mined by equating the coeffi...
of (18) i.e.,

$$\frac{d^n}{dx^n}(x^2-1)^n = \frac{d^n}{dx^n}v = U = C \cdot P_n(x) \qquad (18)$$

The coefficient of x^n in $P_n(x)$ is $\frac{(2n)!}{2^n(n!)^2}$ (obtained by putting $m = 0$ in (15)).

The coefficient of x^n in L.H.S. of (18) arises solely from the n-fold differentiation of the term of highest degree i.e., x^{2n}

$$2n(2n-1)(2n-2)\cdots(2n-(n-1)).$$
$$= (2n)(2n-1)(2n-2)\cdots(n+1)\cdot\frac{n!}{n!}$$
$$= \frac{(2n)!}{n!}$$

Thus $$\frac{(2n)!}{n!} = C\cdot\frac{(2n)!}{2^n(n!)^2}$$

or $$C = 2^n \cdot n! \qquad (19)$$

Putting C from (19) in (18), we get the Rodrigue's formula

$$P_n(x) = \frac{1}{C}U = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}\left\{(x^2-1)^n\right\}$$

Generating Function for Legendre Polynomials

Prove that

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n \cdot P_n(x) \qquad (20)$$

Proof:

$$(1-y)^{-n} = 1 + ny + \frac{n(n+1)}{1\cdot 2}y^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}y^3 + \cdots$$

$$(1-y)^{-\frac{1}{2}} = 1 + \frac{1}{2}y + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}y^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}y^3 + \cdots$$

$$(1-y)^{-\frac{1}{2}} = 1 + \frac{1}{2}y + \frac{1\cdot 3}{2\cdot 4}y^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}y^3 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot y^n + \cdots$$

$$2\cdot 4\cdot 6\cdots 2n$$
$$= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{2\cdot 4\cdots 2n}{2\cdot 4\cdots 2n}$$
$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)\cdot 2n}{2^n\cdot n!\,2^n\cdot n!} = \frac{(2n)!}{2^{2n}(n!)^2}$$

Now using this result, expand

$$(1-2xt+t^2)^{-\frac{1}{2}}$$
$$= \left[1-(2xt-t^2)\right]^{-\frac{1}{2}} = \left[1-\left\{t(2x-t)\right\}\right]^{-\frac{1}{2}}$$
$$= 1 + \frac{2!}{2^2(1!)^2}t\cdot(2x-t) + \frac{4!}{2^4(2!)^2}t^2(2x-t)^2 + \cdots$$
$$+ \frac{(2(n-k))!}{2^{2(n-k)}\left\{(n-k)!\right\}^2}t^{n-k}(2x-t)^{n-k} + \cdots$$
$$+ \frac{(2n)!}{2^{2n}\cdot(n!)^2}t^n(2x-t)^n + \cdots \qquad (21)$$

Coefficients of t^n appear only in the first $(n+1)$ terms. Consider the $(n-k)$th term: t^n arises as product of t^{n-k} and t^k arising out of $(2x-t)^{n-k}$. Thus the coefficient of t^n in $t^{n-k}\cdot(2x-t)^{n-k}$ is the coefficient of t^k in $(2x-t)^{n-k}$

i.e., $$(n-k)_{C_k}(2x)^{(n-k)-k}\cdot(-1)^k$$
$$= \frac{(n-k)!(-1)^k}{k!(n-2k)!}\cdot(2x)^{n-2k} \qquad (22)$$

Therefore the coefficient of t^n is (see 21)

$$\left[\frac{(2n-2k)!}{2^{2n-2k}\{(n-k)!\}^2}\right]\cdot\left[\frac{(n-k)!}{k!(n-2k)!}(-1)^k(2x)^{n-2k}\right]$$
$$= \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}\cdot x^{n-2k}$$

collecting and summing up for k all the coefficients of t^n from the first $(n+1)$ terms, we get

$$\sum_{k=0}^{M}\frac{(-1)^k(2n-2k)!}{2^n(n-k)!k!(n-2k)!}\cdot x^{n-2k} = P_n(x)$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd.

Thus the Legendre polynomials $P_0(x)$, $P_1(x)$, $P_2(x)\cdots P_n(x)\cdots$ appear as coefficients of t^0, t^1, $t^2,\ldots t^n\ldots$ etc. in the expansion of $(1-2xt+t^2)^{-\frac{1}{2}}$. Hence $(1-2xt+t^2)^{-\frac{1}{2}}$ is the generating function of the Legendre polynomials i.e.,

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}P_n(x)\cdot t^n \qquad (20)$$

Result 1: $P_n(1) = 1$ for any n.
Put $x = 1$ in (20). Then

$$\sum_{n=0}^{\infty}P_n(1)t^n = (1-2t+t^2)^{-\frac{1}{2}}$$
$$= \left((1-t)^2\right)^{-\frac{1}{2}} = (1-t)^{-1}$$
$$= 1 + t + t^2 + \cdots + t^n + \cdots$$

Equating the coefficients of t^n on both sides

$$P_n(1) = 1 \qquad \text{for any } n.$$

Result 2: $P_n(-1) = (-1)^n$ for any n.
Put $x = -1$ in (20). Then

$$\sum_{n=0}^{\infty}P_n(-1)t^n = (1+2t+t^2)^{-\frac{1}{2}}$$
$$= \left[(1+t)^2\right]^{-\frac{1}{2}} = (1+t)^{-1}$$
$$= 1 - t + t^2 - \cdots + (-1)^n t^n + \cdots$$

Equating the coefficients of t^n, $P_n(-1) = (-1)^n$.

Result 3:

$$P_n(0) = \begin{cases} 0, & \text{when } n \text{ is odd} \\ (-1)^{\frac{n}{2}}\cdot\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & \text{if } n \text{ is even} \end{cases}$$

Put $x = 0$ in (20). Then

$$\sum_{n=0}^{\infty}P_n(0)t^n = (1+t^2)^{-\frac{1}{2}}$$
$$= 1 - \frac{1}{2}t^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)}{1\cdot 2}t^4 + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{1\cdot 2\cdot 3\cdots n}t^{2n} + \cdots$$
$$= 1 - \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \cdots + \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot t^{2n} + \cdots$$