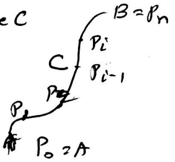
UNIT-5

COMPLEX ANALYSIS -2

Introduction

Line integral:- If $f(z)$ is a continuous function defined at all points on the curve C between A and B, dividing the curve C into n parts such that



$A=P_0, P_1, P_2, \dots P_n=B$. Let $\delta z_i = z_i - z_{i-1}$

and ξ_i is any point on the arc $P_{i-1}P_i$

then the integral of $f(z)$ along c is called as the line integral of $f(z)$ given by

$$\int_C f(z)dz = \underset{\substack{n\to\infty \\ \delta z_i \to 0}}{Lt} \left[\sum_{i=1}^{n} f(\xi_i)\delta z_i\right]$$

Remarks ① If c is a closed curve then the line integral of $f(z)$ is denoted by $\oint_C f(z)dz$

② The line integral of a the complex valued function is expressed as the sum of the line integrals of real valued functions.

i.e, $\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

Properties of line integral

(1) $\oint_C [f(z) \pm g(z)]\,dz = \oint_C f(z)\,dz \pm \oint_C g(z)\,dz$

(2) $\oint_C k\, f(z)\,dz = k \oint_C f(z)\,dz$

(3) If c_1 and c_2 are two parts of a curve C

then $\oint_C f(z)\,dz = \oint_{c_1+c_2} f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$

(4) $\oint_C f(z)\,dz = -\oint_{-C} f(z)\,dz$

PART A: Cauchy's theorem (or Cauchy's integral theorem)

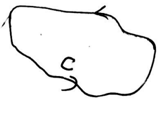
Statement: If $f(z)$ is analytic function and $f'(z)$ is continuous inside and on the boundary of simple closed curve C then

$$\oint_C f(z)\,dz = 0$$

Proof: * Given $f(z) = u + iv$ is analytic function

∴ CR equations are satisfied by u and v

$u_x = v_y$ and $v_x = -u_y$



* $\oint_C f(z)\,dz = \oint_C (u+iv)(dx + i\,dy)$

$$= \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy)$$ 3.

* By Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

* $\therefore$

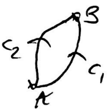
$$\oint_C f(z)\,dz = \iint_R \left[\frac{\partial}{\partial x}(-v) - \frac{\partial}{\partial y}(u) \right] dx\,dy + i \iint_R \left[\frac{\partial}{\partial x}(u) - \frac{\partial}{\partial y}(v) \right] dx\,dy$$

$$= \iint_R (0)\,dx\,dy + i \iint_R (0)\,dx\,dy$$

using CR equations

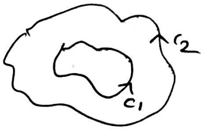
$\therefore$ $\oint_C f(z)\,dz = 0.$

Cor 1: If A and B are any two points then $\int_A^B f(z)\,dz$ is independent of the path joining A and B



$$\cancel{\oint} \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Cor 2: If $f(z)$ is analytic inside and on the boundary of the region between two curves C_1 and C_2 (C_1 contained in C_2) then i.e., $\oint_{C} f(z)\,dz = \oint_{C_2} f(z)\,dz$



Cor 3: If $C_1, C_2, \dots C_n$ are closed curves inside a closed curve C and $f(z)$ is analytic in the region between C and C_i, $i = 1, 2, \dots n$

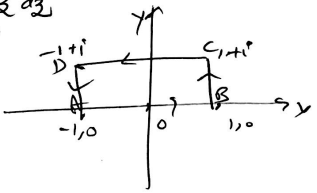
then $$\oint_{C} f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \dots + \oint_{C_n} f(z)\,dz$$

Examples: $\boxed{E1}$ Verify Cauchy's theorem for $f(z) = z^3$ taken over the following paths

(a) rectangle with vertices $z = -1, 1, 1+i, -1+i$

(b) triangle with vertices $(1,2)$, $(3,2)$ and $(1,4)$

Solution: (a) To find $\oint_{ABCD} z^3\,dz$



$$\oint_{ABCD} z^3\,dz = \int_{AB} z^3\,dz + \int_{BC} z^3\,dz + \int_{CD} z^3\,dz + \int_{DA} z^3\,dz.$$

Along AB: $y = 0$, $-1 \le x \le 1$, $dy = 0$

Along BC: $x = 1$, $dx = 0$, $y \to 0$ to 1

Along CD: $y=1,\ dy=0$, $1 \le x \le -1$,

Along DA: $x=-1,\ dx=0$, $1 \le y \le 0$

$\therefore$ LHS $= \int_{-1}^{1} x^3\,dx + \int_{0}^{1} (1+iy)^3\, i\,dy + \int_{1}^{-1} (x+i)^3\,dx$

$+ \int_{1}^{0} (-1+iy)^3\, i\,dy$

$\int_{-1}^{1} x^3\,dx = 0$ $\quad \because$ Integrand is odd function

$\int_{0}^{1} (1+iy)^3\, i\,dy = \frac{(1+iy)^4}{4\times i}\, i\,dy \Big|_0^1 = \frac{1}{4}\left[(1+i)^4 - 1\right]$

$\int_{1}^{-1} (x+i)^3\,dx = \frac{(x+i)^4}{4} \Big|_1^{-1} = \frac{1}{4}\left[(-1+i)^4 - (1+i)^4\right]$

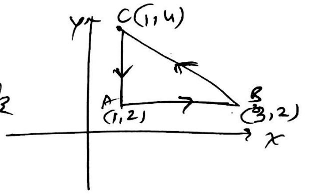
$\int_{1}^{0} (-1+iy)^3\, i\,dy = i\,\frac{(-1+iy)^4}{4i} \Big|_1^0 = \frac{1}{4}\left[(-1)^4 - (-1+i)^4\right]$

LHS $= \oint_C f(z)\,dz = \oint_C z^3\,dz = 0$

(b) Solution $\quad f(z) = \frac{1}{z}$ is analytic except at $z=0$ and $z=0$ is outside the given triangle

$\therefore$ TST $I = \oint_C f(z)\,dz = 0$

$I = \int_{AB} z^3\,dz + \int_{BC} z^3\,dz + \int_{CA} z^3\,dz$



Along AB: $y = 2$, $dy = 0$, $x \to 1$ to 3

Along BC: $y - 4 = -1(x-1)$ or $y - 4 = -x + 1$

$\therefore y = 5 - x$

$x \to 3$ to 1

Along CA: $x = 1$, $dx = 0$, $y \to 4$ to 2

$$\int_{AB} z^3 dz = \int_1^3 (x+2i)^3 dx = \left.\frac{(x+2i)^4}{4}\right|_1^3$$

$$= \frac{(3+2i)^4}{4} - \frac{(1+2i)^4}{4} \quad -(1)$$

$$\int_{BC} z^3 dz = \int_3^1 [x + i(5-x)]^3 (dx - i\,dx)$$

$$= (1-i)\int_3^1 [x(1-i) + 5i]^3 dx$$

$$= (1-i) \left.\frac{(x(1-i)+5i)^4}{4(1-i)}\right|_3^1$$

$$= -\frac{(3+2i)^4}{4} + \frac{(1+4i)^4}{4} \quad -(2)$$

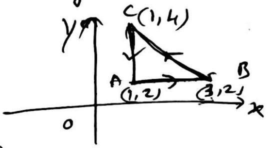
$$\int_{CA} z^3 dz = \int_4^2 (1+iy)^3 i\,dy = i\left.\frac{(1+iy)^4}{4i}\right|_4^2 = \frac{(1+2i)^4}{4} - \frac{(1+4i)^4}{4} \quad -(3)$$

$$\oint_{ABC} z^3 dz = 0 \quad \text{using (1), (2) and (3)}$$

$\boxed{E\,2}$ Verify Cauchy's theorem for the integral of $\frac{1}{z}$ taken over the triangle with vertices $(1,2)$, $(3,2)$, $(1,4)$

Solution $f(z) = \frac{1}{z}$ is analytic inside and on the boundary of the triangle

$\therefore \oint_{ABC} f(z)dz = 0$



$$\oint_{ABC} f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CA} f(z)dz.$$

Along AB:- $y = 2$, $1 \le x \le 3$

$$\int_{AB} f(z)dz = \int_1^3 \frac{dz}{z} = \int_1^3 \frac{dx}{x+2i} = \log(x+2i)\Big|_1^3$$

$$= \log_e\left(\frac{3+2i}{1+2i}\right) \quad (1)$$

Equation of BC is $y - 2 = \frac{2}{-2}(x-3)$

or $y - 2 = -x+3 \Rightarrow y = 5-x$

$dy = -dx$

$x \to 3$ to 1

$$\int_{BC} f(z)dz = \int_3^1 \frac{dx - i\,dx}{x + i(5-x)} = (1-i)\int_3^1 \frac{dx}{x(1-i)+5i}$$

$$= (1-i)\,\frac{\log\left[(1-i)x + 5i\right]}{1-i}\Bigg|_3^1$$

$$= \log_e\left(\frac{1+4i}{3+2i}\right) \quad (2)$$

Along CA $\quad x=1 \; dx=0; \; 4 \le y \le 2$

$$\int_{CA} \frac{dz}{z} = \int_4^2 \frac{i\,dy}{1+iy} = i\,\frac{\log(1+iy)}{i}\Big|_4^2$$

$$= \log(1+2i) - \log(1+4i) \quad -③$$

$$\oint_{ABC} \frac{dz}{z} = \log(3+2i) - \log(1+2i) + \log(1+4i) - \log(3+2i) + \log(1+2i) - \log(1+4i)$$

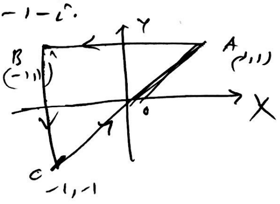
from ①, ② and ③

$$= 0$$

③ Verify cauchy's theorem for $f(z) = e^{iz}$ along the boundary of the triangle with vertices $1+i$, $-1+i$ and $-1-i$.

Solution

$$\oint_{ABC} f(z) = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz$$



Along AB :- $\quad y=1 \; dy=0 \quad x \to 1$ to -1

$$\int_{AB} e^{iz}\,dz = \int_1^{-1} e^{i(x+i)}\,dx = \int_{-1}^{1} e^{ix-1}\,dx$$

$$= \frac{e^{-i-1} - e^{i-1}}{i} \quad -①$$

Along BC : $\quad x=-1, \; dx=0, \quad y \to 1$ to -1

$$\int_{BC} e^{iz}\,dz = \int_1^{-1} e^{i[-1+iy]}\, i\,dy$$

$$= i\int_{1}^{-1} e^{-i-y} = i \left.\frac{e^{-i-y}}{-1}\right]_{1}^{-1} = \frac{e^{-i+1} - e^{-i-1}}{i} \quad \text{—(2)}$$

Along CA: Equation of CA is $y = x$, $dy = dx$

$x \to -1$ to 1

$$\int_{CA} e^{iz} dz = \int_{-1}^{1} e^{i(1+i)x} (1+i) dx$$

$$= (1+i) \left.\frac{e^{(i-1)x}}{i(1+i)}\right|_{-1}^{1} = \frac{e^{i-1} - e^{-i+1}}{i} \quad \text{—(3)}$$

From (1), (2) and (3)

$$\oint_{ABC} e^{iz} dz = 0$$

$\boxed{E4}$ Verify Cauchy's theorm for $f(z) = z^2$ over the square with vertices $-1 \pm i$ and $1 \pm i$.

<u>Assignment</u>