UNIT-4

COMPLEX ANALYSIS-1

Introduction

① **Function of a complex variable:**—

If for every value of $z = x+iy$ there corresponds one or more values of w, then w is said to be a function of a complex variable z denoted by $w = f(z)$.

Eg: $\tan z$, $\sinh z$ etc.

(a) **Single valued function:**-

If for every value of z, there corresponds only one value for w, then w is said to be a single valued function of z.

Eg: $w = z^2$, $w = \cos^{-1} z$, $\log z$ etc

(b) **Multivalued function:**-

If for each value of z, there corresponds more than one value of z, then w is called as a multivalued ~~fn~~ function of z.

Eg: $w = \sqrt{z}$, $\sqrt[3]{z}$ etc.

② **Limit:**- A single valued function $f(z)$ is said to tend to a limit l, if for every $\epsilon > 0$ (however small), there corresponds $\delta > 0$, such that $|f(z) - l| < \epsilon$, whenever $|z - z_0| < \delta$

denoted by $\underset{z \to z_0}{Lt} [f(z)] = l$.

Note:- The limit must exist for every path as $z \to z_0$. $\underset{z \to 0}{Lt} \left(\frac{\bar{z}}{z}\right)$ does not exist.

(3) **Continuity:-** A function $f(z)$ is said to be continuous at a point $z = z_0$ if

$$\underset{z \to z_0}{Lt} f(z) = f(z_0)$$

(4) **Derivative:-** A function $f(z)$ is said to be differentiable if the derivative of $f(z)$ i.e., $\frac{d}{dz}[f(z)]$ give by $\frac{dw}{dz} = \underset{\delta z \to 0}{Lt} \left[\frac{f(z+\delta z) - f(z)}{\delta z}\right]$ exists. i.e., limit on RHS is unique irrespective of the path along which $\delta z \to 0$.

(5) **Analytic function:-** (or regular function or holomorphic function).

A function $f(z)$ is said to be analytic at a point $z = a$ if it is differentiable at $z = a$ and also in the neighbourhood of $z = a$, i.e., in the open disk $|z| \leq a$.

Note:- $|z - z_0| = R$ represents the circle with centre at z_0 and radius R.

(6) Singular point :- If a function fails (or ceases) to be analytic at a point $z = a$, then $z = a$ is called singularity of $f(z)$.

PART A :-

[D1] Derivation of Cauchy-Riemann equations in cartesian form. or to obtain CR equations in cartesian form. [State and Prove CR equations in Cartesian form]

Proof :- Given that $w = f(z) = u(x,y) + iv(x,y)$

Statement: is an analytic function to show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad [\text{or } u_x = v_y \text{ and } v_x = -u_y]$$

$f(z) = u + iv$ i.e., $f(x+iy) = u(x,y) + i\,v(x,y)$ — (1)

Differentiating (1) partially w.r.to x

$$f'(z)(1) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Differentiating (1) partially w.r.t. y

$$f'(z)(i) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$\therefore f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad (3)$$

Since $f(z)$ is analytic function, $f'(z)$ exists and unique

$\therefore$ From (2) and (3)

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Note: CR equations are necessary conditions for $f(z)$ to be analytic function.

$\boxed{D2}$ To show that real and imaginary parts of an analytic function

$f(z) = u(x,y) + iv(x,y)$ [cartesian form]

are harmonic functions:

Note: u and v are said to be harmonic conjugates of each other)

Proof:- To prove that $u(x,y)$ is harmonic function and $v(x,y)$ is harmonic functions.

i.e. to show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\nabla^2 u = 0$

and $\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial y} = 0$ & $\nabla^2 v = 0$

Since $f(z)$ is analytic function, $u(x,y)$ and $v(x,y)$ satisfy CR equations

i.e., $u_x = v_y$ —(1)

and $v_x = -u_y$ —(2)

Differentiating ① partially w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$$

by CR equations

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

To prove that $V(x,y)$ is harmonic function

Differentiating ② partially w.r.to ~~y~~ x

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \text{ using ②}$$

$$= -\frac{\partial^2 v}{\partial y^2}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2} \quad \text{or} \quad V_{xx} + V_{yy} = 0$$

$\boxed{Q_3}$ To show that real and imaginary parts of an analytic function $f(z) = U(x,y) + iV(x,y)$, equated to constants, form orthogonal trajectories of each other.

Proof:- To prove that $U(x,y) = c_1$ and $V(x,y) = c_2$ are orthogonal trajectories.

i.e., to prove that $m_1 m_2 = -1$

$u(x,y) = c_1$

$du = 0$

$\Rightarrow du = \boxed{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0}$

$\therefore m_1 = \frac{dy}{dx} = \frac{-u_x}{V_y}$

$V(x,y) = c_2$

$dV = 0$

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$\therefore \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$

$\Rightarrow m_2 = \frac{dy}{dx} = -\frac{V_x}{V_y}$

$m_1 m_2 = \left(-\frac{U_x}{V_y}\right)\left(-\frac{V_x}{V_y}\right)$

$= \left(\frac{-\cancel{V_y}}{\cancel{V_y}}\right) - \frac{\cancel{(-U_y)}}{\cancel{V_y}}$

$\because U_x = V_y$ and $V_x = -U_y$, by CR equations

[By data, $f(z) = u + iv$ is analytic function]

$m_1 \times m_2 = -1$

$\boxed{D4}$ To obtain CR equations in polar form

State and prove CR equations in polar form

Statement: If $f(z) = U(r,\theta) + iV(r,\theta)$ is an analytic function then to prove that $U_r = \frac{1}{r}V_\theta$ and $V_r = -\frac{1}{r}U_\theta$.

Proof: $f(z) = U(r,\theta) + iV(r,\theta)$

Let $z = re^{i\theta}$

$\therefore f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ — ①

Differentiating ① partially w.r.to r

$$f'(re^{i\theta})e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{— ②}$$

Differentiating ① partially w.r.to θ

$$f'(re^{i\theta})ire^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\therefore f'(re^{i\theta})e^{i\theta} = \frac{1}{ir}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{r}\left[\frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{r}\left[-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right] \quad \text{③}$$

Since $f(z)$ analytic function, $f'(z)$ exists and unique $\therefore$ From ② and ③

$$u_r + iv_r = \frac{1}{r}\left[-iu_\theta + v_\theta\right]$$

$$\Rightarrow \boxed{u_r = \frac{1}{r}v_\theta \quad \text{and} \quad v_r = -\frac{1}{r}u_\theta}$$

equating real and imaginary parts.

Q5 To show that real and imaginary parts of an analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ are harmonic functions, [hence they are called as harmonic conjugates of each other].

Proof: Given $f(z) = u(r,\theta) + i\,v(r,\theta)$ is analytic function, i.e., CR equations are satisfied by $u(r,\theta)$ and $v(r,\theta)$

$\therefore\ u_r = \frac{1}{r} v_\theta \ \rightarrow (1)$ and $v_r = -\frac{1}{r} u_\theta \ \rightarrow (2)$

TPT $\nabla^2 u = 0$ and $\nabla^2 v = 0$

i.e., TST $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \ \rightarrow (3)$

~~here~~ [here $\nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ is Laplacian operator in polar form]

Differentiating (1) partially w.r.to r

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) = \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial v}{\partial \theta}$$

(Using product rule)

From (2) $\rightarrow u_\theta = -r v_r$

Differentiating both the sides partially w.r.to θ

$$u_{\theta\theta} = -r v_{\theta r}$$

(3) becomes $\nabla^2 u = \cancel{\frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial\theta}} - \cancel{\frac{1}{r^2}\frac{\partial v}{\partial\theta}} + \cancel{\frac{1}{r}\left(\frac{1}{r} v_\theta\right)} + \frac{1}{r^2}\left(-r v_{\theta r}\right)$

$= 0.$

TST $\nabla^2 V = 0$ — 9 ÷

Differentiating ② partially w.r.to r (to find V_{rr})

$$\frac{\partial^2 V}{\partial r^2} = -\left[\frac{1}{r} u_{r\theta} + u_\theta \left(\frac{-1}{r^2}\right)\right]$$

Differentiating ① partially w.r.to. θ (to find $V_{\theta\theta}$)

$$u_{\theta r} = u_{r\theta} \quad \& \quad V_\theta = r u_r$$

$$V_{\theta\theta} = \frac{\partial}{\partial\theta}(r u_r) = r u_{\theta r}$$

$$\nabla^2 V = -\frac{1}{r} u_{r\theta} + \frac{1}{r^2} u_\theta + \frac{1}{r}\left[-\frac{1}{r} u_\theta\right] + \frac{1}{r^2}\left[r u_{\theta r}\right] = 0$$

$\boxed{D6}$ To prove that real and imaginary parts of an analytic function $f(z) = U(r,\theta) + i\,V(r,\theta)$ equated to constants are orthogonal trajectories of each other.

Proof:- To show that $u(r,\theta) = c_1$ and $V(r,\theta) = c_2$ are orthogonal families of each other.

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$u(r,\theta) = c_1$

$\therefore du = 0$

$du = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta$

$\Rightarrow \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta = 0$

$\therefore m_1 = \frac{dr}{d\theta} = -\frac{u_\theta}{u_r}$

$v(r,\theta) = c_2$

$\Rightarrow dv = 0$

$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$

$\therefore \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta = 0$

& $m_2 = \frac{dr}{d\theta} = -\frac{V_\theta}{V_r}$

$$m_1 \times m_2 = -\frac{u_\theta}{u_r} \times -\frac{V_\theta}{V_r}$$

$$= \frac{-u_\theta}{\frac{1}{r}u_\theta} \times -\left(\frac{V_\theta}{-\frac{1}{r}u_\theta}\right)$$

using CR equations,
Since $f(z) = u + iv$ is analytic
by data

$$m_1 \times m_2 = -r^2$$