axis
Imaginary Axis

Matlab



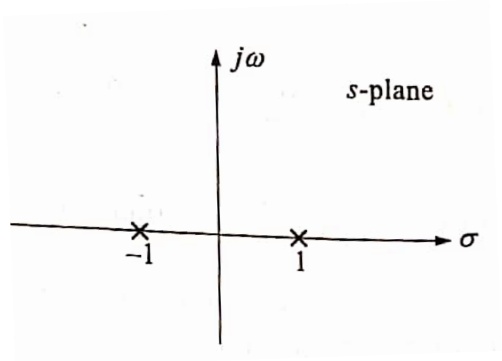
Problem

Using Nyquist stability criterion, Investigate the stability of a closed-loop system whose open-loop transfer function is given by,

$$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$$

Solution:

Step 1: Plot the poles of G(s)H(s) on the s-plane



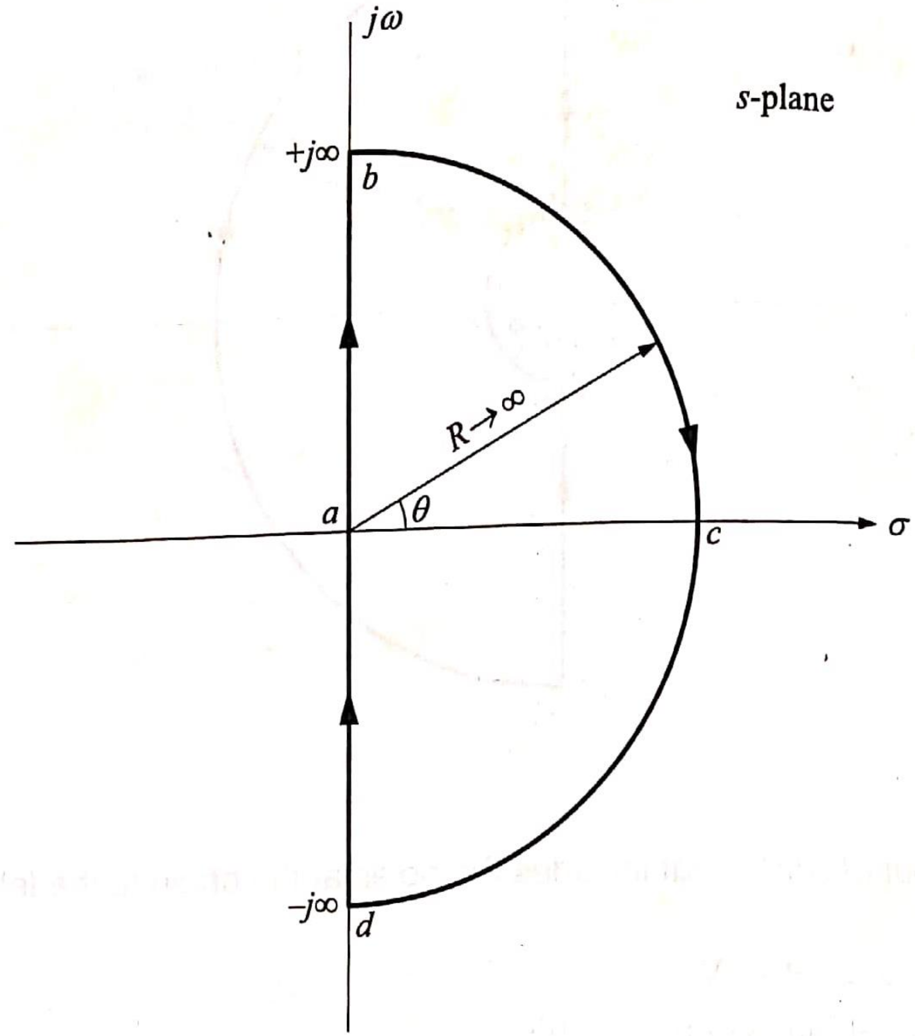
Since one pole lie on the right side of the s-plane,
P = 1

Step 2: To find the image of the contour ‘abcda’ in G(s)H(s) plane and N

Section I: path ab

Section II: path bcd

Section III: path da



Section I : To find the image of path ab (Polar Plot):

$$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$$

put s = jω

$$G(j\omega)H(j\omega) = \frac{(j\omega+2)}{(j\omega+1)(j\omega-1)}$$

$$= \frac{\{\sqrt{\omega^2+4} \angle \tan^{-1}(\frac{\omega}{2})\}}{\{\sqrt{\omega^2+1} \angle \tan^{-1} \omega\} \{\sqrt{\omega^2+1} \angle \tan^{-1}(\frac{\omega}{-1})\}}$$

$$= \frac{\{\sqrt{\omega^2+4} \angle \tan^{-1}(\frac{\omega}{2})\}}{\{(\sqrt{\omega^2+1})^2 \angle \tan^{-1} \omega\} \{\angle 180 - \tan^{-1} \omega\}}$$

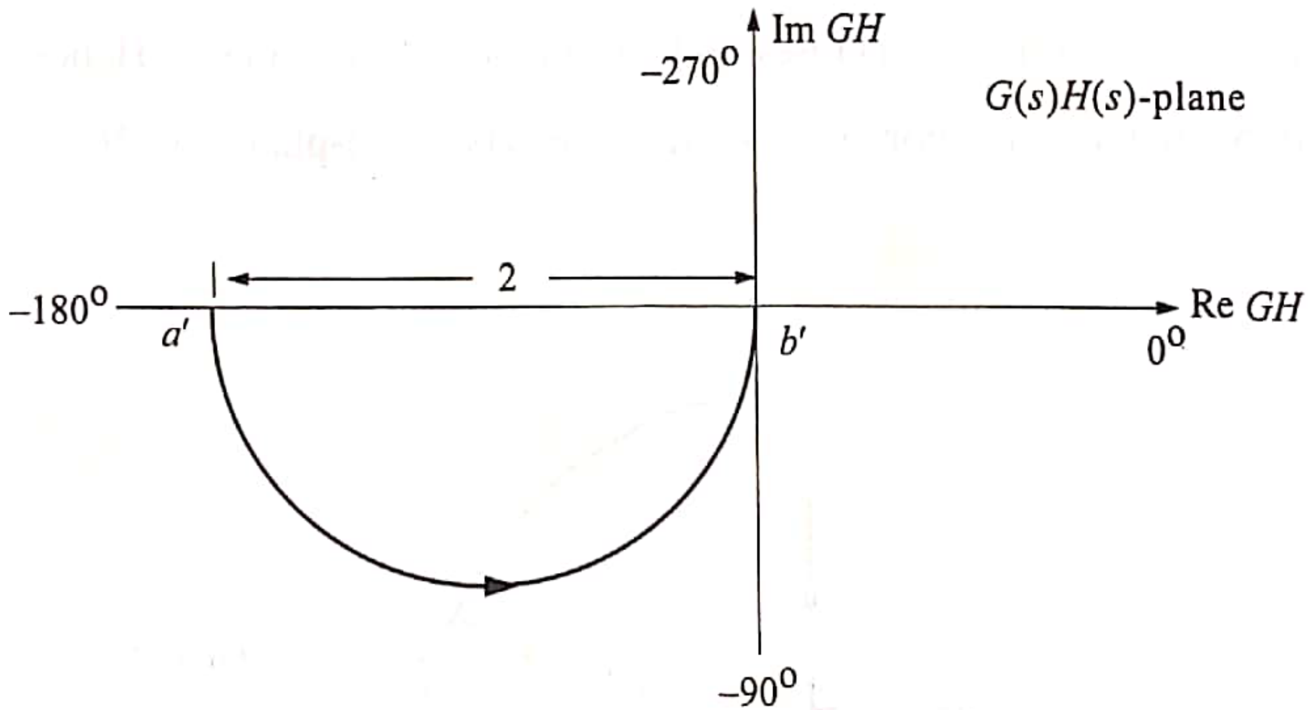
$$M = \frac{\{\sqrt{\omega^2+4}\}}{\omega^2+1} ;$$

$$\emptyset = \angle \tan^{-1}(\frac{\omega}{2}) - \angle \tan^{-1} \omega - 180 + \angle \tan^{-1} \omega$$

$$= \angle \tan^{-1}(\frac{\omega}{2}) - 180$$

$$\lim_{\omega \to 0} M\angle\emptyset = 2\angle\text{-}180 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle\text{-}90 \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

put s = $\lim_{R \to \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim_{R \to \infty} GH(\, Re^{j\theta}) = \lim_{R \to \infty} \frac{(Rej^{\theta}+2)}{(Rej^{\theta}+1)(Re^{j\theta}-1)}$

$$= \lim_{R \to \infty} \frac{Re^{j\theta}}{(Re^{j\theta})(Re^{j\theta})}$$

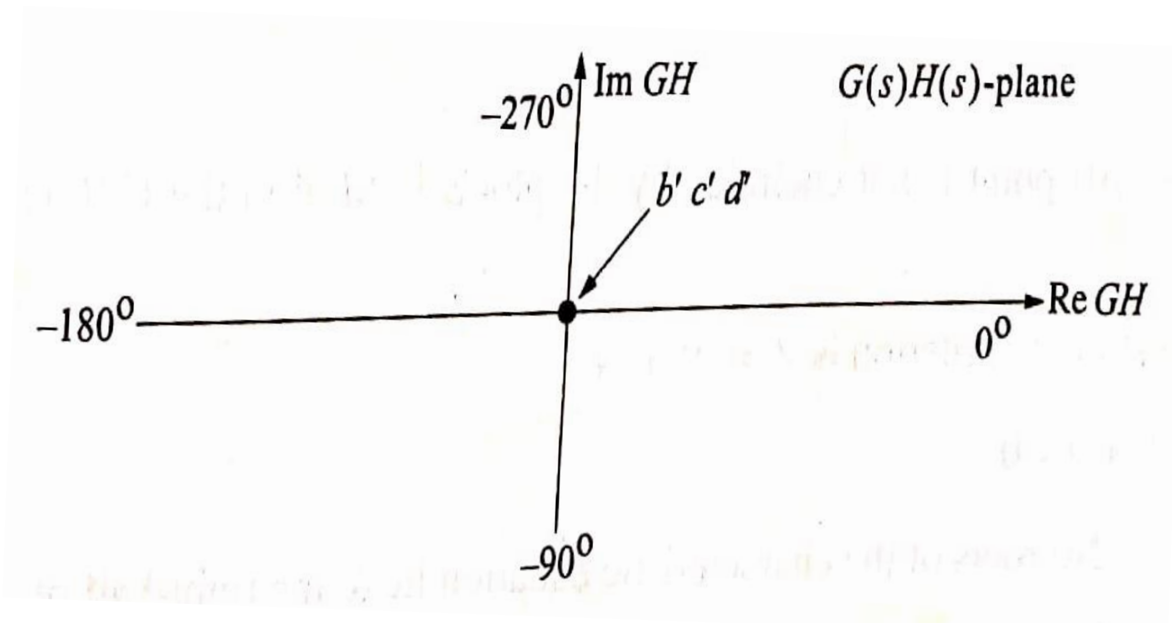
$$= \lim_{R \to \infty} \frac{1}{(Rej^{\theta})}$$

= 0∠-θ

= 0∠-90⟶ 0⟶ 90

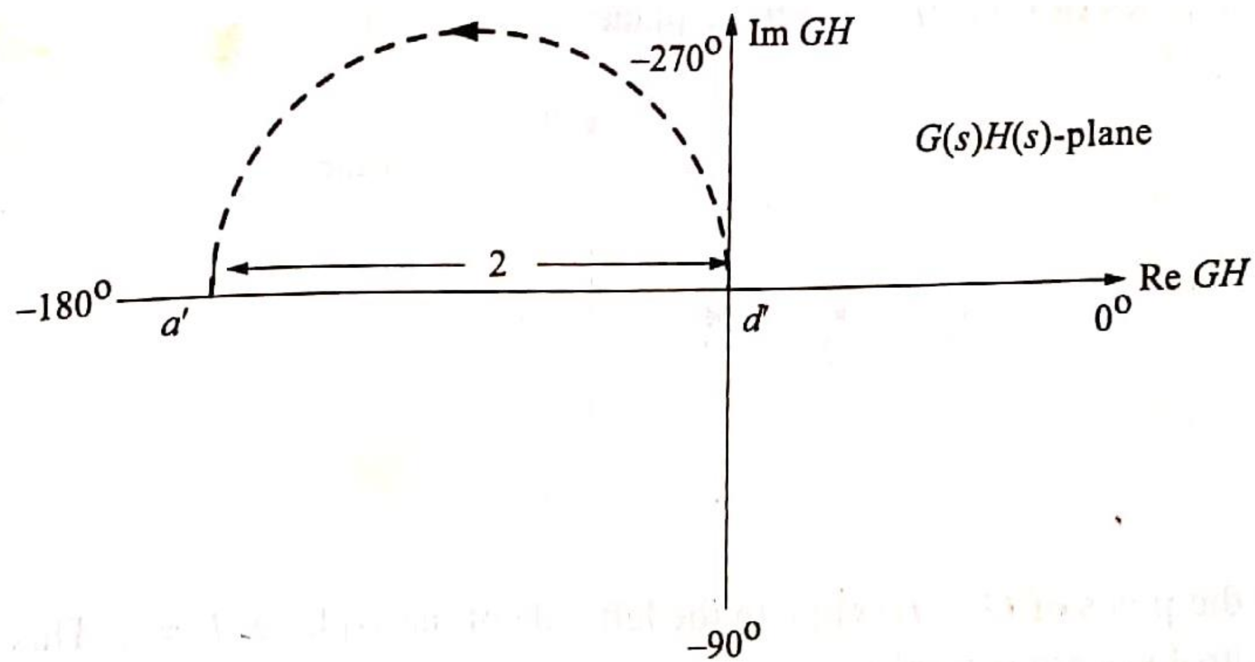
↑b' ↑c' ↑d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.

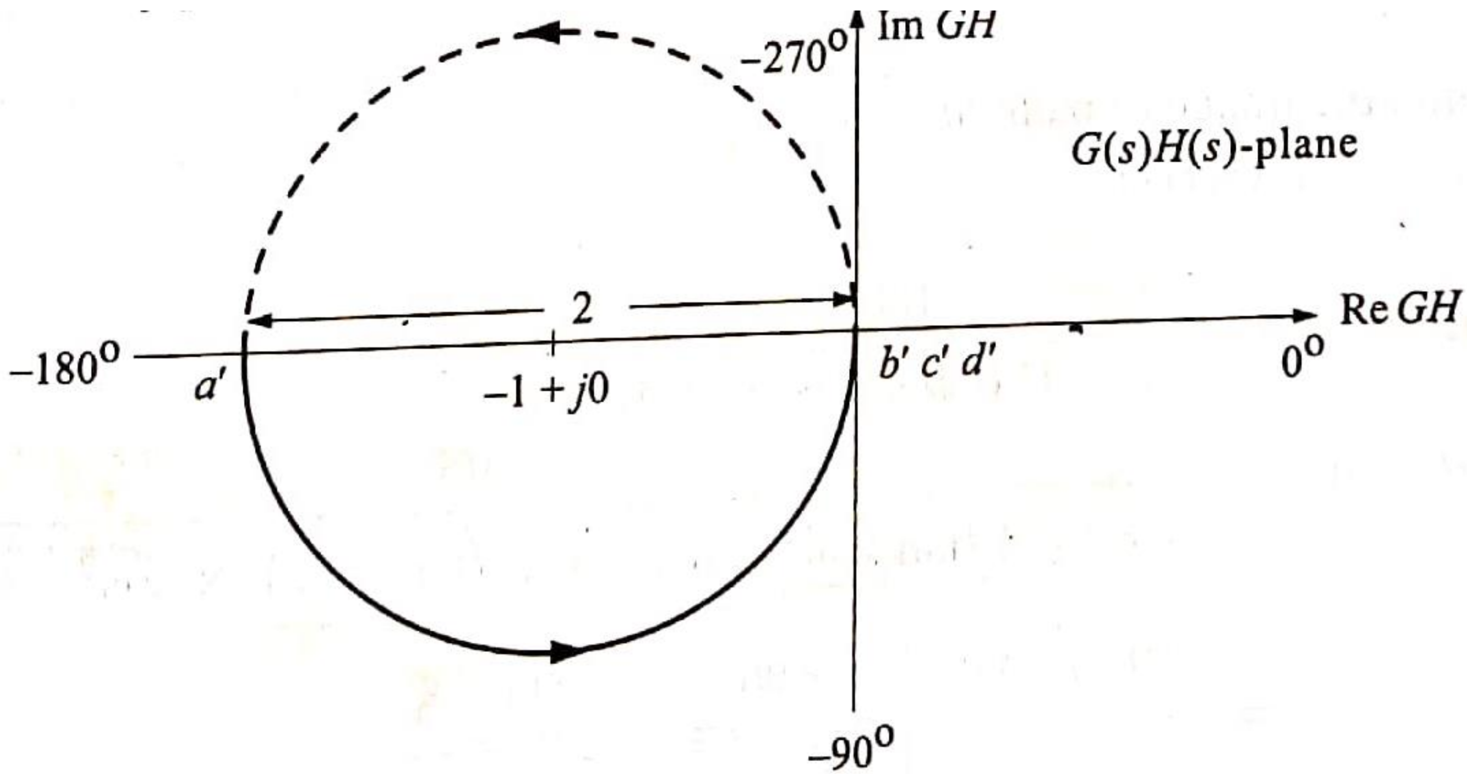


Section III: To find the image of path ‘da’

Path d’a’ is the mirror image of the path a’b’ with respect to real axis.



The complete Nyquist plot is shown below



Since, (-1+j0) point is encircled in anticlockwise direction by the plot a’b’c’d’a’ in the GH(s) plane, N = -1

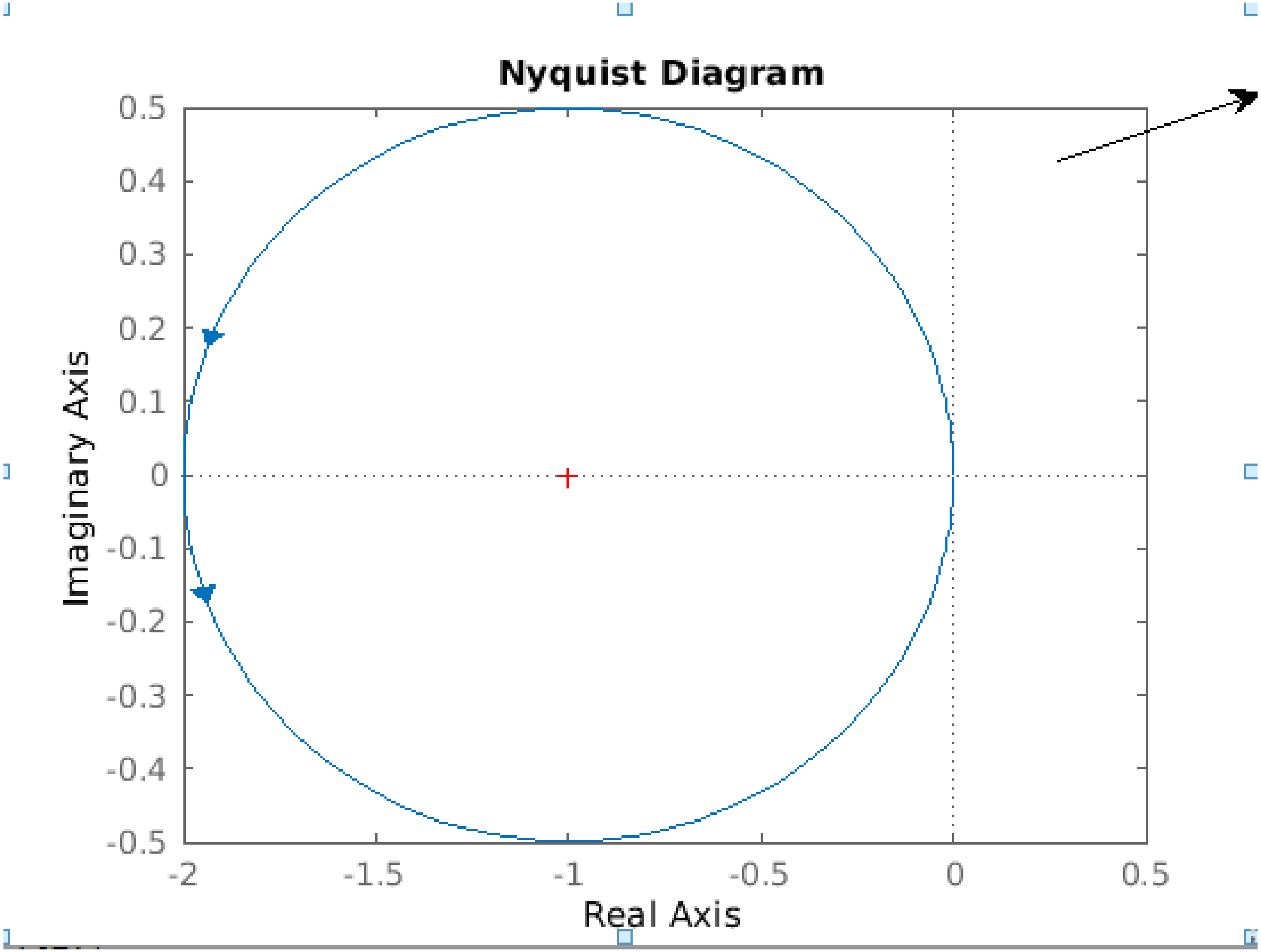
Step 3:

The Nyquist stability criterion is Z = P + N

Hence, Z = 1 - 1 = 0

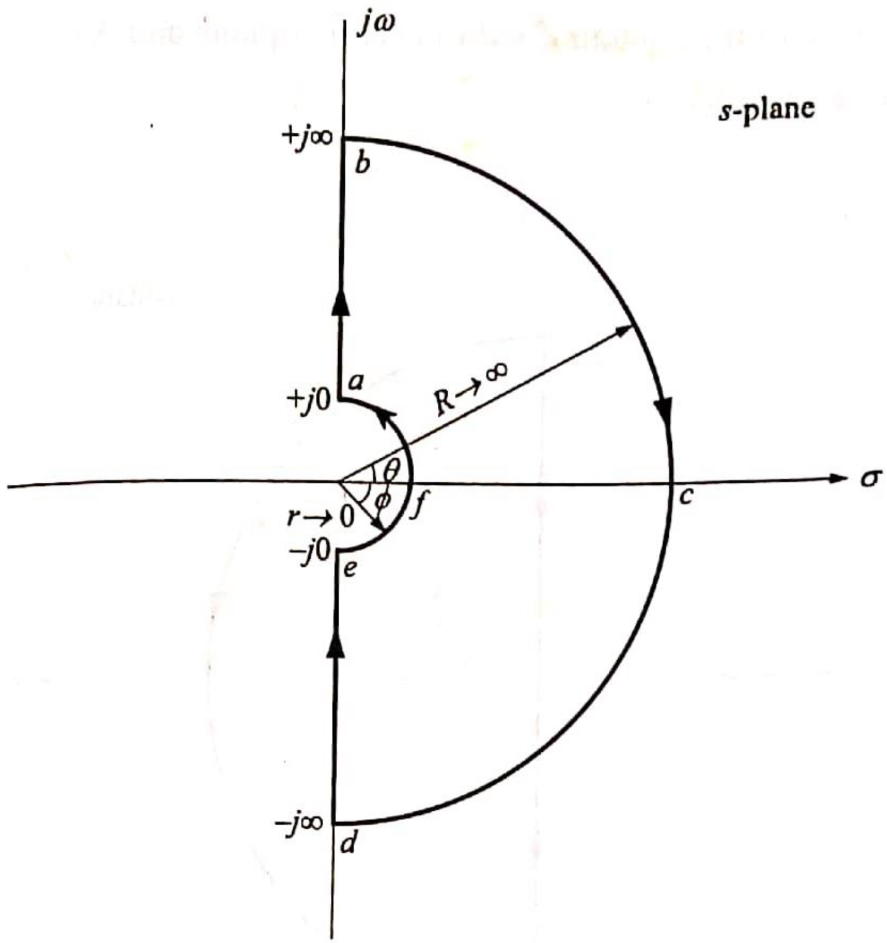
⟹No zeros in the right-half of s-plane i.e No roots of the system lie to the right-half of s-plane. Hence the closed-loop control system is stable.

Using Matlab

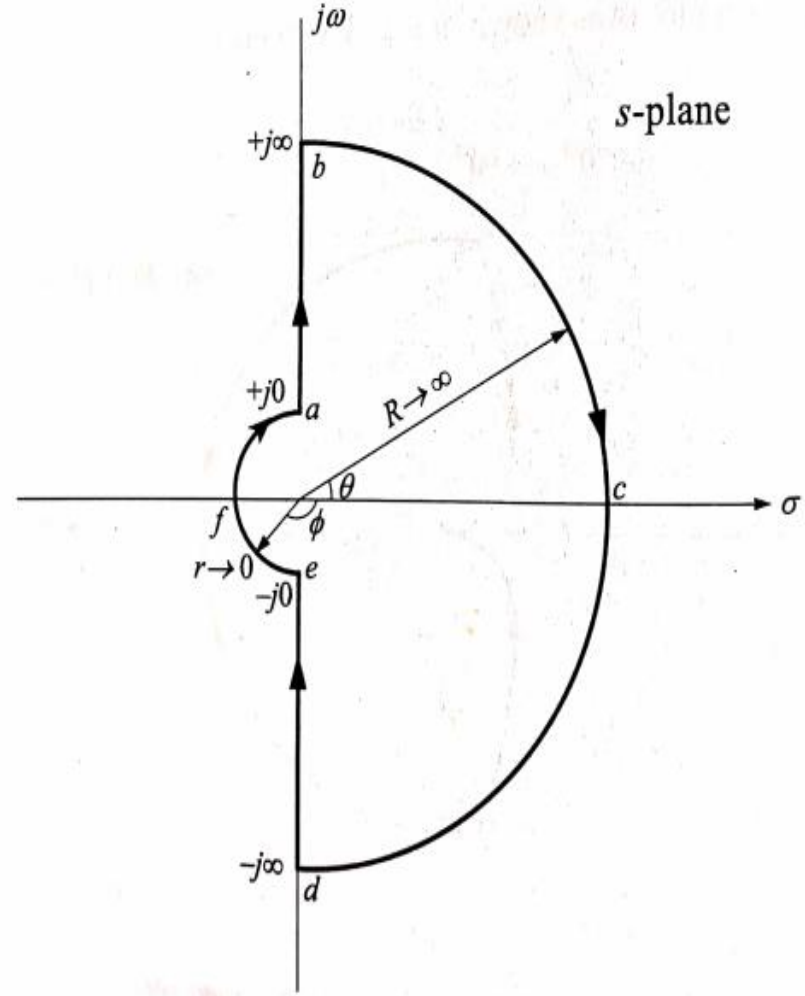


Special Case

If the poles of G(s)H(s) lie at the origin of the s- plane, then they are taken to the left-side of the s- plane (right-side of the s- plane) by drawing an indent ‘efa’ of radius, $r \longrightarrow 0$ as shown. Then find the number of encirclements made by the image of the contour ‘abcdefa’ about (- 1+j0) point on the GH(s)- plane.



jω
s-plane
+j∞
b
a
+j0
R→∞
θ
φ
f
c
σ
r→0
−j0
e
−j∞
d

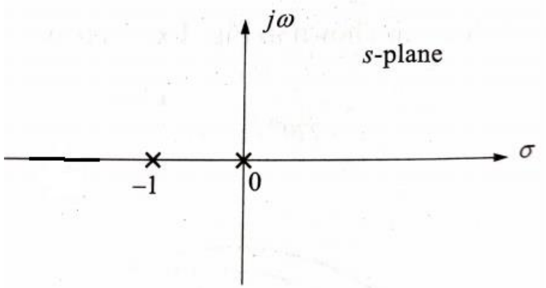


jω
s-plane
+j∞
b
+j0
a
R→∞
θ
c
σ
f
φ
r→0
e
−j0
−j∞
d

Problem

A negative feedback control system is characterized by an open-loop transfer function, G(s)H(s) = $\frac{5}{s(s+1)}$.

Investigate the closed-loop stability of the system using Nyquist stability criterion.



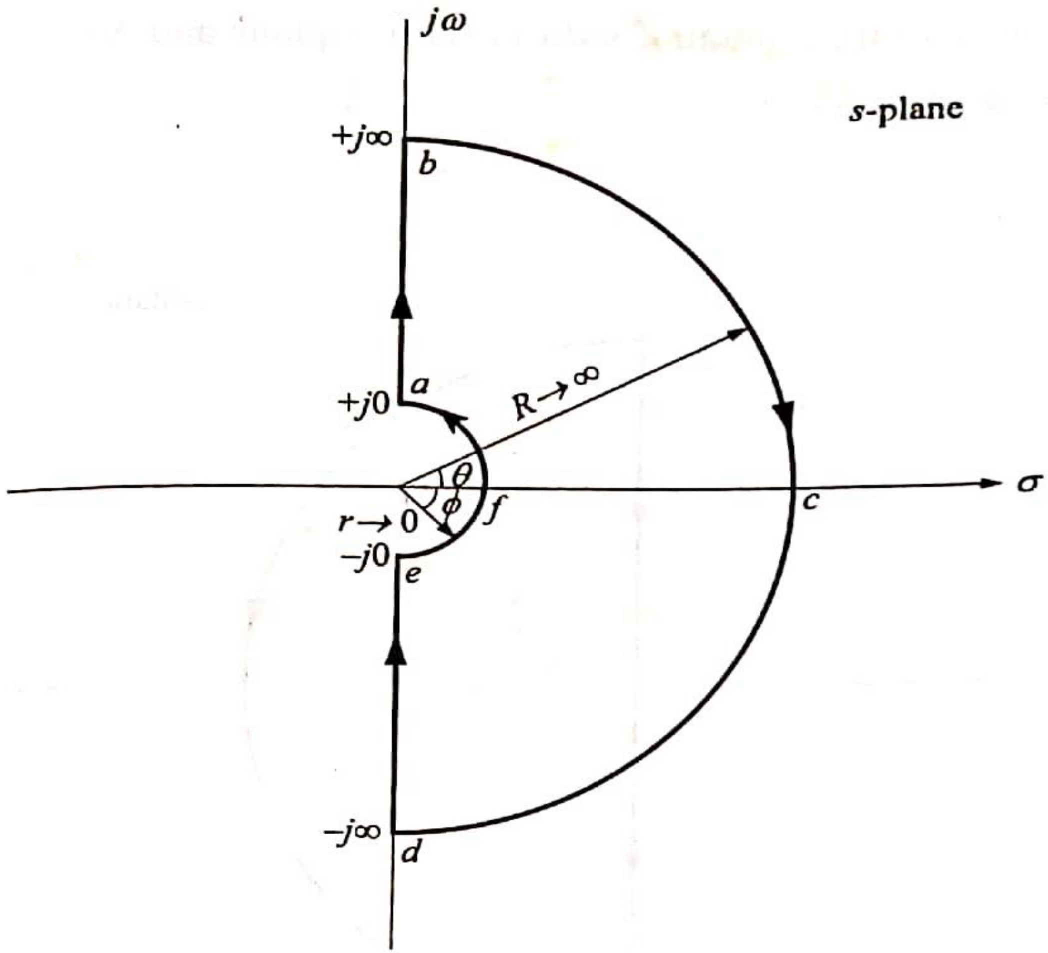
Solution

Step 1: Plot the poles of G(s)H(s) on the s-plane.

The pole at the origin is taken to the left-side of the s-plane by drawing an indent of zero radius around this pole.

Since the pole at the origin is taken to the left-side of the s-plane, P=0.

The contour ‘abcdefa’ that includes at the origin to the left side of the s-plane



Step 2: To find N:

Section I : To find the image of path ab.

$$G(s)H(s) = \frac{5}{s(s+1)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(j\omega+1)}$$

$$= \frac{5}{\omega\angle 90\ \sqrt{(\omega^2+1)}\angle \tan^{-1}\omega}$$

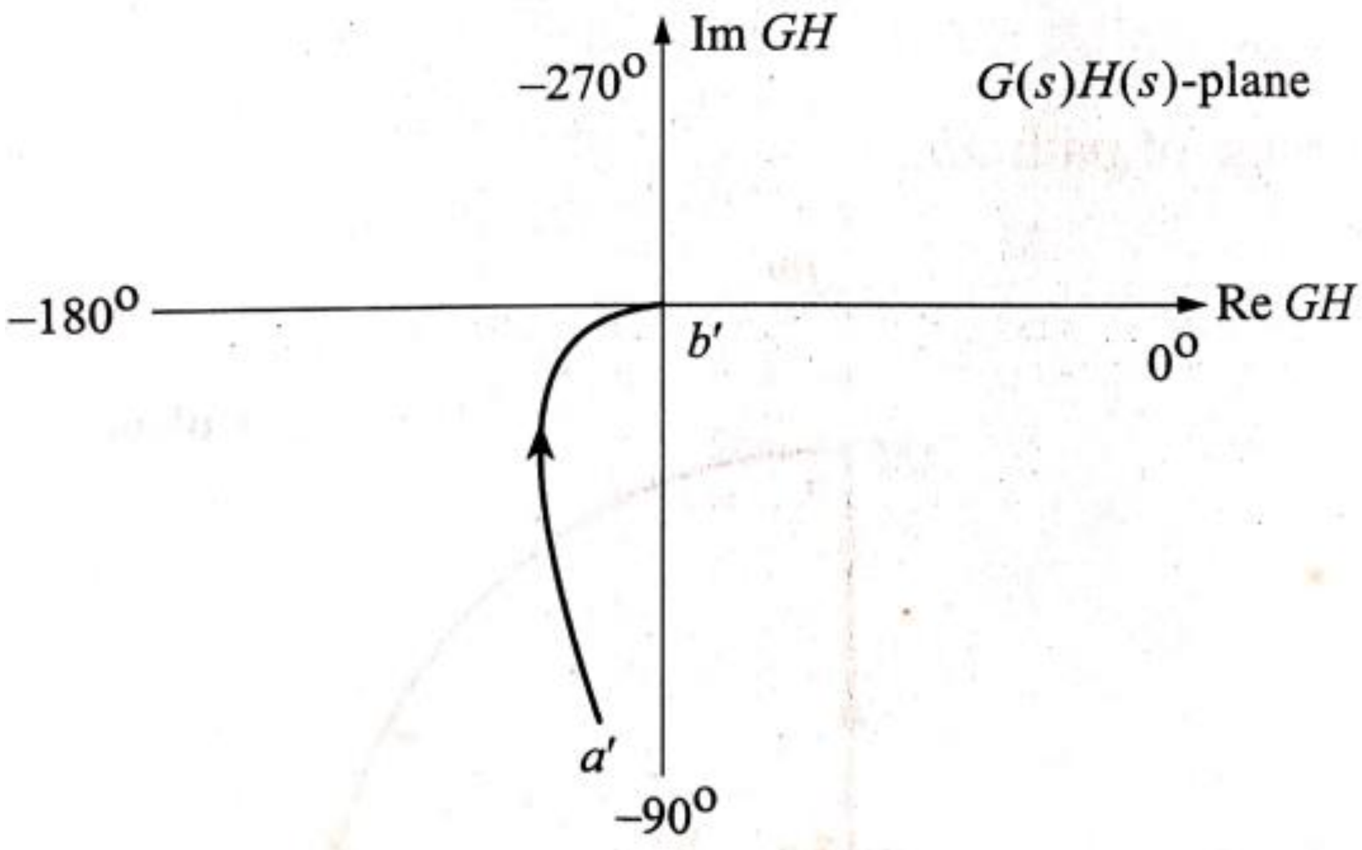
$$= \frac{5}{\omega\sqrt{(\omega^2+1)}\angle 90+\tan^{-1}\omega}$$

$$M = \frac{5}{\omega\sqrt{(\omega^2+1)}}$$

$$\varnothing = -90 - \tan^{-1}\omega$$

$$\lim_{\omega \to 0} M\angle\emptyset = \infty\angle\text{-}90 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle\text{-}180 \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

put s = $Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 $\longrightarrow$ 0 $\longrightarrow$ -90

Then, $\lim_{R\longrightarrow\infty} GH(Re^{j\theta}) = \lim_{R\longrightarrow\infty} \frac{5}{Re^{j\theta}(Re^{j\theta}+1)}$

$$= \lim_{R\longrightarrow\infty} \frac{5}{(Re^{j\theta})(Re^{j\theta})}$$

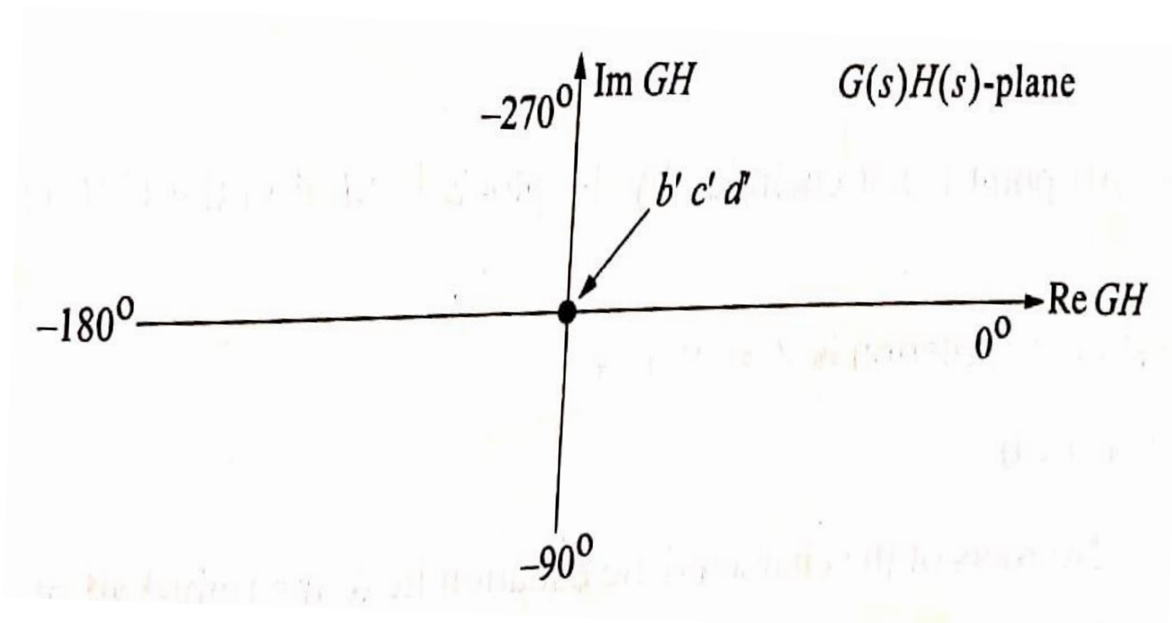
$$= \lim_{R\longrightarrow\infty} \frac{5}{(R^2 e^{j2\theta})}$$

$= 0\angle -2\theta$

$= 0\angle -180 \longrightarrow 0 \longrightarrow 180$

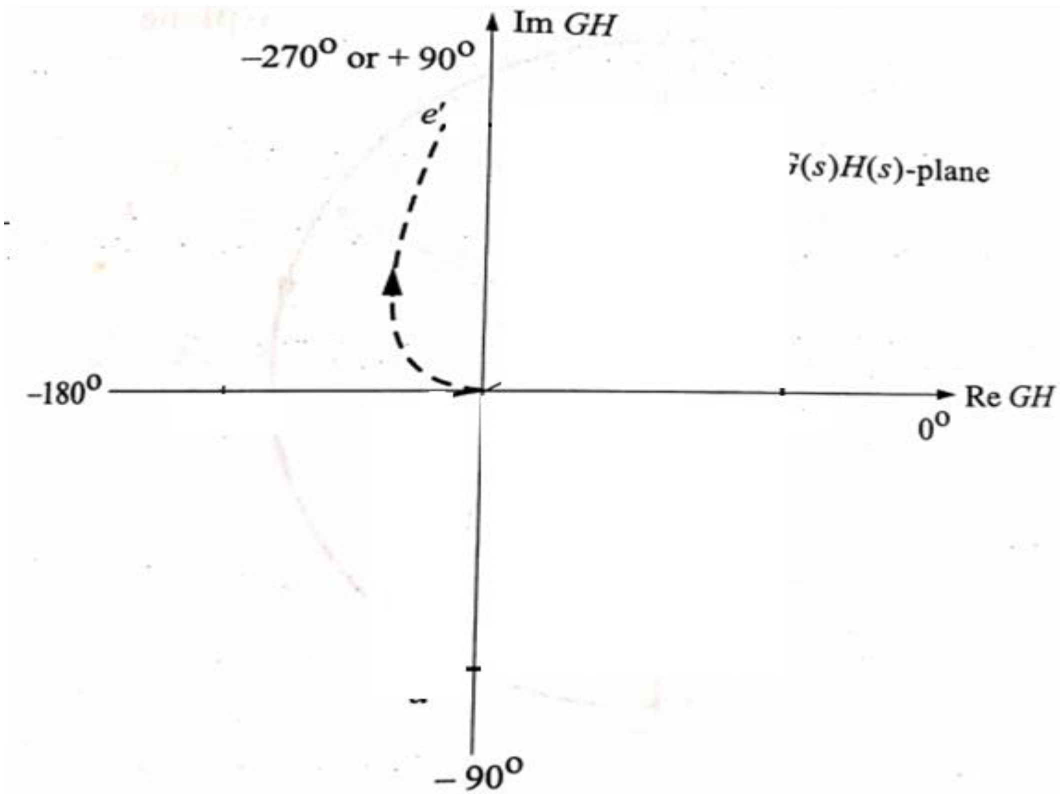
↑b' ↑c' ↑d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.



Section III: To find the image of path ‘de’

Path d’e’ is the mirror image of the path a’b’ with respect to real axis.



Section IV : To find the image of path efa

put s = $\lim_{r \to 0} re^{j\emptyset}$ in G(s)H(s)

Here , Ø changes from -90 $\longrightarrow 0 \longrightarrow +90$

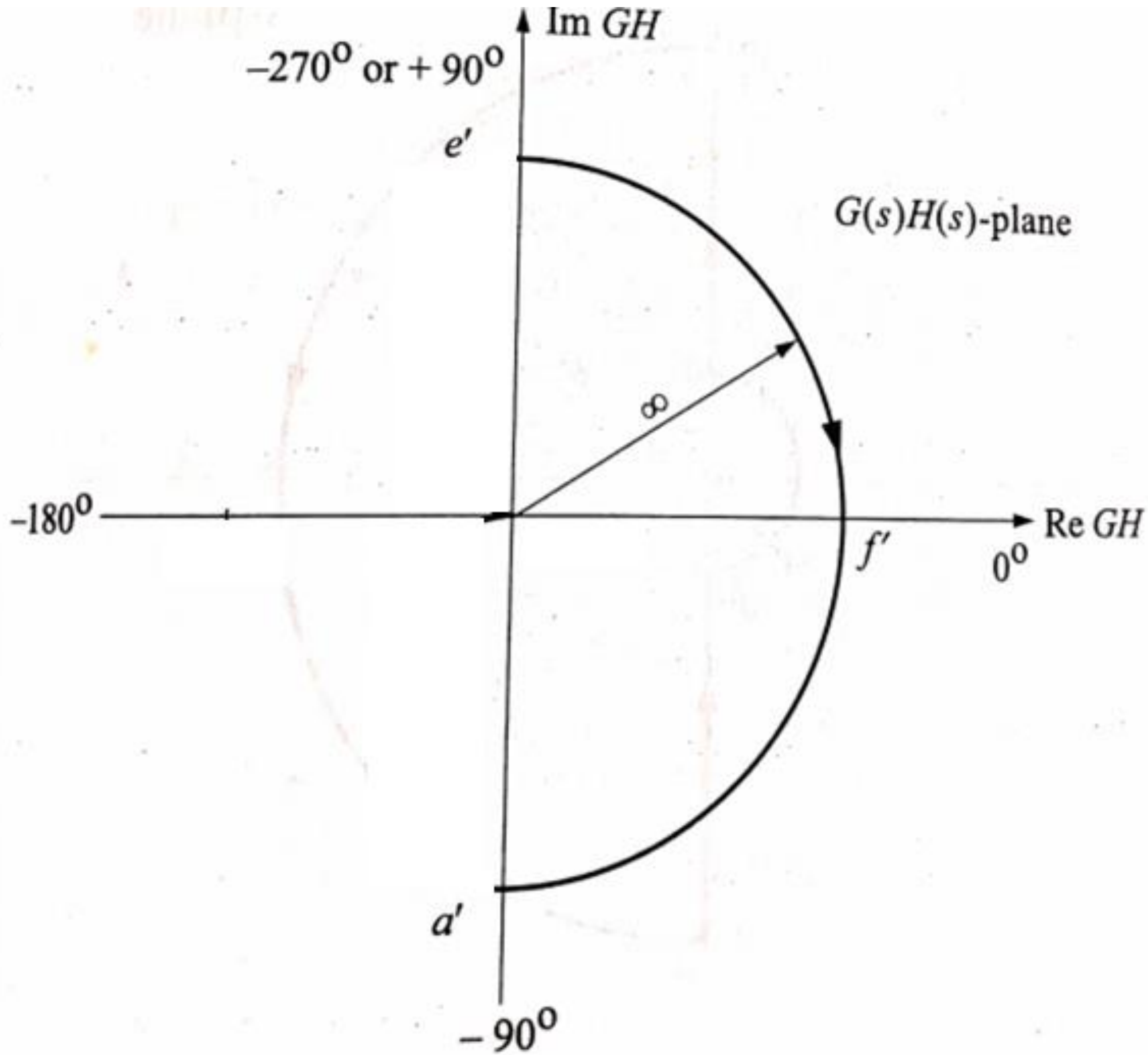
Then, $\lim_{r \to 0} GH(re^{j\emptyset})$ = $\lim_{r \to 0} \frac{5}{re^{j\emptyset}(re^{j\emptyset}+1)}$

$= \lim_{r \to 0} \frac{5}{(re^{j\emptyset})}$

= ∞∠-Ø

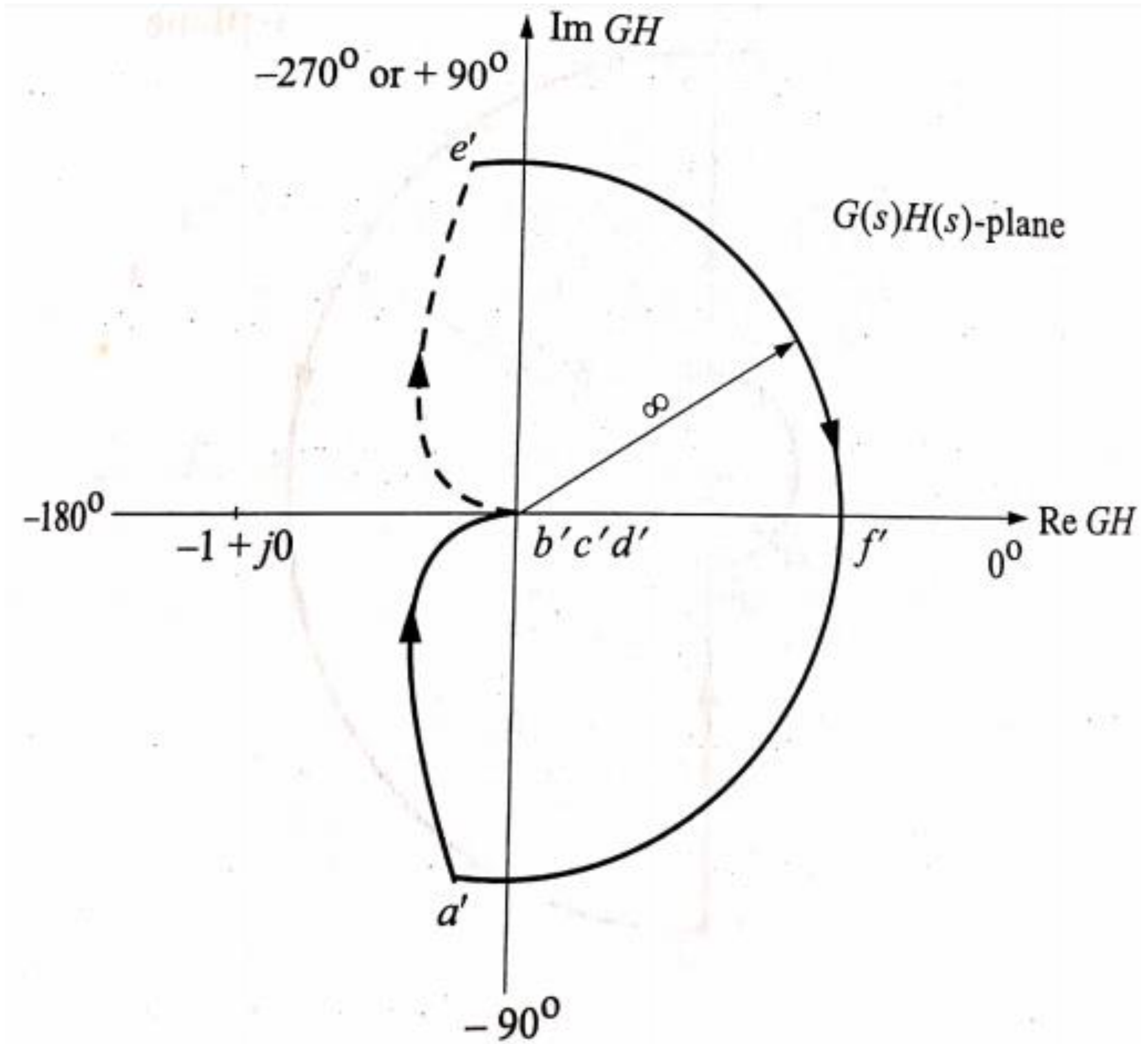
$= \infty\angle 90 \longrightarrow 0 \longrightarrow$ -90

↑e' ↑f' ↑a'



Im GH
–270° or + 90°
e′
G(s)H(s)-plane
∞
–180°
Re GH
f′
0°
a′
– 90°

The complete Nyquist plot is shown



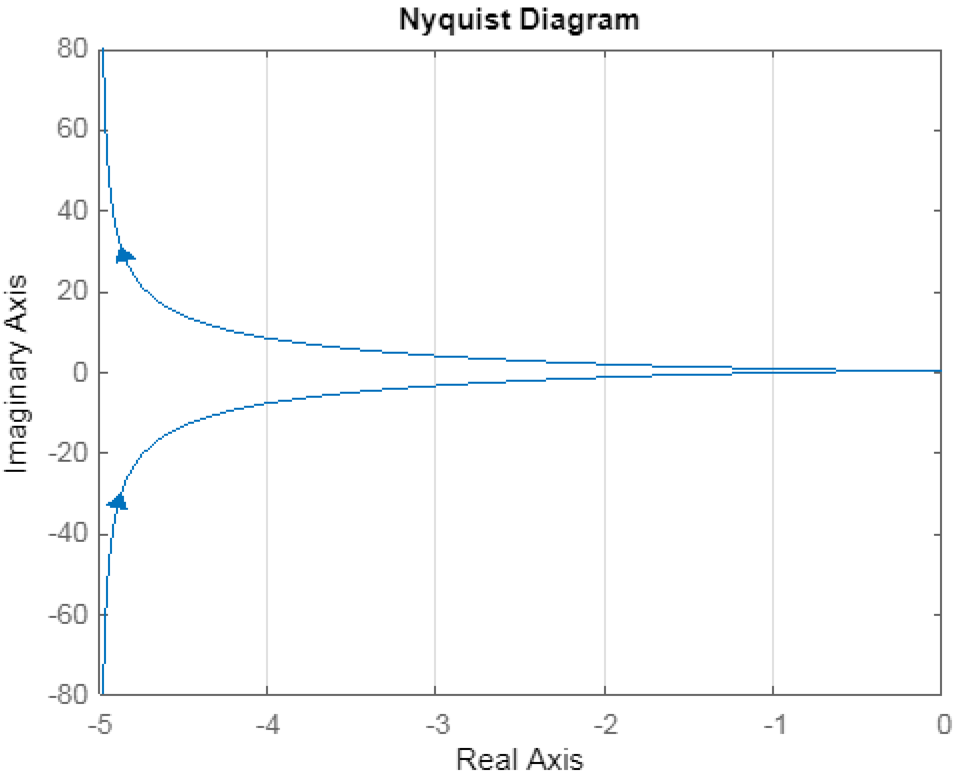
Since $-1+j0$ point is not encircled by the plot ,
$N = 0$

Step 3 :

$Z = P + N = 0 + 0 = 0$

Hence the closed loop system is stable.

Matlab

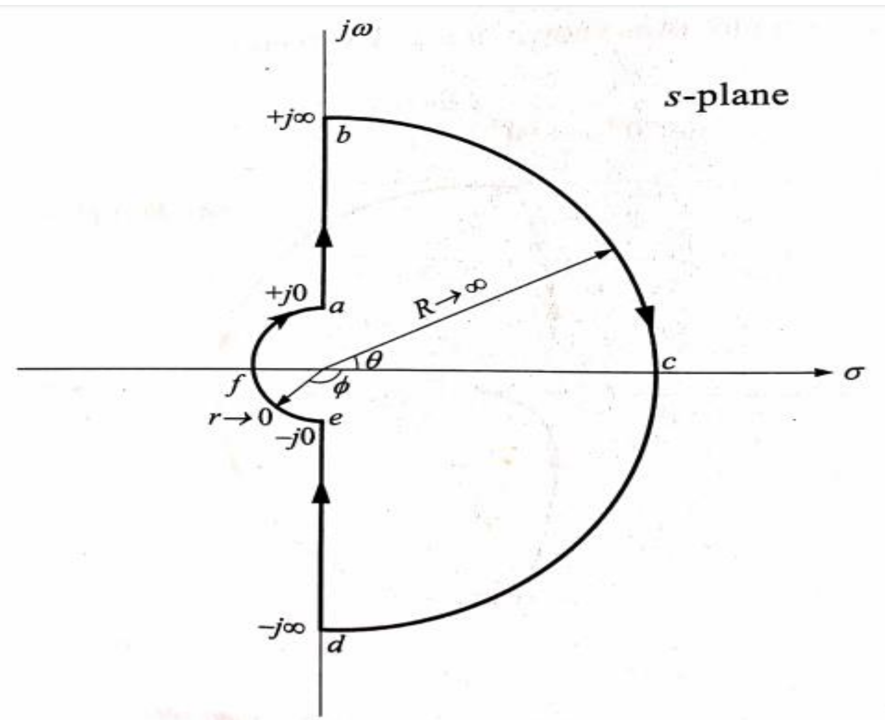


Nyquist Diagram
80
60
40
20
0
-20
-40
-60
-80
Imaginary Axis
-5
-4
-3
-2
-1
0
Real Axis

Let us re work the problem by including the pole at origin to the right-half of the s-plane and the contour ‘abcdefa’

Step 1: P = 1

The mapping of sections ‘ab’, ’bcd’, and ‘de’ on to the G(s)H(s)-plane remains same



Section IV :

Mapping of section 'efa':

put s = $\lim_{r \longrightarrow 0} re^{j\emptyset}$ in G(s)H(s)

Here , Ø changes from -90 ⟶ -180 ⟶ -270

Then, $\lim_{r \longrightarrow 0} GH(re^{j\emptyset})$ = $\lim_{r \longrightarrow 0} \frac{5}{re^{j\emptyset}(rej^{\emptyset}+1)}$

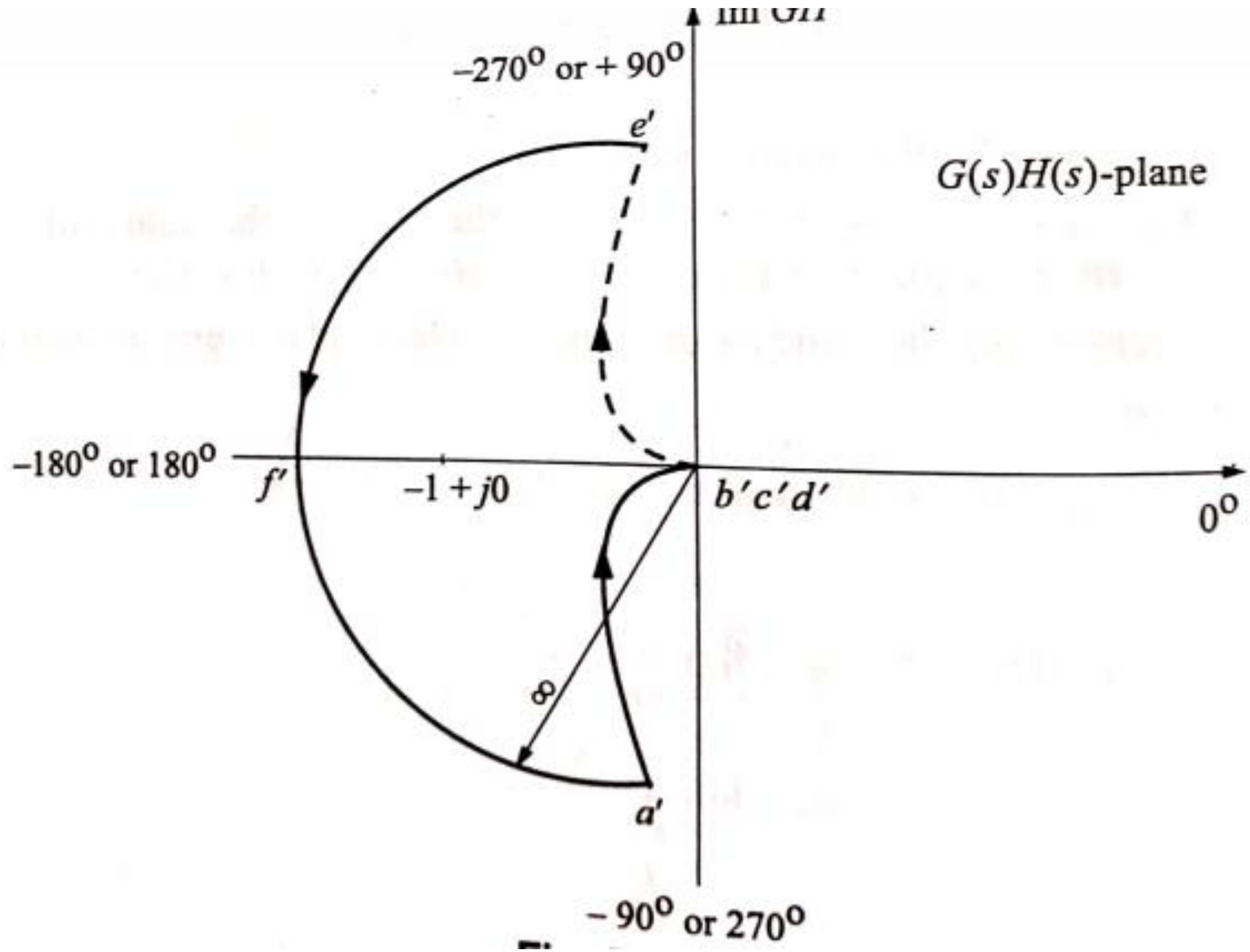
= $\lim_{r \longrightarrow 0} \frac{5}{(rej^{\emptyset})}$

= $\infty\angle -\emptyset$

= $\infty\angle 90 \longrightarrow 180 \longrightarrow 270$

↑e' ↑f' ↑a'

The complete Nyquist plot is shown



Since -1+j0 point is encircled by the plot in anticlockwise direction, $N = -1$

Step 3 :

$Z = 1 - 1 = 0$

Hence the closed loop system is stable.

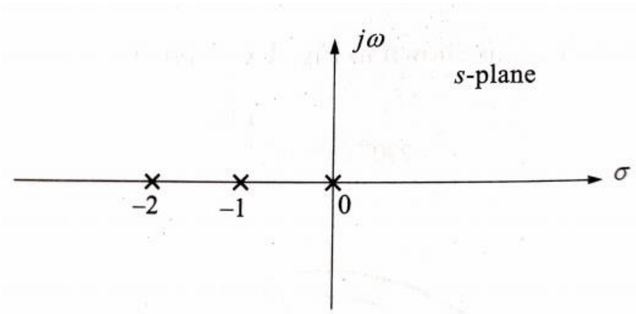
Problem

Sketch the Nyquist plot for G(s)H(s) = $\frac{k}{s(s+1)(s+2)}$

Find the range of k for closed-loop stability

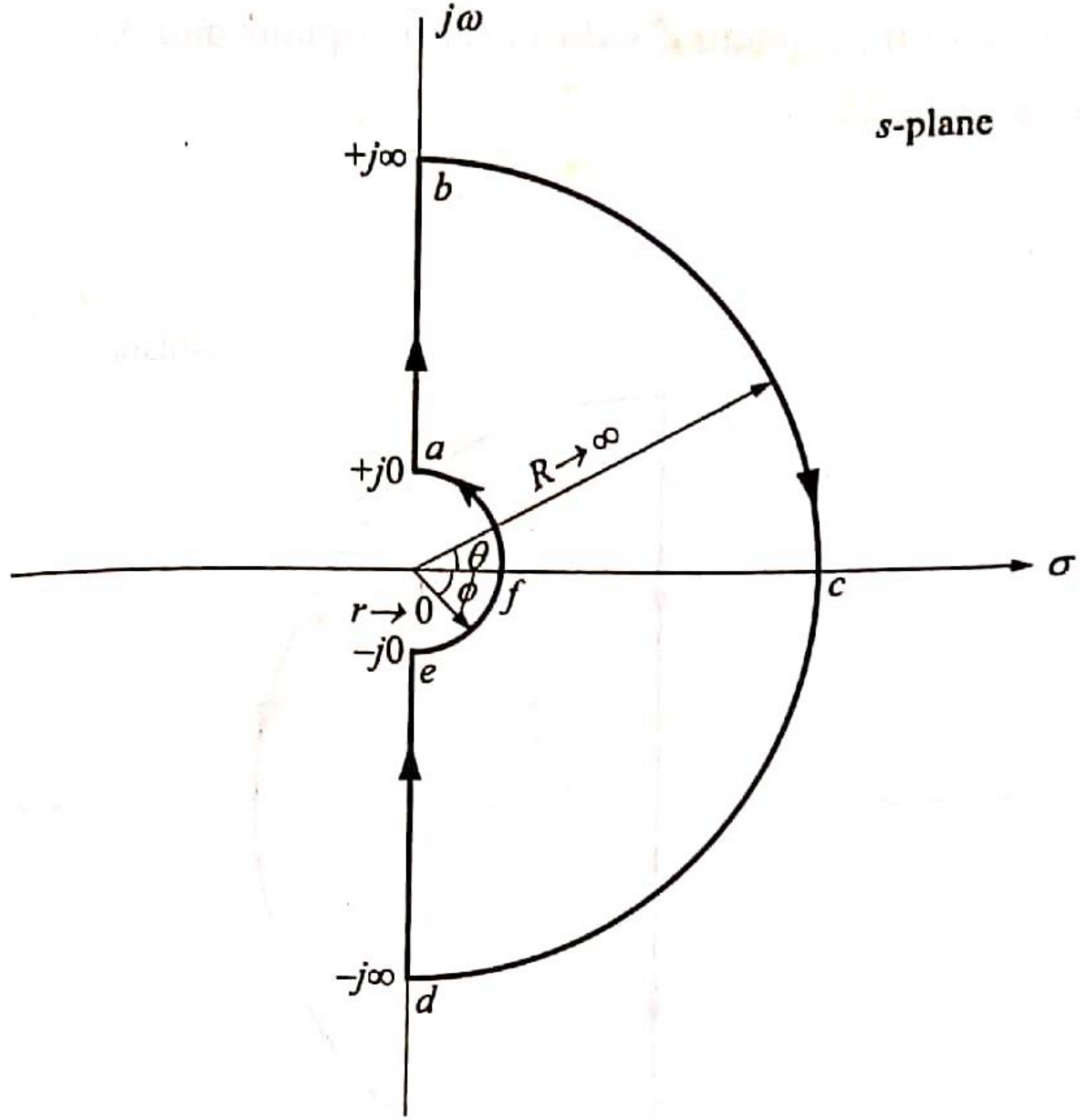
Solution

Step 1: Plot the poles of GH(s) on the s-plane



The pole at the origin is taken to the left-side of the s-plane by drawing an indent of zero radius around this pole.

Since the pole at the origin is taken to the left-side of the s-plane, P=0



$j\omega$
s-plane
$+j\infty$
b
a
$+j0$
$R \to \infty$
θ
ϕ
σ
f
c
$r \to 0$
$-j0$
e
$-j\infty$
d

Step 2: To find N:

Section I : To find the image of path ab.

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{k}{j\omega(j\omega+1)(j\omega+2)}$$

$$= \frac{k}{\{\omega\angle 90\}\{\sqrt{(\omega^2+1)}\angle \tan^{-1}\omega\}\{\sqrt{(\omega^2+4)}\angle \tan^{-1}(\frac{\omega}{2})\}}$$

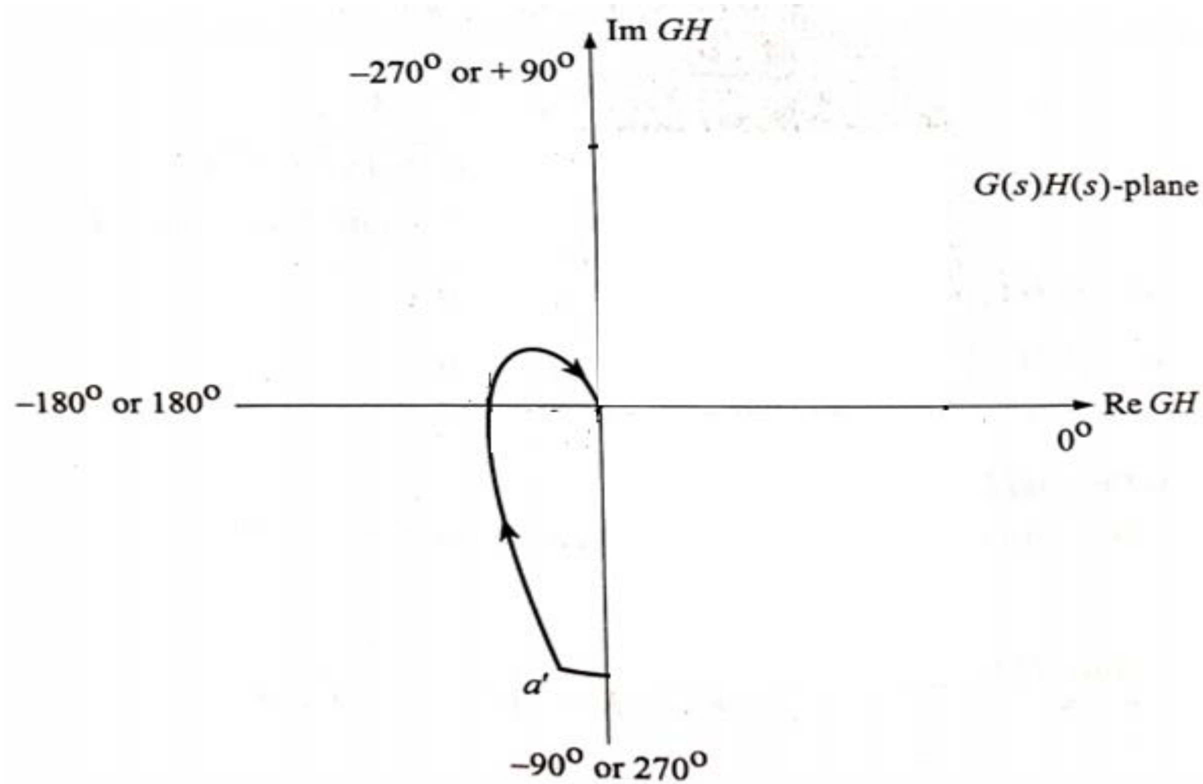
$$= \frac{k}{\omega\sqrt{(\omega^2+1)}\sqrt{(\omega^2+4)}\angle 90+\tan^{-1}\omega+\tan^{-1}(\frac{\omega}{2})}$$

$$M = \frac{k}{\omega\sqrt{(\omega^2+1)}\sqrt{(\omega^2+4)}}$$

$$\emptyset = -90 - \tan^{-1}\omega - \tan^{-1}(\frac{\omega}{2})$$

$$\lim_{\omega \to 0} M\angle\emptyset = \infty\angle\text{-}90 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle\text{-}270 \qquad \text{(point b')}$$



Section II : To find the image of path ‘bcd’

put s = $\lim_{R \longrightarrow \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim_{R \longrightarrow \infty} GH(Re^{j\theta}) = \lim_{R \longrightarrow \infty} \dfrac{k}{Re^{j\theta}(Re^{j\theta}+1)(Rej^{\theta}+2)}$

$= \lim_{R \longrightarrow \infty} \dfrac{k}{(Re^{j\theta})(Re^{j\theta})(Re^{j\theta})}$

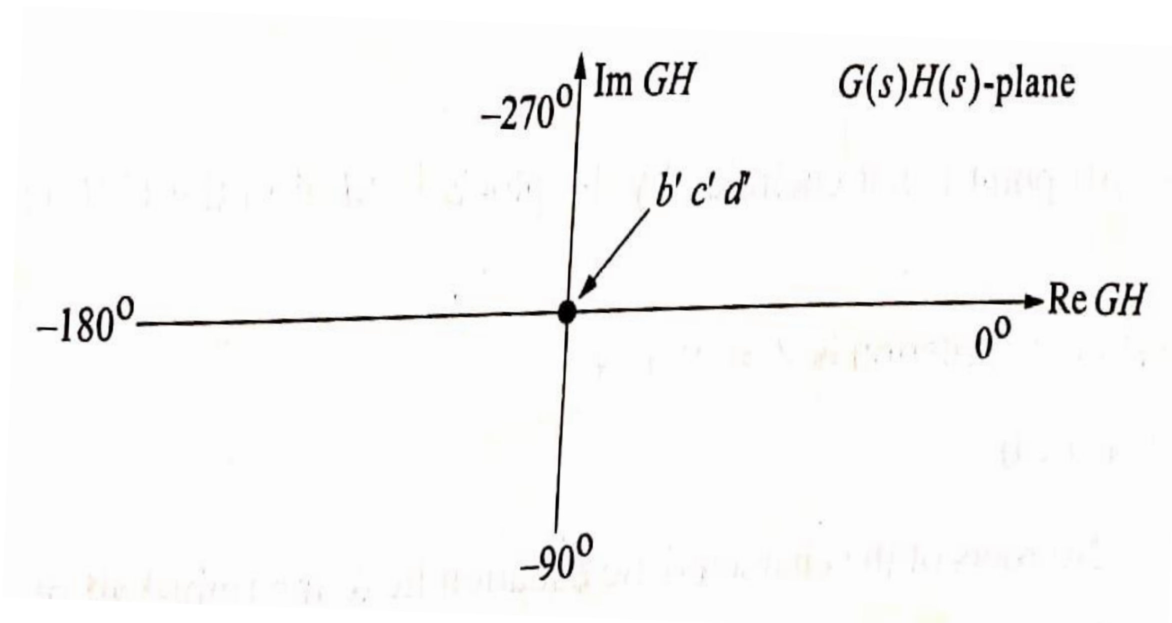
$= \lim_{R \longrightarrow \infty} \dfrac{k}{(R^3 e^{j3\theta})}$

= 0∠-3θ

= 0∠-270⟶ 0⟶ 270

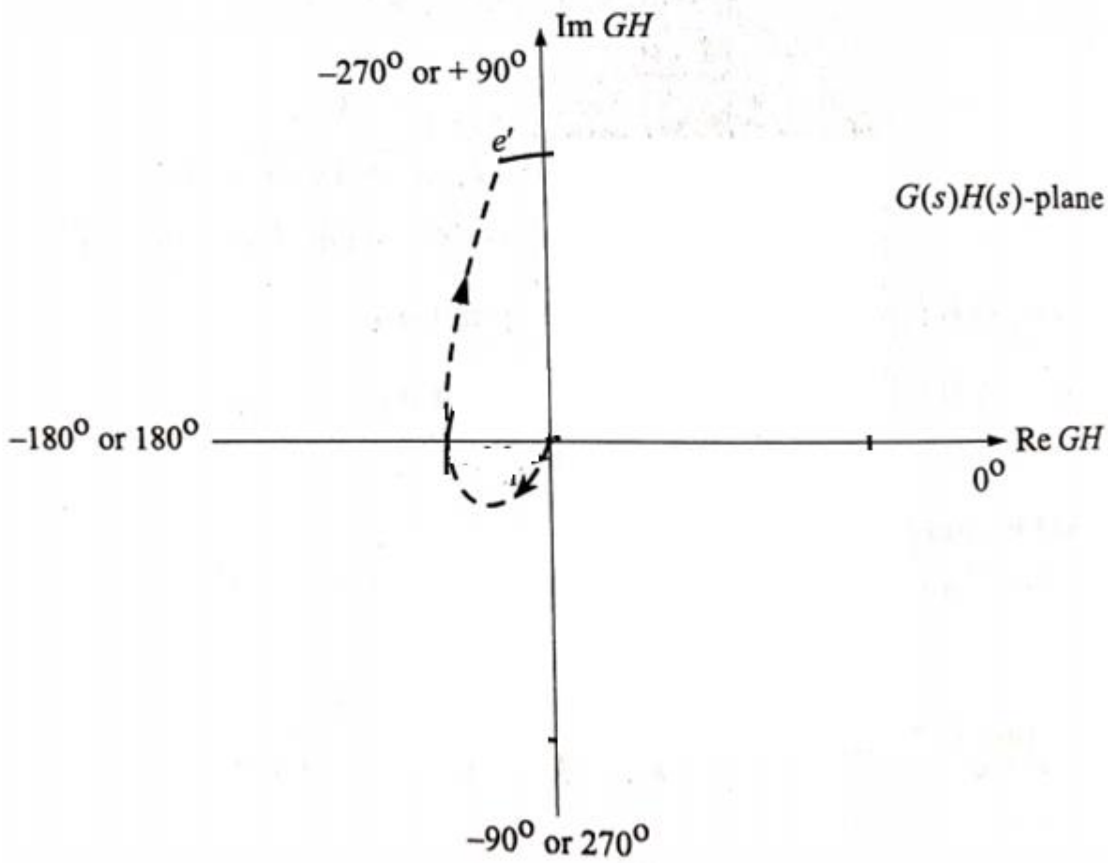
↑b’ ↑c’ ↑d’

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.



Section III: To find the image of path 'de'

Path d'e' is the mirror image of the path a'b' with respect to real axis.



Section IV : To find the image of path efa

$$\text{put } s = \lim_{r \to 0} re^{j\emptyset} \text{ in } G(s)H(s)$$

Here , Ø changes from $-90 \longrightarrow 0 \longrightarrow +90$

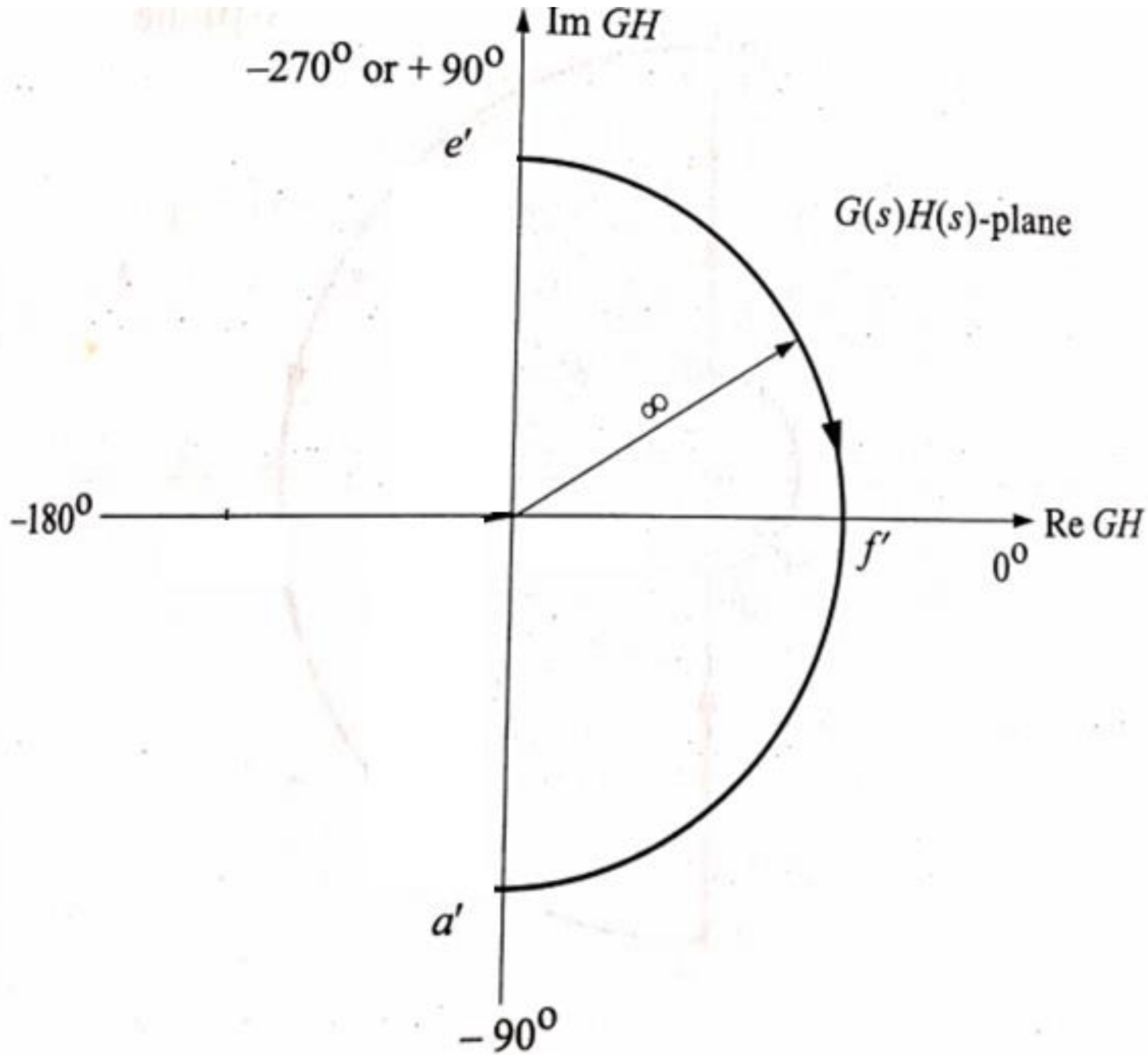
Then, $\lim_{r \to 0} GH(re^{j\emptyset}) = \lim_{r \to 0} \frac{k}{re^{j\emptyset}(re^{j\emptyset}+1)(rej^{\emptyset}+2)}$

$$= \lim_{r \to 0} \frac{k}{(re^{j\emptyset})}$$

$$= \infty\angle -\emptyset$$

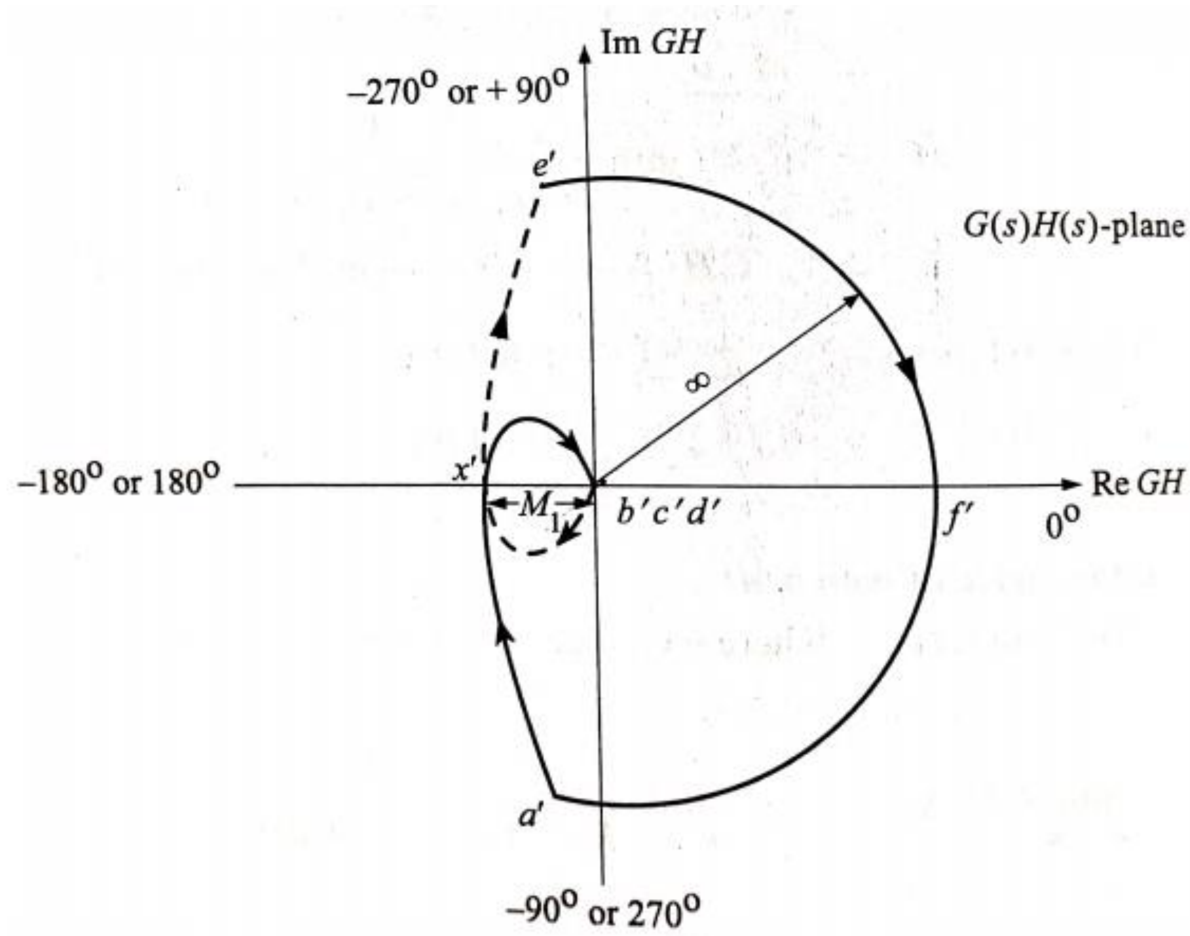
$$= \infty\angle 90 \longrightarrow 0 \longrightarrow -90$$

$$\uparrow e' \qquad \uparrow f' \qquad \uparrow a'$$



Im GH
–270° or + 90°
e′
G(s)H(s)-plane
∞
–180°
Re GH
f′
0°
a′
– 90°

The complete Nyquist plot is shown



To find M_1 :

At point x’, phase = -180

$$\Rightarrow -90 - \tan^{-1}\omega - \tan^{-1}\left(\frac{\omega}{2}\right) = -180$$

$$\tan^{-1}\left(\frac{3\omega}{2-\omega^2}\right) = 0$$

$$2 - \omega^2 = 0$$

$$\omega = \sqrt{2}\ \text{rad/sec}$$

$$M_1 = \left| GH(j\omega) \right|_{\omega = \sqrt{2}}$$

$$= \frac{k}{\omega\sqrt{(\omega^2+1)}\sqrt{(\omega^2+4)}} = \frac{k}{6}$$

Since P is zero , N must be zero for Z to be zero.

N will be zero if and only if -1+j0 is not encircled by the Nyquist plot

For N to be zero, $M_1 < 1$

Hence, $\frac{k}{6} < 1$

$\Longrightarrow k < 6$

Since k is always positive, for closed-loop stability : $0 < k < 6$

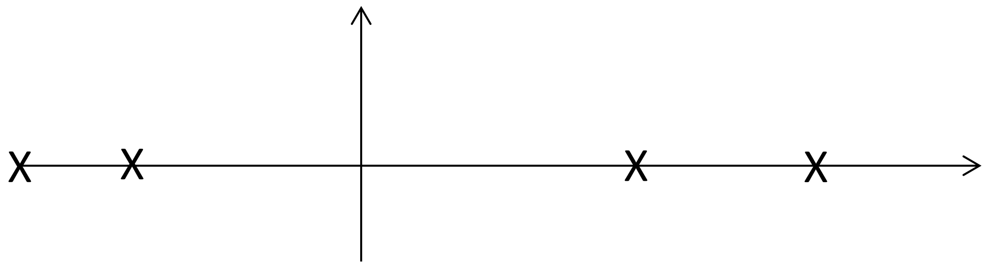
Problem

The open loop transfer function of a negative unity feed back system is given by $\frac{k(s+3)(s+5)}{(s-2)(s-4)}$.

Find the range of k for closed − loop stability

Solution

Step 1 **:** Plot the poles of GH(s) on the s-plane



Since there are 2 poles lies in the right-side of the s-plane, P=2

Step 2: To find N:

Section I : To find the image of path ab:

$$G(s)H(s) = \frac{k(s+3)(s+5)}{(s-2)(s-4)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{k(j\omega+3)(j\omega+5)}{(j\omega-2)(j\omega-4)}$$

$$= \frac{k\,\{\sqrt{(\omega^2+9)}\angle \tan^{-1}(\frac{\omega}{3})\}\,\{\sqrt{(\omega^2+25)}\angle \tan^{-1}(\frac{\omega}{5})\}}{\{\sqrt{(\omega^2+4)}\angle \tan^{-1}(\frac{\omega}{-2})\}\,\{\sqrt{(\omega^2+16)}\angle \tan^{-1}(\frac{\omega}{-4})\}}$$

$$= \frac{k\,\{\sqrt{(\omega^2+9)}\,\sqrt{(\omega^2+25)}\,\}\,\{\angle \tan^{-1}\left(\frac{\omega}{3}\right)+\angle \tan^{-1}(\frac{\omega}{5})\}}{\{\sqrt{(\omega^2+4)}\sqrt{(\omega^2+16)}\,\}\quad \{\angle \tan^{-1}\left(\frac{\omega}{-2}\right)+\angle \tan^{-1}(\frac{\omega}{-4})\}}$$

$$M = \frac{k\,\{\sqrt{(\omega^2+9)}\,\sqrt{(\omega^2+25)}\,\}}{\{\sqrt{(\omega^2+4)}\sqrt{(\omega^2+16)}\,\}}$$

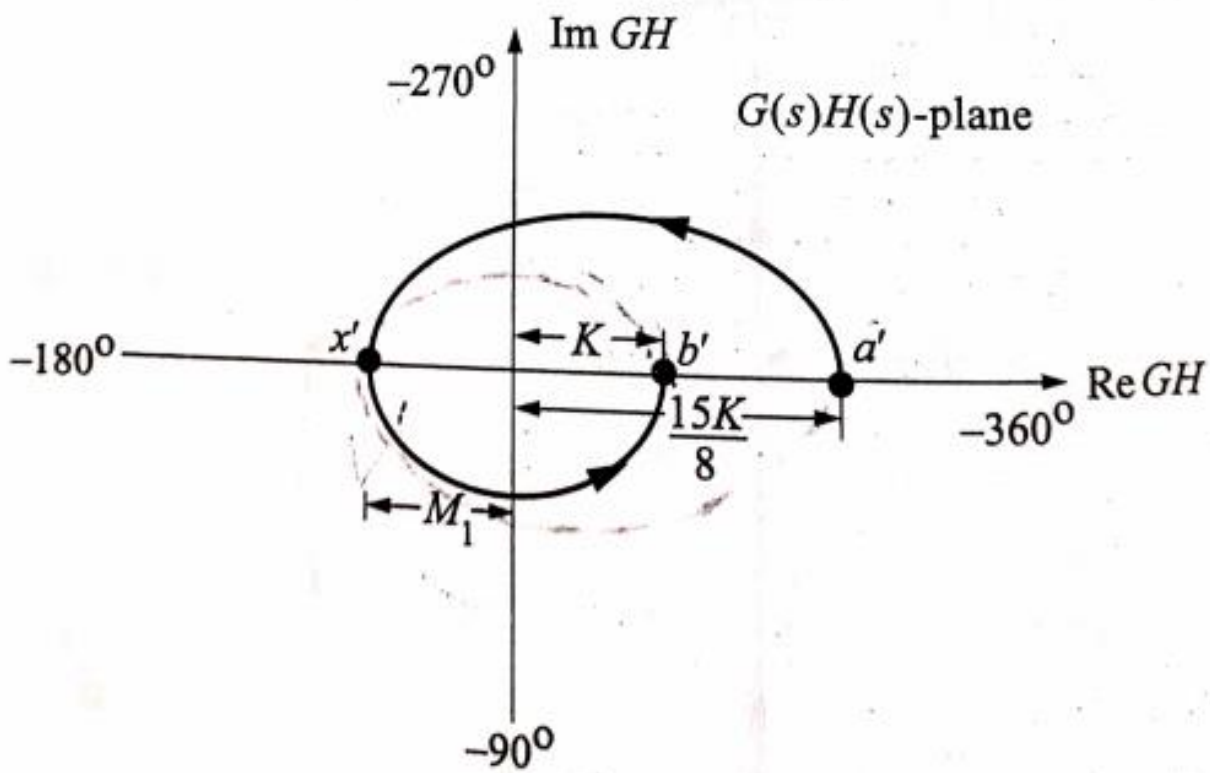
$$\varnothing \;=\angle\tan^{-1}\left(\frac{\omega}{3}\right)+\;\angle\tan^{-1}(\frac{\omega}{5})\}\text{ - }\{\angle\tan^{-1}\left(\frac{\omega}{-2}\right)+\;\angle\tan^{-1}(\frac{\omega}{-4})\}$$

$$=\tan^{-1}\left(\frac{\omega}{3}\right)+\;\tan^{-1}(\frac{\omega}{5})\}\text{ - }\{180-\tan^{-1}\left(\frac{\omega}{2}\right)+\;180-\tan^{-1}(\frac{\omega}{4})\}$$

$$\varnothing \;=\; -\,360+\;\tan^{-1}\left(\frac{\omega}{3}\right)+\;\tan^{-1}(\frac{\omega}{5})\;+\tan^{-1}\left(\frac{\omega}{2}\right)+\;\tan^{-1}(\frac{\omega}{4})$$

$$\lim_{\omega \to 0} M\angle\emptyset = \frac{15k}{8}\angle\text{-}360 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = \text{k}\ \angle 0 \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

$$G(s)H(s) = \frac{k(j\omega+3)(j\omega+5)}{(j\omega-2)(j\omega-4)}$$

put s = $\lim_{R\longrightarrow\infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim_{R\longrightarrow\infty} GH(\, Re^{j\theta}) = \lim_{R\longrightarrow\infty} \frac{k(Re^{j\theta}+3)(Rej^{\theta}+5)}{(Re^{j\theta}-2)(Rej^{\theta}-4)}$

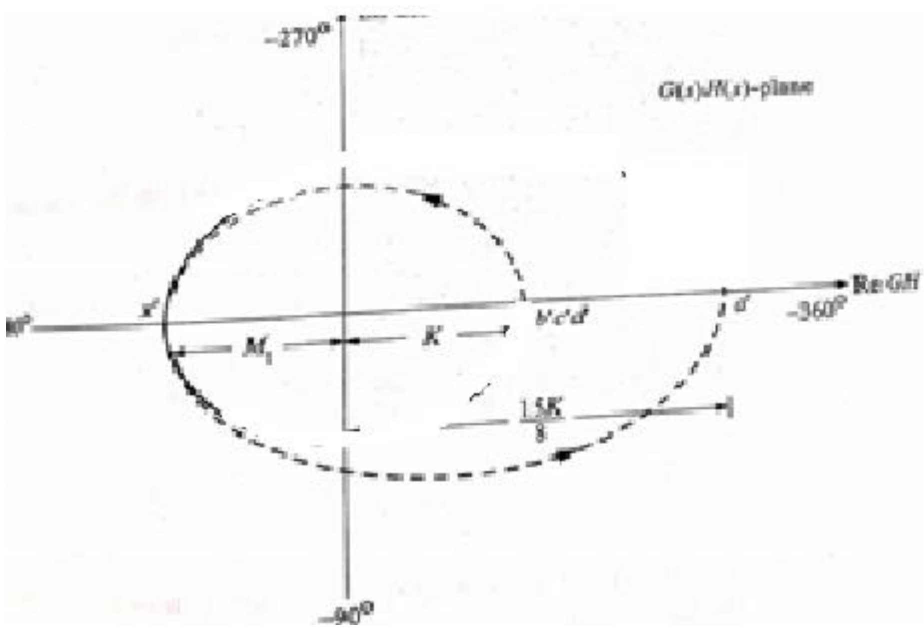
$$= \lim_{R\longrightarrow\infty} \frac{k(Rej^{\theta})(Rej^{\theta})}{(Re^{j\theta})(Re^{j\theta})}$$

$$= \lim_{R\longrightarrow\infty} k$$

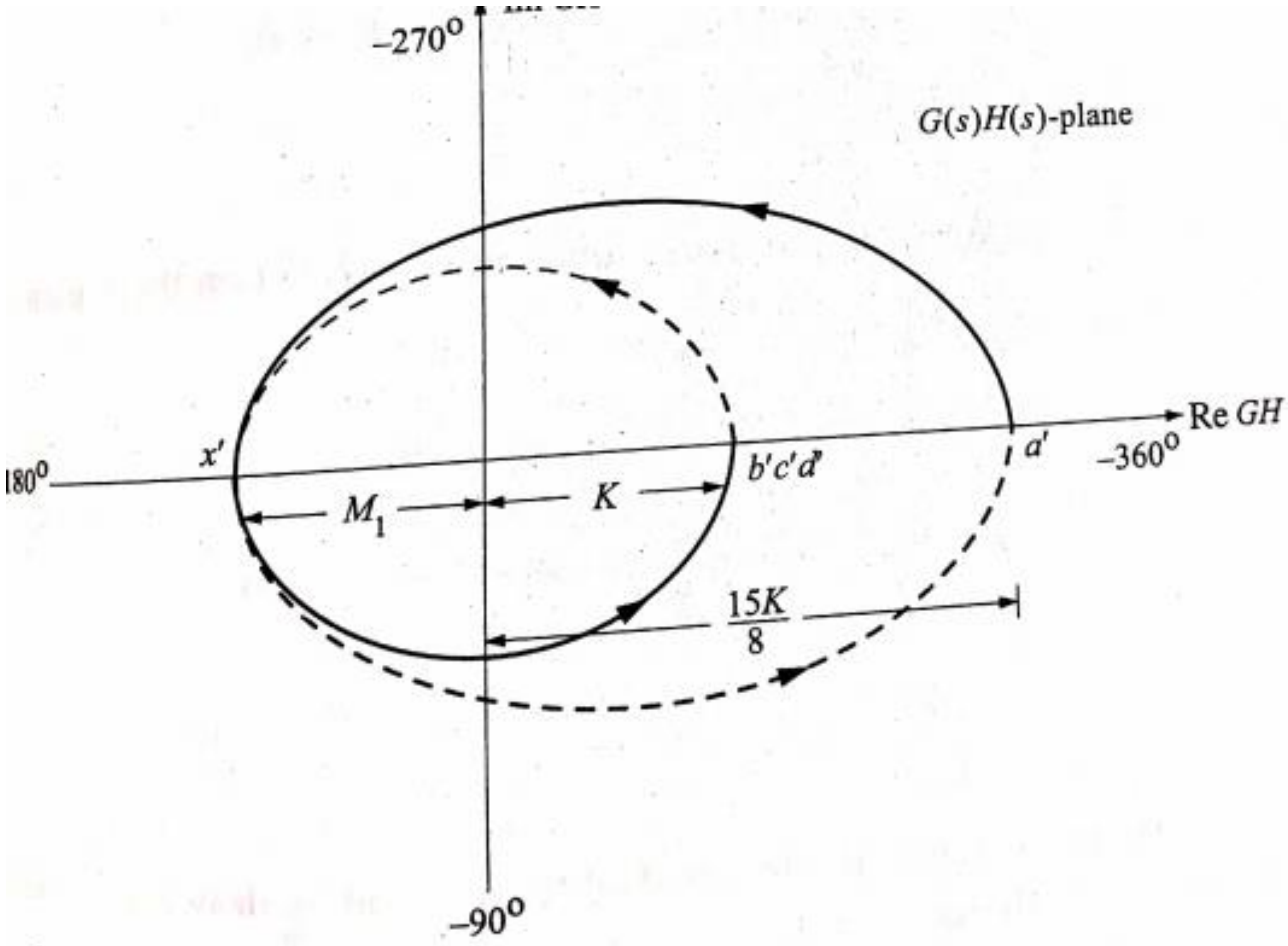
$= k\angle 0$

Section III: To find the image of path ‘de’

Path d’e’ is the mirror image of the path a’b’ with respect to real axis.



The Complete Nyquist plot is shown



To find M_1 :

At point x', phase = -180

$\Longrightarrow$ - 360 + $\tan^{-1}\left(\frac{\omega}{3}\right) + \tan^{-1}(\frac{\omega}{5}) + \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{4}\right)$ = -180

By solving, $\omega = \sqrt{11}$ rad/sec

$$M_1 = \left|GH(j\omega)\right|_{\omega=\sqrt{11}}$$

$$= \frac{k\,\{\sqrt{(\omega^2+9)}\,\sqrt{(\omega^2+25)}\,\}}{\{\sqrt{(\omega^2+4)}\sqrt{(\omega^2+16)}\,\}}$$

= 1.33k

Since P is 2 , N must be -2 for Z to be zero.

N will be -2 if and only if $-1+j0$ is encircled twice in the anticlockwise direction by the Nyquist plot

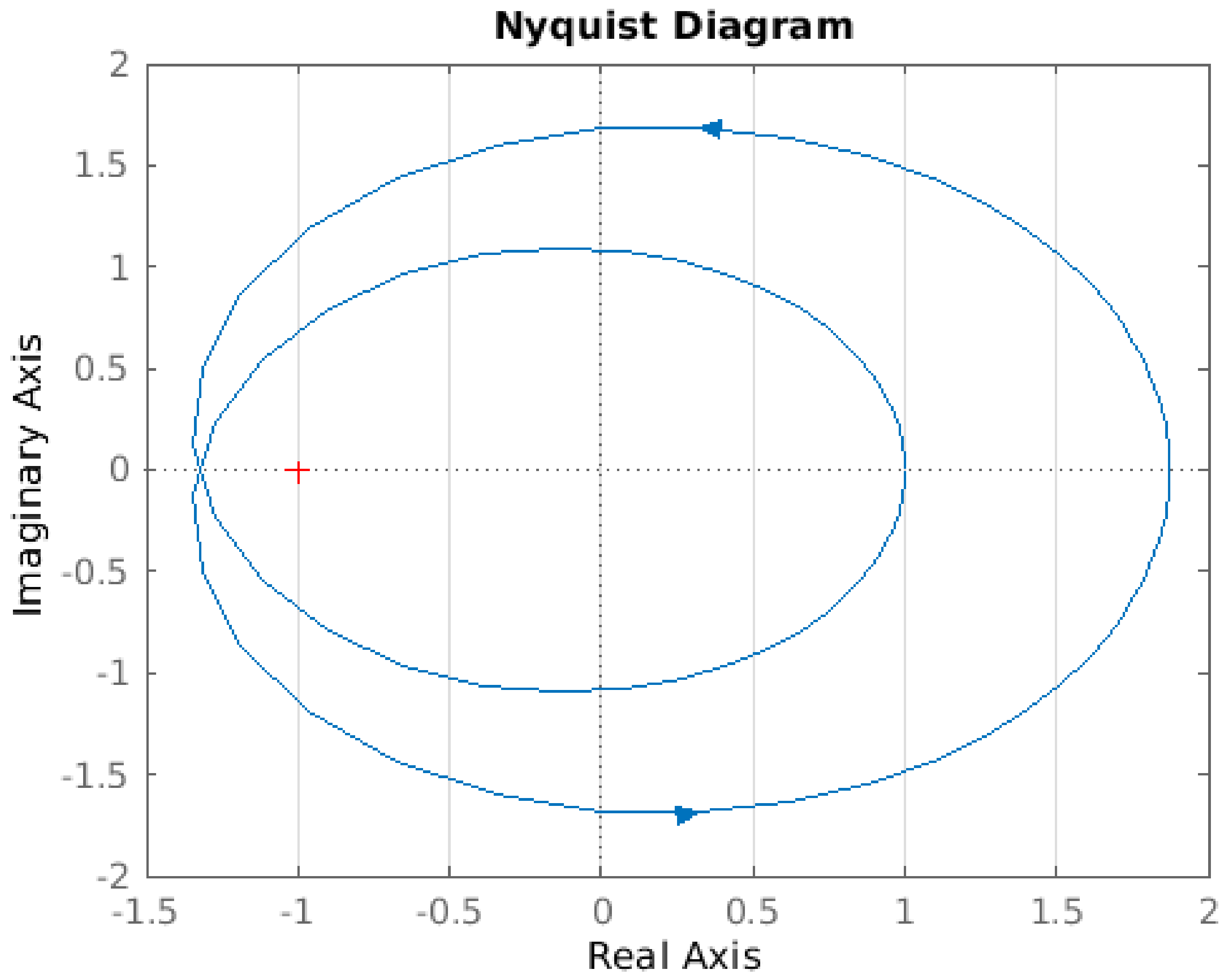
For this to happen, $M_1 > 1$

Hence, $1.33k > 1$

$$\Longrightarrow k > 0.75$$

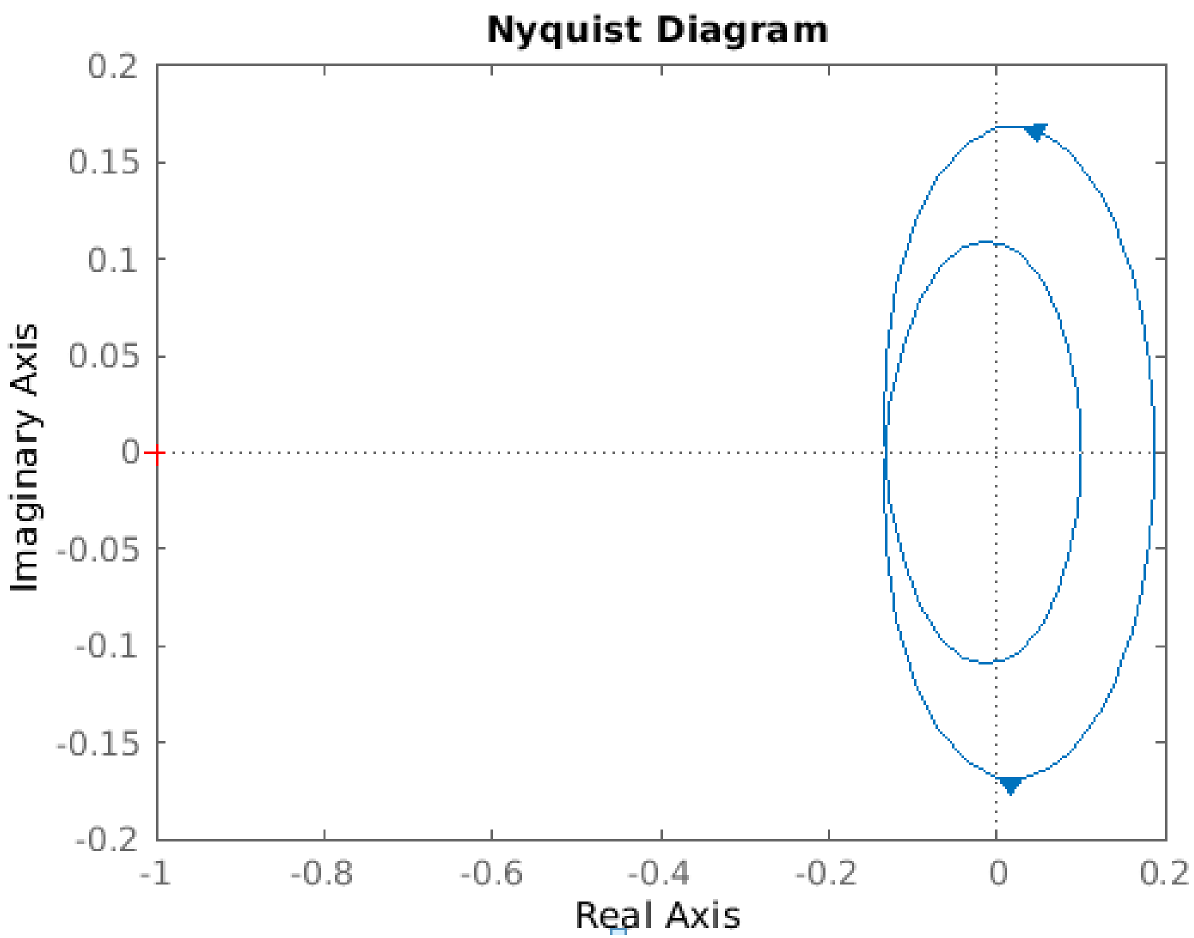
Since k is always positive, for closed-loop stability : $0.75 < k < \infty$

Matlab (k=1)



Nyquist Diagram
Imaginary Axis
Real Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

K=0.1

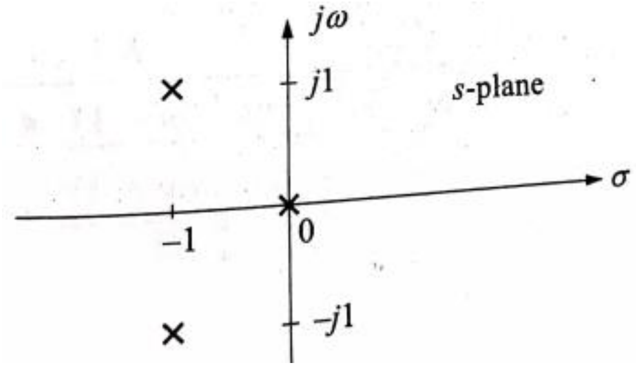

Nyquist Diagram
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
Imaginary Axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
Real Axis

Problem

The open loop transfer function of a negative unity feed back system is given by $\frac{k}{s(s^2+2s+2)}$ .Find the range of k for closed – loop stability

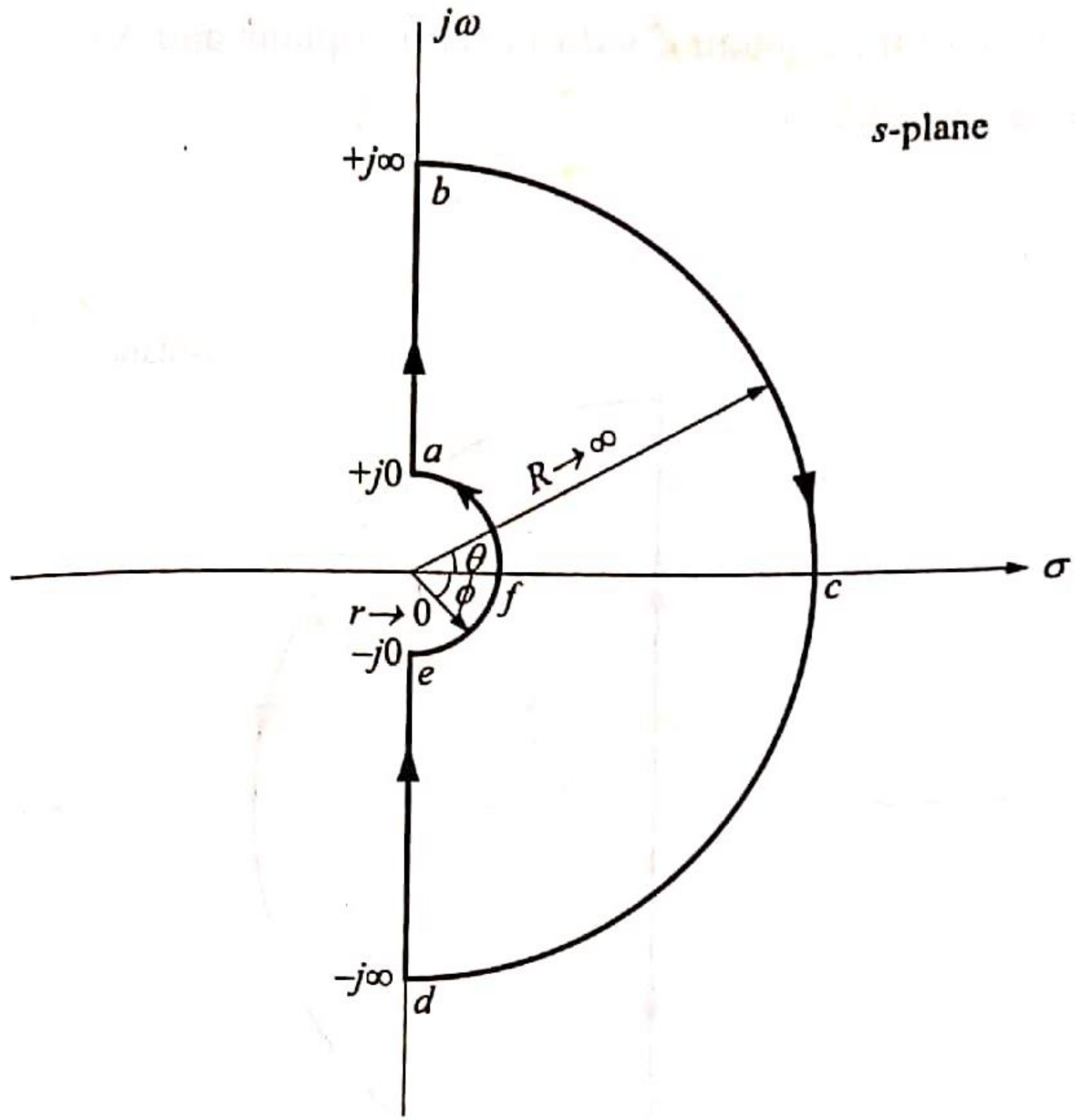
Solution

Step 1 : Plot the poles of GH(s) on the s-plane



The pole at the origin is taken to the left-side of the s-plane by drawing an indent of zero radius around this pole.

Since the pole at the origin is taken to the left-side of the s-plane, P=0



$j\omega$
s-plane
$+j\infty$
b
a
$+j0$
$R \to \infty$
θ
ϕ
σ
f
c
$r \to 0$
$-j0$
e
$-j\infty$
d

Step 2: To find N:

Section I : To find the image of path ab.

$$G(s)H(s) = \frac{k}{s(s^2+2s+2)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{k}{j\omega[(j\omega)^2+2j\omega+2]}$$

$$= \frac{k}{j\omega[-\omega^2+2j\omega+2]}$$

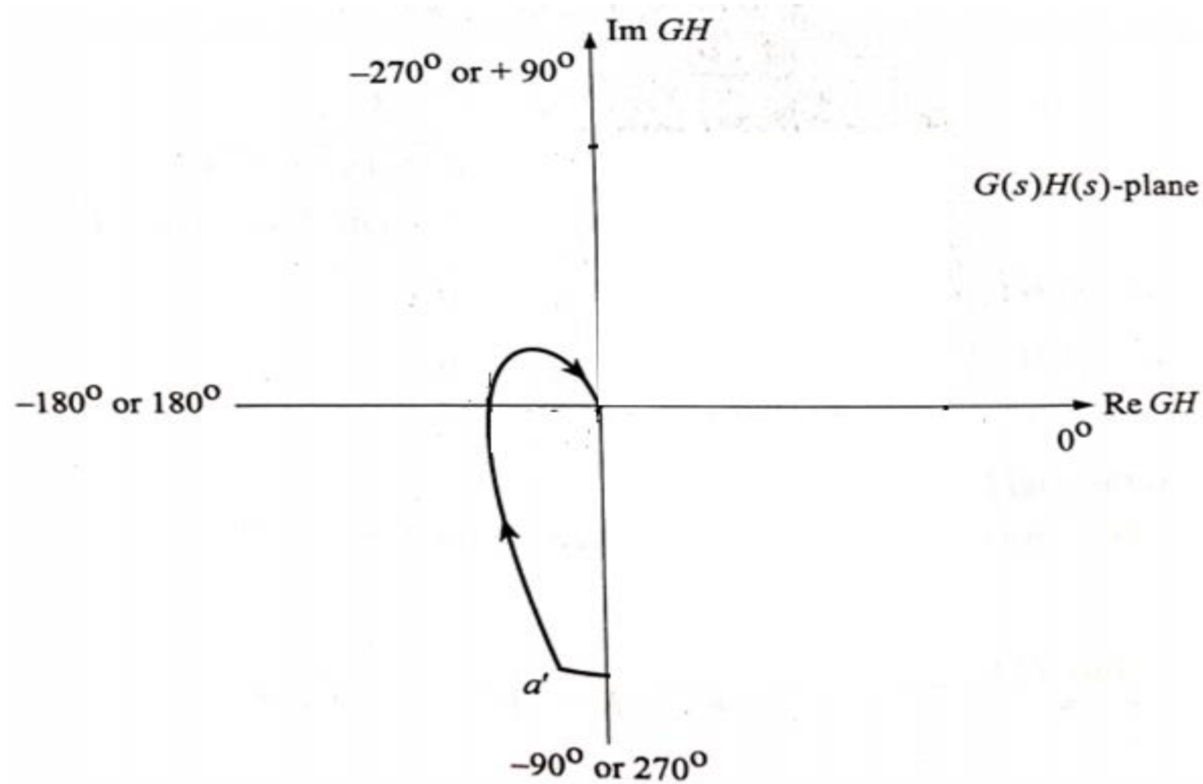
$$= \frac{k}{\omega\angle 90\ \sqrt{(2\omega)^2+(2-\omega^2)^2}\angle\tan^{-1}(\frac{2\omega}{2-\omega^2})}$$

$$M = \frac{k}{\omega\sqrt{(2\omega)^2+(2-\omega^2)^2}}$$

$$\varnothing = -90 - \tan^{-1}(\frac{2\omega}{2-\omega^2})$$

$$\lim_{\omega \to 0} M\angle\emptyset = \infty\angle\text{-}90 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle\text{-}270 \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

put s = $\lim_{R \longrightarrow \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim_{R \longrightarrow \infty} GH(\, Re^{j\theta})$ $= \lim_{R \longrightarrow \infty} \frac{k}{Re^{j\theta}[(Re^{j\theta})^2 + 2Re^{j\theta} + 2]}$

$$= \lim_{R \longrightarrow \infty} \frac{k}{(Re^{j\theta})[(R^2 e^{j2\theta}) + 2Re^{j\theta} + 2]}$$

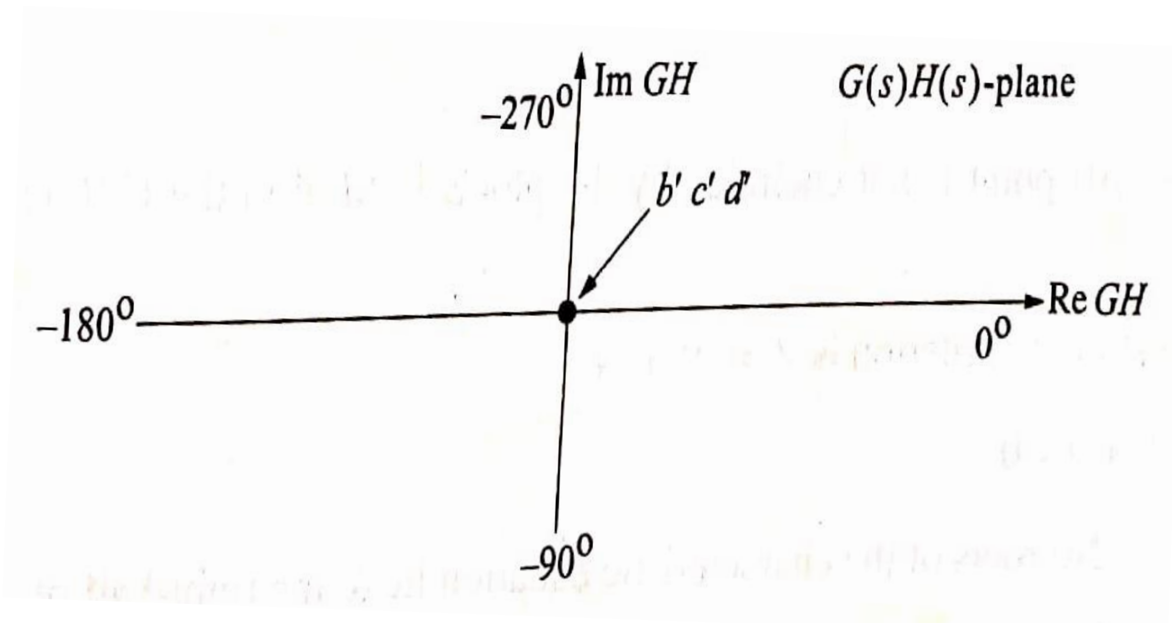
$$= \lim_{R \longrightarrow \infty} \frac{5}{(R^3 e^{j3\theta})}$$

= 0∠-3θ

= 0∠-270⟶ 0⟶ 270

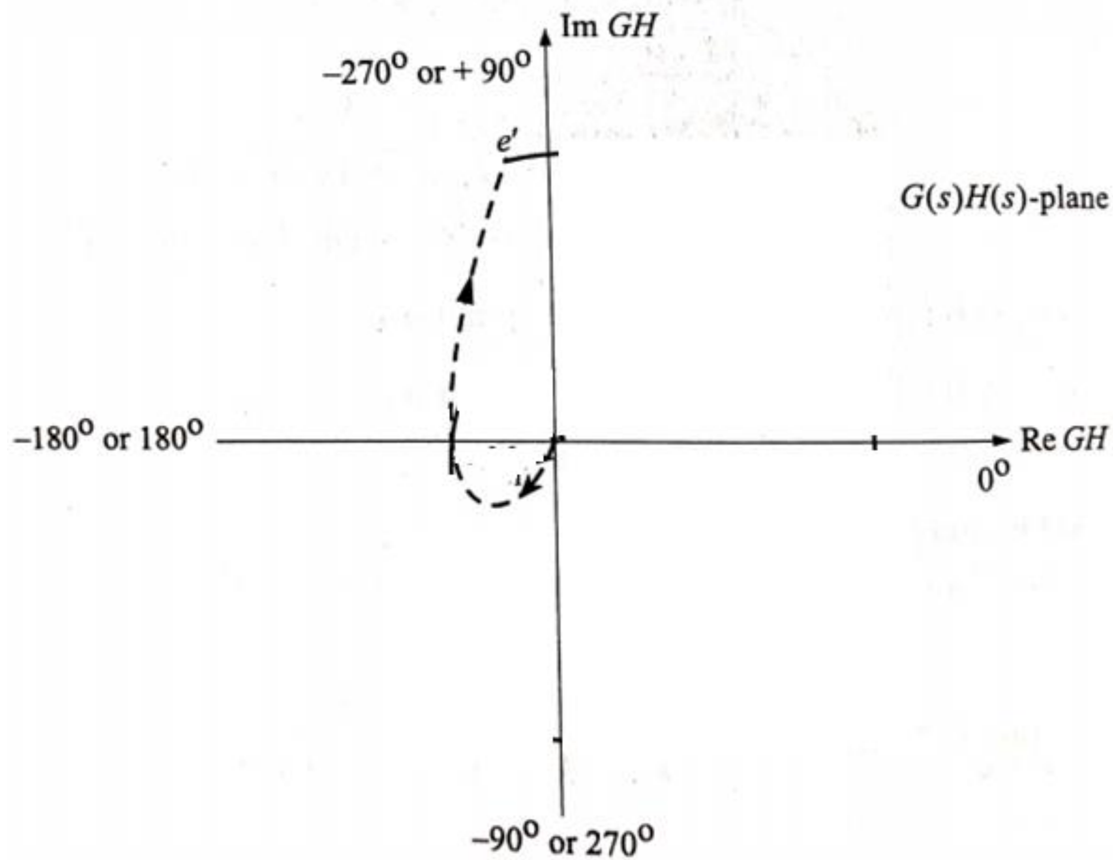
↑b' ↑c' ↑d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.



Section III: To find the image of path 'de'

Path d'e' is the mirror image of the path a'b' with respect to real axis.



Section IV : To find the image of path efa

put s = $\lim_{r \to 0} re^{j\emptyset}$ in G(s)H(s)

Here , ∅ changes from -90 $\longrightarrow$ 0 $\longrightarrow$ +90

Then, $\lim_{r \to 0} GH(re^{j\emptyset}) = \lim_{r \to 0} \frac{k}{re^{j\emptyset}[(re^{j\emptyset})^2 + 2re^{j\emptyset} + 2]}$

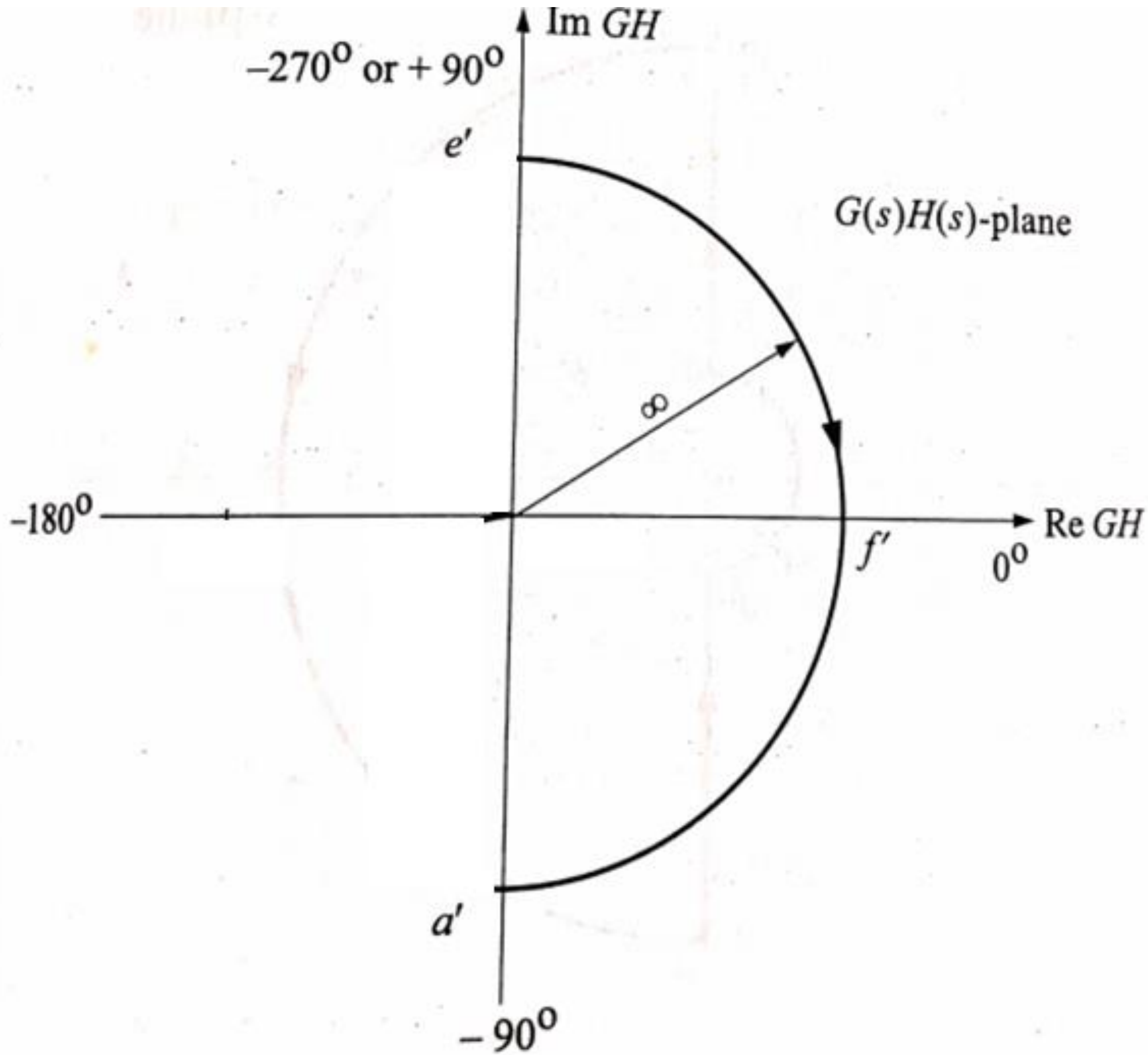
$$= \lim_{r \to 0} \frac{k}{(re^{j\emptyset})[(r^2 e^{j2\emptyset}) + 2re^{j\emptyset} + 2]}$$

$$= \lim_{r \to 0} \frac{k}{(re^{j\emptyset})}$$

$$= \infty\angle{-\emptyset}$$

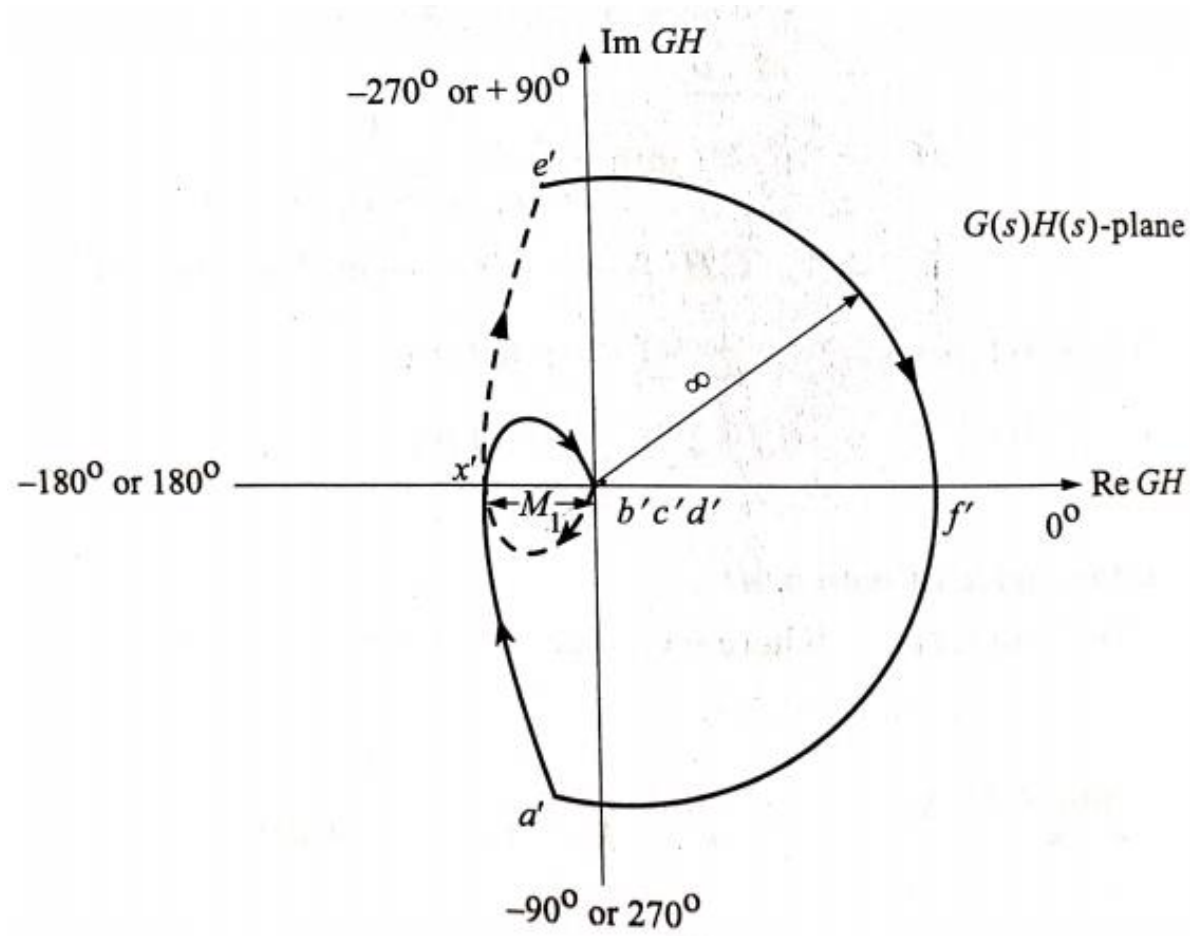
$$= \infty\angle 90 \longrightarrow 0 \longrightarrow -90$$

$\uparrow$e’ $\uparrow$f’ $\uparrow$a’



Im GH
-270° or $+90^{\circ}$
e'
$G(s)H(s)$-plane
∞
-180°
Re GH
f'
0°
a'
-90°

The complete Nyquist plot is shown



To find M_1 :

At point x', phase = -180

$$\Longrightarrow -90 - \tan^{-1}\left(\frac{2\omega}{2-\omega^2}\right) = -180$$

$$\tan^{-1} \frac{2\omega}{2-\omega^2} = -90$$

$$2 - \omega^2 = 0$$

$$\omega = \sqrt{2} \text{ rad/sec}$$

$$M_1 = |GH(j\omega)|_{\omega = \sqrt{2}}$$

$$= \frac{k}{\omega\sqrt{(2\omega)^2+(2-\omega)^2}} = \frac{k}{4}$$

Since P is zero , N must be zero for Z to be zero.

N will be zero if and only if -1+j0 is not encircled by the Nyquist plot

For N to be zero, $M_1 < 1$

Hence, $\frac{k}{4} < 1$

$\Longrightarrow k < 4$

Since k is always positive, for closed-loop stability : $0 < k < 4$