classmate
Date
Page

UNIT - 2
Joint Probability Distribution

Let X and Y be a random variables with range set $\{x_1, x_2 \ldots x_n\}$ and $\{y_1, y_2 \ldots y_n\}$.

$P(X,Y)$ is called a joint probability distribution function if it satisfies the following condition:

① $P(x_i, y_j) \geq 0 \quad \forall x_i \; \forall y_j$

② $\sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) = 1$

Here $P(x_i, y_j)$ gives the probability for the simultaneous occurance of the outcomes of x_i and y_j.
The set $[x_i, y_j, P(x_i, y_j)]$ is known as joint probability distribution of 2 random variable x and y.
Generally the distributions are represented in the form of a table known as joint probab table

Y \ X	x_1	x_2	....	x_n	sum
y_1	$P(x_1,y_1)$	$P(x_2,y_1)$	....	$P(x_n y_1)$	$P(y_1)$
y_2	$P(x_1,y_2)$			$P(x_n, y_2)$	$P(y_2)$
:	:				:
ym	$P(x_1,y_m)$			$P(x_n y_m)$	$P(y_m)$
sum	$P(x_1)$	$P(x_2)$	....	$P(x_n)$	1

classmate

Date

Page

→ Marginal distribution of X and Y:
(M.D)

M.D $f(x)$ of X is the probability distribution of X alone obtained by summing $P(X,Y)$ over the values of y.

$$f(x) = \sum P(x_i, y_j) \quad x_i \text{ is fixed}$$

M.D $g(y)$ of Y is the probability distribution of Y alone obtained by summing $P(X,Y)$ over the values of x.

$$g(y) = \sum P(x_i, y_j) \quad y_j \text{ is fixed}$$

→ Mean:

$$E(X) = \sum x_i P(x_i) = \sum x f(x)$$

expectation or mean

$$E(Y) = \sum y_j P(y_j) = \sum y\, g(y)$$

$$E(XY) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j P(x_i, y_j)$$

$$V(X) = \sum (x-\mu)^2 P(x) = \sum x^2 P(x) - \mu^2$$

variance

$$V(X) = E(X^2) - [E(X)]^2$$

$$\text{llly } V(Y) = E(Y^2) - [E(Y)]^2$$

S.D $\sigma = \sqrt{V}$

Covariance $\rho(X,Y) = \dfrac{\text{Cov}(X,Y)}{\sigma_x \sigma_y}$, $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

cov → coefficient of variation.

$$* \quad E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$$

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→ Stochastic Independant :

The two random variables x & y are said to be stochastic independant if

$$P(x,y) = P(x) \cdot P(y)$$

Also, if x & y are independant variables,

→ $COV(X,Y) = 0$

→ $E(XY) = E(X) \cdot E(Y)$

↳ Expection or mean of XY

$E(X)$ → Expectation of X or mean of x

$E(Y)$ → " " Y or " Y

Q1) Find i) marginal distribution of $P(x)$ & $P(y)$
ii) $E(X)$ and $E(Y)$ iii) $COV(X,Y)$
iv) $\rho(x,y)$ for the following joint distribution
v) IS X & Y Independant random variables

i)

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

→ Marginal distribution of X

X	1	5
P(X)	$\frac{1}{8}+\frac{1}{4}+\frac{1}{8}=\frac{1}{2}$	$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$

Sum of $P(X)$ should be 1

Then correct

→ marginal D of Y

Y	-4	2	7
P(Y)	$\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$	$\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$	$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$

$|||^{ly}$ $\Sigma P(Y)=1$

ii) $E(X) = ?$ $E(Y) = ?$

$$E(X) = \sum x_i P(x_i) = x_1 P(x_1) + x_2 P(x_2)$$
$$= \frac{1}{2} \times 1 + 5 \times \frac{1}{2}$$
$$= \frac{6}{2} = \underline{\underline{3}}$$

$$E(Y) = \sum y_i (P(y_i)) = y_1 P(y_1) + y_2 P(y_2) + y_3 P(y_3)$$
$$= \frac{3}{\cancel{8}4} \times -4^{2} + 2 \times \frac{3}{\cancel{8}4} + 7 \times \frac{1}{4}$$
$$= \frac{-6 + 3 + 7}{4} = \frac{4}{4} = \underline{\underline{1}}$$

iii) $\text{COV}(X, Y) = E(XY) - E(X)E(Y)$

$$E(XY) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j P(x_i, y_j)$$

$$= x_1 y_1 P(x_1, y_1) + x_1 y_2 P(x_1, y_2) + x_1 y_3 P(x_1, y_3) + x_2 y_1 P(x_2, y_1) + x_2 y_2 P(x_2, y_2) + x_2 y_3 P(x_2, y_3)$$
$$= 1 \times -4 \times \frac{1}{\cancel{8}4} + 2 \times \frac{1}{4} + 7 \times \frac{1}{8}$$
$$+ 5 \times -4 \times \frac{1}{4} + (5)(2)\frac{1}{\cancel{8}4} + (7)(5)\left(\frac{1}{8}\right)$$

$$= \frac{12}{8} = \frac{3}{2} //$$

$$\therefore \text{COV}(X, Y) = \frac{3}{2} - 3 \times 1$$
$$= \frac{3 - 6}{2} = \frac{-3}{2} //$$

iv) $E(X^2) = \sum_{i=1} x_i^2 P(x_i) = 1\left(\frac{1}{2}\right) + 25\left(\frac{1}{2}\right)$
$$= \underline{\underline{13}}$$

53+32

$$E(Y^2) = \sum_{j=1}^{m} y_j^2 P(y_j) = 16 \times \frac{3}{8} + 4 \times \frac{3}{8} + 49 \times \frac{1}{4} = \frac{158}{8}$$

$$\sigma_x = \sqrt{E(X^2) - [E(x)]^2}$$

$$= \sqrt{\cancel{\tfrac{188}{8}}\ 13 - 9}$$

$$= \cancel{\tfrac{188}{8}} \quad \sqrt{4} = 2$$

$$\sigma_y = \sqrt{\frac{158}{8} - 1^2}$$

$$= \sqrt{\frac{150}{8}} \quad = 4.33$$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_x \sigma_y}$$

$$= \frac{-3/2}{2 \times 4.33} \quad = \underline{\underline{-0.1732}}$$

v) → Dependant (-vely correlated)

Q4 The distributions of ~~2~~ independent variables x & y defined on the sample space ~~are~~ is given by the following table:

X	0	1
P(x)	0.2	0.8

Y	1	2	3
P(Y)	0.1	0.4	0.5

Find the i) joint probability distribution of X & Y
ii) Evaluate co-variance of X & Y

classmate
Date
Page

i) X & Y are independant R.V

$\Rightarrow P(x,y) = P(x) \cdot P(y)$

$P(x=0, y=1) = P(x=0) \cdot P(y=1)$
$= (0.2)(0.1)$

$P(x=0, y=2) = P(x=0) \cdot P(y=2)$
$= (0.2)(0.4)$
$= 0.08$

X\Y	1	2	3
0	0.02	0.08	0.1
1	0.08	0.32	0.4

$P(x=0, y=3) = P(x=0) \cdot P(y=3)$
$= (0.2)(0.5)$
$= 0.1$

$P(x=1, y=1) = P(x=1) \cdot P(y=1) = (0.8)(0.1)$
$P(x=1, y=2) = P(x=1) \cdot P(y=2) = (0.8)(0.4)$
$P(x=1, y=3) = P(x=1) \cdot P(y=3) = (0.8)(0.5)$

ii) $\text{cov}(x,y) = E(XY) - E(X)E(Y)$

$E(X) = \sum x_i P(x_i)$
$= x_1 P(x_1) + x_2 P(x_2)$
$= 0(0.2) + 1(0.8)$
$E(X) = \underline{\underline{0.8}}$

$E(Y) = \sum y_i P(y_i)$
$= y_1 P(y_1) + y_2 P(y_2)$
$= 1(0.1) + 2(0.4) + 3(0.5)$
$= 0.1 + 0.8 + 1.5$
$E(Y) = \underline{\underline{2.4}}$

$E(XY) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_i P(x_i, y_i)$

$= x_1 y_1 P(x_1, y_1) + x_1 y_2 P(x_1, y_2) + x_1 y_3 P(x_1, y_3)$
$+ x_2 y_1 P(x_2, y_1) + x_2 y_2 P(x_2, y_2)$
$+ x_2 y_3 P(x_2, y_3)$
$= 0 \times 1 \times 0.02 + 0 \times 2 \times 0.08 + 0 \times 3 \times 0.1$
$+ 1 \times 1 \times 0.08 + 1 \times 2 \times 0.32 + 1 \times 3 \times 0.4$
$= 0.08 + 0.64 + 1.2 = \underline{\underline{1.92}}$

$$\text{cov}(x,y) = 1.92 - (0.8)(2.4)$$
$$= \underline{\underline{0}}$$

Q13 A joint probability function for 2 discrete random variables x and y is given by $f(x,y) = c(2x+y)$ where x & y are defined as $0 \le x \le 2$, $0 \le y \le 3$ (only integral values). Find i) the value of constant c ii) $P(x=2, y=1)$ iii) $P(x \ge 1, y \le 2)$ iv) $P(x+y \ge 1)$

$X = \{0,1,2\}$
$Y = \{0,1,2,3\}$

X \ Y	0	1	2	3
0	0	c	2c	3c
1	2c	3c	4c	5c
2	4c	5c	6c	7c

i) i) Given $P(x,y) \ge 0$
$\Rightarrow c \ge 0$

ii) $\Sigma f(x,y) = 1$
$c + 2c + 3c + 2c + 3c + \cdots + 6c + 7c = 1$
$42c = 1$

$$\boxed{c = \frac{1}{42}}$$

ii) $P(x=2, y=1) = 5c = \frac{5}{42}$

iii) $P(x \ge 1, y \le 2)$
$P(x=1, y=0) + P(x=1, y=1) + P(x=1, y=2) +$
$P(x=2, y=0) + P(x=2, y=1) + P(x=2, y=2)$

$= 24c = \underline{\underline{\frac{24}{42}}}$

classmate

Date

Page

iv) $P(x+y>1)$

$P(x=0, y=2) + P(x=0, y=3) + P(x=1, y=1)$
$P(x=1, y=2) + P(x=1, y=3) + P(x=2, y=0)$
$P(x=2, y=1) + P(x=2, y=2) + P(x=2, y=3)$

$= 39C = \frac{39}{42}$

Or

$P(X+Y>1) = 1 - P(X+Y \le 1)$

Q) A coin is tossed 3 times. Let X be equal to 0 or 1 according as a head or a tail occurs on the first toss. Let Y be the total numbers of heads which occur. Determine i) Marginal distribution of X & Y
ii) Joint probability distribution of X & Y
iii) correlation b/w X & Y

S = {HHH, HHT, HTH, HTT, THT, TTH, TTT, THH}

$X = 0 \rightarrow$ head 1st toss
$= 1 \rightarrow$ tail

$Y = 0, 1, 2, 3 \rightarrow$ Range space

Marginal distribution of X

X	0	1
P(X)	0.5	0.5

M.D of Y

Y	0	1	2	3
P(Y)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

J.P.D $P(x,y)$

$x \backslash y$	0	1	2	3
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0

THXY

$$\rho(x,y) = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{V_x}$$

$$V_x = E(x^2) - [E(x)]^2$$

$$\sigma_y = \sqrt{V_y}$$

$$V_y = E(y^2) - [E(y)]^2$$

$$E(x) = \Sigma x_i P_i(x)$$
$$= 0 \times 0.5 + 1 \times 0.5 = 0.5$$

$$E(x^2) = \Sigma x_i^2 P_i(x_i)$$
$$= 0 \times 0.5 + 1^2 \times 0.5 = 0.5$$

$$E(y) = \Sigma y_i P_i(y)$$
$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$
$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = 1.5$$

$$E(y^2) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8}$$

$\Sigma y_i^2 P_i(y)$

$$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$$

$$V_x = 0.5 - (0.5)^2 = 0.5 - 0.25 = 0.25$$

$$V_y = 3 - (1.5)^2 = 0.75$$

$\sigma_x^2 = V_x = \frac{1}{4}$ $\quad \sigma_x = \sqrt{\frac{1}{4}} = 0.5$

$\sigma_y^2 = V_y = \cancel{0.866}\ 0.75$ $\quad \sigma_y = \sqrt{0.75} = 0.866$

$COV(x,y) = E(XY) - E(X)E(Y)$
$= \frac{1}{2} - \left(\frac{1}{2}\right)\left(\frac{12}{8}\right) = \cancel{\frac{4}{24}} = -\frac{1}{4}$

$\rho(x,y) = \frac{COV(x,y)}{\sigma_x \sigma_y} = \frac{\left(-\frac{1}{4}\right)}{(0.5)(0.866)}$

$= -0.5773$

Q5) Two cards are ~~randoms~~ selected at a random from a box which contains 5 cards no. 1, 1, 2, 2, and 3. Find the joint P.D of X & Y where X denotes the sum and Y is the maximum of the two numbers. Also find correlation of X, Y.

X = Sum of 2 no. cards
Y = Max of 2 cards drawn

$X = \{2, 3, 4, 5\}$
$Y = \{1, 2, 3\}$

SS = { ~~(1,1), (2,2)~~ ~~(1,2)~~
(1,1), (1,2) (1,2)
(1,3) (1,2) (1,2)
(1,3) (2,2) (2,3)
(2,3) }

J.P.D

X \ Y	1	2	3
2	1/10	0	0
3	0	4/10	0
4	0	1/10	2/10
5	0	0	2/10

$\rho(x,y) = \frac{COV(x,y)}{\sigma_x \sigma_y}$

$$E(x) = \Sigma x P(x) = \frac{2}{10} + \frac{12}{10} + \frac{12}{10} + \frac{10}{10} = \frac{36}{10}$$

$$E(y) = \Sigma y P(y) = \frac{1}{10} + \frac{10}{10} + \frac{12}{10} = \frac{23}{10}$$

$$E(xy) = \Sigma xy P(x,y)$$
$$= 2\times1\left(\frac{1}{10}\right) + 3\times2\left(\frac{4}{10}\right) + 4\times2\left(\frac{1}{10}\right)$$
$$+ \frac{4\times3\times2}{10} + \frac{5\times3\times2}{10}$$
$$= \frac{2}{10} + \frac{24}{10} + \frac{8}{10} + \frac{24}{10} + \frac{30}{10}$$
$$= \frac{88}{10}$$

$$\Rightarrow COV(x,y) = \frac{88}{10} - \frac{36}{10} \times \frac{23}{10}$$
$$E(XY) - E(X)E(Y) = 0.52$$

$$\sigma_x = \sqrt{V_x}$$
$$E(x^2) = \Sigma x^2 P(x)$$
$$= \frac{4}{10} + \frac{36}{10} + \frac{48}{10} + \frac{50}{10} = \frac{138}{10}$$
$$13.8$$

$$\sigma_y = \sqrt{V_y}$$
$$E(y^2) = \Sigma y^2 P(y)$$
$$= \frac{1}{10} + \frac{20}{10} + \frac{36}{10} = 5.7$$

$$V(x) = E(x^2) - [E(x)]^2$$
$$= 13.8 - (3.6)^2$$
$$V(x) = 0.84$$
$$\sigma_x = \sqrt{0.84} = 0.9165$$
$$V(y) = E(y^2) - [E(y)]^2 = 5.7 - (2.3)^2$$
$$= 0.41$$

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$\sigma_y = \sqrt{0.41} = 0.64$

$\rho = \dfrac{+0.52}{\cancel{0.41} \, 0.64 \times \cancel{0.84} \, 0.965} = +0.8865$

Q6) Two marbles are selected at random from a box containing 3 blue, 2 red & 3 green marbles. If X is the no. of blue marbles & Y is the no of red marbles selected, find M.D of X & Y & J.P.D. of X & Y.

Total 8.

→ X = no of blue marbles   Y = no of red marbles.

$X = \{0, 1, 2\}$
$Y = \{0, 1, 2\}$

2 marbles can be drawn out of 8 in $^8C_2 = 28$ ways

M.D of X

X	0	1	2
P(x)	$\frac{^5C_2}{^8C_2}$		

$\boxed{\dfrac{^3C_1 \cdot {^5C_1}}{^8C_2}}$ $\dfrac{^3C_2}{^8C_2}$

$\boxed{^nC_r = \dfrac{n!}{(n-r)!\,r!}}$

M.D of Y

Y	0	1	2
P(Y)	$\frac{^6C_2}{^8C_2}$	$\frac{^2C_1 \cdot {^6C_1}}{^8C_2}$	$\frac{^2C_2}{^8C_2}$

Joint P.D

X\Y	0	1	2
0	$^3C_2/^8C_2$		
1	$\frac{^3C_1 \cdot {^3C_1}}{^8C_2}$	$\frac{^3C_1 \cdot {^2C_1}}{^8C_2}$	0
2	$^3C_2/^8C_2$	0	0

$\boxed{\dfrac{^2C_1 \cdot {^3C_1}}{^8C_2}}$ $\boxed{\dfrac{^2C_2}{^8C_2}}$

classmate
Date
Page

→ Probability Vector:

A vector $V = (v_1\ v_2\ \ldots\ v_n)$ is said to be a probability vector if

i) $v_i \geq 0$

ii) $\Sigma v_i = 1$

Eg. $V = (1, 0)$

$u = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$

A vector whose components are non-negative but their sum is not one can be converted into a probability vector by dividing each component by the sum of each component.

$w = (1, 2, 3)$ $\quad \Sigma w_i = 1+2+3 = 6$

$= \left(\frac{1}{\Sigma w_i}, \frac{2}{\Sigma w_i}, \frac{3}{\Sigma w_i}\right)$

$= \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right)$

→ Stochastic Matrix:

A square matrix of order m is said to be a stochastic matrix if each row is a probability vector.

→ Regular Stochastic Matrix:

A stochastic matrix P is said to be regular if all the entries of some power P^m are positive (not including 0).

$P = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ P- stochastic matrix

$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 0.375 & 0.25 \\ 0.125 & 0.5 & 0.375 \end{bmatrix}$ $\quad P^3 = \begin{bmatrix} 1 & 0 & 0 \\ \_\_ & & \\ \_\_ & & \end{bmatrix}$ row never change

$\therefore$ Not a regular stochastic matrix

classmate

Date

Page

Note: ① A stochastic matrix P has a unique fixed probability vector $V = (V_1\ V_2 \cdots V_n)$ such that

$V \times P = V$

② P^2, P^3 so on approaches to some matrix B whose rows are each ~~of~~ the fixed probability vector

Q1) Verify that the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}$ is a regular stochastic matrix.

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0.125 & 0.3125 & 0.562 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.1562 & 0.6406 & 0.2032 \\ 0.125 & 0.3125 & 0.5625 \end{bmatrix}$$

→ Regular stochastic matrix (Each entry is positive ie >0)

Q2) Find the fixed probability vector for the Stochastic matrix $P = \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$ And also find the matrix B for which each row is a fixed probability vector. (P.V)

$$\underset{(1\times m)(m\times m)}{VP} = \underset{(1\times m)}{V}$$

$V = (V_1, V_2)$ be a fixed P.V

$V_i > 0$

$\Sigma V_i = 1$ ie $V_1 + V_2 = 1 \rightarrow ③$

$VP = V$

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

$\frac{V_1}{3} + \frac{V_2}{4} = V_1$ $\qquad$ $\frac{2V_1}{3} + \frac{3V_2}{4} = V_2$

$-\frac{2}{3}V_1 + \frac{V_2}{4} = 0 \rightarrow ①$ $\qquad$ $\frac{2V_1}{3} - \frac{V_2}{4} = 0 \rightarrow ②$

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classmate
Date
Page

① & ② are same ∴ use ① or ② with ③ & solve.

$$-\frac{2}{3}v_1 + \frac{1}{4}[1 - v_1] = 0$$

$$-\frac{11 v_1}{\cancel{12}\,3} = \frac{-1}{\cancel{4}}$$

$$\boxed{v_1 = \frac{3}{11}}$$

$$v_2 = 1 - v_1 = 1 - \frac{3}{11} = \frac{8}{11}$$

$$v = (v_1, v_2) = \left(\frac{3}{11}, \frac{8}{11}\right) = (0.2727, 0.7272)$$

$$P = \begin{bmatrix} \frac{1}{3} & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.2777 & 0.7222 \\ 0.2708 & 0.7291 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.2731 & 0.7268 \\ 0.2725 & 0.7274 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.2727 & 0.7272 \\ 0.2727 & 0.7272 \end{bmatrix} = B$$

each row is equal to the fixed P.V i.e v.

$$\boxed{B = P^4}$$

Q3) Find the fixed P.V of regular stochastic matrix $P = \begin{bmatrix} 1/2 & \frac{1}{4} & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}_{3\times3}$

$m = 3$

$$\therefore V = (v_1 \quad v_2 \quad v_3)_{1\times3}$$

$$VP = V$$

$v_i \geq 0$

$v_1 + v_2 + v_3 = 1$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$\frac{v_1}{2} + \frac{v_2}{2} = v_1 \longrightarrow (1)$$

$$\frac{v_1}{4} + v_3 = v_2 \longrightarrow (2)$$

$$\frac{v_1}{4} + \frac{v_2}{2} = v_3 \rightarrow (3)$$

$$(1) \rightarrow \quad \frac{v_2}{2} = \frac{v_1}{2} \qquad \boxed{v_1 = v_2}$$

$$v_3 = v_2 - \frac{v_2}{4}$$

$$\boxed{v_3 = \frac{3v_2}{4} = \frac{3v_1}{4}}$$

$$\therefore \quad v_1 + v_1 + \frac{\cancel{2}3v_1}{\cancel{3}4} = 1$$

$$\frac{11}{4} v_1 = 1$$

$$\therefore \boxed{v_1 = \frac{4}{11}} \quad \boxed{v_2 = \frac{4}{11}} \quad \boxed{v_3 = \frac{3}{11}}$$

$$\boxed{V = \left[\frac{4}{11}, \frac{4}{11}, \frac{3}{11} \right]}$$

$\log_1(9523) = $ error $\lim_{x\to 0} x^x$ $10 = \log_{10}$ error $\log(1)$

$1^x \neq 9523$

→ Stochastic Process:

A process in which sequence of experiments exist is called stochastic process ~~the~~ and the process is denoted by ~~X(t)~~ X_t, $t \in T$ (time)

The values assumed by the random variable X_t is called states & the set of all possible values is called state space of the process

If the state space is finite or discrete then the process is called a chain.

→ A stochastic process in which the state space is discrete and the outcome of any trial depends utmost upon the outcomes of immediately preceding trial and not upon any other previous outcomes is called a markov chain.

→ Suppose if (a_i, a_j) be the ordered pair of states where a_j occurs immediately after a_i occurs, then the number P_{ij} represents the probability that the system changes from i state to j state and the corresponding matrix formed is called transition probability.

→ The entry P_{ij} in TPM (transition prob matrix) is the probability that the system moves from state a_i to a_j in one step.

Similarly if system moves from a_i to a_j in n steps, then the probability is called n-step transition probability. denoted by $p^{(n)}$.

The n-step transition matrix is same as the nth power of the one step transition matrix.

→ Evaluation of n-step transition probability:

let $P_i = (P_1, P_2 \cdots P_n)$ be the probability distribution of some state a_i. At time $t=0$, the process begins and the corresponding probability vector is given by $P_i^{(0)} = [P_1^{(0)}\ P_2^{(0)} \cdots P_n^{(0)}]$ is called initial probability distribution.

The probability distribution after one step

$$P^{(1)} = P^{(0)} \cdot P$$

after 2-steps

$$P^{(2)} = P^{(1)} \cdot P$$
$$= P^{(0)} \cdot P \cdot P$$
$$= P^{(0)} \cdot P^2$$

In general

$$P^{(n)} = P^{(0)} \cdot P^n$$

→ steady state and stationary vector:

Suppose $\lim_{n\to\infty} P^{(n)} = V$ where V is a non-negative independant of initial distribution & $\Sigma V_k = 1$ then the row matrix V is called steady state distribution of the process or a fixed probability vector of the chain.

A markov chain is said to be irreducable if if its transition matrix is a regular stochastic matrix.

Q) A software engineer goes to his work place everyday by motorbike or car. He never goes by bike on 2 consecutive days but if he goes by car on a day then he is equally likely to go by car or by bike on the next day. Find the transition matrix for the change of the mode of transport i.e. ... If car is used on the first day of a week, find the probability that- i) bike is used, ii) car is used on the 5th day.

States - bike , car
(A) (B)

$$P - \begin{array}{c|cc} & A & B \\ \hline A & 0 & 1 \\ B & \frac{1}{2} & \frac{1}{2} \end{array}$$

$\uparrow$ Present day; $A \to B$ next day

$$p^{(0)} = \begin{array}{cc} A & B \\ (0 & 1) \end{array}$$

$p^{(4)} = p^{(0)} \cdot p^4 \rightarrow$ fifth-day

$$p^4 = \begin{bmatrix} 0.375 & 0.625 \\ 0.3125 & 0.6875 \end{bmatrix}$$

$$p^{(4)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.375 & 0.675 \\ 0.3125 & 0.6875 \end{bmatrix}$$

$$= \begin{array}{cc} A & B \\ \left[0.3125 \right. & \left. 0.6875 \right] \end{array}$$

P(bike on fifth day) = 0.3125
P(car " ") = 0.6875

Q) A student changes his mobile every year. If he has a samsung, he changes ~~to~~ over to an apple, if he has apple, he changes to one plus. If he has oneplus, he is just likely to change over to samsung, apple or oneplus. If he had a [1] in 2019, find the probability that he will have a mobile of

i) samsung in 2021
ii) one plus in 2021
iii) Apple in 2022
iv) One plus in 2022.
v) In the long run, how often will he have oneplus.

States - A - apple
B - samsung
C - one plus

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

$P^{(0)}$ $P^{(1)}$ $P^{(2)}$
2019 2020 2021

$$P^{(0)}_{2019} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$P^{(2)}_{2021} = P^{(0)} P^2$$

$$P^2 = \begin{bmatrix} 0.3333 & 0.3333 & 0.3333 \\ 0 & 0 & 1 \\ 0.4444 & 0.1111 & 0.4444 \end{bmatrix}$$

$$P^{(2)}_{2021} = \begin{bmatrix} 0.44 & 0.11 & 0.42 \end{bmatrix}$$

i) P(samsung in 2021) = 0.11
ii) P(oneplus in 2021) = 0.42

$\frac{0}{2}$

Date
Page

$$p^{(3)}_{2022} = p^{(0)} \, p^3$$

$$= (0 \; 0 \; 1) \begin{bmatrix} 0.44 & 0.11 & 0.44 \\ 0.3333 & 0.37 & 0.33 \\ 0.2592 & 0.1481 & 0.5925 \end{bmatrix}$$

$$p^{(3)} = \begin{bmatrix} 0.2592 & 0.1481 & 0.5925 \\ \cancel{0.33} & \cancel{0.33} & \cancel{0.33} \end{bmatrix}$$

iii) $P(\text{Apple in } 2022) = 0.259$

iv) $P(\text{Oneplus in } 2022) = 0.5925$

v) $\lim_{n \to \infty} p^n = v$

Let $v = (v_1 \quad v_2 \quad v_3)$ $\quad v_i > 0 \quad v_1 + v_2 + v_3 = 1$

$$vP = v$$

$$[v_1 \; v_2 \; v_3] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = [v_1 \quad v_2 \quad v_3]$$

$$v_2 + \frac{v_3}{3} = v_1 \rightarrow (1)$$

$$\cancel{v_1 +} \frac{v_3}{3} = v_2 \rightarrow (2) \qquad v_3 = 3v_2$$

$$v_1 + \frac{v_3}{3} = v_3 \rightarrow (3)$$

Solving (1), (2), 3,

$$\boxed{v_1 = \frac{1}{3} \qquad v_2 = \frac{1}{6} \qquad v_3 = \frac{1}{2}}$$

$$\cancel{v_3 = \frac{1}{2} = 0.5}$$

$$\boxed{v = \left[\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right]}$$

Q3] 3 boys A, B, C are throwing a ball to each other. A always throws the ball to B. B always throws the ball to C. But C is just as likely to throw the ball to B as to A. If C was the first person to throw ball, find the probability that for the 4th throw i) A has the ball
ii) B has the ball
iii) C has the ball.
iv) In long run, how often A, B, C will throw the ball. have

$$P = \text{Throw} \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$$P^{(0)} = \begin{pmatrix} \overset{A}{0} & \overset{B}{0} & \overset{C}{1} \end{pmatrix}$$

$$P^{(3)} = P^{(0)} P^3$$

$$P^3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$P^{(3)} = (0 \quad 0 \quad 1) \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$P^{(3)} = \begin{pmatrix} \overset{A}{0.25} & \overset{B}{0.25} & \overset{C}{0.5} \end{pmatrix}$$

i) 0.25
ii) 0.25
iii) 0.5
iv) $VP = V$

$$[V_1 \quad V_2 \quad V_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = [V_1 \; V_2 \; V_3]$$

$v_1 = 0.2$
$v_2 = 0.4$
$v_3 = 0.4$

Q) In a certain city, the weather on a day is reported as sunny, cloudy, or rainy. If a day is sunny the probability that the next day is sunny is 70%, cloudy is 20%, rainy = 10%. If a day is cloudy, prob that next day is sunny is 30%, cloudy is 20%, rainy is 50%. If a day is rainy, prob that next day is sunny is 30%, cloudy is 30%, rainy is 40%. If a sunday is sunny, find the probability the wednesday is rainy.

$$P = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} S\,A \\ C\,B \\ R\,C \end{array} & \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

$$p^{(0)} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$p^{(3)} = p^{(0)} \cdot p^3$$
$$= \begin{bmatrix} 0.532 & 0.221 & 0.247 \end{bmatrix}$$

Rainy sunday = 24.7%.

Q) A sales man territory consists of 3 cities A, B & C. He never sells in the same city on successive days. If he sells in city A then the next day he sells in city B however if he sells in either B or C, then next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities.

$$P \quad \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & \frac{2}{3} & 0 & \frac{1}{3} \\ C & \frac{2}{3} & \frac{1}{3} & 0 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 2x & 0 & x \\ 2x & x & 0 \end{bmatrix}$$

$x \geq 0$

$2x + x = 1$

$3x = 1$

$\boxed{x = \frac{1}{3}}$

$$[V_1 \ V_2 \ V_3] \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = [V_1 \ V_2 \ V_3]$$

$\frac{2V_2}{3} + \frac{2V_3}{3} = V_1 \rightarrow (1)$

$V_1 + \frac{V_3}{3} = V_2 \rightarrow (2)$

$\frac{V_2}{3} = V_3 \rightarrow (3)$

$\therefore V_2 = 3V_3$ substitute in 1 & 3

$\cancel{\frac{8V_3}{3}}$

$2V_3 + \frac{2V_3}{3} = V_1$

$V_1 = 3V_3 - \frac{V_3}{3}$

$V_1 = \frac{8V_3}{3}$

$V_1 + V_2 + V_3 = 1$

$\therefore V_1 = \frac{2}{5} \quad V_2 = \frac{9}{20} \quad V_3 = \frac{3}{20}$

$$V = (V_1 \ V_2 \ V_3) = \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \left(\frac{2}{5}, \frac{9}{20}, \frac{3}{20}\right) \end{array}$$

→ In the long run.

Q) A housewife buys 3 brands of soaps A B C every week. She never buys the same brand on successive weeks. If she buys brand A in a week, she buys brand B in the next week. If she buys the brand other than A in a week, then in the next week, she is 3 times as likely to buy brand A. Suppose she bought brand B in first week, find the probability of buying brand C in the 4th week.

States A, B, C

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ p^{(0)} = & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array}$$

$p^{(3)} = p^{(0)} P^3$

$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.1875 & 0.8125 & 0 \\ 0.6093 & 0.1875 & 0.2031 \\ 0.6093 & 0.2031 & 0.1875 \end{bmatrix}$

$= \begin{bmatrix} \cancel{0.75} & \cancel{0} & \cancel{0.25} \\ 0.6093 & 0.1875 & 0.2031 \end{bmatrix}$

P(buying brand C in 4th week) = ~~25%~~ 20.31%

Q) A players luck follows a pattern. If he wins a game the probability of winning the next game is 0.6. However if he loses a game, the probability of losing the next one is 0.7. If there is an even chances of winning for the first game. i) find the probability of winning second game ii) probability of winning 3rd game iii) In the long run, how often will he win

States - W L

$$P = \begin{array}{c c} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

$P^{(0)} = [0.5 \quad 0.5]$

0.3
+0.15

0.2+0.35

i) $P^{(1)} = P^{(0)} \cdot P^{1}$

$= [0.45 \quad 0.55]$

P(winning second game) = 0.45

ii) $P^{(2)} = P^{(0)} P^{2}$

$= [0.5 \quad 0.5] \begin{bmatrix} 0.48 & 0.52 \\ 0.39 & 0.61 \end{bmatrix}$

$= [0.435 \quad 0.565]$

P(winning 3rd game) = 0.435

iii) $[V_1 \quad V_2] \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = [V_1 \quad V_2]$

$\boxed{VP = V}$

$0.6V_1 + 0.3V_2 = V_1$

$0.4V_1 - 0.3V_2 = 0 \rightarrow ①$

$0.4V_1 + 0.7V_2 = V_2$

$0.4V_1 - 0.3V_2 = 0$

$V_1 + V_2 = 1 \rightarrow ③$

$V_1 = \frac{3}{7} \qquad V_2 = \frac{4}{7}$

$V = [0.4285 \quad 0.5714]$

↳ P(long run win) = 0.4285

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Q) Prove that the markov chain whose transition probability matrix $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducable and also find the corresponding stationary vector.

$$P^2 = \begin{bmatrix} 0.5 & 0.1666 & 0.333 \\ 0.25 & 0.583 & 0.166 \\ 0.25 & 0.333 & 0.4166 \end{bmatrix}$$

~~Stationary~~ Stationary Vector $= \begin{bmatrix} 0.3333 & 0.3701 & 0.2964 \\ 0.3333 & 0.3705 & 0.2961 \\ 0.3333 & 0.3704 & 0.2962 \end{bmatrix}$

$VP = V$

$$V = \begin{bmatrix} \frac{1}{3} & \frac{10}{27} & \frac{8}{27} \end{bmatrix}$$