B. M. S. COLLEGE OF ENGINEERING, BANGALORE-560 019
DEPARTMENT OF MATHEMATICS

Fourth Semester B.E. Course-(AS/ME/EEE/ECE/ET/ML/CIVIL/EIE)
Course Title: Complex Analysis, Probability and Statistical Methods
Course Code: 22MA4BSCPS

UNIT1: COMPLEX ANALYSIS

1. Is the function $u(x,y)=2xy+3xy^2-2y^3$ harmonic. **Ans**: u is not harmonic
2. If $f'(z)=0$ then show that $f(z)$ is constant.
3. If $f(z)$ is an analytic function with constant modulus show that $f(z)$ is constant.
4. If $f(z)$ is a holomorphic function of z, show that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2=|f'(z)|^2$
5. If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$
6. If $f(z)$ is an analytic function of z, prove that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|R\,f(z)|^2=2|f'(z)|^2$
7. Determine the analytic function (or imaginary part of the function) whose real part is
   a) $\log\sqrt{(x^2+y^2)}$ **Ans**: $\log z$
   b) $\frac{y}{(x^2+y^2)}$ **Ans**: $\frac{i}{z}$
   c) $u=e^{2x}(x\cos 2y-y\sin 2y)$ **Ans:** $ze^{2z}+ic$
   d) $y+e^x\cos y$ **Ans**: e^2-iz
   e) $x\sin x\cos h\,y-y\cos x\sin h\,y$ **Ans**: $z\sin z$.
8. Find the regular function (or the real part of the function) whose imaginary part is
   a) $\frac{(x-y)}{(x^2+y^2)}$ **Ans**: $\frac{(1+i)}{z}+c$
   b) $e^x\sin y$ **Ans**: e^z
   c) $-\sin x\sinh y$ **Ans**: $\cos z+c$
   d) $e^{-x}(x\sin y-y\cos y)$ **Ans**: $\bar{z}\,e^{-\bar{z}}+c$
   e) $e^{-x}(x\cos y+y\sin y)$ **Ans**: $1+ize^{-2}$
9. If $f(z)=u+iv$ is an analytic function of z, find $f(z)$ if
   a) $u+v=\sin x\cosh y+\cos x\sinh y$ **Ans**: $\sin z+c/(1+i)$
   b) $2u+v=e^x(\cos y-\sin y)$ **Ans**: $\frac{(1+3i)}{5}e^z+c$
   c) $u-v=2xy+x^2-y^2+x-y$ **Ans**: $z-iz^2+c$

10. If $\varphi + i\psi$ represents the complex potential of an electrostatic field where $\psi = (x^2 - y^2) + \frac{x}{x^2 + y^2}$, find the complex potential as a function of the complex variable z and hence determine φ.

Ans: $-2xy + \frac{y}{x^2 + y^2} + c$

11. Find analytic function $f(z) = u(r, \theta) + i\,v(r, \theta)$ such that $v(r, \theta) = r^2 \cos 2\theta - r\cos\theta + 2$.

Ans: $u = -r^2 \sin 2\theta + r\sin\theta + c,\ \ f(z) = i\left(r^2 e^{2i\theta} - re^{i\theta}\right) + c + 2i$

12. Find the analytic function $f(z) = u + iv$, given $v = (r - 1/r)\sin\theta,\ r \neq 0$.

Ans: $\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta + c$.

13. Verify that the given function is harmonic and find its harmonic conjugate. Express $u + iv$ as analytic function $f(z)$:

a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. **Ans:** $v = 3x^2y - y^3 + 6xy$.

b) $u = x^2 - y^2 - y$ **Ans**: $v = 2xy + x + y,\ f(z) = z^2 + iz + c$

c) $v = x^2 - y^2 + \frac{x}{x^2 + y^2}$ **Ans**: $u = -2xy + \frac{y}{x^2 + y^2} + c,\ w = i\left(z^2 + \frac{1}{2}\right) + c$

d) $u = 3xy^2 - x^3$ **Ans**: $v = y^3 - 3x^2y + c,\ f(z) = -z^3 + ic$

e) $v = y^2 - x^2$ **Ans**: $u = 2xy + c,\ f(z) - iz^2 + c$

f) $u = e^{-x}(x\sin y - y\cos y)$ **Ans**: $v = e^{-x}(y\sin y + x\cos y) + c,\ f(z) = iz\,e^{-z}$

g) $u = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$ **Ans**: $v = \frac{-2xy}{\left(x^2 + y^2\right)^2},\ f(z) = \frac{1}{z^2} + c$

h) $u = 3x^3y + 2x^2 - y^3 - 2y^2$ **Ans**: not harmonic

i) $u = -e^{-2xy}\sin(x^2 - y^2)$ **Ans**: $v = -e^{-2xy}\cos(x^2 - y^2) + c,\ f(z) = -i\,e^{iz^2} + ci$.

j) $u(r, \theta) = -r^3 \sin 3\theta$. **Ans**: $v = r^3 \cos 3\theta + c$

14. Show that $U(x, y) = e^u \cos v,\ V(x, y) = e^u \sin v$ are harmonic conjugate of each other if $f(z) = u + iv$ is analytic.

15. Find the orthogonal trajectories of the family of curves $x^3y - xy^3 = c = \text{constant}$.

Ans: $x^4 + y^4 - 6x^2y^2 = \text{constant}$

16. Find the orthogonal trajectories of the family of curves $r^2 \cos 2\theta = c_1$. **Ans**: $v = r^2 \sin 2\theta$

17. Discuss the transformation $w = z^2$.

18. Discuss the transformation

$$w = z + \frac{k^2}{z}, \qquad (z \neq 0).$$

Complex Integration:

1. Evaluate $\int_0^{1+i}\left(x^2+iy\right)dz$ along the paths $y=x$ and $y=x^2$.
2. Evaluate $\int_0^{2+i}\left(\bar{z}\right)^2 dz$ along

   i) the line $y=\dfrac{x}{2}$. ii) the real axis to 2 and then vertically to $2+i$.
3. Evaluate $\int_0 |z|^2\, dz$ over the curve made up of the vertices (0, 0), (1,0) (1,1) and (0,1).
4. Show that $\int_c (z-a)^n\, dz = \begin{cases} 0 : n\neq -1 \\ 2\pi i : \text{if } n=-1 \end{cases}$ C is the circle $|z-a|=r$.

Cauchy's Theorem:

1. Verify the Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle having vertices $-1,\ 1,\ 1+i,\ -1+i$.
2. Verify the Cauchy's theorem for the integral of $1/z$ taken over the boundary of the triangle having vertices (1, 2), (1, 4) & (3, 2).
3. Verify the Cauchy's theorem for the function $f(z)=z^2$ over the boundary of square having vertices (0,0), (1,0), (1,1) and (0,1).
4. Show that $\int_c |z|^2\, dz = i-1$ where C the square having vertices (0, 0), (1, 0), (1, 1) & (0, 1). Give the reason for Cauchy's theorem not being satisfied.
5. Verify the Cauchy's for the function $f(z)=ze^{-z}$ over the unit circle with origin as the center.

Cauchy's Integral formula and Generalized Cauchy's Integral formula:

1. Evaluate $\int_C \dfrac{z^2-z+1}{z-1}$ where C is the circle $(i)\ |z|=1$ $(ii)\ |z|=\dfrac{1}{2}$
2. Evaluate $\int_C \dfrac{cos(\pi z)}{z^2-1}dz$ around a rectangle with vertices $2\pm i, -2\pm i$
3. Evaluate

   a. $\int_C \dfrac{e^{2z}dz}{(z-1)(z-2)}$ where $C: |z|=3$

   b. $\int_C \dfrac{sin^2 z}{\left(z-\pi/6\right)^3}dz$ where C: $|z|=1$

   c. $\int_C \dfrac{e^{2z}}{(z+1)^4}dz$ where C: $|z|=2$.

   d. $\int_C \dfrac{e^z}{\left(z^2+\pi^2\right)^2}dz$ where C: $|z|=4$.

   e. $\oint_C \dfrac{z^3-2z+1}{(z-i)^2}dz$ where C: $|z|=2$.

   j. $\oint_C \dfrac{z}{z^2-3z+2}dz$ where C: $|z-2|=\dfrac{1}{2}$.

   k. $\oint_C \dfrac{e^z dz}{(z+1)^2}$ where C: $|z-1|=3$.

   l. $\oint_C \dfrac{log\, z}{(z-1)^3}dz$ where C: $|z-1|=1/2$

   m. $\oint_C \dfrac{z+4}{z^2+2z+5}dz$ where $C: |z+1-i|=2$.

   n. $\int_C \dfrac{e^{3z}}{z^2}dz$ where $C: |z|=1$.

f. $\oint_C \frac{e^{-z}}{(z-1)(z-2)^2} dz$ where C: $|z| = 3$.

g. $\int_C \frac{3z^2+7z+1}{z+1} dz$ where C: $|z| = \frac{1}{2}$.

h. $\int_C \frac{2z^2+1}{z^2+z} dz$ where C: $|z| = \frac{1}{2}$.

i. $\int_C \frac{z^2+1}{z^2(2z+1)} dz$ where C: $|z| = 1$.

o. $\int_C \frac{z^2+z+1}{(z-2)^3} dz$ where $C: |z| = 3$.

p. $\int_C \frac{e^{\pi z}}{(2z-i)^3} dz$ where $C: |z| = 1$.

q. $\int_C \frac{dz}{(z^2+4)^2}$ where $C: |z-i| = 2$.

r. $\int_C \frac{e^{2z}}{(z+1)^2(z-2)} dz$ where $C: |z| = 3$.

4. Evaluate $\int_C \frac{dz}{z^2-4}$ over the circle i) $|z|=1$ ii) $|z|=3$ iii) $|z+2|=1$.

5. Evaluate $\int_C \frac{e^z}{z+i\pi} dz$ over the circle i) $|z| = 2\pi$ ii) $|z| = \frac{\pi}{2}$.

6. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ over the circle i) $|z|=3$ ii) $|z| = \frac{1}{2}$ iii) $|z+2|=3$.

7. Evaluate $\int_C \frac{z^2+1}{z^2-1} dz$ over the C i) $|z+1|=1$ ii) $|z-1|=1$.