subject: Fields and waves (FAW)

staff : Bhavana

SEM : III SEM (2^{nd} year)

Vector Analysis

Scalor : The term scalar refers to a quantity whose value may be represented by a single number (positive or negative)

Eg: * Temperature on any point in a geographical area
* Speed of a vehicle on road
* Pressure at any point in a liquid in vessel etc

Vector : A vector quantity has both magnitude and direction in space. There may be n-dimensional space.

Eg: * Velocity (car's velocity is 70 km/hr , south)
* If a person steps 1 step forward and 1 step backward then his velocity is zero.
* Accelaration due to gravity
* Force applied on door to push

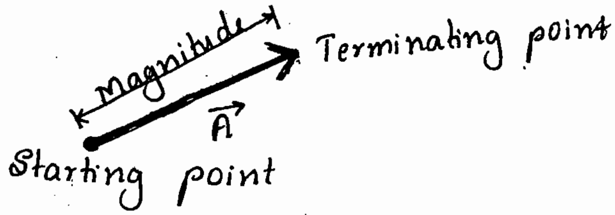
Fields : Field is a physical quantity that takes on different values at different locations.

Scalar Fields : Eg: The temperature throughout the bowl of soup

Vector Fields : Eg: Voltage grdient in a cable

The value of a field varies in general with both position & time.

In two-dimension vector can be represented by a straight line with an arrow in a plane. The length of the segment is magnitude , and arrow d indicates direction



<u>Vector Algebra</u> : It includes addition, subtraction, scaling multiplication of vectors.

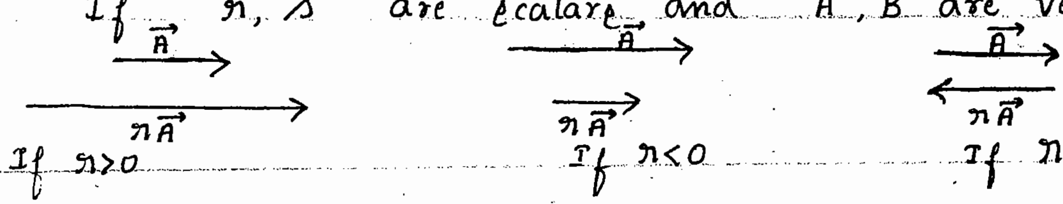
<u>Scaling of Vectors</u>

→ It is multiplication of a vector by scalar.

→ Scaling changes magnitude but the direction remains same.

→ When the scalar is negative it reverses the direction.

If r, s are scalars and $\vec{A}$, $\vec{B}$ are vectors

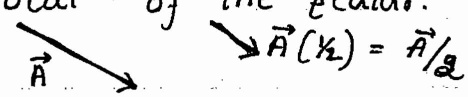


→ Scaling of a vector obeys associative law and distributive law

$$(r+s)(\vec{A}+\vec{B}) = r(\vec{A}+\vec{B}) + s(\vec{A}+\vec{B})$$

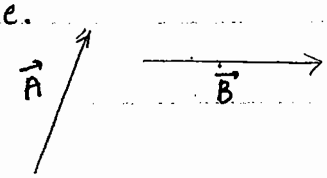
$$(r+s)(\vec{A}+\vec{B}) = r\vec{A} + r\vec{B} + s\vec{A} + s\vec{B}$$

→ Division of a vector by a scalar is just multiplication by the reciprocal of the scalar.

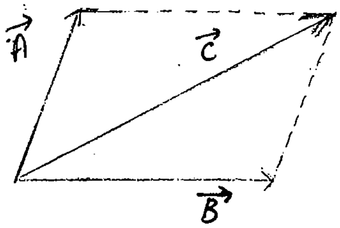


<u>Addition of Vectors</u>

Vector addition can be done using parallelogram rule.

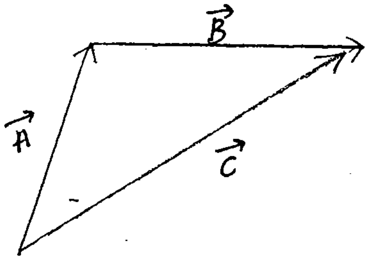


If vector $\vec{A}$ and $\vec{B}$ are to be added then, move one of two vectors parallel to itself without changing its direction to origin of other vector Construct the parallelogram.



Vector $\vec{C}$ is the result of addition of 2 vectors i.e, $\vec{C} = \vec{A} + \vec{B}$.

Two vectors can be added by beginning the second vector from head of first and completing triangle. i.e, Head to tail rule of addition of vectors.

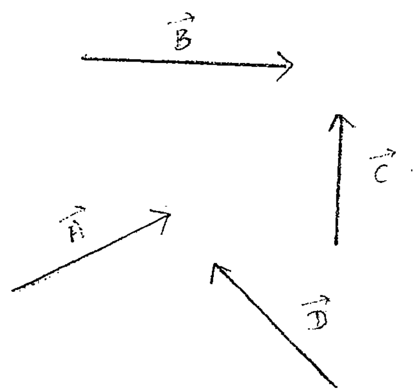


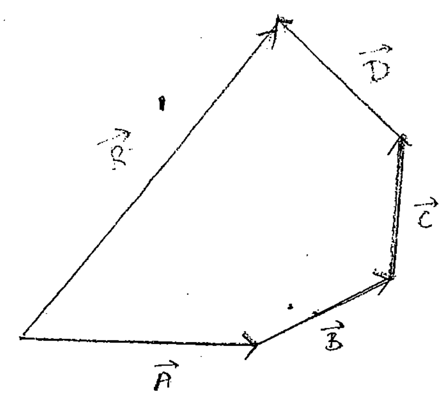
$\vec{A} + \vec{B}$ $\quad$ $\vec{B} + \vec{A}$

Vector addition also obeys associative law

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

If $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ are vectors

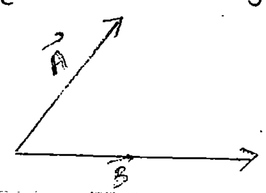


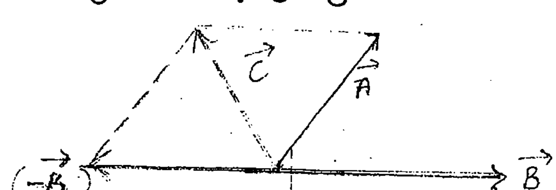


The resultant of all vectors $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$

Subtraction of Vectors

This can be obtained from vector addition. If $\vec{B}$ is to be subtracted from $\vec{A}$ then $\vec{C} = \vec{A} + (-\vec{B})$ where $(-\vec{B})$ is reverse of $\vec{B}$ by multiplying with -1.





Coordinate Systems

The coordinate systems provide specific lengths, directions, angles, projections to describe a vector accurately.

The three simple coordinate systems are

① Cartesian rectangular coordinate system

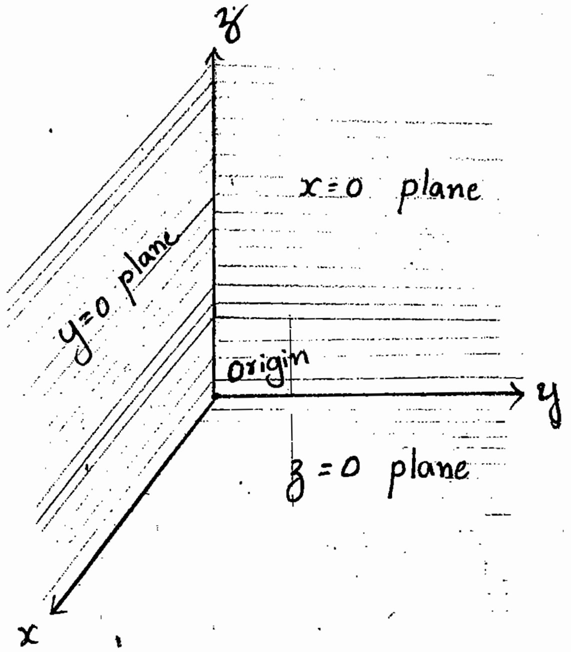
② Circular cylindrical coordinate system

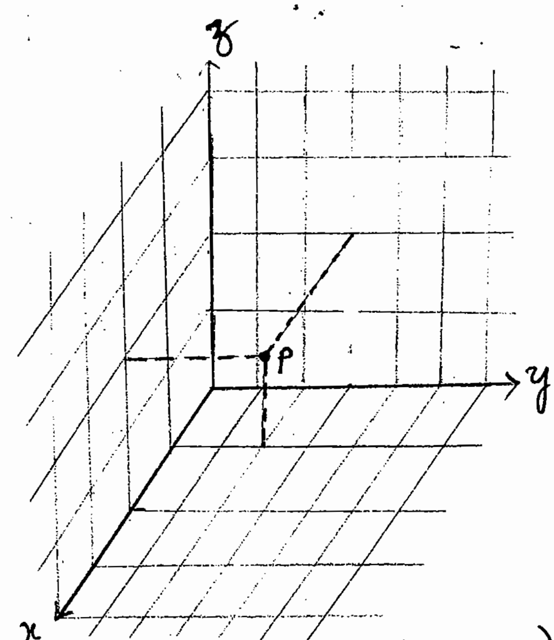
③ Spherical coordinate system.

① Rectangular Coordinate System

→ There are three coordinate axes mutually at right angles to each other and call them as x, y and z.

→ A pt is located by giving its x, y and z coordinates. These are respectively the distances from the origin to the intersection of a perpendicular dropped from the point to x, y and z planes where $x=0$, $y=0$ and $z=0$ planes





Point P has coordinates (2, 2, 2)

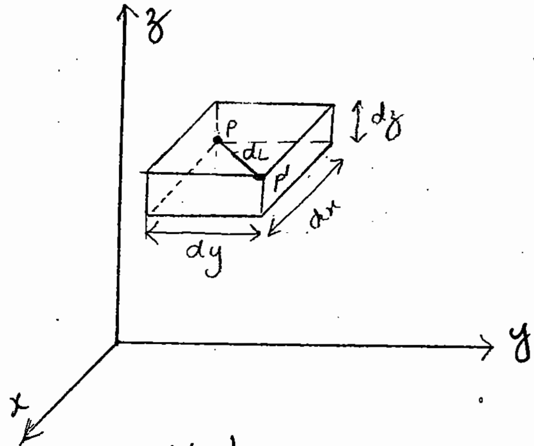
If we visualize three planes intersecting at general point P whose coordinates are x, y and z, we may increase each coordinate value by a differential amount

It will form three slightly displaced planes at P' whose coordinates are $(x+dx, y+dy, z+dz)$

The six planes define parallelpiped whose volume is
$dv = (dx\ dy\ dz)$

The surfaces have differential areas ds of dxdy, dydz, dzdx

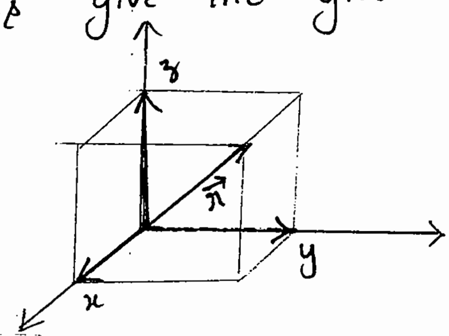
The distance dL from P to P' is the diagonal of parallelpiped and has length of $\sqrt{(dx)^2+(dy)^2+(dz)^2}$. P is located at invisible corner



Vector Components and Unit Vectors

Let us consider a vector $\vec{r}$ extending outward from the origin

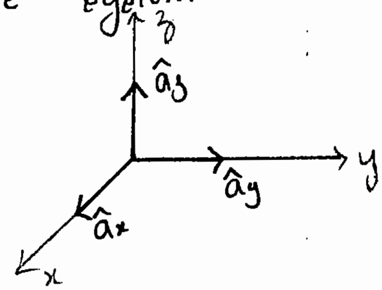
Logically this vector can be identified by giving component vectors along three coordinate axes. The vector sum of component vectors give the given vector $\vec{r}$



i.e, the component vectors have magnitudes which depend on given vector R but they each have a constant direction.

Unit vectors have unit magnitude and directed along the coordinate axes in the direction of increasing coordinate values.

Unit vectors can be denoted as $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ in rectangular coordinate system



A vector $\vec{r}_p$ pointing from origin to point P(1,2,3) is written $\vec{r}_p = \hat{a}_x + 2\hat{a}_y + 3\hat{a}_z$ and $\vec{r}_q$ pointing from origin to point Q(2,-2,1) is written as $\vec{r}_q = 2\hat{a}_x - 2\hat{a}_y + \hat{a}_z$

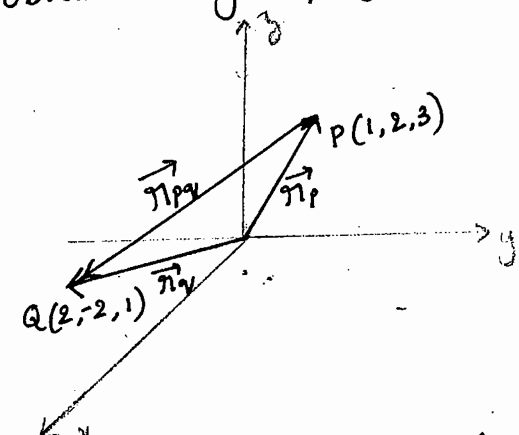
The vector from P to Q can be obtained by applying the rule of addition of vectors

i.e,

$$\vec{r}_p + \vec{r}_{pq} = \vec{r}_q$$

$$\vec{r}_{pq} = \vec{r}_q - \vec{r}_p$$

|||'ly $$\vec{r}_{qp} = \vec{r}_p - \vec{r}_q$$



Any vector $\vec{B}$ then may be described by $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

The magnitude of B is $|B| = \sqrt{B_x^2 + B_y^2 + B_z^2}$

|||'ly unit vector in given direction of vector B is

$$\vec{a}_B = \frac{B}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{B}{|B|}$$

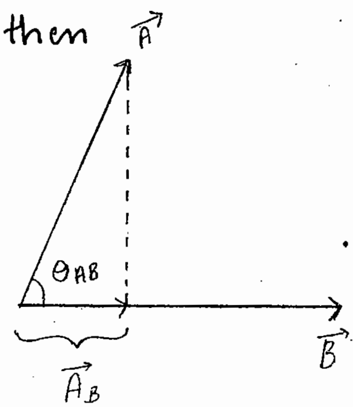
The Dot Product

The dot product or scalar product is defined as the product of the magnitude of $\vec{A}$, the magnitude of $\vec{B}$ and the cosine of smaller angle between them

$$\vec{A} \cdot \vec{B} = |A|\,|B| \cos \theta_{AB}$$

The geometrical term projection is also used in dot product

If $\vec{A}_B$ is projection of $\vec{A}$ on $\vec{B}$ then



$$|\vec{A}_B| = \cos \theta_{AB}\, |\vec{A}|$$

$$\vec{A} \cdot \vec{B} = |\vec{A}|\, |\vec{B}| \cos \theta_{AB}$$

$$\vec{A} \cdot \vec{B} = |\vec{B}|\, |\vec{A}_B|$$

Properties of Dot Product

(a) If 2 vectors are parallel to each other, $\theta = 0^\circ$ then

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|$$

(b) If 2 vectors are perpendicular then $\theta = 90^\circ$

$$\vec{A} \cdot \vec{B} = 0$$

(c) Dot product obeys commutative law

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

(d) Dot product obeys distributive law

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

(e) The dot product of a vector with itself is the square of magnitude of that vector

$$\vec{A} \cdot \vec{A} = |\vec{A}|\,|\vec{A}| \cos(0) = |\vec{A}|^2$$

(f) If $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ are unit vectors in cartesian coordinate system. All these vectors are mutually perpendicular to each other then

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$

(g) Any unit vector dotted with itself is unity

$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$$

(h) If $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$\vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z$

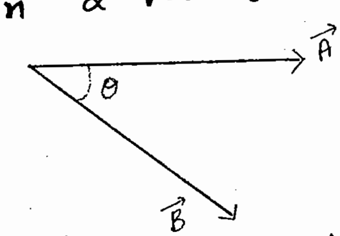
Then $\vec{A} \cdot \vec{B}$ has 9 scalar terms among them 6 terms equal to zero because of property (f)

$$\begin{aligned}\vec{A} \cdot \vec{B} = {} & A_x B_x(\hat{a}_x \cdot \hat{a}_x) + \cancel{A_x B_y(\hat{a}_x \cdot \hat{a}_y)}^{0} + \cancel{A_x B_z(\hat{a}_x \cdot \hat{a}_z)}^{0} \\ & + \cancel{A_y B_x(\hat{a}_y \cdot \hat{a}_x)}^{0} + A_y B_y(\hat{a}_y \cdot \hat{a}_y) + \cancel{A_y B_z(\hat{a}_y \cdot \hat{a}_z)}^{0} \\ & + \cancel{A_z B_x(\hat{a}_z \cdot \hat{a}_x)}^{0} + \cancel{A_z B_y(\hat{a}_z \cdot \hat{a}_y)}^{0} + A_z B_z(\hat{a}_z \cdot \hat{a}_z)\end{aligned}$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

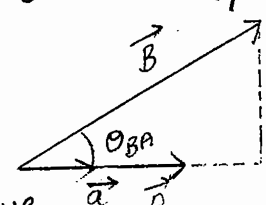
Applications of Dot product

① Finding angle between 2 vectors $\quad \theta = \cos^{-1}\left\{\frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|}\right\}$



② To find component of a vector in a given direction

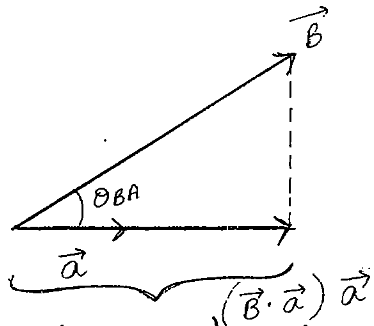
If we need component of $\vec{B}$ in the direction specified by unit vector $\hat{a}$



· If $0 \le \theta_{BA} \le 90°$ Sign of component is positive

[illegible] Sign of component is negative

③ Similarly component vector of vector B in the direction of unit vector $\vec{a}$ by just multiplying the component by $\vec{a}$ i.e,



Hence component of a vector in any direction becomes the problem of finding unit vector in that direction.

Note: ① Component of a vector in given direction → scalar

② Component vector in a given direction → Vector

④ Physical work done by a constant force can be expressed as dot product of 2 vectors

$$W = |\vec{F}|\, d \cos\theta = \vec{F}\cdot\vec{d}$$

If force varies along with path then total work done is

$$W = \int \vec{F}\cdot\vec{dL}$$

Vector Field

Vector field is vector function of a position vector. The magnitude and direction of the function will change as we move throughout the region.

In rectangular coordinate system vector field should be a function of x, y and z.

If position vector is $\vec{r}$ vector field $\vec{G}$ can be written as $\vec{G}(r)$

Eg: Velocity vector $\vec{V} = V_x(r)\,\hat{a}_x + V_y(r)\,\hat{a}_y + V_z(r)\,\hat{a}_z$

Each component V_x, V_y and V_z may be function of three variables

Cross Product

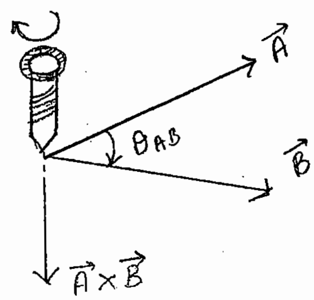
The cross product $\vec{A} \times \vec{B}$ (A cross B) is a vector, the magnitude of $\vec{A} \times \vec{B}$ is equal to the product of the magnitudes of $\vec{A}, \vec{B}$ and the sine of the smaller angle between $\vec{A}$ & $\vec{B}$. The direction of $\vec{A} \times \vec{B}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$

The direction of resultant $\vec{A} \times \vec{B}$ is along one of 2 possible perpendiculars which is in the direction of advance of a right handed screw as $\vec{A}$ is turned into $\vec{B}$.

$$\vec{A} \times \vec{B} = \vec{a}_N \, |\vec{A}| \, |\vec{B}| \, \sin(\theta_{AB})$$

$a_N \rightarrow$ Unit vector

$N \rightarrow$ Stands for Normal



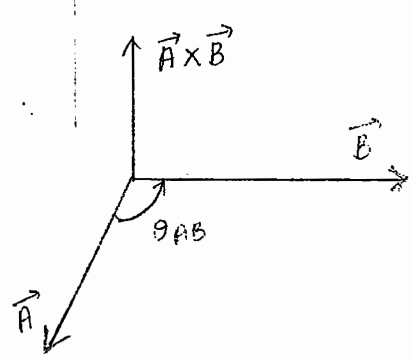
Properties of Cross Product

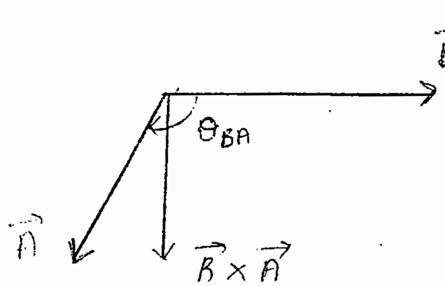
① The commutative law is not applicable for cross product

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

Reversing the order of vector reverses the resultant vector

i.e,
$$[\vec{A} \times \vec{B}] = -[\vec{B} \times \vec{A}]$$





(2) The cross product is not associative

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

(3) Cross product is distributive

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(4) If two vectors are in same direction i.e, $\theta = 0^\circ$ then the cross product is zero

(5) The cross product of a vector with itself is 0

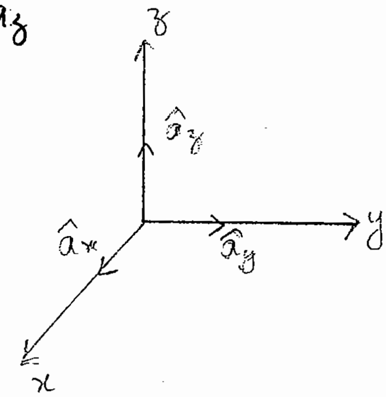
$$\vec{A} \times \vec{A} = 0$$

(6) Cross product of unit vectors

$$\hat{a}_x \times \hat{a}_y = |a_x|\,|a_y| \sin(90^\circ)\, \vec{a}_N$$

$$\sin(90^\circ) = |a_x| = |a_y| = 1 \quad \& \quad \vec{a}_N = \hat{a}_z$$

Hence
$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$
$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$
$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$



(7) Cross product in determinant form

Consider two vectors $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$\vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z$

$$\vec{A} \times \vec{B} = A_x B_x \underbrace{(\hat{a}_x \times \hat{a}_x)}_{0} + A_x B_y (\hat{a}_x \times \hat{a}_y) + A_x B_z (\hat{a}_x \times \hat{a}_z) +$$
$$A_y B_x (\hat{a}_y \times \hat{a}_x) + A_y B_y \underbrace{(\hat{a}_y \times \hat{a}_y)}_{0} + A_y B_z (\hat{a}_y \times \hat{a}_z) +$$
$$A_z B_x (\hat{a}_z \times \hat{a}_x) + A_z B_y (\hat{a}_z \times \hat{a}_y) + A_z B_z \underbrace{(\hat{a}_z \times \hat{a}_z)}_{0}$$

$$= A_x B_y (\hat{a}_z) + A_x B_z (-\hat{a}_y) + A_y B_x (-\hat{a}_z) + A_y B_z (\hat{a}_x) +$$
$$A_z B_x (\hat{a}_y) + A_z B_y (-\hat{a}_x)$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\, \hat{a}_x + (A_z B_x - A_x B_z)\, \hat{a}_y + (A_x B_y - A_y B_x)\, \hat{a}_z$$

In determinant form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Circular Cylindrical Coordinates

→ It is a three dimensional coordinate system.

→ It consists of a point which is located in a plane by giving its distance ρ from the origin.

→ An angle ϕ between the line from the point to the origin and an arbitrary radial line taken as $\phi = 0$

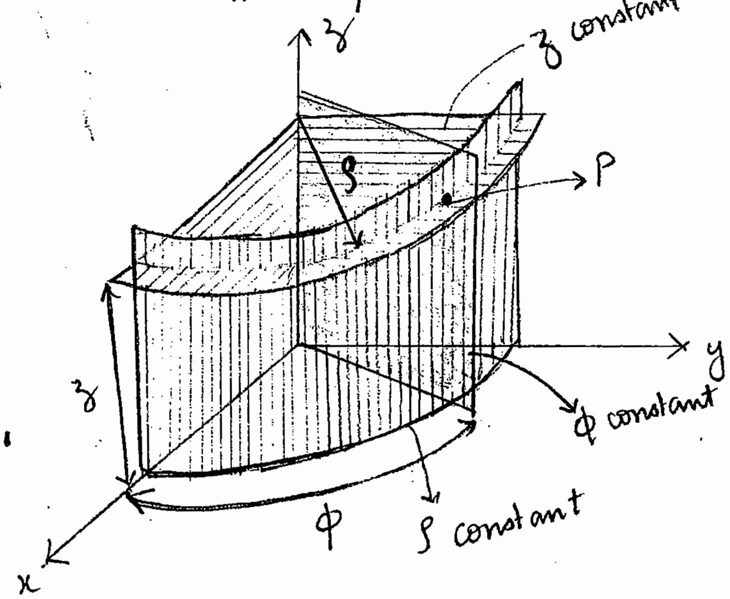
→ The distance z of the point from an arbitrary $z=0$ reference plane which is perpendicular to the line $\rho = 0$.

→ Any point on cylindrical coordinate system can be considere as intersection of three mutually perpendicular planes.

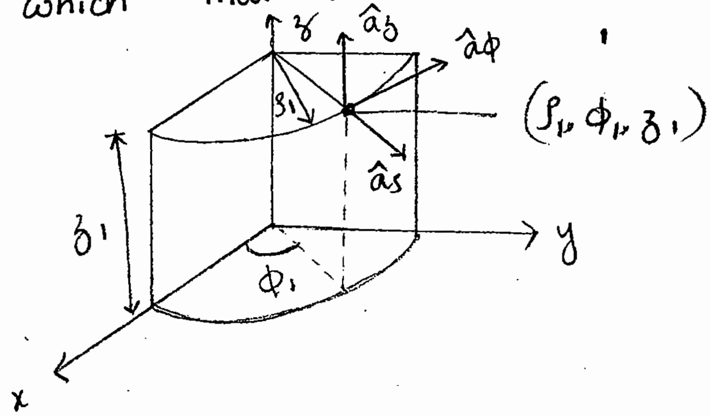
These planes are * a circular Cylinder ρ = constant

* a plane where ϕ = constant

* a plane where z = constant

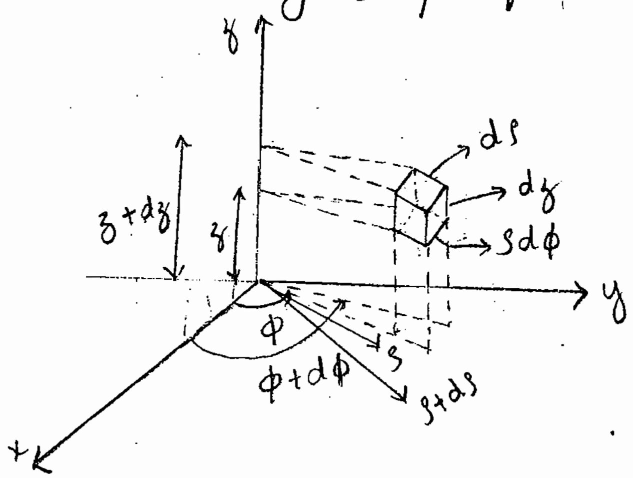


∴ Unit vector can be considered directed towards the increasing coordinate values and are perpendicular to the surface on which that coordinate value is constant.



A differential volume in cylindrical coordinates may be obtained by increasing ρ, ϕ and z by differential increments $d\rho$, $d\phi$ and dz.

The two cylinders of radius ρ and $\rho + d\rho$
The radial planes at angles ϕ and $\phi + d\phi$
Two horizontal planes at elevation z and $z + dz$
give a small volume having shape of truncated wedge



This forms rectangular parallelpiped with sides of length $d\rho$, dz, $\rho d\phi$
The surfaces have areas $\rho d\phi\, d\rho$, $d\rho dz$, $\rho d\phi\, dz$
Volume becomes $\rho d\phi\, d\rho\, dz$.

The relationship between rectangular and cylindrical systems.

$$x = \rho \cos\phi \qquad\qquad \rho = \sqrt{x^2+y^2}$$
$$y = \rho \sin\phi \qquad \& \qquad \phi = \tan^{-1}(y/x)$$
$$z = z \qquad\qquad z = z$$

Using above equations scalar functions in one coordinate system can be easily converted to other

If $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$ and we need vector in cylindrical coordinates as $\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$

To find desired component of a vector we find dot product of the vector and unit vector in desired direction.

i.e, If A_ρ is scalar component of vector $\vec{A}$ in ρ direction then

$$A_\rho = \vec{A} \cdot \hat{a}_\rho = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \cdot \hat{a}_\rho = A_x \hat{a}_x \cdot \hat{a}_\rho + A_y \hat{a}_y \cdot \hat{a}_\rho + \cancel{A_z \hat{a}_z \cdot \hat{a}_\rho}$$

$$\boxed{A_\rho = A_x \hat{a}_x \cdot \hat{a}_\rho + A_y \hat{a}_y \cdot \hat{a}_\rho}$$

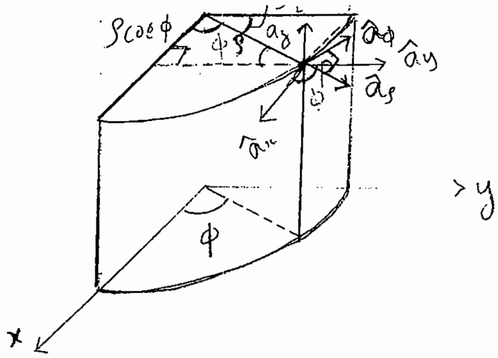
$$A_\phi = \vec{A} \cdot \hat{a}_\phi = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \cdot \hat{a}_\phi = A_x \hat{a}_x \cdot \hat{a}_\phi + A_y \hat{a}_y \cdot \hat{a}_\phi + \cancel{A_z \hat{a}_z \cdot \hat{a}_\phi}$$

$$\boxed{A_\phi = A_x \hat{a}_x \cdot \hat{a}_\phi + A_y \hat{a}_y \cdot \hat{a}_\phi}$$

$$A_z = \vec{A} \cdot \hat{a}_z = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \cdot \hat{a}_z = A_z$$

$$\boxed{A_z = A_z}$$

In above equations the dot products can be solved by applying definition of dot product. Since these are unit vectors the dot product will be cosine of angle between two vectors



	a_ρ	a_ϕ	a_z
$a_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$a_y \cdot$	$\sin\phi$	$\cos\phi$	0
$a_z \cdot$	0	0	1

The Spherical Coordinate System

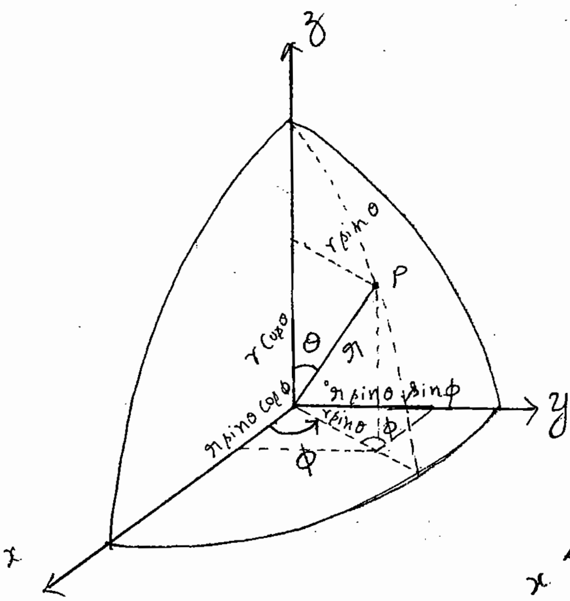
→ The first coordinate in spherical coordinate system is the distance from the origin to any point as r.

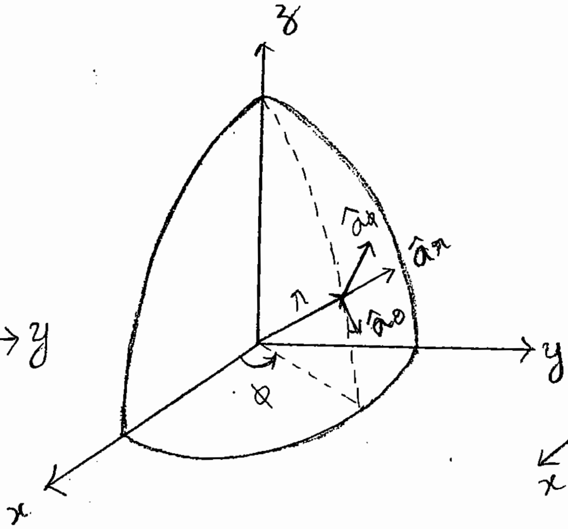
→ The second coordinate is the angle θ between the z axis and the line drawn from the origin to the point P

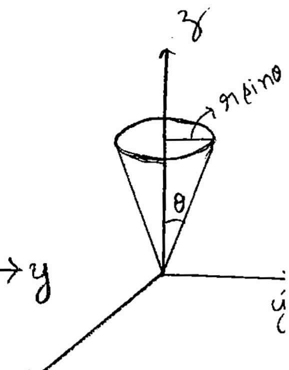
→ The surface θ = constant is a cone. The surface of sphere and surface of cone are everywhere perpendicular to each other. The intersection of θ = Constant and surface of sphere forms a circle of radius $r\sin\theta$.

→ The third coordinate φ is also an angle between the x axis and the projection of point P on z=0 plane

→ Unit vectors may be defined at any point. Each unit Vector is $\perp^{lar}$ to one another and oriented in the increasing direction of Coordinates

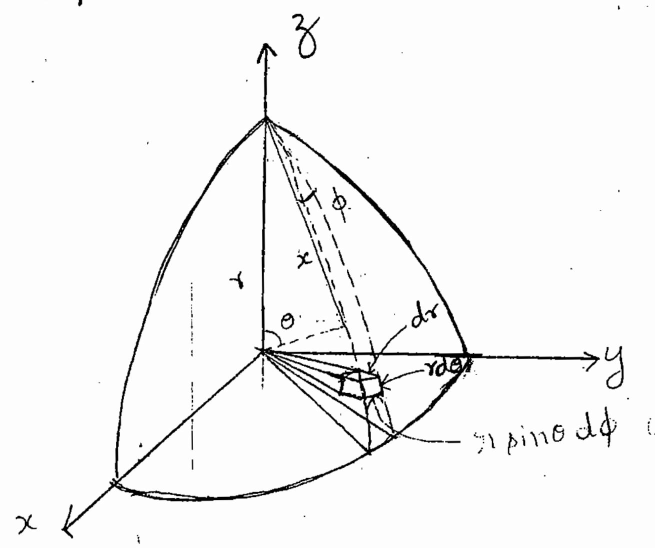


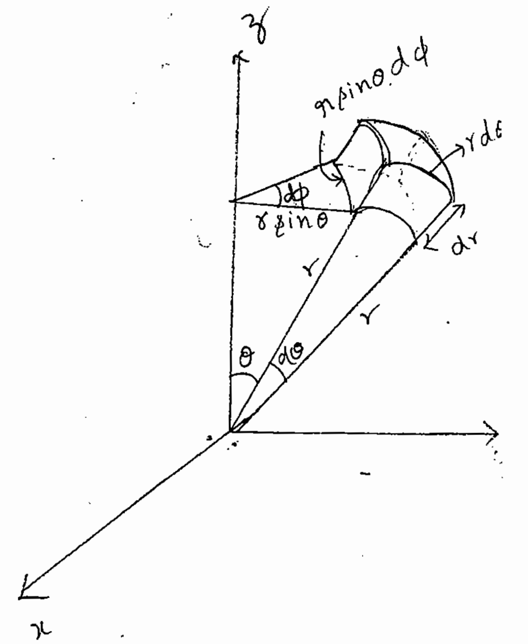




θ = Constant

→ A differential volume element may be constructed in spherical coordinate by increasing r, θ and φ by dr, dθ and dφ as shown in figure.





→ The volume is $r^2 \sin\theta \, dr \, d\theta \, d\phi$

→ The transformations of scalars from rectangular to spherical coordinate system is

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

The transformation in the reverse direction is achived as (9)

$$r = \sqrt{x^2+y^2+z^2} \qquad (r \geq 0)$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

$$\phi = \tan^{-1}(y/z)$$

The transformation of vectors requires the determination of the products of the unit vectors in rectangular & spherical coordinates

	$\hat{a}_r$	$\hat{a}_\theta$	$\hat{a}_\phi$
$\hat{a}_x \cdot$	$\sin\theta \cos\phi$	$\cos\theta \sin\phi$	$-\sin\phi$
$\hat{a}_y \cdot$	$\sin\theta \sin\phi$	$\cos\theta \sin\phi$	$\cos\phi$
$\hat{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Unit - 1

Introduction to Electrostatics

Coulomb's Law and Electric field Intensity.

Coulomb stated that the force between two very small objects separated in vacuum or free space by distance R which is large compared to their size is proportional to the charge on each and inversely proportional to the square of distance between them.

$$F = k\frac{Q_1 Q_2}{R^2} \quad (\text{measured in } N)$$

Q_1 & Q_2 → Positive or negative quantity of charges

R → Separation measured in meters

k → Proportionality constant

This will be achieved if $k = \frac{1}{4\pi\epsilon_0}$

ϵ_0 permittivity of free space (measured in F/m^2)

$$\epsilon_0 = 8.854 \times 10^{-12} = \frac{1}{36\pi} 10^{-9}\ F/m$$

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

Vector form of above equation can be written by considering the force acting along the line joining the 2 charges. Force is repulsive if they have same sign and attractive if opposite sign.

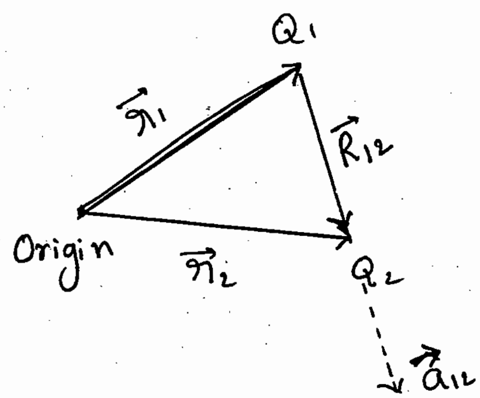
Let the vector $\vec{r}_1$ locates Q_1 and $\vec{r}_2$ locates Q_2 then

$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$ directed line segment from Q_1 to Q_2

i.e, $$\vec{F}_2 = \frac{Q_1 Q_2}{4\pi \epsilon_0 R_{12}^2} \vec{a}_{12}$$

$\vec{F}_2$ is the force on Q_2 and $\vec{a}_{12}$ is the unit vector in the direction of $\vec{R}_{12}$

$$\vec{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$



Electric Field Intensity.

If Q_1 is the charge fixed in a position and Q_t is test charge moving around Q_1. Then Q_1 will exert force on Q_t

This force is given by $\vec{F}_t = \dfrac{Q_1 Q_t}{4\pi\epsilon_0 |\vec{R}_{1t}|^2} \vec{a}_{1t}$

Force/unit charge is $\dfrac{\vec{F}_t}{\vec{Q}_t} = \dfrac{Q_1}{4\pi\epsilon_0 |R_{1t}|^2} \vec{a}_{1t}$

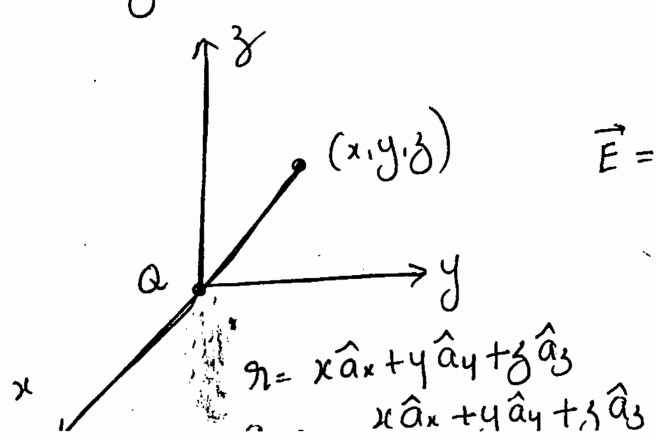
The quantity $\dfrac{Q_1}{4\pi\epsilon_0 |R_{1t}|^2} \vec{a}_{1t}$ is function of only Q_1 and directed line segment from Q_1 to Q_t. This describes a vector field known as Electric Field Intensity.

The unit of electric field intensity is N/c i.e, Force/unit charge is also measured in V/m.

Electric field intensity $\vec{E} = \vec{F}_t / Q_t$

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2} \hat{a}_{1t}$$

If charge Q is at the origin and $\vec{E}$ at (x, y, z) is



$$\vec{E} = \frac{Q}{4\pi\epsilon_0 [x^2+y^2+z^2]} \frac{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}{[x^2+y^2+z^2]^{3/2}}$$

If charge is not at the origin of coordinate system, if a charge Q located at $r' = x'\hat{a}_x + y'\hat{a}_y + z'\hat{a}_z$ and we need its field intensity at point $r = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ by expressing R as $r - r'$

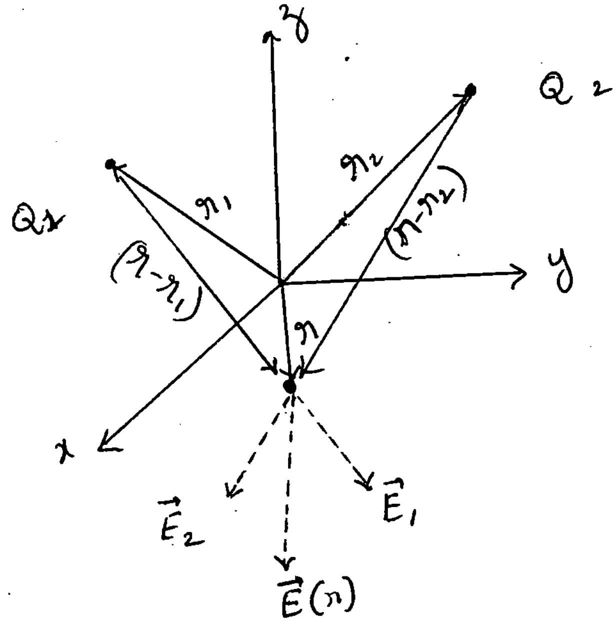
$$E(r) = \frac{Q}{4\pi\epsilon_0 (|r-r'|)^2} \frac{(r-r')}{|r-r'|} = \frac{Q\,(r-r')}{4\pi\epsilon_0 (r-r')^3}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{(x-x')\hat{a}_x + (y-y')\hat{a}_y + (z-z')\hat{a}_z}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}$$

→ Coulomb forces are linear i.e, electric field intensity due to two point charges Q_1 at r_1 and Q_2 at r_2 is sum of forces on Q_t caused by Q_1 and Q_2 acting alone

$$\vec{E}(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|^2}\vec{a}_1 + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|^2}\vec{a}_2$$

where a_1 and a_2 are unit vectors in the direction $(r-r_1)$ and $(r-r_2)$



The field ...

$$\vec{E}(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|^2}\vec{a}_1 + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|^2}\vec{a}_2 + \cdots + \frac{Q_n}{4\pi\epsilon_0 |r-r_n|^2}\vec{a}_n$$

$$\vec{E}(r) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |r-r_m|^2}\vec{a}_m \quad V/m$$

Field due to a continous volume charge distribution

The space filled with large number of charges with smooth continous distribution described by a volume charge density.

Volume charge density is denoted by ρ_v (C/m^3)

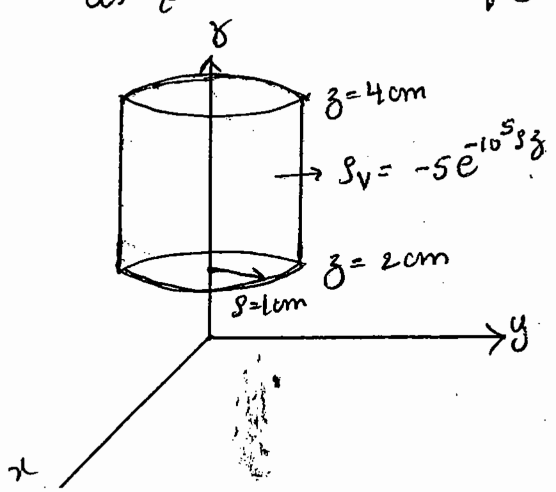
The small amount of charge ΔQ in small volume ΔV is

$$\Delta Q = \rho_v \Delta V$$

The total charge within some finite volume is

$$Q = \int_{vol} \rho_v \, dv$$

Eg: The total charge contained in 2cm length electron beam as shown in the figure is



The total charge

$$Q = \int_{vol} \rho_v \, dv$$

$$= \int_{z=0.02}^{0.04}\int_{\rho=0}^{0.01}\int_{\phi=0}^{2\pi} \rho_v \, \rho \, d\rho \, d\phi \, dz$$

$$Q = \int_{z=0.02} \int_{\rho=0} \int_{\phi=0} -5\times10^{-6}\, e^{-10^5 \rho z}\; \rho \, d\rho\, d\phi\, dz$$

$$= \int_{z=0.02}^{0.04} \int_{\rho=0}^{0.01} -5\times10^{-6}\, e^{-10^5 \rho z}\; \rho\, d\rho\; (2\pi)$$

$$= -10\pi\times10^{-6} \int_{z=0.02}^{0.04} \int_{\rho=0}^{0.01} e^{-10^5 \rho z}\; \rho\, d\rho\, dz$$

$$= -\pi\times10^{-5} \int_{\rho=0}^{0.01} \frac{-1}{10^5 \rho}\; e^{-10^5 \rho z}\Big]_{0.02}^{0.04} \rho\, d\rho$$

$$= \frac{\pi\times10^{-5}}{10^5} \int_{\rho=0}^{0.01} \left(e^{-10^3 \rho \cdot 4} - e^{-10^3 2\rho}\right) d\rho$$

$$= 10^{-10}\pi \left[\int_{\rho=0}^{0.01} e^{-4000\rho}\, d\rho - \int_{\rho=0}^{0.01} e^{-2000\rho}\, d\rho \right]$$

$$= 10^{-10}\pi \left\{ \left(\frac{-1}{4000}\right) e^{-4000\rho}\Big]_0^{0.01} - \left(\frac{-1}{2000}\right) e^{-2000\rho}\Big]_0^{0.01} \right\}$$

$$= 10^{-10}\pi \left\{ \frac{-1}{4000}\left[e^{-40} - e^{0}\right] + \frac{1}{2000}\left[e^{-20} - e^{0}\right] \right\}$$

$$= 10^{-10}\pi \left[\frac{1}{\dots} - \frac{1}{\dots} \right] = 0.0785 \text{ pC}$$ electron

Field of a Line Charge

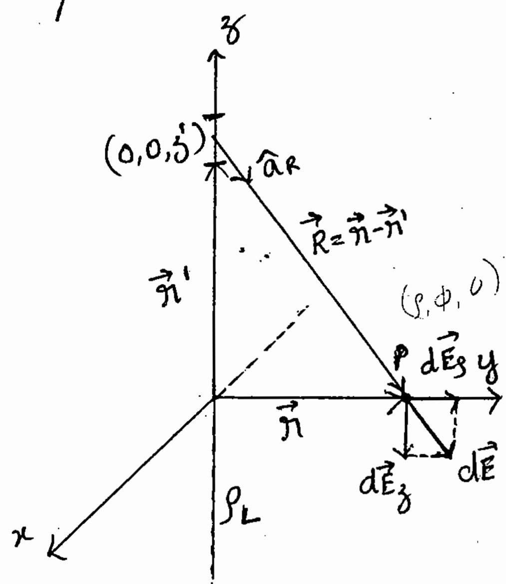
→ The charge distribution in a charged conductor with small radius can be treated as line charge density ρ_L C/m

→ If electron motion is steady and uniform and if we ignore magnetic field for a moment these electron beam may be considered as stationary electrons.

· Consider a straight line charge extending along the z-axis in a cylindrical coordinate system from $-\infty$ to ∞ we desire the electric field intensity $\vec{E}$ at any point from a uniform line charge density

· Symmetry should be considered first in order to determine two specific factors

1) with which coordinates the field does not vary
2) which component of the field are not present.



Lets analyze first factor

From figure, as we move around the line charge varying ϕ while keeping ρ and z constant the line charge appears the same from every angle i-e, azimuthal symmetry & no field component may vary with ϕ

Again if ρ and ϕ are maintained constant while moving z along the line charge the line charge will not vary i.e, axial symmetry leads to fields which are not functions of z.

If ϕ and z are ... Hence
to coulomb's law field become weaker as ρ increases
field varies only with ρ.

→ Each incremental length of line charge acts as point charge and produces an incremental contribution to the electric field.

→ No incremental length produces electric field in ϕ direction hence $\vec{E}_\phi = 0$

→ E_z component is also zero because of presence of charges which are of equal distances above and below the point, this will cancel the field.

→ Therefore there is only $\vec{E}_\rho$ that varies along with ρ coordina

Choose a point $P(0,y,0)$ on y-axis, the incremental charge $dQ = \rho_L dz'$ we h field at P because of incremental

$$d\vec{E} = \frac{\rho_L \, dz' (\vec{r}-\vec{r}')}{4\pi\epsilon_0 \, |\vec{r}-\vec{r}'|^3}$$

$$\vec{r} = y\hat{a}_y = \rho\hat{a}_\rho$$

$$\vec{r}' = z'\hat{a}_z$$

$$\vec{r} - \vec{r}' = \rho\hat{a}_\rho - z'\hat{a}_z$$

$$d\vec{E} = \frac{\rho_L \, dz' \, (\rho\hat{a}_\rho - z'\hat{a}_z)}{4\pi\epsilon_0 \, [\rho^2 + z'^2]^{3/2}}$$

Only $\vec{E}_\rho$ component is present we may simplify

$$dE_\rho = \frac{\rho_L \, dz' \, \rho}{4\pi\epsilon_0 \, [\rho^2 + z'^2]^{3/2}}$$

(5)

$$E_\rho = \int_{-\infty} \frac{\rho_L \, \rho \, dz'}{4\pi\epsilon_0 \left[\rho^2 + z'^2\right]^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho \, dz'}{\left[\rho^2 + z'^2\right]^{3/2}}$$

Let $z' = \rho \tan\theta$

$dz' = \rho \sec^2\theta \, d\theta$

if $z' = -\infty \Rightarrow \theta = -\pi/2$

$z' = +\infty \Rightarrow \theta = \pi/2$

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\rho^2 \sec^2\theta \, d\theta}{\left(\rho^2 + \rho^2 \tan^2\theta\right)^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\rho^2 \sec^2\theta}{\rho^3 \sec^3\theta} \, d\theta$$

$$= \frac{\rho_L}{4\pi\epsilon_0 \rho} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{\rho_L}{4\pi\epsilon_0 \rho} \sin\theta \Big]_{-\pi/2}^{\pi/2} = \frac{\rho_L}{4\pi\epsilon_0 \rho} \left[\sin(\pi/2) - \sin\left(\frac{-\pi}{2}\right)\right]$$

$$\boxed{E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}}$$

$$d\vec{E} = \frac{\rho_s \, dy'}{2\pi\epsilon_0 \sqrt{x^2+y'^2}} \frac{(x\hat{a}_x - y'\hat{a}_y)}{\sqrt{x^2+y'^2}}$$

$$d\vec{E} = \frac{\rho_s \, dy' \, (x\hat{a}_x - y'\hat{a}_y)}{2\pi\epsilon_0 \, (x^2+y'^2)}$$

$y'\hat{a}_y = 0$ as Fields will cancel because of +ve and negative y-axis. Field along x-direction

$$dE_x = \frac{\rho_s \, dy'}{2\pi\epsilon_0} \frac{x}{(x^2+y'^2)}$$

$$E_x = \int_{-\infty}^{\infty} \frac{\rho_s \, dy'}{2\pi\epsilon_0} \frac{x}{(x^2+y'^2)}$$

$$E_x = \frac{\rho_s}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x}{(x^2+y'^2)} dy'$$

$y' = x \tan\theta$

$dy' = x \sec^2\theta \, d\theta$

$$E_x = \frac{\rho_s}{2\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{x^2 \sec^2\theta \, d\theta}{(x^2 + x^2\tan^2\theta)}$$

$$= \frac{\rho_s}{2\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} d\theta$$

$$E_x = \frac{\rho_s}{2\epsilon_0}$$

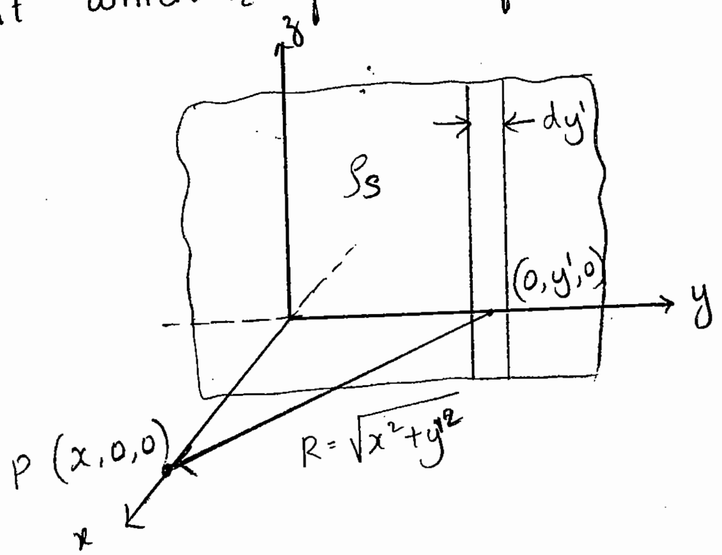
$$\boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0} \hat{a}_x}$$

Field of a Sheet of Charge

(6)

→ This is charge configuration in the infinite sheet of charge having uniform density of $\rho\ C/m^2$, is known as surface charge density. Eg: Strip transmission line parallel plate capacitor.

→ Let us place a sheet of charge along yz plane. Considering symmetry the field will not vary with y & z then the field $\vec{E_y}$ and $\vec{E_z}$ will cancel. Hence only $\vec{E_x}$ is present which is function of x alone



Let us use the field of infinite line charge by dividing the infinite sheet charge into differential width stripe

$\rho_L = \rho_S\, dy'$

The distance from point $P(x,0,0)$ on x-axis is $R = \sqrt{x^2+y'^2}$ differential width stripe to

dE_x is the Contribution of differential width stripe to E_x at P.

Electric flux density. (7)

→ Electric flux is line of force around the charge. This line of force starts from positive charge and terminate on the negative charge.

→ Michael Faraday had a pair of concentric metallic spheres. The outer sphere consists of 2 hemispheres which can be clamped together

→ Faraday dismantled outer sphere and he charged inner sphere with some positive charges
Then the outer sphere joined together with dielectric about 2cm between inner and outer sphere.

→ He discharged outer sphere momentarily by connecting it to ground

→ Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed in the inner sphere.

→. Faraday concluded that there was some sort of displacement from inner sphere to the outer sphere which was independent of the medium.
This is refered as displacement flux or simply electric flux.
Thus total number of lines of force in any perticular electric field is called flux.

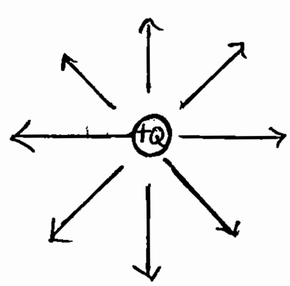
→ If electric flux is denoted as ψ ...
on the inner sphere by Q then

$$\psi = Q \quad (\text{unit } C)$$

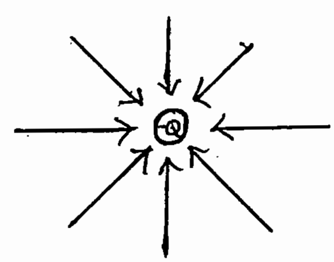
Properties of flux Lines -

1. Electric flux lines start from positive charge and terminate on negative charge
2. If negative charge is absent then flux lines terminate at
3. There will be more number of lines if electric field is strong.
4. The flux lines will be parallel & never cross other flux lines
5. Flux lines are independent of medium in which charges are placed.
6. The lines always enter and leave the charge surface normally.

To ∞

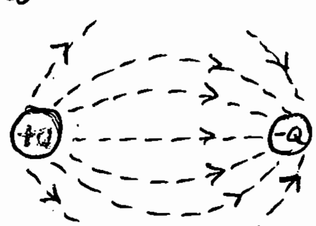


Flux lines +Q



From ∞

Flux lines -Q



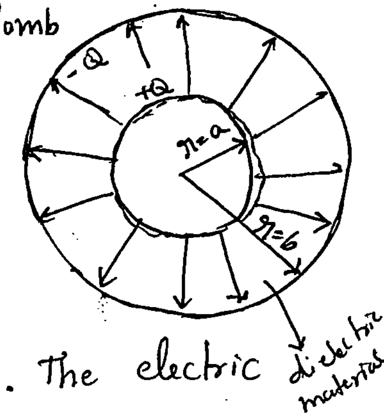
Flux lines +Q to -Q

∴ If inner sphere of radius 'a' and outer sphere of radius 'b' with charges +Q and -Q respectively is considered

At the surface of inner sphere Ψ Coulomb of electric flux are produced by charge Q Coulomb

The density of the flux at this surface is

$$\frac{\Psi}{4\pi a^2} \quad \text{or} \quad \frac{Q}{4\pi a^2} \; C/m^2$$



The flux density is denoted by letter $\vec{D}$. The electric flux density is a vector field

→ The direction of $\vec{D}$ is direction of flux lines and magnitude is given by the number of flux lines crossing a surface normal to the lines divided by surface area

$$\vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\,\hat{a}_r$$

$$\vec{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\,\hat{a}_r$$

at radial distance r where $a \leq r \leq b$

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a}_r$$

W.K.T

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r$$

Therefore $\boxed{\vec{D} = \epsilon_0 \vec{E}}$ for free space

For general volume charge distribution

$$\vec{E} = \int_{vol} \frac{\rho_v \, dv}{4\pi\epsilon_0 R^2} \hat{a}_R$$

$$\vec{D} = \int_{vol} \frac{\rho_v \, dv}{4\pi r^2} \hat{a}_r$$

Gauss's Law.

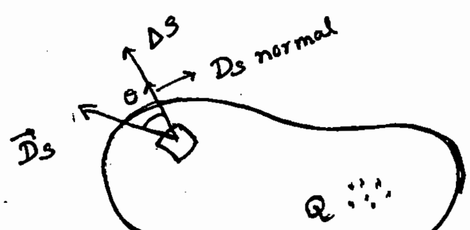
The generalization of Faraday's experiment leads to Gauss's law.

+Q Coulomb of any inner conductor would produce an induced charge of -Q Coulomb on the surrounding conducting surface

i.e, Gauss's law can be defined as

"The electric flux passing through any closed surface is equal to the total charge enclosed by that surface."

Let us imagine distribution of charges as shown, surrounded by a closed surface of any shape.



If the total charge is Q, then Q Coulomb of electric flux will pass through the enclosing surface

At every point on the surface the electric flux density Vector $\vec{D}$ will have some value $\vec{D}_s$. Subscript s indicates that $\vec{D}$ must be evaluated at surface. $\vec{D}_s$ will vary in magnitude and direction from one point to other

→ ΔS is nearly a portion of plane surface. ΔS is a Vector quantity as it includes both magnitude & direction in space i.e, $\Delta \vec{S}$

→ Let $D_{s,norm}$ is the normal to surface $\Delta \vec{S}$ at point P.

The total flux crossing ΔS is

$$\Delta \Psi = \text{flux crossing } \Delta \vec{S} = D_{s,norm} \Delta S = D_s \cos\theta \, \Delta S$$

$$= \vec{D}_s \cdot \Delta \vec{S}$$

The total flux passing through closed surface is

$$\Psi = \int d\psi = \oint_{\substack{\text{Closed} \\ \text{Surface}}} \vec{D}_s \cdot d\vec{S} = \text{charge enclosed} = Q$$

The charge enclosed might be

Several point charges $Q = \sum Q_n$

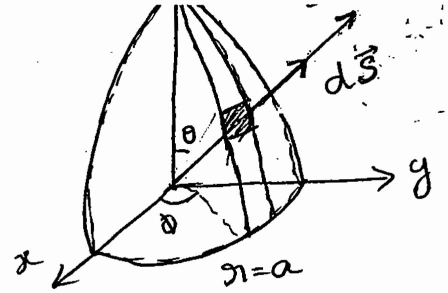
Line charge $Q = \int \rho_L \, dL$

Surface charge $Q = \int \rho_s \, ds$

Volume charge $Q = \int \rho_v \, dv$

Let's find the charge enclosed in sphere with radius a with $\vec{E} = \dfrac{Q}{4\pi \epsilon_0 r^2} \vec{a}_r$

Proof : WKT $E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$



$\vec{D} = \epsilon_0 \vec{E}$

$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r$

At surface $d\vec{s}$ of sphere $\vec{D}_s = \frac{Q}{4\pi a^2} \hat{a}_r$

$ds = a^2 \sin\theta \, d\theta \, d\phi$

$d\vec{s} = a^2 \sin\theta \, d\theta \, d\phi \, \vec{a}_r$

$\vec{D}_s \cdot d\vec{s} = \frac{Q}{4\pi a^2} \vec{a}_r \cdot a^2 \sin\theta \, d\theta \, d\phi \, \vec{a}_r$

$d\psi = \vec{D}_s \cdot d\vec{s} = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$

$\psi = \oint_S \vec{D}_s \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$

$= \frac{Q}{4\pi} (2\pi) \int_{\theta=0}^{\pi} \sin\theta \, d\theta$

$= \frac{Q}{4\pi} (2\pi) \left. -\cos\theta \right]_0^{\pi}$

$= \frac{Q}{\cancel{4\pi}} (\cancel{2\pi}) (\cancel{2})$

$\underline{\underline{\psi = Q}}$

Applications of Gauss's Law (Symmetrical Charge distributions)

Gauss's law is another way of stating Coulomb's law

Gauss's law can be used to find $\vec{E}$ & $\vec{D}$ for symmetrical charge distributions like point charge, infinite line charge, an infinite sheet charge and spherical distribution of charges

- It also help us to find total charge present inside the closed surface

- Gauss's law holds good for all closed surfaces symmetrical or non symmetrical distribution of charges, but Gauss's law we can find $\vec{E}$ & $\vec{D}$ only for symmetrical charge distributions

→ Evaluating $\vec{E}$ & $\vec{D}$ still can be made easy if the surface we select satisfies two conditions.

① $\vec{D}_s$ is everywhere either normal or tangential to the closed surface, so that $\vec{D}_s \cdot d\vec{s}$ is either 0 or $D\,ds$ respectively

② If $\vec{D}_s \cdot d\vec{s}$ is not zero then D_s is constant over a portion of closed surface. These assumptions allows us to replace the dot product with the product of scalars D_s and ds.

If we select symmetrical surface then both conditions will be easily satisfied.

① Field Intensity because of point charge

→ Let us consider a point charge Q at origin of spherical coordinate system

→ The surface is spherical surface. Ds is everywhere normal to the surface. Ds has some value at all points in the surface

$$Q = \oint_S \vec{D_s} \cdot d\vec{s} = \oint_{Sphere} D_s \, ds$$

$$= D_s \oint_{Sphere} ds = D_s \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta$$

$$Q = D_s (4\pi r^2)$$

$$D_s = \frac{Q}{4\pi r^2}$$

Ds is directed radially outwards

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2} \vec{a_r}} \qquad \boxed{\vec{E} = \frac{Q}{4\pi \epsilon_0 r^2} \vec{a_r}}$$

② Field Intensity of a Infinite line charge

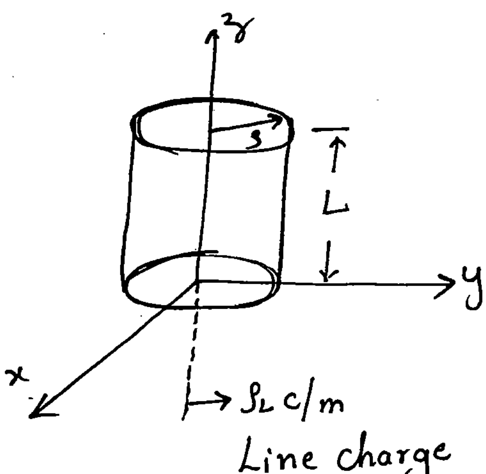
Consider an infinite line charge of density ρ_L c/m lying along z-axis from $-\infty$ to ∞

Consider Guassian surface as the right circular cylinder with z-axis with radius ρ. The length of cylinder is L

→ The flux density at any point on the surface of cylinder is directed radially outwards. $\vec{a}_r$ (H)

→ Only radial component of D is present which varies only with ρ.

→ If we consider Gaussian surface as cylindrical surface then D_ρ will be everywhere normal.



Applying Gauss's law

$$Q = \oint_{cyl} \vec{D_s} \cdot d\vec{s} = D_s \int_{sides} ds + 0 \int_{top} ds + 0 \int_{bottom} ds$$

Since $\vec{D}$ has only radial component and no component along $\vec{a}_z$ & $-\vec{a}_z$

$$\oint_{top} \vec{D_s} \cdot d\vec{s} = \oint_{bottom} \vec{D_s} \cdot d\vec{s} = 0$$

then $$Q = D_s \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = 2\pi \rho D_s L$$

$$D_\rho = \frac{Q}{2\pi \rho L}$$

The total charge enclose in L length line charge

$$Q = \rho_L (L)$$

then $$\boxed{\vec{D} = \frac{\rho_L \vec{a}_\rho}{2\pi \rho}}$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi \epsilon_0 \rho} \vec{a}_\rho \ \text{V/m}}$$

③ Field Intensity of Coaxial Cable

→ To find electric field intensity of coaxial cable is difficult from standpoint of coulomb's law.

→ Consider coaxial cylindrical conductors with inner conduct radius a and outer conductor radius b each infinite in length.

→ Let ρ_s be charge distribution on outer surface of inner Conductor

→ Considering symmetry, only D_ρ component is present and it is function of only ρ

Consider a right circular cylinder of radius ρ and length L. where $a<\rho<b$.

From discussion of line charge

$$Q = D_\rho 2\pi\rho L \longrightarrow ①$$

Total charge on a length L of inner conductor is

$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_s \, a \, d\phi \, dz.$$

$$Q = 2\pi \rho_s a L \longrightarrow ②$$

Equating ① & ②

$$D_\rho \cancel{2\pi}\rho \cancel{L} = \cancel{2\pi} \rho_s a \cancel{L}$$

$$D_\rho = \frac{\rho_s a}{\rho}$$

$$\boxed{\vec{D} = \frac{\rho_s a}{\rho} \hat{a}_\rho} \qquad a<\rho<b$$

ρ_s can be expressed in terms of ρ_L

$$\rho_L = \frac{\rho_s \times \text{surface area}}{\text{Total length}} = \frac{\rho_s \times 2\pi a \cancel{L}}{\cancel{L}}$$

$$\rho_L = (2\pi a)\rho_s$$

$$\vec{D} = \frac{\cancel{a}\rho_L}{(2\pi \cancel{a})\rho}\hat{a}_\rho = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

$$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho}$$

Every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of outer cylinder the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{s(\text{inner cyl})}$$

$$2\pi b L \rho_{s(\text{outer cyl})} = -2\pi a L \rho_{s(\text{inner cyl})}$$

$$\rho_{s(\text{outer cyl})} = \frac{a}{b}\rho_{s(\text{inner cyl})}$$

If $\rho > b$ then total charge is zero i.e,

$$D_\rho = 0 \qquad \rho > b$$

$$D_\rho = 0 \qquad \rho < a$$

Application of Gauss's law : Differential volume element.

→ Gauss law can be applied to the non symmetrical surface.

→ For non symmetric surface simple Gaussian surface cannot be chosen such that normal component of $\vec{D}$ is either 0 or constant.

→ This problem can be solved by choosing very small surface such that D is almost constant over the surface and the small change in $\vec{D}$ may be adequately represented by using the first two terms of Taylor's series expansion for D.

→ Let us consider any point P located in a rectangular cordinate system

Let the value of $\vec{D}$ at the point P may be expressed as

$$\vec{D_0} = \vec{D}_{x0}\,\hat{a}_x + \vec{D}_{y0}\,\hat{a}_y + \vec{D}_{z0}\,\hat{a}_z$$

→ We choose as our closed surface the small rectangular box centered at P having sides of length Δx, Δy and Δz and apply Gauss's law

$$\oint_s \vec{D}\cdot d\vec{s} = Q$$

Integral can be divided on six faces one over each face.

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

Consider the first of these in detail.
Since surface element is very small, $\vec{D}$ is essentially constant

$$\int_{front} = \vec{D}_{front} \cdot \Delta \vec{S}_{front}$$

$$= \vec{D}_{front} \cdot \Delta y \, \Delta z \, \hat{a}_x$$

$$= D_{x,front} \, \Delta y \, \Delta z$$

We have to approximate only value of D_x at this face. Front face is at distance $\frac{\Delta x}{2}$ from P.

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

D_{x0} is value of D_x at P. Since D_x in general also varies with y and z, partial derivative must be used to represent rate of change of D_x with x

$$\int_{front} = \left[D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right] \Delta y \, \Delta z$$

Considering integral over back surface

$$\int_{back} = \vec{D}_{back} \cdot \Delta \vec{S}_{back}$$

$$= \vec{D}_{back} \cdot (-\Delta y \, \Delta z \, \hat{a}_x)$$

$$= - D_{x,back} \, \Delta y \, \Delta z$$

$$D_{x,back} = D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{back} = \left[- D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right] \Delta y \, \Delta z$$

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly same process

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

These results are collected to yield

$$\oint_S \vec{D} \cdot d\vec{s} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] \Delta x \, \Delta y \, \Delta z$$

$$\oint \vec{D} \cdot d\vec{s} = Q = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] \Delta V$$

Divergence

The expression for charge enclosed by volume ΔV becomes exact by allowing the volume element ΔV to shrink to zero

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta V} = \frac{Q}{\Delta V}$$

as a limit $$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \lim_{\Delta V \to 0} \frac{\oint_S \vec{D} \cdot d\vec{s}}{\Delta V} = \lim_{\Delta V \to 0} \frac{Q}{\Delta V}$$

The last term is the volume charge density ρ_v

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta V \to 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta V} = \rho_v$$

We can write it as two separate equations (14)

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \lim_{\Delta v \to 0} \frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v} \rightarrow (a)$$

and

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \rho_v \rightarrow (b)$$

Equation (a) involves no charge density. This operation is known as "divergence", divergence of any vector is defined as the outflow of flux from fr small closed surface per unit volume as the volume shrinks to zero.

$$\text{div } D = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] \longrightarrow (c)$$

The positive divergence for any vector quantity indicates a source of that vector quantity and negative divergence indicates a sink.

Equation (b) is result of applying definition of divergence to differential volume in rectangular coordinate system.

Similary divergence of cylindrical coordinate system is

$$\text{div } D = \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}\right]$$

in Spherical system

$$\text{div } D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Maxwell's first equation

With the concept of divergence for given Gauss's law

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

per unit volume

$$\frac{\oint \vec{D} \cdot d\vec{s}}{\Delta V} = \frac{Q}{\Delta V}$$

As the volume shrinks to zero

$$\lim_{\Delta V \to 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta V} = \lim_{\Delta V \to 0} \frac{Q}{\Delta V}$$

i.e,

$$\boxed{\text{div } \vec{D} = \rho_v}$$

This is first of Maxwell's four equations, it states that electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density. This equation is also known as point form of Gauss's law.

The vector operator ∇ and the divergence theorem

Divergence is an operation on vector yielding scalar, same as dot product. It is possible to find something which may be dotted formally with $\vec{D}$ to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

This can be accomplished by dot product i.e, we define operator ∇ as a vector operator

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

Considering $\nabla \cdot \vec{D}$

$$\cdot\vec{D} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z\right)$$

$$\nabla\cdot\vec{D} = \left[\frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}D_z\right]$$

This divergence of $\vec{D}$ i.e,

$$\text{div } \vec{D} = \nabla\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

From Gauss's law

$$\oint_S \vec{D}\cdot d\vec{s} = Q$$

$$Q = \int_{vol} \rho_v \, dv$$

$$\int_{vol} \nabla\cdot D \, dv = \int_{vol} \rho_v \, dv$$

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = \int_{vol} \nabla\cdot\vec{D} \, dv}$$

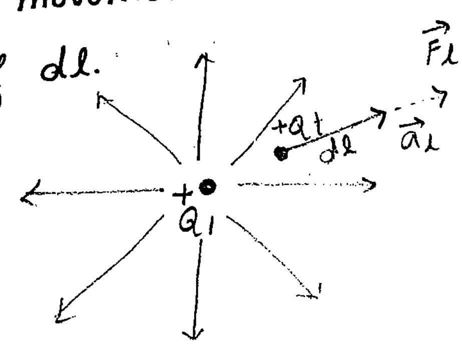
Energy & Potential.

Electric scalar Potential can be used to obtain electric field intensity $\vec{E}$. This is another method of obtaining vector field $\vec{E}$ from electric scalar potential.

Energy Expended in moving a point charge in an Electric Field

Electric field intensity is defined as the force on a unit test charge at that point at which we want to find value of $\vec{E}$.

Consider a positive charge Q_1 and its electric field $\vec{E}$. If a positive test charge Q_t is placed in this field, it will move due to force of repulsion. Let movement of charge Q_t is dl. The direction in which the movement has taken is denoted by $\hat{a}_L$ in the direction of dl.



The force exerted by field $\vec{E}$ on Q_t is

$$\vec{F} = Q_t \vec{E} \quad N$$

But the component of a vector in the direction of the unit vector is the dot product of the vector with that unit vector

$$F_l = \vec{F} \cdot \hat{a}_L = Q_t \vec{E} \cdot \hat{a}_L \quad N$$

If we wish to move charge in the direction of field by same distance dl, our energy expenditure turns to be -ve. i.e,

$$F_{applied} = -F_l = -Q_t \vec{E} \cdot \hat{a}_L \; N$$

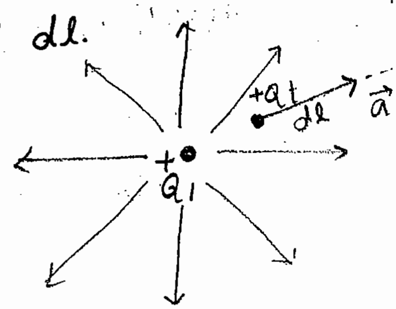
Energy & Potential.

Electric scalar Potential can be used to obtain electric field intensity $\vec{E}$. This is another method of obtaining vector field $\vec{E}$ from electric scalar potential.

Energy Expended in moving a point charge in an Electric Field

Electric field intensity is defined as the force on a unit test charge at that point at which we want to find value of $\vec{E}$.

Consider a positive charge Q_1 and its electric field $\vec{E}$. If a positive test charge Q_t is placed in this field, it will move due to force of repulsion. Let movement of charge Q_t is dl. The direction in which the movement has taken is denoted by $\hat{a}_L$ in the direction of dl.



The force exerted by field $\vec{E}$ on Q_t is

$$\vec{F} = Q_t \vec{E} \ N$$

But the component of a vector in the direction of the unit vector is the dot product of the vector with that unit vector

$$F_l = \vec{F} \cdot \hat{a}_L = Q_t \ \vec{E} \cdot \hat{a}_L \ N$$

If we wish to move charge in the direction of field by same distance dl, our energy expenditure turns to be $-v$ i.e,

$$F_{applied} = -F_l = -Q_t \ \vec{E} \cdot \hat{a}_L \ N$$

Mathematically the differential work done by external source moving Q is

$$dW = (F_{applied}) \times (dl) = (-Q_t\ \vec{E} \cdot \hat{a}_L)(dl)$$

$$\boxed{dW = -Q_t\ \vec{E} \cdot d\vec{L}}$$

Total work done if a charge is moved from initial position to the final position against direction of electric field is

$$W = \int_{\text{Initial position}}^{\text{Final Position}} dW =$$

$$\boxed{W = -Q \int_{\text{Initial}}^{\text{Final}} \vec{E} \cdot d\vec{L} \quad J} \text{ units}$$

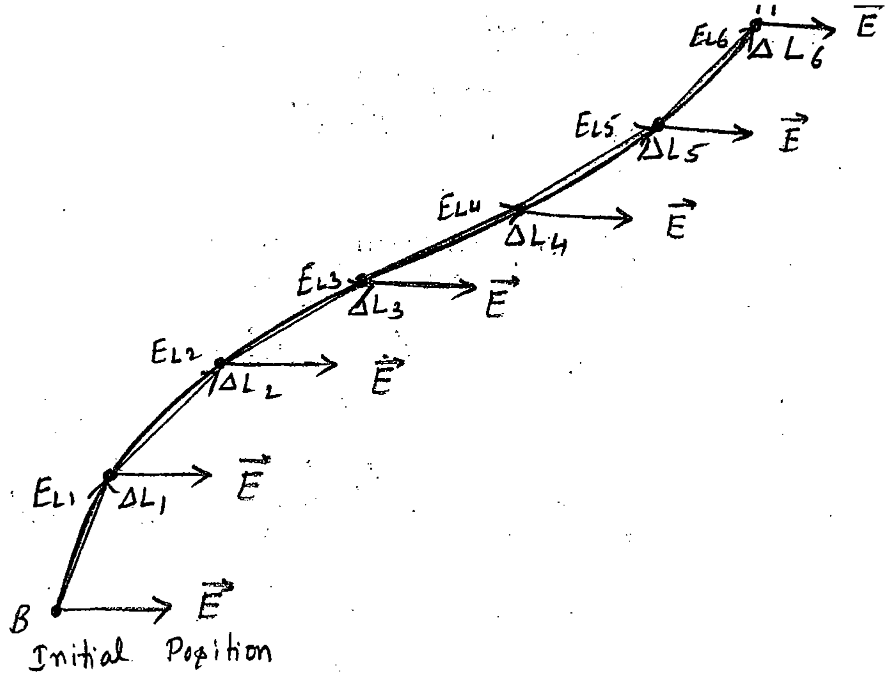
Line Integral

Consider a charge Q is moved from initial position B to final position A against the electric field $\vec{E}$ then

$$W = -Q \int_{B}^{A} \vec{E} \cdot d\vec{L}$$

$\vec{E} \cdot d\vec{L}$ gives component of $\vec{E}$ along $d\vec{L}$ direction.

Mathematically this involves choosing any arbitrary path B to A, break up into a large number of very small segments, multiply the component of the field along each segment by the length of the segment, then add the result of all the segments



A path has been chosen from B to A, and a uniform electric field is chosen.

The path is divided into 6 line segments $\Delta L_1, \Delta L_2 \dots \Delta L_6$ and components of $\vec{E}$ along each segment are denoted by $E_{L1}, E_{L2} \dots E_{L6}$. The work involved in moving a charge Q from B to A is then

$$W = -Q\left[E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \dots + E_{L6}\Delta L_6\right]$$

Using vector notation

$$W = -Q\left[\vec{E}_1 \cdot \vec{\Delta L_1} + \vec{E}_2 \cdot \vec{\Delta L_2} + \dots + \vec{E}_6 \cdot \vec{\Delta L_6}\right]$$

Since we have assumed uniform electric field $\vec{E} = \vec{E}_1 = \vec{E}_2 = \dots$

$$W = -Q\,\vec{E} \cdot \left[\vec{\Delta L_1} + \vec{\Delta L_2} + \dots + \vec{\Delta L_6}\right]$$

The sum $\left[\vec{\Delta L_1} + \vec{\Delta L_2} + \dots + \vec{\Delta L_6}\right]$ is result obtained by parallelogram law of addition. the sum is vector directed fro B to A i.e., $\vec{L}_{BA}$ therefore

$$\boxed{W = -Q\,\vec{E} \cdot \vec{L}_{BA}}$$

From the integral expression

$$W = -Q\int_B^A \vec{E}\cdot d\vec{L}$$

The electric field is uniform hence

$$W = -Q\,\vec{E}\cdot\int_B^A d\vec{L}$$

Last integral is $\vec{L}_{BA}$

$$\boxed{W = -Q\,\vec{E}\cdot\vec{L}_{BA}}$$ uniform $\vec{E}$.

While solving the problem it is necessary to select $d\vec{L}$ according to coordinate systems selected. The expression for $d\vec{L}$ in three coordinate systems are

Cartesian	$d\vec{L} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$
Cylindrical	$d\vec{L} = dr\,\hat{a}_r + r\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$
Spherical	$d\vec{L} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$

Potential Difference and Potential

→ The work done in moving a point charge Q from point B to A in the electric field $\vec{E}$ is given by

$$W = -Q\int_B^A \vec{E}\cdot d\vec{L}$$

If Q is selected as unit charge then from the above equation we get the work done in moving unit charge from B to A.

This work done in moving unit charge from point B to A in the field $\vec{E}$ is called potential difference between the points B and A. i.e,

$$\text{Potential difference} = V_{AB} = -\int_B^A \vec{E}\cdot d\vec{L} \quad \text{Volts or J/C}$$

From the integral expression

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L}$$

The electric field is uniform hence

$$W = -Q\, \vec{E} \cdot \int_B^A d\vec{L}$$

Last integral is $\vec{L}_{BA}$

$$\boxed{W = -Q\, \vec{E} \cdot \vec{L}_{BA}}$$ uniform $\vec{E}$.

While solving the problem it is necessary to select $d\vec{L}$ according to coordinate systems selected. The expression for $d\vec{L}$ in three coordinate systems are

Cartesian $\quad d\vec{L} = dx\, \hat{a}_x + dy\, \hat{a}_y + dz\, \hat{a}_z$

Cylindrical $\quad d\vec{L} = dr\, \hat{a}_r + r\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z$

Spherical $\quad d\vec{L} = dr\, \hat{a}_r + r\, d\theta\, \hat{a}_\theta + r \sin\theta\, d\phi\, \hat{a}_\phi$

Potential Difference and Potential

→ The work done in moving a point charge Q from point B to A in the electric field $\vec{E}$ is given by

$$W = -Q \int_B^A \vec{E} \cdot d\vec{L}$$

If Q is selected as unit charge then from the above equation we get the work done in moving unit charge from B to A.

This work done in moving unit charge from point B to A in the field $\vec{E}$ is called potential difference between the points B and A. i.e,

$$\text{Potential difference} = V_{AB} = -\int_B^A \vec{E} \cdot d\vec{L} \quad \text{Volts or } J/C$$

→ If B is initial position and A is the final point then the potential difference is denoted as V_{AB} which indicates the potential difference between the points A and B. & charge is moved from B to A.

→ If V_{AB} is positive then work done by external source in moving the unit charge from B to A is against direction of $\vec{E}$

→ One volt potential difference is one joule of work done in moving unit charge from one point to other in the field $\vec{E}$

$$1\,\text{volt} = \frac{1\,\text{joule}}{1\,\text{coulomb}}$$

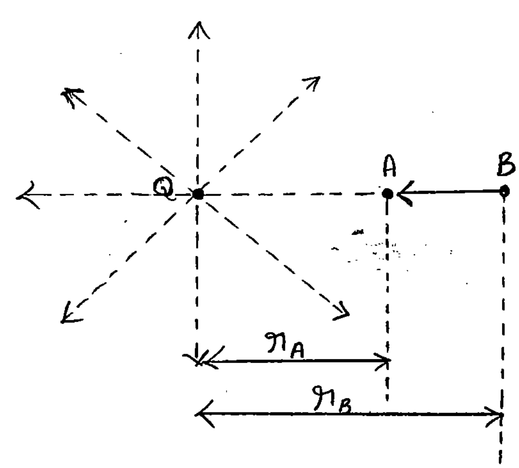
→ It is convenient to express absolute potentials than potential difference. Absolute potentials are measured with respect to the specified reference position. Such reference position is assumed to be at zero potential. For practical circuits such zero potential is selected as ground.

The potential field of a point charge

Consider a point charge located at the origin of a spherical coordinate system, producing $\vec{E}$ radially in all the directions

The field $\vec{E}$ due to a point charge Q at distance r from origin is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r$$



Consider a unit charge which is placed at a point at radial distance r_B from the origin. It is moved against the direction of $\vec{E}$ from point B to point A. The point A is at radial distance r_A from the origin

The differential length in spherical coordinate system is

$$d\vec{L} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$

The potential difference between A & B is V_{AB} where

$$V_{AB} = -\int_B^A \vec{E}\cdot d\vec{L}$$

$B = r_B$ & $A = r_A$

$$V_{AB} = -\int_{r_B}^{r_A} \left(\frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r\right)\cdot\left(dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi\right)$$

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$= \frac{-Q}{4\pi\epsilon_0}\int_{r_B}^{r_A} \frac{1}{r^2}\,dr$$

$$V_{AB} = \frac{-Q}{4\pi\epsilon_0}\left(\frac{-1}{r}\right)_{r_B}^{r_A}$$

$$V_{AB} = \frac{+Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \text{ V}$$

→ Absolute potential can be defined by considering a reference point. The simplest possibility is to let $V=0$ at infinity. If $r = r_B$ recede to infinity the potential at r_A becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

→ Thus the expression for potential at any distance r from point charge Q at the origin is

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}}$$

→ If charge Q_1 is at $\vec{r}_1$ and potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0 |\vec{r}-\vec{r}_1|}$$

→ If charges Q_1 is at $\vec{r}_1$ and Q_2 is at $\vec{r}_2$ then potential at r is

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0 |\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\vec{r}-\vec{r}_2|}$$

In general potential arising from n point charges is

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0 |\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\vec{r}-\vec{r}_2|} + \cdots + \frac{Q_n}{4\pi\epsilon_0 |\vec{r}-\vec{r}_n|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |\vec{r}-\vec{r}_m|}$$

→ If number of charges are infinite then

$$V(\vec{r}) = \int_{vol} \frac{\rho_v(r')\, dv'}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

In above expression $\rho_v(r')\,dv'$ represents differential amount of charge located at r'. The distance $|r-r'|$ is the distance from source point to the field point.

→ If charge distribution is in form of line charge then

$$V(\vec{r}) = \int \frac{\rho_L(r')\, dL'}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

→ If charge distribution is over surface

$$V(\vec{r}) = \int \frac{\rho_s(r')\, ds'}{4\pi\epsilon_0\, |\vec{r}-\vec{r}'|}$$

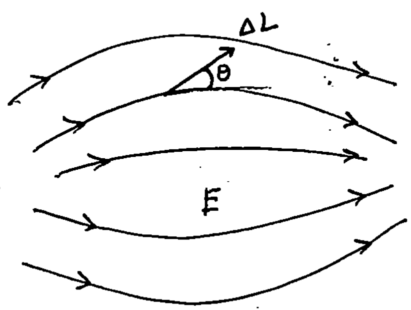
Potential Gradient

we know the line integral relationship

$$V = -\int \vec{E}\cdot d\vec{L}$$

If the above expression is applied to short element length $\Delta\vec{L}$ leading to incremental potential difference ΔV.

$$\Delta V = -\vec{E}\cdot\Delta\vec{L}$$



Consider a general region of space in which $\vec{E}$ and V both changes as we move from point to point

$\Delta V = -\vec{E}\cdot\Delta\vec{L}$ tells us to choose an incremental vector element of length $\Delta\vec{L} = \Delta L\,\hat{a}_L$ and multiply its magnitude by the component of $\vec{E}$ in the direction of $\hat{a}_L$ to obtain small potential difference between the final and initial points of $\Delta\vec{L}$

If θ is angle between $\Delta \vec{L}$ and $\vec{E}$ then

$$\Delta V = -E \, \Delta L \cos\theta$$

$$\frac{\Delta V}{\Delta L} = -E \cos\theta$$

Applying limit and considering derivative $\lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \frac{dv}{dL}$

$$\frac{dv}{dL} = -E \cos\theta$$

Maximum positive increment of potential ΔV_{max} will occur when $\Delta \vec{L}$ points in the direction opposite to $\vec{E}$ i.e., $\cos\theta = -1$ or

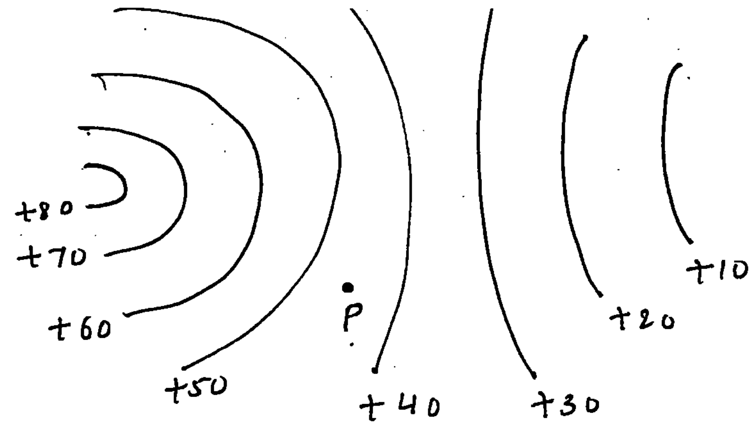
$$\left.\frac{dV}{dL}\right|_{max} = E$$

Two characteristics of the relationship between E and V at any point can be defined.

(a) Magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance

(b) This maximum value is obtained when the direction of the distance increment is opposite to $\vec{E}$.

The rate of change of potential with respect to the distance is called the potential gradient.



A potential field by equipotential surfaces.

If we consider a point P in equipotential surface. We desire information about electric field intensity $\vec{E}$.

The magnitude of $\vec{E}$ is given by maximum rate of change of V with distance. From above potential field towards left field is varying (increasing) rapidly, therefore the electric field will be oppositely directed.

Mathematically let $\hat{a}_N$ be the unit vector normal to the equipotential surface directed towards higher potential.

$$\left\langle \vec{E} = -\frac{dv}{dL}\Big|_{max} \hat{a}_N \right\rangle$$

The above equation is physical interpretation of the process of finding the electric field intensity from the potential

The operation on V by which $\vec{E}$ is obtained is known a gradient i.e,

$$\boxed{\vec{E} = -\text{grad } V}$$

V is a unique function of x, y & z then

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \longrightarrow (a)$$

But we also have

$$dV = -\vec{E} \cdot d\vec{L} = -E_x dx - E_y dy - E_z dz \longrightarrow (b)$$

Comparing above two expressions

$$\vec{E}_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

These results may be combined vectorially

$$\boxed{\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right]}$$

we know that $\boxed{\vec{E} = -\text{grad } V}$ thus

$$\boxed{\text{grad } V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z}$$

The vector operator $\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$ can be used

$$\boxed{\vec{E} = -\nabla V}$$

Energy Density in the Electrostatic field.

we know that when a unit positive charge is moved from infinity to a point in a field the work is done by the external source and energy is expended.

To hold the charge at a point in an electrostatic field an external source has to to work. This energy gets stored in the form of potential energy.

when the external source is removed potential energy gets converted to kinetic energy.

→ Consider an empty space where there is no electric field at all.

→ charge Q_1 is moved from infinity to a point in the space say P_1. This requires no work as there is no electric field

→ The charge Q_2 is to be placed at P_2, but now there is a field due to charge Q_1 and Q_2 is required to move against the field of Q_1.

Potential = Work done per unit charge

$$V = \frac{W}{Q}$$

work done to position Q_2 at $P_2 = V_{2,1} Q_2$

where $V_{2,1}$ = potential at P_2 due to Q_1

If charge Q_3 is to be moved from ∞ to P_3 then

work done to position Q_3 at $P_3 = V_{3,1} Q_3 + V_{3,2} Q_3$

similarly work done to position Q_4 at $P_4 = V_{4,1} Q_4 + V_{4,2} Q_4 + V_{4,3} Q_4$

The total work done to position all the charges

$$W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \cdots \longrightarrow (c)$$

Total work done is nothing but potential energy in the system of charges.

Consider $$Q_3 V_{3,1} = Q_3 \frac{Q_1}{4\pi\epsilon_0 R_{13}} = Q_1 \frac{Q_3}{4\pi\epsilon_0 R_{31}} = Q_1 V_{1,3}$$

where R_{13} and R_{31} are scalar distance betweeen Q_1 & Q_3

$Q_1 V_{1,3}$ is equivalent to $Q_3 V_{3,1}$

Hence by replacing each term in the expression of W_E

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \cdots \longrightarrow (d)$$

Adding equations (c) & (d)

$$2W_E = Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \cdots) + Q_2(V_{2,1} + V_{2,3}, + V_{2,4} + \cdots) + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \cdots) + \cdots \longrightarrow (e)$$

Each sum of potentials in parantheses is the combined potential due to all the charges except for the charge at the point where this combined potential is being found.

i.e., $V_{1,2} + V_{1,3} + V_{1,4} + \cdots = V_1$

where V_1 is potential at location of Q_1 due to charges Q_2, Q_3

hence from above expressions

$$2W_E = Q_1V_1 + Q_2V_2 + \cdots$$

$$2W_E = \sum_{m=1}^{N} Q_m V_m$$

$$\boxed{W_E = \frac{1}{2}\sum_{m=1}^{N} Q_m V_m} \longrightarrow (f)$$

The expression for energy stored in a region of continous charge distribution is obtained by replacing each charge by $\rho_v\, dv$ in equation (f) the summation becomes integral

$$W_E = \frac{1}{2}\int_{vol} (\rho_v\, dv)V \longrightarrow (g)$$

By Maxwell's first equation $\nabla \cdot \vec{D} = \rho_v$

By vectore identity

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot \nabla V$$

$$\nabla\cdot(V\vec{D}) = V\rho_v + \vec{D}\cdot\nabla V$$

But $\vec{E} = -\nabla V$

So,
$$\nabla\cdot(V\vec{D}) = V\rho_v - \vec{D}\cdot\vec{E}$$

$$\rho_v V = \nabla\cdot(V\vec{D}) + \vec{D}\cdot\vec{E} \longrightarrow (h)$$

From equations (g) and (h) we have

$$W_E = \frac{1}{2}\int_{vol}\left(\nabla\cdot(V\vec{D}) + \vec{D}\cdot\vec{E}\right)dv$$

$$W_E = \frac{1}{2}\int_{vol}\nabla\cdot(V\vec{D})\,dv + \frac{1}{2}\int_{vol}\vec{D}\cdot\vec{E}\,dv$$

Using divergence theorem i.e, $\oint_S \vec{D}\cdot d\vec{s} = \int_{vol}\nabla\cdot\vec{D}\,dv$ changed to closed surface the first volume integral can be changed to closed surface integral. i.e.,

$$W_E = \frac{1}{2}\oint_S (V\vec{D}\cdot d\vec{s}) + \frac{1}{2}\int_{vol}\vec{D}\cdot\vec{E}\,dv$$

In the above expression surface integral is zero, surrounding the universe is approaching zero at the rate $\frac{1}{r}$ and $\frac{1}{r^2}$ but the surface area is increasing at the rate as r^2 consequently in the limit $r\to\infty$ the increasing integration becomes zero hence

$$\boxed{W_E = \frac{1}{2}\int_{vol}\vec{D}\cdot\vec{E}\,dv = \frac{1}{2}\int_{vol}\epsilon_0 E^2\,dv}$$

Current & Current Density

→ Electric charges in motion constitute electric current. The unit of current is Ampere (A). 1 ampere current is said to be flowing across a surface when 1 Coulomb of charge is passing across surface in one second.

Current is symbolized as I

$$I = \frac{dQ}{dt}$$

Current density is a vector represented by $\vec{J}$, measured in ampere / square meter (A/m^2)

The increment of current ΔI crossing an incremental surface ΔS normal to the current density is

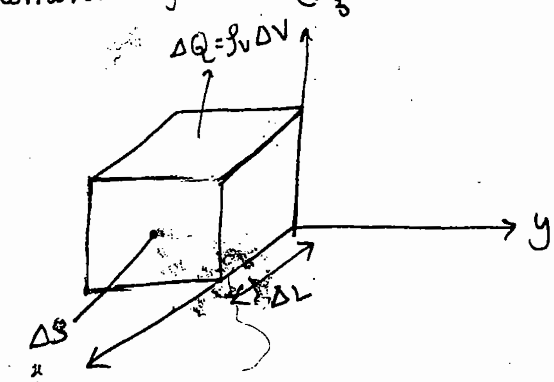
$$\Delta I = J_N \, \Delta S$$

If current density is not normal to ΔS

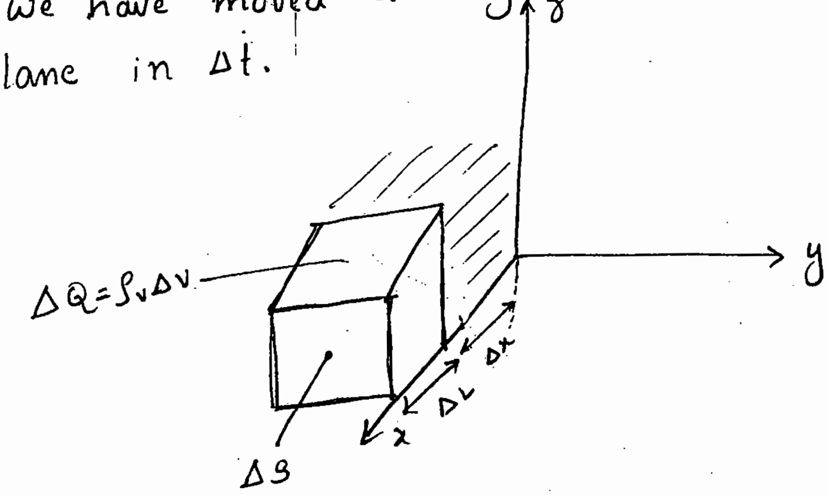
$$\Delta I = \vec{J} \cdot \Delta \vec{S}$$

Total current $\quad I = \int_S \vec{J} \cdot d\vec{S}$

Consider element of charge $\Delta Q = \rho_v \Delta V = \rho_v \Delta S \, \Delta L$ as shown below



Let the charge is moving in x-direction with velocity $\vec{V}$ and thus velocity has only x component V_x i.e., we have moved a charge $\Delta Q = \rho_v \Delta S \Delta x$ through a reference plane in Δt.



In the time interval Δt the element of charge moved through distance Δx in direction of x-axis as shown in above figure. The resultant current is

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

Now $\Delta Q = \rho_v \Delta S \Delta x$ and

$$\Delta I = \frac{\rho_v \Delta S \Delta x}{\Delta t}$$

$\frac{\Delta x}{\Delta t} = V_x$ velocity in x direction

$$\Delta I = \rho_v \Delta S V_x$$

In terms of current density

$$J = \rho_v V_x$$

In general $\vec{J} = \rho_v \vec{V}$

$\vec{V} \rightarrow$ Velocity vector

The principle of conservation of charges states that charges can neither be created nor destroyed although equal amount of positive & negative charges may be simultaneously created, obtained by separation, destroyed, or lost by recombination.

The continuity equation follows this principle when we consider any region bounded by a closed surface
The current through the closed surface is

$$I = \oint_S \vec{J} \cdot d\vec{s}$$

This outward flow of charge must be balanced by a decrease of positive charge within the closed surface
If the charge inside the closed surface is denoted by Q_i,
then rate of decrease is $-dQ_i/dt$

$$I = \oint \vec{J} \cdot d\vec{s} = -\frac{dQ_i}{dt}$$

Above equation is integral form of continuity equation, the point form is obtained by following steps.

using divergence theorem $\oint_S \vec{J} \cdot d\vec{s} = \int_{vol} (\nabla \cdot \vec{J})\, dv$

Enclosed charge Q_i is volume integral of the charge density

$$\int_{vol} (\nabla \cdot \vec{J})\, dv = -\frac{d}{dt} \int_{vol} \rho_v \, dv$$

If we keep surface constant, the derivative becomes partial derivative i.e.,

$$\int_{vol} (\nabla \cdot \vec{J})\, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t}\, dv$$

This expression is true for any volume however small,

$$(\nabla \cdot \vec{J})\,\Delta v = -\frac{\partial \rho_v}{\partial t}\,\Delta v$$

$$\boxed{(\nabla \cdot \vec{J}) = -\frac{\partial \rho_v}{\partial t}}$$

This equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

Metallic Conductors

→ The range of energies that an electron may possess in an atom is known as the energy band.

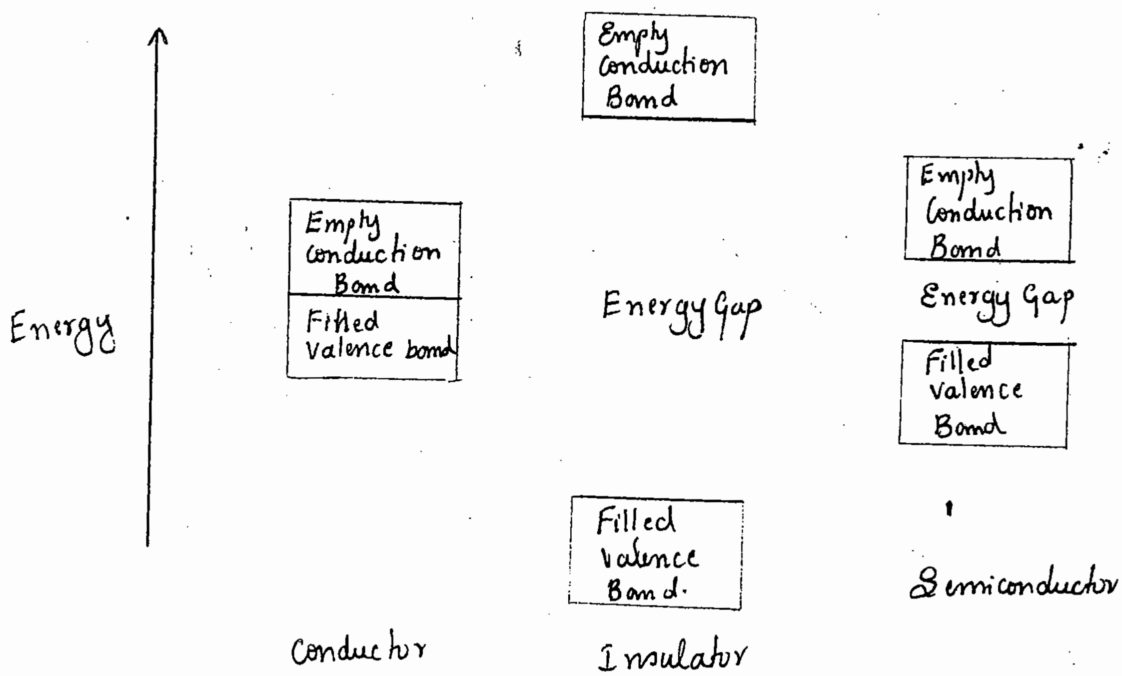
→ According to the quantum theory, only certain discrete energy levels or energy states are permissible in a given atom.

∴ electrons are located in valence band.

→ In conductors the conduction and valence bonds are overlapped, additional kinetic energy may be given to the valence electron by an external source resulting in electron flow.

→ In case of insulators gap exists between the valence band and the conduction band, the electron cannot accept energy in smaller amount. if it accepts the insulator breaks down

→ In semiconductors an intermediate condition occurs when only a small "forbidden region" separates 2 bonds. Small amount of energy in the form of heat, light or an electric field may raise the energy of the electrons at the top of the filled band and drag to conduction bond



Energy bond Structure in three different types of materials

Conductors

→ In conductors the free electrons move under the influence of electric field
If electric field is $\vec{E}$ and an electron having charge $Q = -e$ will experience a force $\vec{F} = -e\vec{E}$ ⟶ ①

→ In free space the electron continously increases its velocity
In crystalline material because of collisions with thermally excited crystalline lattice structure constant velocity is attained
This velocity v_d is known as drift velocity. The drift velocity is linearly related with field intensity and mobility of electron in the given material.

$$\vec{v}_d = -\mu_e \vec{E} \longrightarrow ②$$

mobility is measured in terms of m^2/Vs and $\vec{E}$ as V/m
we know that $\vec{J} = \rho_v \vec{v}$ substituting for $\vec{v}$

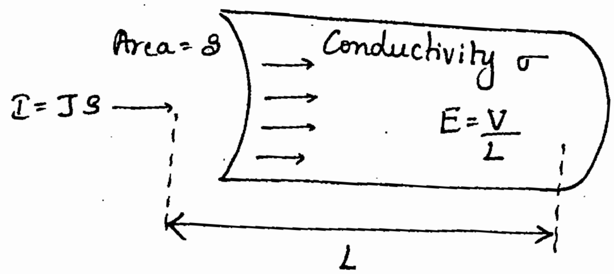
$$\vec{J} = -\rho_e \mu_e \vec{E} \longrightarrow ③$$

→ The relation between $\vec{E}$ and $\vec{J}$ is also given as.

$$\vec{J} = \sigma \vec{E} \longrightarrow ④$$

where σ is conductivity measured in Siemens/meter.
1 Siemen is the basic unit of conductance in the SI system and is defined as 1 ampere/volt.
Later the unit of conductance called mho i.e, ℧
From equation ③ & ④ $\sigma = -\rho_e \mu_e$

Let us assume J and E are uniform



$$I = \int_S \vec{J}\cdot d\vec{S} = JS$$

$$V_{ab} = -\int_b^a \vec{E}\cdot d\vec{L}$$

$$V_{ab} = -\vec{E}\cdot\vec{L}_{ba}$$

or

$$V = EL$$

Thus $J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$

$$V = \frac{L}{\sigma S} I$$

$$\boxed{V = RI}$$

where R is resistance of the cylinder above equation is known as Ohm's law.

Conductor Properties & Boundary Conditions

Property 1: If number of electrons are placed interior of a conductor, as there is no positive charge to neutralize the electrons begin to accelarate away from each other. This continues until the electrons reach the surface of the conductor. Outward progress of electrons stops as material surrounding the conductor is insulator. No charge may remain within the conductor.

Property 2: No current may flow during the static condition, the electric field intensity within the conductor is zero.

Summarizing for electrostatics no charge and no field exists within a conducting material.

The charge may appear on the surface as surface charge density. The external fields are related to the charge on the surface of the conductor.

Boundary Conditions between conductor & free space

When an electric field passes from one medium to other medium it is important to study the conditions at the boundary between the two media.

The conditions existing at boundary of the two media when field passes from one medium to other are called boundary conditions.

There are 2 cases of boundary conditions.

1) Boundary between conductor and free space
2) Boundary between two dielectrics with different properties

We know from Maxwell's equations that

$$\oint \vec{E} \cdot d\vec{L} = 0 \quad \text{and} \quad \oint \vec{D} \cdot d\vec{S} = Q$$

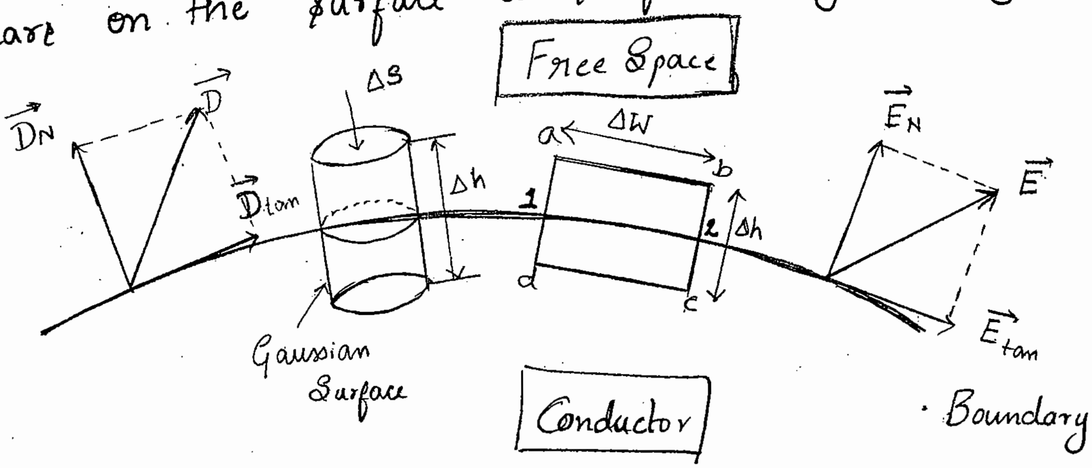
The field intensity and flux density can be ...
into two components namely tangential to boundary and normal to boundary hence at any point on boundary

$$\vec{E} = \vec{E}_{tan} + \vec{E}_N$$

$$\vec{D} = \vec{D}_{tan} + \vec{D}_N.$$

For ideal conductors it is known that

1. The field intensity and flux density inside the conductor is zero
2. No charge exists inside the conductor, the charge appears on the surface as surface charge density.



Consider the conductor free space boundary as shown in the above figure.

We know that $\oint \vec{E} \cdot d\vec{L} = 0$

i.e, workdone in carrying a unit positive charge around a closed path is zero.

Consider a rectangular closed path a-b-c-d-a traced in clockwise direction. $\oint \vec{E} \cdot d\vec{L}$ can be divided into 4 parts

$$\oint \vec{E} \cdot d\vec{L} = \int_a^b \vec{E} \cdot d\vec{L} + \int_b^c \vec{E} \cdot d\vec{L} + \int_c^d \vec{E} \cdot d\vec{L} + \int_d^a \vec{E} \cdot d\vec{L}$$

The closed contour is placed such that two sides a-b and c-d are parallel to tangential direction and b-c and d-a are parallel to normal direction of the surface

Let height and width of elementary rectangle is Δh and Δw respectively. Thus $\Delta h/2$ is in the conductor and $\Delta h/2$ is in free space.

The portion c-d is in the conductor where $\vec{E} = 0$ hence

$$\int_a^b \vec{E} \cdot d\vec{L} + \int_b^c \vec{E} \cdot d\vec{L} + \int_d^a \vec{E} \cdot d\vec{L} = 0$$

as Δw is small $\vec{E}$ over it can be assumed constant

hence above integral can be written as

$$\int_a^b \vec{E} \cdot d\vec{L} = \vec{E} \int_a^b d\vec{L} = \vec{E}\,(\Delta w)$$

$\vec{E}$ is along tangential direction to the boundary

$\vec{E} = \vec{E}_{tan}$ $\qquad E_{tan} = |\vec{E}_{tan}|$

$$\int_a^b \vec{E} \cdot d\vec{L} = E_{tan}\,(\Delta w)$$

b-c is parallel to the normal component so we have E_N along this direction. Let $E_N = |\vec{E}_N|$

Over the small height Δh, E_N can be assumed constant and can be taken out of integration

$$\int_b^c \vec{E}\cdot d\vec{L} = \vec{E}\int_b^c d\vec{L} = E_N\int_b^c d\vec{L}$$

Out of b-c, b-2 is in free space and 2-c is in the conductor where $\vec{E}=0$

$$\int_b^c d\vec{L} = \int_b^2 d\vec{L} + \int_2^c d\vec{L} = \frac{\Delta h}{2}+0 = \frac{\Delta h}{2}$$

$$\int_b^c \vec{E}\cdot d\vec{L} = E_N\left(\frac{\Delta h}{2}\right) \qquad \text{--- (8)}$$

Similarly for path d-a the condition is same as for the path b-c, only direction is opposite.

$$\int_d^a \vec{E}\cdot d\vec{L} = -E_N\left(\frac{\Delta h}{2}\right) \qquad \text{--- (9)}$$

Substituting in

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L} = 0$$

$$E_{tan}\Delta W + E_N\left(\frac{\Delta h}{2}\right) - E_N\left(\frac{\Delta h}{2}\right) = 0$$

$$E_{tan}\Delta W = 0$$

but $\Delta W \neq 0$ as finite

$$\boxed{E_{tan} = 0}$$

Thus tangential component of electric field intensity is zero at the boundary between conductor and free space.

$$\vec{D} = \epsilon_0 \vec{E} \quad \text{for free space}$$

$$D_{tan} = \epsilon_0 E_{tan} = 0$$

$$\boxed{D_{tan} = 0}$$

The tangential component of electric field intensity is zero at the boundary between conductor and free space.

To find normal component of $\vec{D}$ and $\vec{E}$ select a closed Gausian surface in the form of right circular cylinder. Its height is Δh and it is placed such that $\Delta h/2$ is in the conductor and $\Delta h/2$ is in the free space

According to Gauss's law $\oint_S \vec{D} \cdot d\vec{s} = Q$

The surface integral must be evaluated over three surfaces

$$\int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} + \int_{lateral} \vec{D} \cdot d\vec{s} = Q$$

The bottom surface is inside the conductor where $\vec{D} = 0$ hence above equation reduces to

$$\int_{top} \vec{D} \cdot d\vec{s} + \int_{lateral} \vec{D} \cdot d\vec{s} = Q$$

The lateral surface area is $2\pi r\,\Delta h$ where r is the radius of cylinder. Because D_N is tangential to surface & $\vec{D}_N \cdot d\vec{s} = 0$

The corresponding integral is zero

$$\int_{top} \vec{D}\cdot d\vec{s} + \cancel{\int_{lateral} \vec{D}\cdot d\vec{s}}^{\,0} = Q$$

$$\int_{top} \vec{D}\cdot d\vec{s} = D_N \int_{top} d\vec{s} = D_N \Delta S$$

From Gauss's law

$$D_N \Delta S = Q$$

But at the boundary charges exist as surface charge density $Q = \rho_S \Delta S$ by comparing above two equations

$$D_N \cancel{\Delta S} = \rho_S \cancel{\Delta S}$$

$$\boxed{D_N = \rho_S}$$

$$\boxed{E_N = \frac{\rho_S}{\epsilon_0}}$$

The electric flux leaves the conductor in a direction normal to the surface, and value of electric flux density is numarically equal to the surface charge density.

Zero tangential electric field intensity is the fact that a conductor surface is an equipotential surface

To summarize the concepts which apply to conductors in electrostatic fields

1. The static electric field intensity inside a conductor is zero

2. The static field intensity at the surface of the conductor is everywhere directed normal to that surface.

3. The conductor surface is an equipotential surface.

Dielectric Material

→ Insulating materials or dielectric materials differ from conductor. There wil be no free charge that can be transported within them to produce conduction current. All charges are confined to molecular or lattice sites by coulomb forces.

→ By applying electric field has the effect of displacing the charges slightly, leading to the ensemble of electric dipoles.

→ The extent to which this occurs (polarization) is measured by the relative permittivity or dielectric constant.

→ The charge displacement principle constitutes an energy storage mechanism that is used in construction of capacitor.

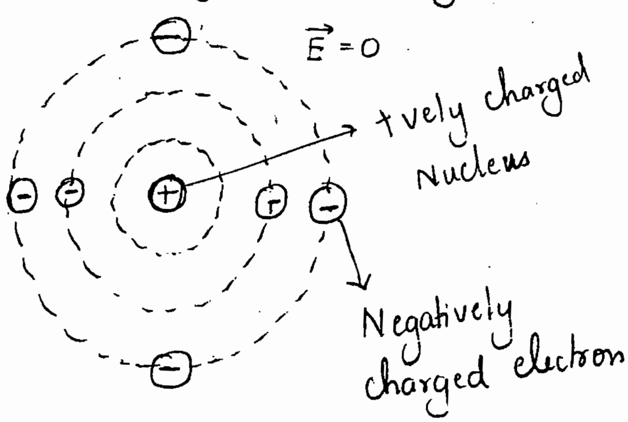
→ The dielectric in an electric field can be viewed as a free space arrangement of microscopic electric dipoles which are composed of positive and negative charges whose centers do not coincide.

They are not free charges and they cannot contribute to the conduction process.

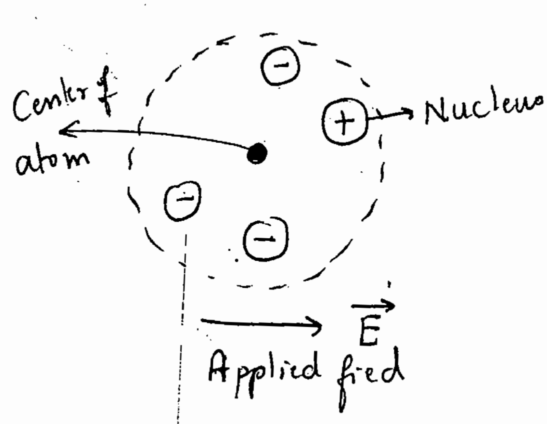
→ The dipoles are known as bound charges, which can be considered as any other sources of the electrostatic field.

→ The common characteristic of dielectric material (whether they are solid, liquid or gas, no crystalline in nature) is their ability to store energy.

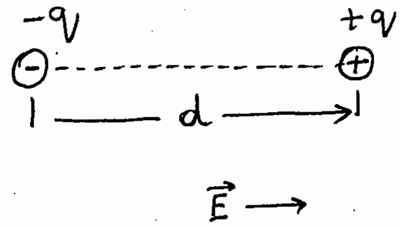
The storage of energy takes place by means of shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces. (Similar to lifting a weight or stretching a spring i.e., potential energy)



Unpolarized atom of dielectric



Polarized atom



Equivalent dipole

Mathematical Expression for Polarization.

When a dipole is formed due to polarization, there exists an electric dipole moment $\vec{p}$

$$\vec{p} = Q\,\vec{d}$$

Q is magnitude of one of the two charges

$\vec{d}$ distance vector from -ve to positive charge

If there are n dipoles per unit volume, the number of dipoles in volume ΔV is $n\Delta V$, and total dipole moment is

$$\vec{p}_{total} = \sum_{i=1}^{n\Delta V} Q_i \vec{d_i}$$

If dipoles are randomly oriented $\vec{P}_{total}$ is zero but if dipoles are aligned in the direction of applied $\vec{E}$ then $\vec{P}_{total}$ has significant value.

The polarization $\vec{P}$ is defined as the total dipole moment per unit volume

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n\Delta V} Q_i \vec{d_i}}{\Delta V} \quad C/m^2$$

It can be seen that the unit of polarization is same as that of flux density $\vec{D}$. The increase in polarization leads to increase in flux density in a dielectric medium.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

For isotropic and linear medium, the linear relationship between $\vec{P}$ & $\vec{E}$ is

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

where χ_e is dimensionless quantity called electric susceptibility of the material. using this relation

$$\vec{D} = \epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E} = (\chi_e + 1)\, \epsilon_0 \vec{E}$$

The expression within parenthesis defined as

$\epsilon_r = \chi_e + 1$ thus

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E} = \epsilon \vec{E}$$

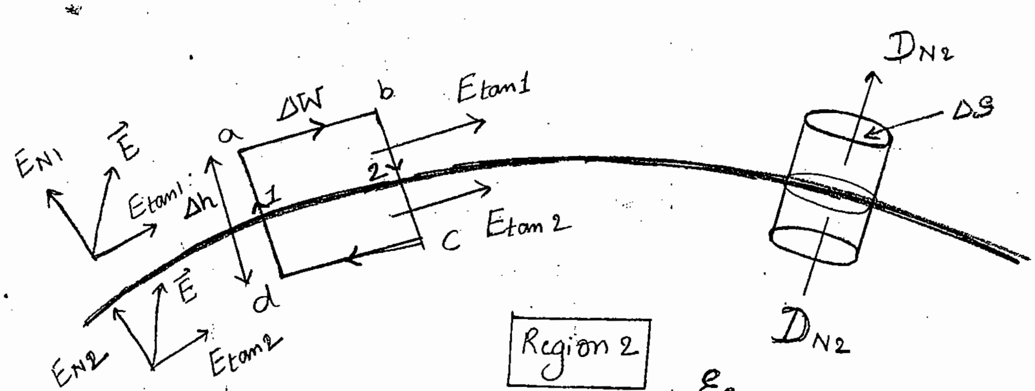
$$\boxed{\epsilon = \epsilon_0 \epsilon_r}$$

ϵ is known as permittivity

ϵ_r is relative permittivity or dielectric constant of the material. (dimensionless)

Boundary condition for perfect dielectric material:

Consider the boundary between two perfect dielectrics. One dielectric has permittivity ε_1 while the other has permittivity ε_2.



The $\vec{E}$ and $\vec{D}$ are to be obtained by resolving each into two components, tangential to the boundary and normal to the surface

Consider a closed path abcda rectangular in shape having elementary height Δh and elementary width ΔW. It is placed such that $\Delta h/2$ is in dielectric 1 while the remaining is in dielectric 2.

The integral over closed path abcda is

$$\oint \vec{E}\cdot d\vec{L} = 0$$

$$\therefore \int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L} = 0 \longrightarrow (a)$$

Let E_1 and E_2 be the field intensity in dielectric 1 and dielectric 2 respectively.

These electric fields have both normal & tangential components i.e.,

$$\vec{E_1} = \vec{E_{1t}} + \vec{E_{1N}}$$
$$\vec{E_2} = \vec{E_{2t}} + \vec{E_{2N}}$$

Let $\quad |\vec{E_{1t}}| = E_{tan1}, \quad |\vec{E_{2t}}| = E_{tan2}$

$\quad |\vec{E_{1N}}| = E_{1N}, \quad |\vec{E_{2N}}| = E_{2N}$

From equation (a) and above

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^a \vec{E}\cdot d\vec{L} = 0$$

$$\int_a^b \vec{E}\cdot d\vec{L} + \int_b^2 \vec{E}\cdot d\vec{L} + \int_2^c \vec{E}\cdot d\vec{L} + \int_c^d \vec{E}\cdot d\vec{L} + \int_d^1 \vec{E}\cdot d\vec{L} + \int_1^a \vec{E}\cdot d\vec{L} = 0$$

$$\left[E_{tan1}\,\Delta W\right] + \cancel{\left[-E_{N1}\frac{\Delta h}{2}\right]} + \cancel{\left[-E_{N2}\frac{\Delta h}{2}\right]} + \left[-E_{tan2}\,\Delta W\right] + \cancel{\left[E_{N2}\frac{\Delta h}{2}\right]} + \cancel{\left[E_{N1}\frac{\Delta h}{2}\right]} = 0$$

$$\Delta W\left[E_{tan1} - E_{tan2}\right] = 0$$

ΔW cannot be equal to zero hence

$$\boxed{E_{tan1} = E_{tan2}}$$

Thus the tangential components of field intensity at the boundary in both the dielectrics remain same i.e., electric field intensity is continous across the boundary.

W.K.T $\vec{D} = \varepsilon \vec{E}$

If D_{tan1} and D_{tan2} are tangential components of electric flux density in dielectric 1 and dielectric 2

$$D_{tan1} = \varepsilon_1 E_{tan1} \qquad D_{tan2} = \varepsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}\,\varepsilon_0}{\varepsilon_{r2}\,\varepsilon_0} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

Thus the tangential components of $\vec{D}$ undergoes some changes across the interface hence $\vec{D}$ is discontinous across the boundary.

To find normal components let us use Gauss's law

Consider a Gassian surface in the form of right circular cylinder, placed in such a way that half of its lies in dielectric 1 while the remaining half in dielectric 2

$$\oint \vec{D}\cdot d\vec{S} = Q$$

$$\int_{Top} \vec{D}\cdot d\vec{S} + \int_{bottom} \vec{D}\cdot d\vec{S} + \int_{lateral} \vec{D}\cdot d\vec{S} = Q.$$

$$\int_{top} \vec{D}\cdot d\vec{S} + \int_{bottom} \vec{D}\cdot d\vec{S} + \int_{\substack{lateral\\Top}} \vec{D}\cdot d\vec{S} + \int_{\substack{lateral\\bottom}} \vec{D}\cdot d\vec{S} = Q$$

$$D_{N1}\,\Delta S - D_{N2}\,\Delta S - \cancel{D_{tan1}\frac{\Delta h}{2}\rho\Delta\phi} + \cancel{D_{tan1}\frac{\Delta h}{2}\rho\Delta\phi}$$

$$-\cancel{D_{tan2}\frac{\Delta h}{2}\rho\Delta\phi} + \cancel{D_{tan2}\frac{\Delta h}{2}\rho\Delta\phi} = Q$$

$$D_{N1}\,\Delta S - D_{N2}\,\Delta S = Q$$

$$D_{N1} - D_{N2} = \frac{Q}{\Delta S} = \rho_s$$

$$D_{N1} - D_{N2} = \rho_s$$

There is no charge available in perfect dielectric. As all the charges are bound charges and are not free, ~~hen~~ the surface charge density can be assumed zero

$$\rho_s = 0$$

$$\boxed{D_{N1} = D_{N2}}$$

Hence Normal component of flux density $\vec{D}$ is continous at the boundary between two perfect dielectrics.

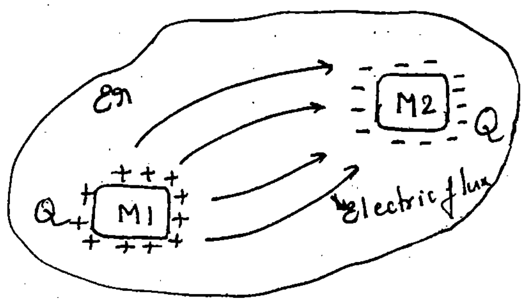
Now $\quad D_{N1} = \varepsilon_1 E_{N1} \qquad D_{N2} = \varepsilon_2 E_{N2}$

$$\boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

The normal component of electric field intensity $\vec{E}$ are inversely proportional to the relative permitivities of two media.

Capacitance

→ Consider M1 & M2 two conducting materials placed in dielectric medium having permittivity ε_r. The material M_1 carries a positive charge and M_2 carries negative charge. The total charge of the system is zero.



→ In conductors charges reside on the surface. Such two conducting surfaces carrying equal and opposite charge placed in dielectric medium is called capacitive system giving rise to capacitance.

→ The flux is directed from M_1 to M_2

→ Work must be done to carry a positive unit charge from M2 to M1. i.e, potential difference between M_1 & M_2 is V_{12}.

→ The ratio of magnitude of charge on any one of the conductor & potential difference between two conductor is defined as capacitance

$$C = \frac{Q}{V_{12}}$$

In general $C = \frac{Q}{V}$ Farads

$$1 \text{ Farad} = \frac{1 \text{ coulomb}}{1 \text{ Volt}}$$

$\therefore$ As charges exists only on surface of conductor we can use Gauss's law to find total charges

$$Q = \oint \vec{D} \cdot d\vec{S} = \oint_S \varepsilon_0 \varepsilon_r \vec{E} \cdot d\vec{S} = \oint_S \varepsilon \vec{E} \cdot d\vec{S}$$

→ V is the work done in moving a unit positive charge from negative to positive surface

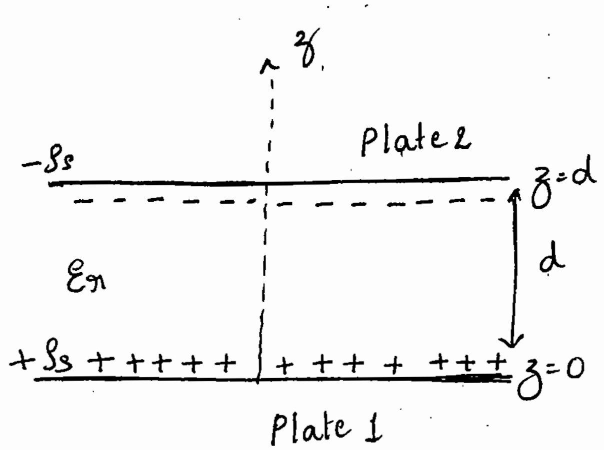
$$V = -\int \vec{E} \cdot d\vec{L} = -\int_{-}^{+} \vec{E} \cdot d\vec{L}$$

Hence $$C = \frac{Q}{V} = \frac{\oint_S \varepsilon \vec{E} \cdot d\vec{S}}{-\int_{-}^{+} \vec{E} \cdot d\vec{L}}$$

→ Capacitance is not dependent on $\vec{E}$, $\vec{D}$, charge & V

→ Capacitance depends on physical dimensions of the system and properties of dielectric such as permittivity of dielectric.

Parallel plate capacitor



Consider two plates separated by 'd'. Let S be area of cross section of plates

The total charge $Q = \rho_s S$ Coulomb

Magnitude of charge on any one plate

Assuming plate 1 to be infinite sheet charge

$$\vec{E}_1 = \frac{\rho_s}{2\varepsilon} \hat{a}_N = \frac{\rho_s}{2\varepsilon} \hat{a}_z \text{ V/m}$$

$\vec{E}_1$ is normal at the boundary without any tangential component for plate 2

$$\vec{E}_2 = \frac{-\rho_s}{2\varepsilon}\,\hat{a}_N = \frac{-\rho_s}{2\varepsilon}(-\hat{a}_z) = \frac{\rho_s\,\hat{a}_z}{2\varepsilon}$$

In between plates

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_s}{\varepsilon}\,\hat{a}_z \text{ V/m}$$

The potential difference is

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{L} = -\int_{\text{Upper}}^{\text{Lower}} \frac{\rho_s}{\varepsilon}\,\hat{a}_z \cdot d\vec{L}$$

$$V = -\int_{z=d}^{0}\left(\frac{\rho_s}{\varepsilon}\,\hat{a}_z\right)\cdot\left(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z\right) = -\int_{z=d}^{0}\frac{\rho_s}{\varepsilon}\,dz$$

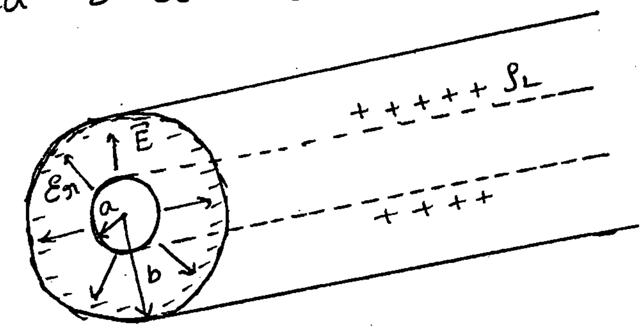
$$V = -\frac{\rho_s}{\varepsilon}(-d) = \frac{\rho_s d}{\varepsilon}$$

$$C = Q/V = \frac{\rho_s S}{\frac{\rho_s d}{\varepsilon}} = \frac{\varepsilon S}{d} \text{ F}$$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r S}{d}} \text{ F}$$

Capacitance of coaxial cable

Consider coaxial cable or coaxial capacitor. Let 'a' be inner radius and 'b' be outer radius



The two concentric conductors are separated by dielectric & the length of cable is 'L' m.

The inner conductor carries the charge density ρ_L C/m, on its surface and $-\rho_L$ C/m exists on the outer conductor

$$Q = \rho_L L$$

Assuming cylindrical coordinate system, $\vec{E}$ will be radially outwards from inner to outer conductor. for infinite line charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\,\hat{a}_\rho$$

$\vec{E}$ is directed from inner conductor to outer conductor.

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{L} = -\int_{\rho=b}^{a}\left(\frac{\rho_L}{2\pi\varepsilon\rho}\,\hat{a}_\rho\right)\cdot(d\rho\,\hat{a}_\rho)$$

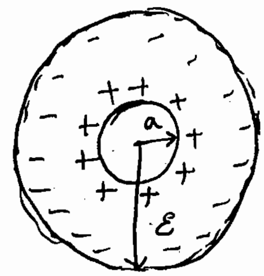
$$V = \frac{-\rho_L}{2\pi\varepsilon}\ln(\rho)\Big]_b^a = \frac{+\rho_L}{2\pi\varepsilon}\ln(b/a)$$

$$\boxed{C = Q/V = \frac{+2\pi\varepsilon L}{\ln(b/a)}}\ \mathrm{F}$$

Spherical Capacitor

Consider a spherical capacitor formed by 2 concentric spheres of radius a & b. Let radius of outer sphere is 'b' and inner sphere is 'a' i.e., $a < b$.

The electric field of sphere with charge Q is given by



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r \ \text{V/m}$$

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{L} = -\int_{r=b}^{a} \left(\frac{Q}{4\pi\varepsilon r^2}\,\hat{a}_r\right)\cdot\left(dr\,\hat{a}_r\right)$$

$$= -\int_{r=b}^{a} \frac{Q}{4\pi\varepsilon r^2}\,dr$$

$$= -\frac{Q}{4\pi\varepsilon}\left.\frac{-1}{r}\right]_b^a$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \ \text{F}$$

$$\boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \ \text{F}}$$

Introduction to Magnetostatics

→ Electrostatic field exists due to the static charges, the magnetic field exists due to a permanent magnet, which is a natural magnet.

But in electromagnetic engineering a link between electric and magnetic field is required to be studied. That link will be absent with magnetic field due to a natural magnet.

→ The scientist Oersted discovered that when the charges in motion they are surrounded by a magnetic field. Thus flow of charges constitute an electric current. Thus the current carrying conductor is always surrounded by a magnetic field.

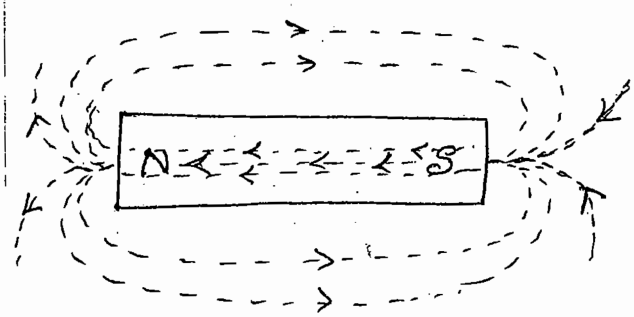
If such current is steady (time invarient) then magnetic field produced is also steady magnetic field

→ The study of steady magnetic field existing in a given space produced due to the flow of direct current through a conductor is called magnetostatics.

The various concepts like e.m.f induced, force experienced by a conductor, motoring action, transformer action etc are dependent on the magnetostatics.

Magnetic Field and its properties

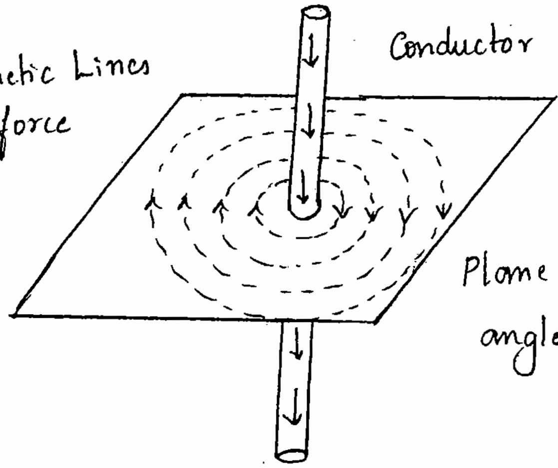
A permanent magnet has two poles north (N) & south (S)
The region around a magnet within which the influence
of the magnet can be experienced is called magnetic field
Such a field is represented by imaginary lines around the
magnet which are called magnetic lines of force.
These lines are introduced by Scientist Michael Faraday.
The direction of lines are always from North pole to
South pole. These lines are also called magnetic flux lines



Parmanent magnet and magnetic lines of forces.

Magnetic field due to current carrying conductor

when a straight conductor carries a direct current, it
produces a magnetic field around it all along its
length. The lines of force in such a case are in the
form of concentric circles in the planes at right angles
to the conductor.
The direction of concentric circles depends on the direction
of current flowing in the conductor.



Magnetic field due to conductor carrying direct current.

A right hand thumb rule is used to determine the direction of magnetic field around a conductor carrying a direct current. It states that hold the current carrying current in the right hand such that the thumb pointing in the direction of current and parallel to the conductore then curled fingere point in the direction of magnetic flux linee around it

Right hand thumb rule.

→ The quantitative measure of strongness or weakness of the magnetic field is given by 'magnetic field intensity.

→ The magnetic field intensity is measured as the force experienced by a unit north pole of one weber strength when placed at that point.

→ The magnetic flux lines are measured in webers (Wb) while magnetic field intensity is measured in N/wb or Amperes/metre. It is denoted as $\vec{H}$

→ The total magnetic lines of force crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density. It is denoted as $\vec{B}$ and is a vector quantity. It is measured in wb/m^2 which is also called Tesla (T)

→ In electrostatics $\vec{E}$ & $\vec{D}$ are related to each other through permittivity ε of the region.

In magnetostatics $\vec{B}$ and $\vec{H}$ are related through the property of the region in which current carrying conductor is placed. It is called permeability denoted as μ.

It is the ability with which the current carrying conductor forces the magnetic flux through the region around it. For the free space permeability is denoted as μ_0 and its value is $4\pi \times 10^{-7}$ Henry/meter.

For any other region relative permeability is specified as μ_r and $\mu = \mu_0 \mu_r$

Thus $\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}$

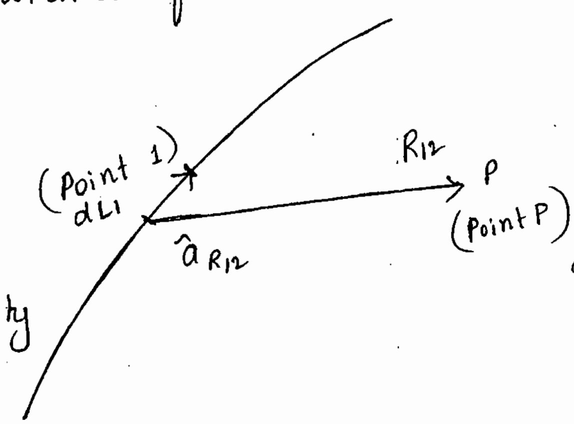
for free space $\vec{B} = \mu_0 \vec{H}$.

Biot Savart Law.

→ Consider a conductor carrying direct current I and a steady magnetic field produced around it.

→ The Biot - Savart law allows us to obtain the differential magnetic field intensity $d\vec{H}$, produced at a point P, due to a differential vector length of the filament $d\vec{L}$

→ The law of Biot - Savart states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of distance from the differential element to the point P.



The Biot - Savart's law expresses the magnetic field intensity dH_2 produced by a differential current element $I_1 dL_1$

$$d\vec{H}_2 = \frac{I_1 d\vec{L}_1 \times \hat{a}_{R12}}{4\pi R_{12}^2}$$

$$d\vec{H} = \frac{I\, d\vec{L} \times \hat{a}_r}{4\pi R^2} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi R^3}$$

Mathematically

$$d\vec{H} \propto \frac{I\,dL \sin\theta}{R^2}$$

$$d\vec{H} = \frac{k\, I\,dl \sin\theta}{R^2}$$

where k = proportionality constant $k = 1/4\pi$

$$d\vec{H} = \frac{I\,dl \sin\theta}{4\pi R^2}$$

If $\hat{a}_R$ is the unit vector in the direction from differential current element to point P. then

$$d\vec{L} \times \hat{a}_R = dL|\hat{a}_R| \sin\theta = dL \sin\theta$$

$$d\vec{H} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi R^3} \quad A/m$$

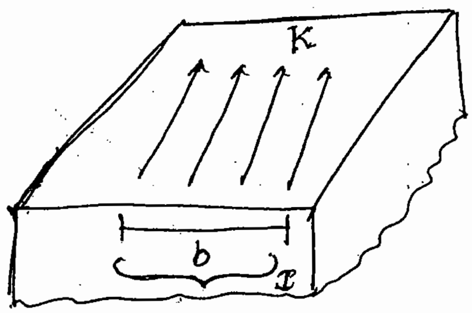
It is impossible to check experimentally the law of Biot Savart law as expressed in above equation. because the differential current element cannot be isolated. Hence integral form of above Biot Savart's law can be verified experimentally

$$\vec{H} = \oint \frac{I\, d\vec{L} \times \hat{a}_R}{4\pi R^2}$$

The Biot-Savart law may also be expressed in terms of distributed sources such as current density J and surface current density K

Biot - Savart Law Interms of Distributed Sources

Consider a surface carrying a uniform current over its surface as shown in the figure..



Then the surface current density is denoted as $\vec{K}$ and measured in A/m. Thus for uniform current density the current I in any width b is given by $I = kb$ where b is $\perp^{lar}$ to direction of current flow.

If dS is the differential surface area considered of a surface having current density $\vec{K}$ then

$$I\,d\vec{L} = \vec{K}\,dS$$

If current density in a volume of given conductor is $\vec{J}$ measured in A/m^2 then for differential volume dv

$$I\,d\vec{L} = \vec{J}\,dv$$

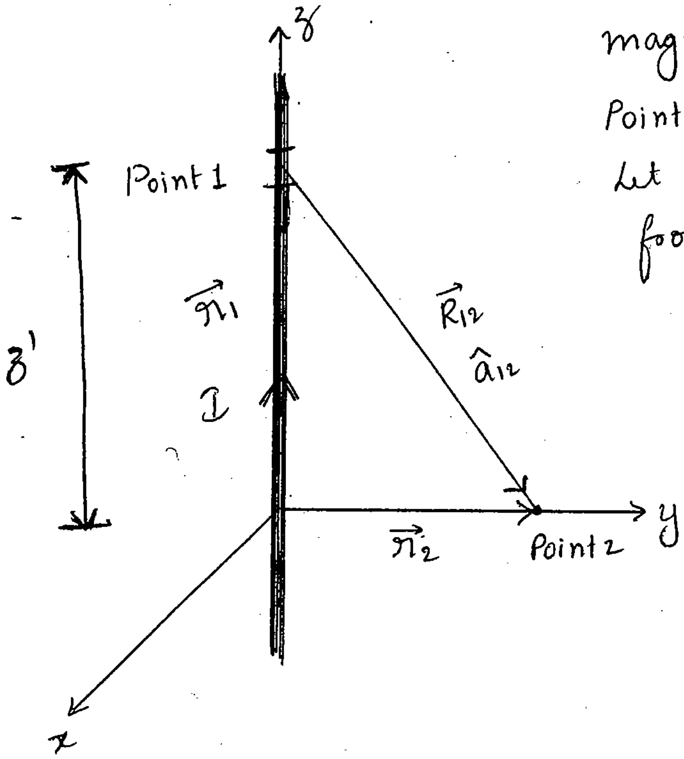
Hence Biot Savart's law can be expressed as

$$\vec{H} = \int_S \frac{\vec{K} \times \hat{a}_R\,dS}{4\pi R^2} \quad A/m$$

$$\vec{H} = \int_{vol} \frac{\vec{J} \times \hat{a}_R\,dv}{4\pi R^2} \quad A/m$$

$\vec{H}$ due to Infinitely Long Straight Conductor

Consider an infinitely long straight conductor as shown in the figure below. It carries current I Amperes. The conductor is placed in z-axis



At point 2 we need to determine magnetic field intensity.
Point 2 is chosen in z=0 plane
Let point 2 is at $\vec{r}_2$ distance from origin. where

$$\vec{r}_2 = \rho\, \hat{a}_\rho$$

Consider a small differential element on conductor at point 1 at distance $\vec{r}_1$ from origin where $\vec{r}_1 = z'\,\hat{a}_z$

So $\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 = \rho\hat{a}_\rho - z'\hat{a}_z$ and $\hat{a}_{12}$ is unit vector along $\vec{R}_{12}$

$$\hat{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\rho\hat{a}_\rho - z'\hat{a}_z}{\sqrt{\rho^2 + z'^2}}$$

and $d\vec{L} = dz'\,\hat{a}_z$

From Biot Savart's Law at point 2 the magnetic field intensity due to dz' element is given by

$$d\vec{H} = \frac{I\, d\vec{L} \times \vec{a}_R}{4\pi R^2} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi R^3}$$

$$d\vec{H_2} = \frac{I\,(dz'\,\hat{a}_z) \times (\rho\,\hat{a}_\rho - z'\,\hat{a}_z)}{4\pi\,(\rho^2 + z'^2)^{3/2}}$$

Since the current is directed toward increasing valu of z' the limits are $-\infty$ & ∞ on the integral

$$d\vec{H_2} = \frac{I\, dz'\,\rho\,\hat{a}_\phi - 0}{4\pi\,(\rho^2 + z'^2)^{3/2}}$$

$$d\vec{H_2} = \frac{I\, dz'\,\rho\,\hat{a}_\phi}{4\pi\,(\rho^2 + z'^2)^{3/2}}$$

$$\vec{H_2} = \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{dz'\,\rho\,\hat{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

Here $\hat{a}_\phi$ changes with coordinate ϕ but not with ρ or z. So, $\hat{a}_\phi$ is a constant and can be removed out of integral as integration is w.r.t z'

$$\vec{H_2} = \frac{I\,\hat{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\, dz'}{(\rho^2 + z'^2)^{3/2}}$$

(6)

∴ Since it is even function.

$$\vec{H_2} = \frac{2 I \hat{a}_\phi}{4\pi} \int_0^\infty \frac{\rho \, dz'}{(\rho^2+z'^2)^{3/2}}$$

Substitute $z' = \rho \tan\theta$ $\quad dz' = \rho \sec^2\theta \, d\theta$

$$(\rho^2 + \rho^2 \tan^2\theta)^{3/2} \Rightarrow \rho^3 \sec^3\theta$$

$$z' = 0 \Rightarrow \theta = \tan^{-1}(0) = 0$$

$$z' = \infty \Rightarrow \theta = \tan^{-1}(\infty) = \pi/2$$

$$\vec{H_2} = \frac{2 I \hat{a}_\phi}{4\pi} \int_0^{\pi/2} \frac{\rho^2 \sec^2\theta \cdot d\theta}{\rho^3 \sec^3\theta}$$

$$= \frac{2 I \hat{a}_\phi}{4\pi} \int_0^{\pi/2} \frac{1}{\rho \sec\theta} \, d\theta$$

$$= \frac{I \hat{a}_\phi}{2\pi\rho} \int_0^{\pi/2} \cos\theta \, d\theta$$

$$\boxed{\vec{H_2} = \frac{I \hat{a}_\phi}{2\pi\rho}}$$

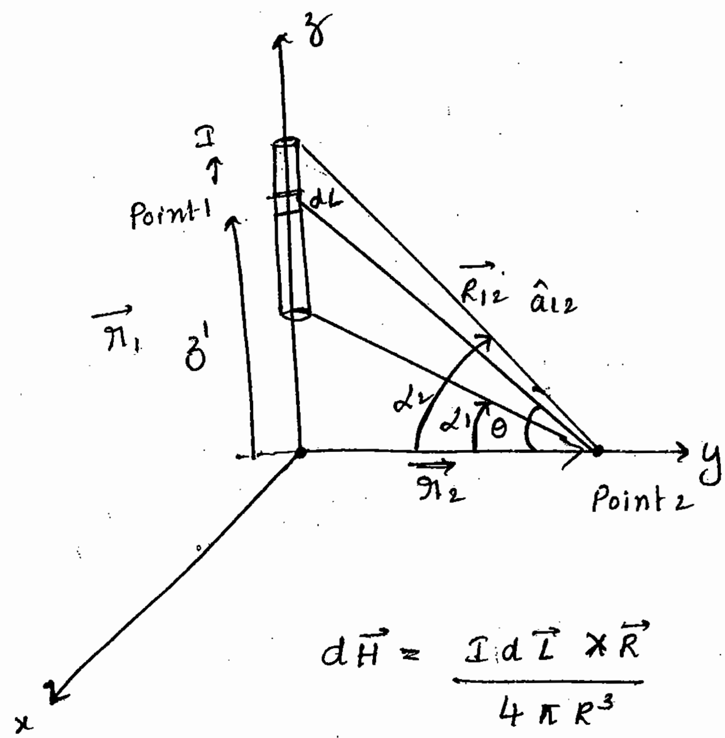
From above expression the magnitude of magnetic field intensity is not function of ϕ or z. It is inversely proportional to ρ which is perpendicular distance from the point to conductor.

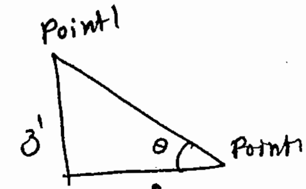
The direction of $\vec{H}$ is tangential i.e, circumferential along $\hat{a}_\phi$

The magnetic flux lines are circumferential

Magnetic field intensity due to finite length current element

Consider a finite-length current element as in figure below





$\tan\theta = z'/\rho$

$z' = \rho\tan\theta$

$$d\vec{H} = \frac{I\,d\vec{L} \times \vec{R}}{4\pi R^3}$$

$$d\vec{H} = \frac{I\,dz'\,\hat{a}_z \times (\rho\hat{a}_\rho - z'\hat{a}_z)}{4\pi\,(\rho^2+z'^2)^{3/2}}$$

$$d\vec{H} = \frac{I\rho\,dz'\,\hat{a}_\phi}{4\pi(\rho^2+z'^2)^{3/2}}$$

Substitute $z' = \rho\tan\theta$

$dz' = \rho\sec^2\theta\,d\theta$

$(\rho^2+z'^2)^{3/2} = \rho^3\sec^3\theta$

$$d\vec{H} = \frac{I\rho\cdot\rho\sec^2\theta\,d\theta\,\hat{a}_\phi}{4\pi\,(\rho^3\sec^3\theta)} = \frac{I\,\hat{a}_\phi}{4\pi\rho}\cos\theta\,d\theta$$

$$\vec{H} = \frac{I\,\hat{a}_\phi}{4\pi\rho}\int_{\theta=\alpha_1} \cos\theta\,d\theta = \frac{I\,\hat{a}_\phi}{4\pi\rho}\,\sin\theta\Big]_{\alpha_1}$$

$$\vec{H} = \frac{I}{4\pi\rho}\left[\sin\alpha_2 - \sin\alpha_1\right]\hat{a}_\phi$$

Ampere's Circuit Law.

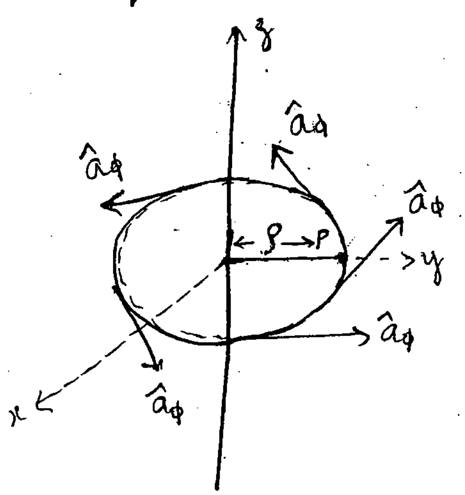
This law is used to solve complex problems in magnetostatics.

Ampere's Circuit law states that

the line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

Mathematically $\oint \vec{H}\cdot d\vec{L} = I$

Proof : Consider a long straight conductor carrying direct current I placed along z-axis as shown in figure.



Point P is at $\perp^{lar}$ distance ρ from conductor. Consider $d\vec{L}$ at point P in $\hat{a}_\phi$ direction

$$d\vec{L} = \rho\,d\phi\,\hat{a}_\phi$$

$\vec{H}$ obtained at point P from Biot-Savart law due to infinitely long conductor is,

$$\vec{H} = \frac{I}{2\pi\rho}\,\hat{a}_\phi$$

Then $\vec{H}\cdot d\vec{L} = \frac{I}{2\pi\rho}\hat{a}_\phi \cdot \rho\, d\phi\, \hat{a}_\phi = \frac{I}{2\pi}d\phi$

Integrating over entire closed path

$$\oint \vec{H}\cdot d\vec{L} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi}d\phi = \frac{I}{2\pi}(2\pi - 0) = I$$

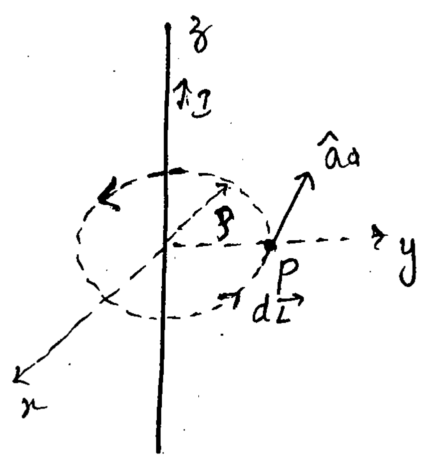
$\therefore \oint \vec{H}\cdot d\vec{L} = I$ Current carried by conductor

→ Ampere's law doesn't depends upon the shape of the path, but the path must be enclosed & is called as Amperian path.

Applications of Ampere's Circuit Law

$\vec{H}$ due to Infinitely long Straight Conductor

Consider an infinite long straight conductor placed along z-axis carrying a direct current I.



From figure at point P

$$\vec{H} = H_\phi\, \hat{a}_\phi$$

Because the $\vec{H}$ depends on ρ and the direction is always tangential to closed path i.e., $\hat{a}_\phi$

$$d\vec{L} = \rho\, d\phi\, \hat{a}_\phi$$

$$\therefore \vec{H}\cdot d\vec{L} = H_\phi\, \hat{a}_\phi \cdot \rho\, d\phi\, \hat{a}_\phi = H_\phi\, \rho\, d\phi$$

From Ampere's ckt law $\oint \vec{H}\cdot d\vec{L} = I$

$$\int_{\phi=0} H_\phi \, \rho \, d\phi = I$$

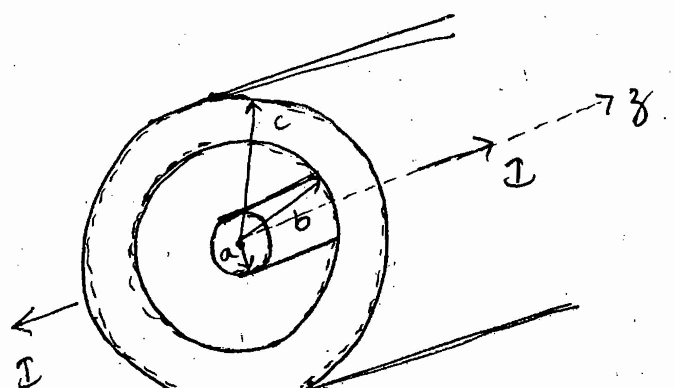
$$H_\phi \, \rho \, (2\pi) = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho} \, \hat{a}_\phi \quad A/m}$$

$\vec{H}$ due to a Co-axial Cable

Consider a co-axial cable as shown in figure below. The inner conductor radius is 'a' carrying current I. The outer conductor is in the form of concentric cylinder whose inner radius is 'b' and outer radius 'c'. This cable is placed along z-axis



The current I is uniformly distributed in the inner conductor, while $-I$ is uniformly distributed in the outer conductor.

→ The space between inner and outer conductor is filled with dielectric say air.
The calculation of $\vec{H}$ is divided corresponding to various regions of the cable.

Region 1

Within the inner conductor $\rho < a$.

Consider a closed path having radius $\rho_1 < a$

The area of cross section enclosed is $\pi \rho^2\ m^2$

The total current flowing is I through the area πa^2

Hence the current enclosed by the closed path is

$$I' = \frac{\pi \rho^2}{\pi a^2} I = \frac{\rho^2}{a^2} I$$

$\vec{H}$ is only in $\hat{a}_\phi$ direction and depends on ρ

$$\vec{H} = H_\phi\ \hat{a}_\phi \quad \& \quad d\vec{L} = \rho\, d\phi\ \hat{a}_\phi$$

$$\therefore \vec{H} \cdot d\vec{L} = H_\phi\, \rho\, d\phi$$

According to Ampere's circuit law

$$\oint \vec{H} \cdot d\vec{L} = I'$$

$$\oint H_\phi \rho\, d\phi = \frac{\rho^2}{a^2} I$$

$$\int_{\phi=0}^{2\pi} H_\phi \rho\, d\phi = \frac{\rho^2}{a^2} I$$

$$H_\phi \rho (2\pi) = \frac{\rho^2}{a^2} I$$

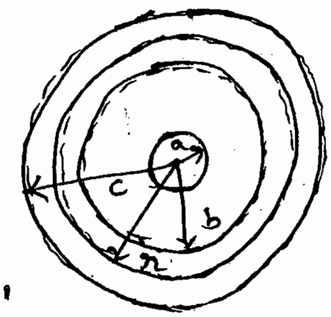
$$\boxed{\vec{H} = \frac{I \rho}{2\pi a^2}\ \hat{a}_\phi}\ A/m$$

Region 2 : within

which encloses the inner conductor carrying direct current I. This is the case of infinitely long conductor along z-axis. Hence $\vec{H}$ in this region is

$$\boxed{\vec{H} = \frac{I}{2\pi\rho}\,\hat{a}_\phi \text{ A/m}}$$

Region 3 : Within outer conductor $b<\rho<c$ Consider a closed path



The current enclosed by the closed path is only the part of current $-I$ in the outer conductor. The total current $-I$ is flowing through cross section $\pi(c^2-b^2)$ the closed path encloses the cross section $\pi(r^2-b^2)$. Hence the current enclosed by the closed path of outer conductor i.e

$$I' = \frac{\pi(\rho^2-b^2)}{\pi(c^2-b^2)}(-I) = \frac{-(\rho^2-b^2)}{(c^2-b^2)}I$$

$I'' = I$ = Current in inner conductor enclosed.

As the closed path also encloses the inner conductor and hence the current I flowing through it.

$$I_{enc} = I' + I'' = -\frac{(\rho^2-b^2)}{(c^2-b^2)}I + I$$

$$= I\left[1 - \frac{(\rho^2-b^2)}{c^2-b^2}\right]$$

$$I_{enc} = I\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

According to Ampere's circuit law

$$\oint \vec{H}\cdot d\vec{L} = I_{enc}$$

$$\vec{H} = 0 H_\phi \hat{a}_\phi \quad \& \quad d\vec{L} = \rho\, d\phi\, \hat{a}_\phi$$

$$\vec{H}\cdot d\vec{L} = H_\phi\, \rho\, d\phi$$

$$\int_{\phi=0}^{2\pi} H_\phi\, \rho\, d\phi = I_{enc}$$

$$2\pi\rho H_\phi = I\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

$$H_\phi = \frac{I}{2\pi\rho}\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

$$\vec{H} = \frac{I}{2\pi\rho}\left[\frac{c^2-\rho^2}{c^2-b^2}\right]\hat{a}_\phi \quad A/m$$

Region 4: Outside the cable $r > c$

$$I_{enc} = +I - I = 0\,A$$

$$\oint \vec{H}\cdot d\vec{L} = 0$$

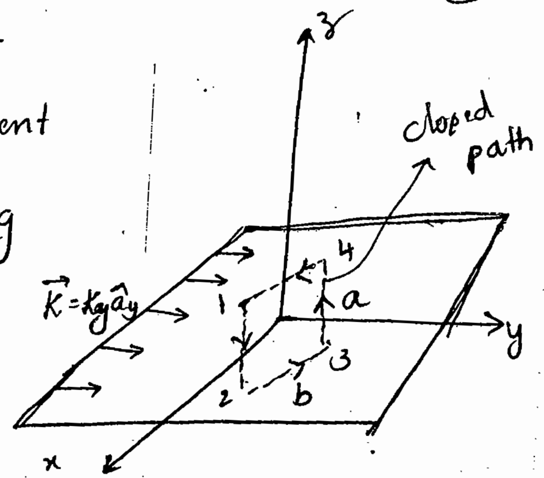
$$\boxed{\vec{H} = 0 \ A/m.}$$

The magnetic field does not exists outside the cable

$\vec{H}$ due to Infinite sheet of

Consider an infinite sheet of current in the $z=0$ plane. The surface current density is $\vec{K}$. The current is flowing in positive y direction hence

$$\vec{K} = K_y \hat{a}_y.$$

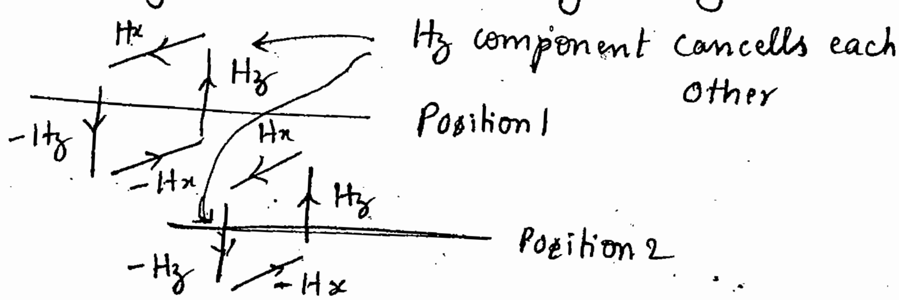


Consider a closed path 1-2-3-4 as shown in the figure. The width of the path is 'b' and the height 'a'.

The current flowing across the distance b is given by

$$I_{enc} = K_y\, b$$

Consider current in $\hat{a}_y$ direction according to right hand thumb rule.



Between any two very closely spaced conductors, the components of $\vec{H}$ in z direction are oppositely directed. All such components cancel each other and $\vec{H}$ cannot have any component in $\hat{a}_z$ direction.

As current flowing in y direction $\vec{H}$ cannot have any component in y direction. So $\vec{H}$ has only component in x direction.

$$\therefore \quad \vec{H} = H_x \hat{a}_x \quad \text{for } z>0$$
$$= -H_x \hat{a}_x \quad \text{for } z<0$$

Applying Ampere's ckt law $\oint \vec{H} \cdot d\vec{L} = I_{enc}$

Evaluating integral over the path 1-2-3-4-1

For path 1-2, $d\vec{L} = dz\, \hat{a}_z$

For path 3-4, $d\vec{L} = dz\, \hat{a}_z$ As $\vec{H}$ is in x direction

$\hat{a}_x \cdot \hat{a}_z = 0$

Hence along paths 1-2 and 3-4 $\oint \vec{H} \cdot d\vec{L} = 0$

Consider path 2-3 along which $d\vec{L} = dx\, \hat{a}_x$

$$\therefore \int_2^3 \vec{H} \cdot d\vec{L} = \int_2^3 (-H_x \hat{a}_x) \cdot (dx\, \hat{a}_x) = H_x \int_2^3 dx = b H_x$$

Consider path 4-1 along which $d\vec{L} = dx\, \hat{a}_x$

$$\therefore \int_4^1 \vec{H} \cdot d\vec{L} = \int_4^1 (H_x \hat{a}_x) \cdot (dx\, \hat{a}_x) = H_x \int_4^1 dx = b H_x$$

$$\therefore \oint \vec{H} \cdot d\vec{L} = 2b H_x$$

Equating this to $I_{enc} = K_y b$

$$2b H_x = K_y b$$

$$H_x = \frac{1}{2} K_y$$

Hence,
$$\vec{H} = \frac{1}{2} K_y \hat{a}_x \quad \text{for } z>0$$
$$= -\frac{1}{2} K_y \hat{a}_x \quad \text{for } z<0$$

In general for an infinite sheet of current density K A/m

(4)

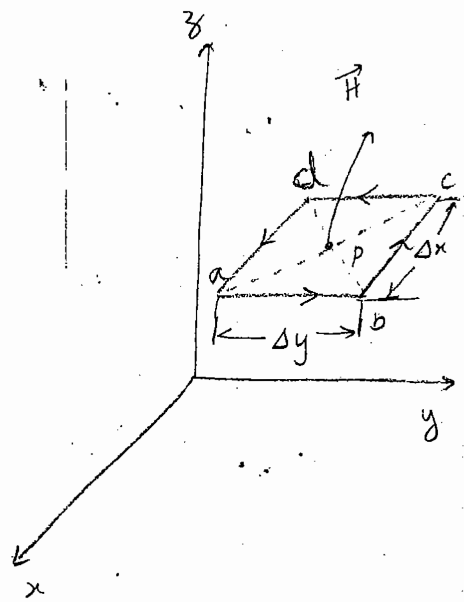
$$\boxed{\vec{H} = \frac{1}{2}\vec{K} \times \hat{a}_N}$$

$\hat{a}_N$ = unit vector normal from the current sheet to the point at which $\vec{H}$ is to be obtained.

Curl

Ampere's circuit law is to be applied to the differential surface element to develop the concept of curl.

Consider a differential surface element having sides Δx and Δy plane. The unknown current has produced $\vec{H}$ at the centre of this incremental closed path



The total magnetic field intensity at the point P which is centre of the small rectangle is

$$\vec{H} = H_{x0}\,\hat{a}_x + H_{y0}\,\hat{a}_y + H_{z0}\,\hat{a}_z$$

To apply Ampere's ckt law to this closed path let us evaluate the closed line integral of $\vec{H}$ about this path in the direction abcda.

According to right hand thumb rule the current is in $\hat{a}_z$ direction

Along path a-b $\quad \vec{H} = H_y \hat{a}_y$ and $d\vec{L} = \Delta y \hat{a}_y$

$\therefore \vec{H} \cdot d\vec{L} = H_y \hat{a}_y \cdot \Delta y \hat{a}_y = H_y \Delta y$

The intensity H_y along a-b can be expressed in terms of H_{yo} existing at P and the rate of change of H_y in x direction with x

$$\therefore \left(\vec{H} \cdot d\vec{L}\right)_{a-b} = \left[H_{yo} + \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}\right] \Delta y$$

Along path b-c $\quad \vec{H} = -H_x \hat{a}_x$ and $d\vec{L} = \Delta x \hat{a}_x$

$\therefore \vec{H} \cdot d\vec{L} = -H_x \Delta x$

$$\left(\vec{H} \cdot d\vec{L}\right)_{b-c} = -\left[H_{xo} + \frac{\Delta y}{2} \frac{\partial H_x}{\partial y}\right] \Delta x$$

Along path c-d $\quad \vec{H} = -H_y \hat{a}_y$ and $d\vec{L} = \Delta y \hat{a}_y$

$\therefore \vec{H} \cdot d\vec{L} = -H_y \Delta y$

$$\left(\vec{H} \cdot d\vec{L}\right)_{c-d} = -\left[H_{yo} - \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}\right] \Delta y$$

Along path d-a $\quad \vec{H} = H_x \hat{a}_x$ and $d\vec{L} = \Delta x \hat{a}_x$

$\therefore \vec{H} \cdot d\vec{L} = H_x \hat{a}_x$

$$\left(\vec{H} \cdot d\vec{L}\right)_{d-a} = \left[H_{xo} - \frac{\Delta y}{2} \frac{\partial H_x}{\partial y}\right] \Delta x$$

The total $\left(\vec{H} \cdot d\vec{L}\right)$ along abcda path ie

$$\vec{H} \cdot d\vec{L} = \cancel{H_{yo} \Delta y} + \frac{\partial H_y}{\partial x} \frac{\Delta x \Delta y}{2} + \frac{\partial H_y}{\partial x} \frac{\Delta x \Delta y}{2} - \cancel{H_{yo} \Delta y} -$$

$$\cancel{H_{xo} \Delta x} - \frac{\partial H_x}{\partial y} \frac{\Delta x \Delta y}{2} - \frac{\partial H_x}{\partial y} \frac{\Delta x \Delta y}{2} + \cancel{H_{xo} \Delta x}$$

$$\oint \vec{H}\cdot d\vec{L} = \frac{\partial H_y}{\partial x}\Delta x\,\Delta y - \frac{\partial H_x}{\partial y}\Delta x\,\Delta y$$

$$\oint \vec{H}\cdot d\vec{L} = \Delta x\,\Delta y\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]$$

This integral must be current enclosed by the differential element according to Ampere's ckt law.

Current enclosed = Current Density normal to Closed path × Area of the closed path.

$$I_{enc} = J_z\,\Delta x\,\Delta y$$

$J_z \rightarrow$ Current density in $\hat{a}_z$ direction as the current enclosed in $\hat{a}_z$ direction

$$\oint \vec{H}\cdot d\vec{L} = \Delta x\,\Delta y\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] = J_z\,\Delta x\,\Delta y$$

$$\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta x\,\Delta y} = \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] = J_z$$

This gives accurate result as closed path shrinks to a point i.e., $\Delta x\,\Delta y$ area ~~d~~ tends to zero

$$\lim_{\Delta x \Delta y \to 0}\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta x\,\Delta y} = \frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial x} = J_z$$

Considering incremental closed path in yz plane we get the current density normal to it i.e., in x direction

i.e.,
$$\lim_{\Delta y \Delta z \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta y \, \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

Considering incremental closed path in zx plane we get the current density normal to it i.e, in y direction

i.e.,
$$\lim_{\Delta z \Delta x \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta z \, \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

In general we can write

$$\lim_{\Delta S_N \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S_N} = J_N \quad \text{where}$$

J_N = Current density normal to the surface ΔS

ΔS_N = Area enclosed by closed line integral

The term on left hand side of the equation is called "curl $\vec{H}$".

The total current density is given by

at point P $\quad \vec{J} = J_x \hat{a}_x + J_y \hat{a}_y + J_z \hat{a}_z$

$$\vec{J} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] \hat{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] \hat{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] \hat{a}_z$$

$$\vec{J} = \text{Curl } \vec{H} = \nabla \times \vec{H}$$

The "curl $\vec{H}$" is indicated by $\nabla \times \vec{H}$ which is cross product of operator del and $\vec{H}$.

Point form of Ampere's Circuit Law

$$\boxed{\text{Curl } \vec{H} = \nabla \times \vec{H} = \vec{J}}$$

This is Second Maxwell's equations.

The third Maxwell's equation is the point form of

$$\oint \vec{E} \cdot d\vec{L} = 0 \qquad \boxed{\nabla \times \vec{E} = 0}$$

Curl in various Coordinate Systems.

1. Cartesian Coordinate System

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\nabla \times \vec{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\hat{a}_z$$

② Cylindrical Coordinate System

$$\nabla \times \vec{H} = \frac{1}{r} \begin{vmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$\nabla \times \vec{H} = \left[\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right]\hat{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\hat{a}_\phi + \left[\frac{\partial rH_\phi}{\partial r} - \frac{\partial H_r}{\partial \phi}\right]\frac{\hat{a}_z}{r}$$

3) <u>Spherical Coordinate System</u>

$$\nabla \times \vec{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\, \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\, H_\phi \end{vmatrix}$$

$$\nabla \times \vec{H} = \frac{1}{r\sin\theta}\left[\frac{\partial (H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right] + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial (r H_\phi)}{\partial r}\right]\hat{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\hat{a}_\phi$$

<u>Significance of a Curl</u>

The curl is a closed line integral per unit area as the area shrinks to a point. It gives circulation density of a vector about a point at which the area is going to shrink. The curl also gives the direction which is along the axis through a point at which curl is defined.

The magnetic field lines produced by the current carrying conductor are rotating in the form of Concentric circles around the conductor. Thus there exists a curl of magnetic field intensity which we have defined as $\nabla \times \vec{H}$

The direction of curl is to be obtained by right hand thumb rule.

If curl of a vector field exists then field is called as rotational. Fn irrotational vector field curl is zero

Magnetic flux and Magnetic flux density (14)

→ In free space magnetic flux density is $\vec{B} = \mu_0 \vec{H}$ where $\vec{B}$ is measured in wb/m^2 or a new ISU Tesla (T) where $1\ T = 1\ wb/m^2$

The constant $\mu_0 = 4\pi \times 10^{-7}$ H/m

where μ_0 is permeability of free space.

→ If magnetic flux is represented by ϕ i-e, flux passing through designated area

$$\left\langle \phi = \int_S \vec{B} \cdot d\vec{S} \quad \text{Webere} \right\rangle$$

Electric flux measured in Coulombs is

$$\left\langle \psi = \oint_S \vec{D} \cdot d\vec{S} = Q \right\rangle$$

→ The magnetic flux lines are closed and do not terminate on a magnetic charge. For a closed surface the number of magnetic flux lines entering must be equal to number of flux lines leaving The single magnetic pole cannot exist. like a single electric charge. Hence the integral $\vec{B} \cdot d\vec{S}$ evaluated over a closed surface is zero

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$

Applying divergence theorem to above equation

$$\oint_S \vec{B}\cdot d\vec{s} = \int_{vol} \nabla\cdot\vec{B}\, dv = 0$$

where dv is the volume enclosed by the closed surface

$$\nabla\cdot\vec{B} = 0$$

The divergence of magnetic flux density is always zero. This is called Gauss's law in differential form for magnetic fields.

The above expression is last Maxwell's equation.

Thus for static electric fields and steady magnetic fields

$$\boxed{\begin{aligned} \nabla\cdot\vec{D} &= \rho_v \\ \nabla\times\vec{E} &= 0 \\ \nabla\times\vec{H} &= \vec{J} \\ \nabla\cdot\vec{B} &= 0 \end{aligned}}$$

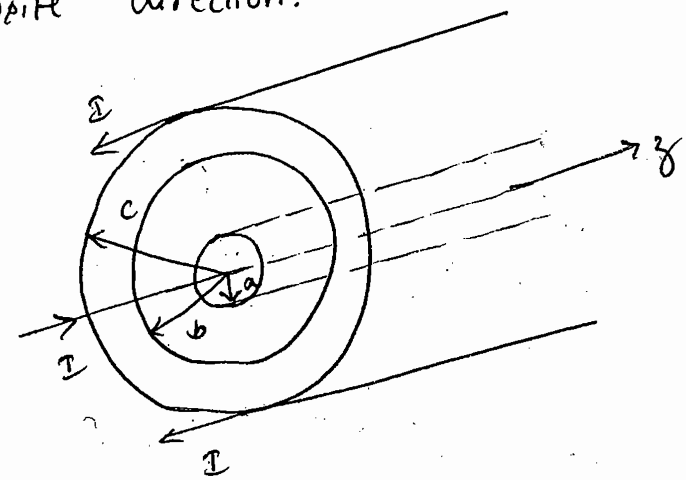
The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

$$\boxed{\begin{aligned} &\oint_S \vec{D}\cdot d\vec{s} = Q = \int_{vol} \rho_v\, dv \\ &\oint \vec{E}\cdot d\vec{L} = 0 \\ &\oint \vec{H}\cdot d\vec{L} = I = \int_S \vec{J}\cdot d\vec{s} \\ &\oint_S \vec{B}\cdot d\vec{s} = 0 \end{aligned}}$$

Application of Flux density and flux ...

15

Consider a coaxial cable such that its axis is along the z-axis. It carries a direct current I which is uniformly distributed in the inner conductor. The outer conductor carries the same current I in opposite direction.



$\vec{H}$ in the region $a < \rho < b$ is

$$\vec{H} = \frac{I}{2\pi\rho}\, \hat{a}_\phi \ \ A/m$$

We are interested in the flux in the region $a < \rho < b$. The cable is filled with air i.e., $\mu = \mu_0$

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi\rho}\, \hat{a}_\phi \ \ wb/m^2$$

Let d be the length of the conductor. The magnetic flux contained between the conductors in a length d is magnetic flux crossing the radial plane from $\rho = a$ to $\rho = b$ and for $z = 0$ to $z = d$.

The magnetic flux is given by

$$\phi = \int_S \vec{B} \cdot d\vec{s}$$

$d\vec{s} = d\rho \, dz$

$$\phi = \int_S \vec{B} \cdot d\vec{s} = \int_S \frac{\mu_0 I}{2\pi\rho} \hat{a}_\phi \cdot d\rho \, dz \, \hat{a}_\phi$$

$$= \int_{z=0}^{d} \int_{\rho=a}^{b} \frac{\mu_0 I}{2\pi\rho} \, d\rho \, dz = \frac{\mu_0 I}{2\pi} \Big[z\Big]_0^d \Big[\ln \rho\Big]_a^b$$

$$\boxed{\phi = \frac{\mu_0 I d}{2\pi} \ln(b/a) \quad wb}$$

<u>The Scalar and Vector Magnetic potentials.</u>

In electrostatics it is seen that there exists a scalar electric potential V which is related to the electric field intensity $\vec{E}$ as $\vec{E} = -\nabla V$

In case of magnetic fields there are two types of potentials can be defined.

1. Scalar magnetic potential denoted as V_m
2. Vector magnetic potential denoted as $\vec{A}$

Two vector identities (Properties of curl) are

$$\nabla \times \nabla V = 0 , \qquad V = \text{Scalar}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 , \qquad \vec{A} = \text{Vector}$$

If V_m is scalar magnetic potential then it must satisfy the equation $\nabla \times \nabla V_m = 0$

But $\vec{H} = -\nabla V_m$

Then from above 2 expressions

$$\nabla \times (-\vec{H}) = 0 \qquad \text{i.e., } \nabla \times \vec{H} = 0$$

$$\text{But} \quad \nabla \times \vec{H} = \vec{J} \qquad \text{hence} \quad \vec{J} = 0$$

Thus scalar magnetic potential V_m can be defined for region where $\vec{J}$ the current density is zero

$$\vec{H} = -\nabla V_m \qquad \text{only for } \vec{J} = 0$$

Similar to $\vec{E}$ and V the relation between $\vec{H}$ & V_m is

$$\left\langle V_{m\,a,b} = -\int_b^a \vec{H} \cdot d\vec{L} \right\rangle$$

We know that

$$\nabla \cdot \vec{B} = \nabla \cdot \mu_0 \vec{H} = 0$$

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

$$\boxed{\nabla^2 V_m = 0} \qquad \text{for } \vec{J} = 0$$

This is Laplace's equation for scalar magnetic potential.

In many magnetic problems involve geometries in which current - carrying conductor occupy a relatively small fraction of the total region of interest, it is also used (scalar magnetic potential) case of permanent magnets.

(b) Vector Magnetic Potential

Vector magnetic potential is denoted as $\vec{A}$ measured in Wb/m

Then any vector has to satisfy

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

But $\nabla \cdot \vec{B} = 0$

$$\vec{B} = \nabla \times \vec{A}$$

Thus curl of vector magnetic potential is the flux density

Now $\quad \nabla \times \vec{H} = \vec{J}$

$$\nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J}$$

Using vector identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

hence

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\vec{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \vec{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right]$$

Thus if vector magnetic potential is known the current density can be obtained.

Consider a differential current element $d\vec{L}$ carrying current I. Then according to Biot Savart's law the vector magnetic potential $\vec{A}$ at a distance R from the differential current element is given by

$$\vec{A} = \oint \frac{\mu_0 I \, d\vec{L}}{4\pi R} \quad \text{Wb/m}$$

For distributed uniform ... by $\vec{K}\, ds$ where $\vec{K}$ is surface current density (17)

$$\vec{A} = \oint_S \frac{\mu_0 \vec{K}\, ds}{4\pi R} \quad Wb/m$$

If volume current density $\vec{J}$ is given in A/m^2 then $I\, d\vec{L}$ can be replaced by $\vec{J}\, dv$

$$\vec{A} = \oint_{Vol} \frac{\mu_0 \vec{J}\, dv}{4\pi R} \quad Wb/m$$

Note: ① Zero reference for $\vec{A}$ is ∞

② No finite current can produce the contributions as $R \to \infty$

Magnetic Forces

The electric field causes a force to be exerted on a charge which may be either stationary or in motion. The steady magnetic field is capable of exerting a force only on a moving charge.

Magnetic field may be produced by moving charges & may exert forces on moving charges magnetic field cannot arise from stationary charges and cannot exert any force on a stationary charge.

Force on a Moving Charge

→ In an electric field force on a charged particle is

$$\boxed{\vec{F_e} = Q\vec{E}} \longrightarrow (1)$$

The force is in the same direction as the electric field intensity (for a positive charge) and is directly proportional to both $\vec{E}$ and Q. If the charge is in motion, the force at any point in its trajectory is given by eq^n (1)

A charged particle in motion in a magnetic field of flux density $\vec{B}$ is found experimentally to experience a force, whose magnitude is proportional to the product of magnitude of charge, its velocity $\vec{v}$, and the flux density $\vec{B}$ and to the sine of angle between vectors $\vec{v}$ & $\vec{B}$

Thus force may be expressed as $\boxed{\vec{F}_m = Q\vec{v}\times\vec{B}} \rightarrow (2)$

→ The direction of $\vec{F_m}$ is perpendicular to the plane containing $\vec{v}$ and $\vec{B}$ both.

→ From eq^n(1) it is clear that the force $\vec{F_e}$ is independent of the velocity of the moving charge Thus the electric force performs work on the charge

→ $\vec{F_m}$ cannot perform work on charge (moving charge) as it is at right angles to the direction of motion of charg $(\vec{F}\cdot d\vec{l} = 0)$

The total force on a moving charge in the presence of both electric and magnetic field is

$$\vec{F} = \vec{F_e} + \vec{F_m} = Q(\vec{E} + \vec{V} \times \vec{B}) \quad N$$

This equation is called Lorentz force equation

Force on a Differential current element

The force exerted on a differential element of charge dQ moving in a steady magnetic field is given by

$$d\vec{F} = dQ\, \vec{V} \times \vec{B} \quad N \longrightarrow (1)$$

The current density $\vec{J}$ can be expressed in terms of a volume charge density as

$$\vec{J} = \rho_v \vec{V} \longrightarrow (2)$$

But the differential element of charge can be expressed in terms of the volume charge density as

$$dQ = \rho_v\, dv \longrightarrow (3)$$

Substituting value of dQ in eq^n (1)

$$d\vec{F} = \rho_v\, dv\, \vec{V} \times \vec{B}$$

$$d\vec{F} = \vec{J} \times \vec{B}\, dv \longrightarrow (4)$$

The relationship between current elements are

$$\vec{J}\, dv = \vec{K}\, ds = I\, d\vec{L} \longrightarrow (5)$$

Then the force exerted on surface current density is

$$d\vec{F} = \vec{K} \times \vec{B}\, ds \longrightarrow (6)$$

The force exerted on a differential current element is

$$d\vec{F} = (I\, d\vec{L} \times \vec{B}) \longrightarrow (7)$$

Integrating equation (4) over the volume ie

$$\boxed{\vec{F} = \int_{vol} \vec{J} \times \vec{B}\, dv}$$

Integrating equation (6) over open or closed surface

$$\boxed{\vec{F} = \int_S \vec{K} \times \vec{B}\, ds}$$

Similary integrating equation (7) over closed path

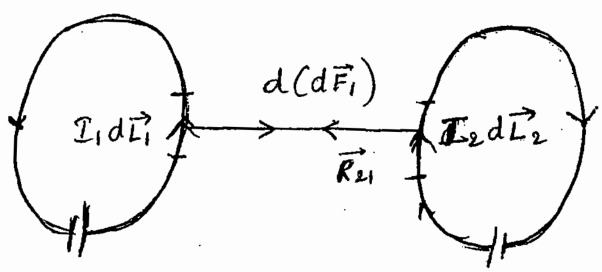
$$\boxed{\vec{F} = \oint I\, d\vec{L} \times \vec{B}}$$

If the conductor is straight and the field $\vec{B}$ is uniform along it, $\boxed{\vec{F} = I\vec{L} \times \vec{B}}$ the magnitude of the force is given by $F = ILB\sin\theta$ where θ is the angle between the direction of current flow and the direction of magnetic flux density.

In electrostatic fields we have studied that a point charge exert a force on another point charge, separated by a distance R.

Consider two current carrying conductors placed parallel to each other. Each conductor produces its own flux around it. When such two conductors are placed closed to each other there exists a force due to interaction of two fluxes.

Consider two current elements $I_1 d\vec{L_1}$ and $I_2 d\vec{L_2}$ as shown in figure.



Both current elements produce their own magnetic fields. As the currents are flowing in the same direction through the elements, the force exerted on element $I_1 d\vec{L_1}$ due to magnetic field $d\vec{B_2}$ produced by current element $I_2 d\vec{L_2}$ is force of attraction

The differential force on a differential current element is

$$d\vec{F} = I\, d\vec{L} \times \vec{B}$$

The differential amount of differential force on $I_1 d\vec{L_1}$ is

$$d(d\vec{F_1}) = I_1 d\vec{L_1} \times d\vec{B_2}$$

From Biot Savart's law

$$d\vec{B_2} = \mu_0 d\vec{H_2} = \mu_0 \left[\frac{I_2 d\vec{L_2} \times \hat{a}_{R21}}{4\pi |\vec{R}_{21}|^2} \right]$$

Substituting value of $d\vec{B_2}$ we can write

$$d(d\vec{F_1}) = \mu_0 \left[\frac{I_1 d\vec{L_1} \times (I_2 d\vec{L_2} \times \hat{a}_{R21})}{4\pi |\vec{R}_{21}|^2} \right]$$

This equation represents force between two current element

By integrating above equation twice the total force $\vec{F_1}$ on current element 1 due to current element 2 is given by

$$\vec{F_1} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\vec{L_1} \times (d\vec{L_2} \times \hat{a}_{R21})}{|\vec{R}_{21}|^2}$$

Similarly

$$\vec{F_2} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\vec{L_2} \times (d\vec{L_1} \times \hat{a}_{12})}{|\vec{R}_{12}|^2}$$

Magnetic Boundary Conditions:

→ The conditions of the magnetic field exists at the boundary of two media when the magnetic field passes from one medium to other are called boundary conditions for magnetic fields.

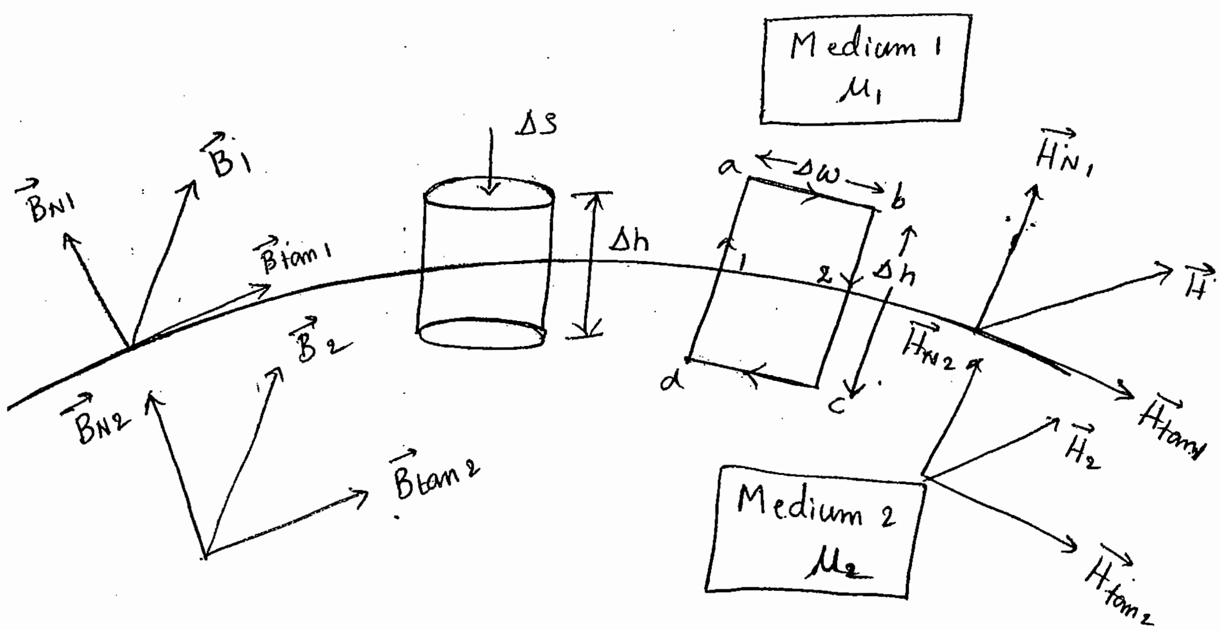
→ The conditions of $\vec{B}$ & $\vec{H}$ are studied at the boundary. The boundary between two different magnetic materials is considered

→ To study conditions of $\vec{B}$ and $\vec{H}$ at the boundary both the vectors are resolved into two components

a) Tangential to boundary

b) Normal to boundary

Consider two materials with different permeabilities μ_1 and μ_2 as shown in the figure



To find the normal component of $\vec{B}$ select a closed Gaussian surface in the form of a right circular cylinder. Let the height of cylinder be Δh and be placed in such a way that $\Delta h/2$ in medium 1 and remaining $\Delta h/2$ is in medium 2.

According to Gauss's law $\oint \vec{B} \cdot d\vec{S} = 0$

The surface integral must be evaluated over three surfaces top, bottom & lateral.

$$\oint_{top} \vec{B} \cdot d\vec{S} + \oint_{lateral} \vec{B} \cdot d\vec{S} + \int_{bottom} \vec{B} \cdot d\vec{S} = 0$$

As we are interested in boundary conditions, reduce Δh to zero the cylinder tends to boundary as $\Delta h \to 0$. Only top & bottom surfaces contribute in the surface integral.

$$\oint_{top} \vec{B} \cdot d\vec{S} = B_{N1} \Delta S$$

$$\oint_{bottom} \vec{B} \cdot d\vec{S} = -B_{N2} \Delta S$$

For Lateral Surface $\oint_{Lateral} \vec{B} \cdot d\vec{S} = 0$

$$B_{N1} \Delta S = B_{N2} \Delta S$$

$$\boxed{B_{N1} = B_{N2}}$$

Normal component of $\vec{B}$ is continous at the boundary

W.K.T. $\vec{B} = \mu \vec{H}$

$$\cancel{\mu_0} \mu_{r1} H_{N1} = \cancel{\mu_0} \mu_{r2} H_{N2}$$

$$\boxed{H_{N1}/H_{N2} = \mu_{r2}/\mu_{r1}}$$

According to Ampere's circuit law

$$\oint \vec{H} \cdot d\vec{L} = I$$

Consider rectangular closed path a-b-c-d-a as shown in figure it is traced in clockwise direction.

$\oint \vec{H} \cdot d\vec{L}$ can be divided into 6 parts

$$\oint \vec{H} \cdot d\vec{L} = \int_a^b \vec{H} \cdot d\vec{L} + \int_b^2 \vec{H} \cdot d\vec{L} + \int_2^c \vec{H} \cdot d\vec{L} + \int_c^d \vec{H} \cdot d\vec{L} + \int_d^1 \vec{H} \cdot d\vec{L} + \int_1^a \vec{H} \cdot d\vec{L}$$

$$\oint \vec{H} \cdot d\vec{L} = (H_{tan1}\, \Delta w) + \cancel{\left(-H_{N1} \frac{\Delta h}{2}\right)} + \cancel{\left(-H_{N2} \frac{\Delta h}{2}\right)} + (-H_{tan2}\, \Delta w) + \cancel{\left(H_{N2} \frac{\Delta h}{2}\right)} + \cancel{\left(H_{N1} \frac{\Delta h}{2}\right)} = I$$

$$\left[H_{tan1} - H_{tan2}\right] \Delta W = I$$

$$H_{tan1} - H_{tan2} = I/\Delta W = K$$

In vector form

$$(\vec{H}_1 - \vec{H}_2) \times \hat{a}_{N12} = \vec{K}$$

$$\boxed{\vec{H}_{tan1} - \vec{H}_{tan2} = \hat{a}_{N12} \times \vec{K}}$$

For tangential $\vec{B}$

$$\boxed{\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K.}$$

(8) (1)

3. Time Varying fields & Maxwell's Equations

→ With knowledge of basic relationships of the electrostatics and steady magnetic field we study time varying fields in this unit.

→ Two concepts are introduced in this unit
(a) Electric field produced by changing magnetic field.
(b) Magnetic field produced by changing electric field.

The first concept is result of experimental research by Michael Faraday. & the second from theoretical efforts of James Clerk Maxwell.

Faraday's Law.

→ In the year 1820, Prof Oersted demonstrated that a compass needle deflected due to an electric current.
After ten years Faraday professed his belief that if a current could produce a magnetic field, then magnetic field should be able to produce a current.

→ According to Faraday's experiment, a static magnetic field can not produce any current flow.
But with time varying fields, an electromotive force induces which may drive a current in a closed path or circuit.

(2) → The induced emf is nothing but voltage that induces from changing magnetic fields or motion of the conductors in a magnetic field.

→ Faraday discovered that the induced e.m.f is equal to the time rate of change of magnetic flux linking with the closed circuit.

Faraday's law can be stated as.

$$\boxed{emf = -\frac{d\phi}{dt} \ V} \longrightarrow (1)$$

Equation (1) implies a closed path, not necessarily closed conducting path, it might include a capacitor, or it might be purely imaginary line in space.

→ The negative sign indicates that the direction of induced e.m.f is such that to produce a current which will produce a magnetic field which will oppose the original field.

This statement that induced e.m.f acts to produce an opposing flux is known as Lenz's Law.

→ If the closed path is that taken by an N-turn filamentary conductor,

$$\boxed{emf = -N\frac{d\phi}{dt}} \longrightarrow (2)$$

ϕ is flux passing through any one of N coincident paths

→ The emf is scalar & we define emf as

$$emf = \oint \vec{E} \cdot d\vec{L} \longrightarrow (3)$$

It is the voltage about a specific closed path, such that if any part of the path is changed, the emf will also change.

The magnetic flux ϕ passing through a specified area is

$$\phi = \int_S \vec{B} \cdot d\vec{S} \longrightarrow (4)$$

Let us assume single turn circuit $N=1$ hence

$$emf = -\frac{d\phi}{dt} \text{ volts} \longrightarrow (5)$$

Using above results

$$emf = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{S}\right) \longrightarrow (6)$$

From equation (3) & (6) we get

$$emf = \oint \vec{E} \cdot d\vec{L} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S} \longrightarrow (7)$$

There are two conditions for the induced emf

(1) The contribution to the total emf made by a changing field within a stationary path.

(2) A moving path in stationary field.

(4) (1) Stationary closed path in a Time varying $\vec{B}$ field:
(Statically induced emf)

→ The closed circuit in which emf is induced is stationary and magnetic flux is sinusoidally varying with time.

→ It is clear that magnetic flux density is the only quantity varying with time. Hence

$$\oint \vec{E} \cdot d\vec{L} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \longrightarrow (8)$$

This is similar to transformer action & emf is called transformer emf. Using Stokes theorem

$$\oint \vec{E} \cdot d\vec{L} = \int_s (\nabla \times \vec{E}) \cdot d\vec{s} \longrightarrow (9)$$

From eq^n (8) & eq^n (9)

$$\int_s (\nabla \times \vec{E}) \cdot d\vec{s} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

Assuming both the surface integrals taken over identical surfaces

$$(\nabla \times \vec{E}) \cdot d\vec{s} = -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

Finally $$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \longrightarrow (10)$$

The above equation represents one of Maxwell's equations for time varying fields, If $\vec{B}$ is not varying with time then $\boxed{\nabla \times \vec{E} = 0}$

(2) Time varying closed path in stationary Magnetic field (Generator emf) (5)

→ The magnetic field is stationary, constant not varying with time while the closed ckt is revolved hence this action is similar to generator action hence

→ This action is similar to generator emf. induced emf is called generator emf.

- Consider charge Q moved in magnetic field $\vec{B}$ at velocity $\vec{V}$ then force on a charge

$$\vec{F} = Q\,\vec{V} \times \vec{B}$$

The motional electric field intensity is force per charge

$$\vec{E}_m = \frac{\vec{F}}{Q} = \vec{V} \times \vec{B}$$

Thus induced emf is

$$\boxed{\oint \vec{E}_m \cdot d\vec{L} = \oint (\vec{V} \times \vec{B}) \cdot d\vec{L}} \longrightarrow (ii)$$

Equation (ii) represents total emf induced when a conductor is moved in a uniform constant magnetic field.

In case if magnetic flux density and conductor both are varying with time then induced emf is

$$\oint \vec{E} \cdot d\vec{L} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint (\vec{V} \times \vec{B}) \cdot d\vec{L}$$

(6) Displacement Current density & Displacement Current

→ For static electromagnetic fields we can write

$$\nabla \times \vec{H} = \vec{J} \longrightarrow (1)$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} \longrightarrow (2)$$

But according to vector identity divergence of the curl of any vector field is zero.

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = 0 \longrightarrow (3)$$

The equation of continuity is $\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \longrightarrow (4)$

From equation (4) it is clear that when $\frac{\partial \rho_v}{\partial t} = 0$ then only equation (4) becomes true.

Thus (3) & eq^n (4) are not compatible for time varying fields. We modify eq^n (1) by adding some unknown term say $\vec{G}$

Then $\nabla \times \vec{H} = \vec{J} + \vec{G}$

Again taking divergence on both sides

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \nabla \cdot \vec{G} = 0.$$

$$\nabla \cdot \vec{G} = \frac{\partial \rho_v}{\partial t}$$

According to Gauss's law

$$\nabla \cdot \vec{G} = \frac{\partial (\nabla \cdot \vec{D})}{\partial t} = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

Comparing both equations $\boxed{\vec{G} = \frac{\partial \vec{D}}{\partial t}} \longrightarrow (5)$

Now we can write

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

→ The above equation is a form we have obtained which does not disagree with the continuity equation. This is second Maxwell's equation for time varying fields.

→ The term $\frac{\partial \vec{D}}{\partial t}$ has the dimensions of current density - Ampere per square meter.

Since it results from time-varying electric flux density Maxwell termed it as displacement current density denoted by $\vec{J}_d$

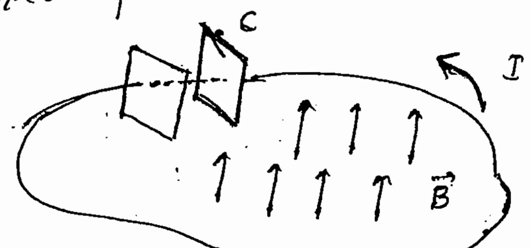
$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d$$

$$\text{Where} \quad \vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

In the above expression $\vec{J} = \sigma \vec{E}$ is conduction current density due to motion of charges.

Significance of displacement current

Consider a filamentary loop connecting the two plates of a parallel-plate capacitor. A time-varying magnetic field inside the closed path produces an emf $V_0 \cos \omega t$ around the closed path



(8) For the filament magnetic field varying sinusoidally with time is applied to produce an emf about the closed path.

i.e., $emf = V_0 \cos \omega t$

Using filamentary circuit theory and assuming filament has negligible inductance and resistance

$$I = -\omega C V_0 \sin \omega t$$

$$I = -\omega \frac{\varepsilon S}{d} V_0 (\sin \omega t)$$

Every where along the loop conductor current I will be there except capacitor. We need to consider displacement current for within the capacitor.

The ~~conductor~~ displacement is found to be

$$I_d = \frac{\partial D}{\partial t} S = -\omega \frac{\varepsilon S}{d} (V_0 \sin \omega t)$$

This is same as that of conduction current in filamentary loop.

Hence for some surfaces the current is entirely conduction current, but for those surfaces passing between capacitor plates the conduction current is zero. and it is displacement current.

→ Physically, capacitor stores charge and hence electric field between capacitor plates. This field will be much greater than the small leakage fields outside.

→ Displacement current is associated with time varying electric fields and therefore exists in all imperfect conductors carrying a time-varying conduction current.

Maxwell's Equations in Point form.

The Maxwell's equations for time varying fields are

$$\boxed{\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}} \longrightarrow (1)$$

~~and~~

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}} \longrightarrow (2)$$

$$\boxed{\nabla \cdot \vec{D} = \rho_v} \longrightarrow (3)$$

$$\boxed{\nabla \cdot \vec{B} = 0} \longrightarrow (4)$$

→ The first two equations we have derived in previous sections.

→ Equation (3) essentially states that charge density is a source or sink of electric flux lines. We can no longer say that all electric flux begins & terminates on charge, because eq^n (1) tells us that the electric field $\vec{E}$ and hence $\vec{D}$ may have circulation if a changing magnetic field is present. Thus electric flux lines may form closed loops. However converse is still true, every coulomb of charge must have one coulomb of electric flux diverging from it.

→ Equation (4) acknowledges the fact that "magnetic charges" or poles will not exists. Magnetic flux is found always in closed loops and never diverges from a point source.

(10) These four equations form the basis of all electromagnetic theory.

Some auxilliary equations relating $\vec{D}, \vec{E}, \vec{B}, \vec{H}, \sigma, \rho_v$ are

$$\boxed{\vec{D} = \varepsilon \vec{E}}$$

$$\boxed{\vec{B} = \mu \vec{H}}$$

$$\boxed{\vec{J} = \sigma \vec{E}}$$

$$\boxed{\vec{J} = \rho_v \vec{V}}$$

are required to define & relate the quantities appearing in Maxwell's equations.

Maxwell's equations in Integral form.

→ Maxwell's equations in integral form are obtained by generalization process of Maxwell's equations.

$$\boxed{\oint \vec{E} \cdot d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}}$$

$$\boxed{\oint \vec{H} \cdot d\vec{L} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}}$$

Integrating eq^n (1) & (2) over surface and applying Stoke's theorem

$$\boxed{\oint_S \vec{D} \cdot d\vec{S} = \int_{vol} \rho_v \, dv}$$

$$\boxed{\oint_S \vec{B} \cdot d\vec{S} = 0}$$

→ Integrating over volume and applying Divergen theorem

Uniform Plane Waves

→ The four Maxwell's equations are for electric and magnetic phenomena

→ The time varying fields can propagate through free space, where there no charges ($\rho=0$) and no electric current ($J=0$) i.e., lossless medium.

→ There were no evidences regarding the existence of electromagnetic waves, at that time Maxwell from his theoretical analysis predicted about existence of EM waves.

→ In equation $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, if $\frac{\partial \vec{B}}{\partial t}$ has non-zero value then $\nabla \times \vec{E}$ also will be non-zero. But $\nabla \times \vec{E}$ is non zero means $\vec{E}$ must be a function of position.

→ Thus a time varying magnetic field gives rise in general to an electric field which varies both in space and time.

→ Similarly a time varying electric field gives rise to a magnetic field that varies both in space and time. This idea is basis for electromagnetic wave propagation

Wave equation in terms of electric and magnetic field

Consider $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \rightarrow (1)$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \rightarrow (2)$

Let us obtain the wave equation in terms of the electric field i.e., equation relating to the spacial coordinates of $\vec{E}$ to its time coordinates i.e., by eliminating $\vec{H}$ between both equations.

Taking curl on both the sides of equation (1)

$$\nabla \times \nabla \times \vec{H} = \nabla \times \vec{J} + \varepsilon \frac{\partial (\nabla \times \vec{E})}{\partial t} \rightarrow (3)$$

From vector identity $\nabla \times \nabla \times \vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}$

i.e.,
$$\nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H} = \nabla \times \vec{J} + \varepsilon \frac{\partial}{\partial t}\left(\frac{-\partial \vec{B}}{\partial t}\right) \rightarrow (4)$$

Consider $\nabla \cdot \vec{H} = \nabla \cdot \frac{\vec{B}}{\mu} = 0.$

From above result and equation (4)

$$-\nabla^2 \vec{H} = \nabla \times \vec{J} - \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{H} - \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2} = -\nabla \times \vec{J}} \rightarrow (5)$$

This is equation of wave in terms of magnetic field intensity in medium with constant μ & ε.

For free space where $\vec{J} = 0$ the above equation reduces to

$$\boxed{\nabla^2 \vec{H} - \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2} = 0} \rightarrow (6)$$

Consider equation and apply curl on both sides

$$\nabla \times \nabla \times \vec{E} = \nabla \times \left[-\mu \frac{\partial \vec{H}}{\partial t}\right]$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

$$\nabla\left(\nabla \cdot \frac{\vec{D}}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$$\nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial \vec{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla\left(\frac{\rho_v}{\varepsilon}\right) + \mu \frac{\partial \vec{J}}{\partial t}} \longrightarrow (7)$$

The above expression is wave equation in terms of Electric field intensity for medium with constant ε & μ.

For free space

$$\boxed{\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \longrightarrow (8)$$

The RHS in eq^n (7) represents sources responsible for the wave field i.e., charges and currents.

The classical wave equation is represented as

$$\nabla^2 \vec{F} - \frac{1}{V^2} \frac{\partial^2 \vec{F}}{\partial t^2} = 0 \longrightarrow (9)$$

The above equation represents a wave travelling with velocity v.

Comparing equations 6, 8 & 9 it is clear that

$$V = \frac{1}{\sqrt{\mu \varepsilon}}$$

With this as reference Maxwell predicted that the free space supports the propagation of electromagnetic waves at speed

$$V = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \ m/s$$

Uniform Plane Waves

Plane waves are waves which posses variation only in the direction of propagation, and their characteristics remain constant across planes normal to the direction of propagation.

In case of electromagnetic waves propagating along x-axis, they are refered to as uniform plane waves if the electric field and magnetic field are independent of y and z-axis but is a function of x & t only.

Non-Existence of field components along the direction of propagation. (Transverse Nature of Electromagnetic wave)

Consider a uniform plane wave propagating along x-direction where there are no free charges.

From Maxwell's equation at any point:

$$\nabla \cdot \vec{D} = \rho_v$$

But $\rho_v = 0$

$$\nabla \cdot (\varepsilon \vec{E}) = 0$$

$\because \varepsilon \neq 0$, $\nabla \cdot \vec{E} = 0$

Expanding $$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \longrightarrow (1)$$

for a uniform plane wave travelling in x-direction, E_y and E_z components remain constant in y-z plane

$$\frac{\partial E_y}{\partial y} = 0 \quad \text{and} \quad \frac{\partial E_z}{\partial z} = 0$$

Then equation (1) can be reframed as

$$\frac{\partial E_x}{\partial x} = 0 \longrightarrow (2)$$

The above equation is satisfied if $E_x = 0$ or $E_x =$ constant

To understand which one of these hold good consider

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \longrightarrow (3)$$

Since $\vec{E}$ is independent of y & z equation (3) can be written as

$$\frac{\partial^2 \vec{E}}{\partial x^2} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \rightarrow$$

In terms of 3 components of $\vec{E}$ the

$$\frac{\partial^2 E_x}{\partial x^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}, \quad \frac{\partial^2 E_y}{\partial x^2} = \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2}, \quad \frac{\partial^2 E_z}{\partial x^2} = \mu\varepsilon \frac{\partial^2 E_z}{\partial t^2}$$

From equation (2) and above first expression

$$\frac{\partial^2 E_x}{\partial x^2} = 0$$

i.e, $\mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} = 0$ Since $\mu \neq 0$ & $\varepsilon \neq 0$

$\frac{\partial^2 E_x}{\partial t^2} = 0$ which requires either

$\frac{\partial E_x}{\partial t} = \text{Constant}$ or $\frac{\partial E_x}{\partial t} = 0$

$\frac{\partial E_x}{\partial t} = \text{constant}$ means, the value of E_x steadily increases with time indefinitely, But if E_x variable present in $\vec{E}$ that must have both increasing and decreasing values over a period of time Hence fails to hold good.

Now, considering other possibility $\frac{\partial E_x}{\partial t} = 0$ to hold good the solutions are $E_x = \text{Constant}$ and $E_x = 0$

Because of period nature of E_x, $E_x = 0$ is the only solution. The same procedure follows for $H_x = 0$.

$E_x = 0$ means, $\vec{E}$ will take orientations depending on E_y and E_z only, i.e., $\vec{E}$ lies only in y-z plane.
Thus with variations in values of $\vec{E_x}$ and $\vec{E_y}$ with passage of time. $\vec{E}$ assumes different orientations in successive y-z planes. Since y-z planes are all at right angles to x-direction, the variation in $\vec{E}$ also occurs only at right angle to x-direction. The same holds good for $\vec{H}$ also.

Thus it can be said that, a uniform plane electromagnetic wave can have its $\vec{E}$ and $\vec{H}$ components only along normal to the direction of propagation, or in other words electromagnetic waves are transverse in nature.

Relation between $\vec{E}$ & $\vec{H}$ for uniform plane wave

We have the Maxwell's equation for curl of $\vec{H}$ as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

If we consider a wave to be travelling in space where there are no free charges or current, then $\vec{J} = 0$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Expanding the terms

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \varepsilon \frac{\partial}{\partial t}\left[E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z\right] \quad \longrightarrow (1)$$

If we consider wave to be uniform plane wave travelling in x-direction, then H_x and E_x components are absent. Since there is no variation in y & z direction $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ operators produce null result.

The remaining factors in equation (1) are

$$-\left[\frac{\partial H_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial H_y}{\partial x}\right]\hat{a}_z = \varepsilon\left[\frac{\partial E_y}{\partial t}\right]\hat{a}_y + \varepsilon\left[\frac{\partial E_z}{\partial t}\right]\hat{a}_z \longrightarrow (2)$$

Equating the components along y-direction of both sides

$$-\frac{\partial H_z}{\partial x} = \varepsilon\frac{\partial E_y}{\partial t}$$

Similarly for z-direction

$$\frac{\partial H_y}{\partial x} = \varepsilon\frac{\partial E_z}{\partial t}$$

The solution for the wave equation for a wave travelling in x-direction is given by.

$$E_y = f_1(x - vt) \longrightarrow (3)$$

$$V = \frac{1}{\sqrt{\mu\varepsilon}}$$

Let

$$u = (x - vt) \longrightarrow (4)$$

$$E_y = f_1(u)$$

$$\frac{\partial E_y}{\partial t} = \frac{\partial f_1(u)}{\partial t} = \frac{\partial f_1(u)}{\partial u} \cdot \frac{\partial u}{\partial t}$$

But from equation (4)

$$\frac{\partial u}{\partial t} = -V$$

$$\frac{\partial f_1(u)}{\partial u} = f_1'$$

$$\frac{\partial E_y}{\partial t} = -v f_1'$$

$$-\frac{\partial H_z}{\partial x} = -\varepsilon v f_1' = -\sqrt{\frac{\varepsilon}{\mu}}\, f_1'$$

$$\partial H_z = \sqrt{\frac{\varepsilon}{\mu}}\, f_1'\, \partial x.$$

Integrating both sides.

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} \int f_1'\, dx + c$$

$$\frac{\partial E_y}{\partial x} = \frac{\partial f_1(u)}{\partial x} = \frac{\partial f_1(u)}{\partial u} \cdot \frac{\partial u}{\partial x} = f_1' \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} = 1$$

$$f_1' = \frac{\partial E_y}{\partial x}$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} \int \frac{\partial E_y}{\partial x}\, \partial x + c$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}}\, E_y + c$$

By comparing dimensions on both sides of the equation, it is clear that c must be a magnetic field. Since it is not confined to x-direction, it is ~~clear~~ t· it is dissociated from the wave motion and hence ignored.

Similar procedure is carried on equation $\frac{\partial H_y}{\partial x} = \varepsilon \frac{\partial E_z}{\partial t}$

then the expression for H_y becomes

$$H_y = -\sqrt{\frac{\varepsilon}{\mu}}\, E_z + c$$

Hence magnetic field components

$$E_y = \sqrt{\frac{\mu}{\varepsilon}}\; H_z$$

$$E_z = \sqrt{\frac{\mu}{\varepsilon}}\; H_y$$

Squaring both sides and adding above expressions.

$$E_y^2 + E_z^2 = \frac{\mu}{\varepsilon}\left(H_y^2 + H_z^2\right)$$

Since $E_x = 0$ and $H_x = 0$ the total electric and magnetic field strength are

$$E = \sqrt{E_y^2 + E_z^2} \quad \& \quad H = \sqrt{H_y^2 + H_z^2}$$

Using the above equation and after taking square root on both sides

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

It is the relation between $\vec{E}$ and $\vec{H}$ for a travelling uniform plane wave.

<u>Orientation between $\vec{E}$ & $\vec{H}$</u>

Consider the dot product $\vec{E} \cdot \vec{H}$.

$$\vec{E} \cdot \vec{H} = \left[E_y\,\hat{a}_y + E_z\,\hat{a}_z\right] \cdot \left[H_y\,\hat{a}_y + H_z\,\hat{a}_z\right]$$

$$\vec{E} \cdot \vec{H} = E_y H_y + E_z H_z$$

$$\vec{E} \cdot \vec{H} = \sqrt{\frac{\mu}{\varepsilon}}\, H_y H_z - \sqrt{\frac{\mu}{\varepsilon}}\, H_y H_z = 0$$

As per the above equation we see that the dot product of $\vec{E} \cdot \vec{H}$ is zero Hence they are at right angles to each other

Characteristic or intrensic Impedance (η)

For vacuum $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

$$\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

$$\frac{E}{H} = \eta = 377\ \Omega \text{ for free space.}$$

The ratio $\left(E/H\right) = \eta$ is called characteristic or intrensic impedance.

① Determine value of 'k' such that following pairs of fields satisfies Maxwell's equations in the region where $\sigma = 0$ & $\rho_v = 0$

a) $\vec{E} = [kx - 100t]\,\hat{a}_y$ V/m

$\vec{H} = [x + 20t]\,\hat{a}_z$ and $\mu = 0.25$ H/m , $\varepsilon = 0.01$ F/m

b) $\vec{D} = 5x\,\hat{a}_x - 2y\,\hat{a}_y + kz\,\hat{a}_z$ $\mu C/m^2$

$\vec{B} = 2\hat{a}_y$ mT and $\mu = \mu_0$ $\varepsilon = \varepsilon_0$

ⓐ For time varying fields

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

We can write

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & [kx - 100t] & 0 \end{vmatrix}$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial z}[kx - 100t]\,\hat{a}_x + \frac{\partial}{\partial x}[kx + 100t]\,\hat{a}_z$$

$$k\hat{a}_z = -\frac{\partial}{\partial t}(\mu \vec{H}) = \mu \vec{H}$$

$$\underline{k = -5 \text{ V/m}^2}$$

b) $\nabla \cdot \vec{D} = \rho_v$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v = 0.$$

$$\frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(kx) = 0$$

$$5 - 2 + k = 0$$

$$\boxed{k = -3 \, \mu C/m^3}$$

---

(2) A parallel plate capacitor with plate area of $5 \, cm^2$ and plate separation of 3mm has voltage $50 \sin(10^3 t)$ V applied to its plates. Calculate displacement current assuming $\varepsilon = 2\varepsilon_0$

$$\vec{D} = \varepsilon \vec{E} = \varepsilon V/d$$

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon V/d\right)$$

$$J_D = \frac{\varepsilon}{d}\frac{\partial V}{\partial t}$$

$$i_D = J_D \cdot \text{Area}$$

$$= \left[\frac{\varepsilon}{d}\frac{dv}{dt}\right](A)$$

$$= \frac{\varepsilon}{d}\left(\frac{dv}{dt}\right) A \cancel{C\frac{dv}{dt}} \qquad A = \frac{\varepsilon A}{d}$$

$$\cancel{i_D = C\frac{dv}{dt}}$$

$$= \frac{(2\varepsilon_0)}{(3 \times 10^{-3})} \frac{d\left[50\sin(10^3 t)\right]}{dt} (5 \times 10^{-2})$$

$$\boxed{i_D = 0.1475 \cos(10^3 t) \, \mu A}$$

Solution of wave equation for Uniform Plane Waves (1)

For an electromagnetic wave propagating in a medium where there are no free charges or currents, the wave equation for electric field is given by

$$\nabla^2 \vec{E} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \longrightarrow (1)$$

Let the wave associated with the above equation be a sinusoidal wave. For undamped variation the wave is given by

$$\vec{E} = \vec{E}_0 e^{j\omega t} \longrightarrow (2)$$

where $\vec{E}$ is instantaneous field at time t
$\vec{E}_0$ is amplitude or maximum value of E
$\omega = 2\pi f$ is angular frequency of wave
$j = \sqrt{-1}$

Differentiating E twice

$$\frac{\partial^2 \vec{E}}{\partial t^2} = j^2 \omega^2 (\vec{E}_0 e^{j\omega t}) = -\omega^2 \vec{E}$$

Using above equation wave equation for lossless medium is

$$\boxed{\nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0}$$

This is called Helmholtz equation.

Wave propagating in good conductor & Dielectrics

(2) If we consider a uniform plane wave propagating in x-direction, then the derivative with respect to y and z vanish

The wave equation becomes

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \omega^2 \mu \varepsilon \vec{E} = 0$$

This is a 3-dimensional equation, the field cannot have component along the direction of propagation, $E_x = 0$.

Among the y and z components, we shall consider the y-component

$$\frac{\partial^2 E_y}{\partial x^2} + \omega^2 \mu \varepsilon E_y = 0$$

$$\text{Let } \beta = \omega\sqrt{\mu\varepsilon}$$

$$\frac{\partial^2 E_y}{\partial x^2} + \beta^2 E_y = 0$$

The solution to this second order equation is given by

$$E_y = C_1 e^{-j\beta x} + C_2 e^{j\beta x}$$

The displacement at any instant t is

$$E_y(t) = E_y e^{j\omega t} = C_1 e^{j(\omega t - \beta x)} + C_2 e^{j(\omega t + \beta x)}$$

This is the solution of the wave equation and it represents the sum of two waves travelling in opposite directions. One is in +ve x-direction away from source (first term) and other in -ve x-direction towards the source (second term)

Wave equations in E and H for a conducting medium (3)

(Lossy Dielectric or Imperfect dielectric)

We have wave equation in $\vec{E}$ for a homogenous and isotropic medium as

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla\left(\frac{\rho_v}{\varepsilon}\right) \longrightarrow (1)$$

Let it be a conducting medium of conductivity σ (i.e., $\sigma\vec{E} = \vec{J}$) but let there be no charges $\rho_v = 0$.

Equation (1) becomes

$$\boxed{\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \sigma\mu \frac{\partial \vec{E}}{\partial t} = 0} \longrightarrow (2)$$

This is wave equation in terms of $\vec{E}$ for conducting medium

Similary wave equation in $\vec{H}$ for Homogenous medium as

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = -\nabla \times \vec{J} \longrightarrow (3)$$

but $\nabla \times \vec{J} = \sigma(\nabla \times \vec{E}) = \sigma\left(\frac{-\partial \vec{B}}{\partial t}\right) = -\mu\sigma \frac{\partial \vec{H}}{\partial t} \longrightarrow (4)$

From (3) & (4)

$$\boxed{\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} = 0} \longrightarrow (5)$$

This is wave equation in $\vec{H}$ for conducting medium.

(4) Uniform Plane wave in conducting medium

Let us consider the wave equation in $\vec{E}$ for conducting medium.

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0 \quad \longrightarrow (1)$$

The solution of above equation can be written as

$$\vec{E} = \vec{E_0}\, e^{j\omega t}$$

Differentiating $\vec{E}$ $\quad \frac{\partial \vec{E}}{\partial t} = j\omega \left(\vec{E_0}\, e^{j\omega t}\right) = j\omega \vec{E} \quad \longrightarrow (2)$

and $\quad \frac{\partial^2 \vec{E}}{\partial t^2} = j^2\omega^2 \left(\vec{E_0}\, e^{j\omega t}\right) = j^2\omega^2 \vec{E} \quad \longrightarrow (3)$

Substituting eqns (2) & (3) in (1)

$$\nabla^2 \vec{E} - \mu\varepsilon \left(j^2\omega^2 \vec{E}\right) - \mu\sigma \left(j\omega \vec{E}\right) = 0$$

$$\nabla^2 \vec{E} - j\omega\mu \left(j\omega\varepsilon + \sigma\right) \vec{E} = 0$$

The above equation can be written as.

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$

$$\gamma^2 = j\omega\mu \left(j\omega\varepsilon + \sigma\right)$$

γ is known as propagation constant.

If we consider a uniform plane wave propagating in x-direction, then considering y-component

$$\frac{\partial^2 E_y}{\partial x^2} + \gamma^2 E_y = 0$$

The solution for this second order differential equation is given by

$$E_y = C_1 e^{-\gamma x} + C_2 e^{\gamma x}$$

The displacement $E_y(t)$ at any instant t is

$$E_y(t) = E_y e^{j\omega t} = C_1 e^{(j\omega t - \gamma x)} + C_2 e^{(j\omega t + \gamma x)}$$

If we ignore the reflected wave then

$$E_y(t) = C_1 e^{(j\omega t - \gamma x)}$$

The propagation constant γ is a complex number hence

$$\gamma = \alpha + j\beta$$

Hence

$$E_y(t) = C_1 e^{[j\omega t - (\alpha + j\beta)x]}$$

$$E_y(t) = C_1 e^{-\alpha x} \cdot e^{j(\omega t - \beta x)}$$

The real part of above equation represents a uniform plane wave travelling in a conducting medium in x-direction. It is a damped wave whose attenuation is $e^{-\alpha x}$.

α is called attenuation constant and β is called phase constant. the values are given by

(6) $\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right\}^{1/2} \geq 0$

$\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right\}^{1/2} \geq 0$

Distinction between conductors and dielectrics.

When an electromagnetic wave propagates in free space it doesn't suffer any loss. But when it travels in any material medium, there will always be some loss due to absorption of energy by the medium (ohmic heating losses of the medium)

The magnitude of loss of energy depends on the medium. If the absorbing medium is low loss dielectric (good dielectric) σ is very small and wave undergoes a small but exponential decrease as it advances in the medium. But exponential decay becomes rapid if σ is very high

We have Maxwell's equation

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

For variation in time

$$\nabla \times \vec{H}\, e^{j\omega t} = \vec{J}\, e^{j\omega t} + \frac{\partial \vec{D}}{\partial t} e^{j\omega t}$$

$$e^{j\omega t}\left[\nabla \times \vec{H} - \vec{J} - j\omega\vec{D}\right] = 0$$

$$\nabla \times \vec{H} - \vec{J} - j\omega\vec{D} = 0$$

$$\nabla \times H = J + j\omega D$$

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E}$$

We know that $\vec{J} = \sigma\vec{E}$ is conduction current density

$\frac{\partial \vec{D}}{\partial t} = j\omega\varepsilon\vec{E}$ is known as displacement current density.

The magnitude of ratio of conduction current density to displacement current density is

$$\left| \frac{\sigma\vec{E}}{j\omega\varepsilon\vec{E}} \right| = \frac{\sigma}{\omega\varepsilon}$$

For non conductors this ratio is very less and for good conductors the ratio will be high

The ratio $\frac{\sigma}{\omega\varepsilon}$ is called loss tangent or dissipation factor. The loss tangent is defined on the basis of fact that the conduction current density leads the displacement current density by phase of $90°$.

For A uniform plane wave propagating in a conducting medium

ⓐ For a good conductor $\frac{\sigma}{\omega\varepsilon} \gg 1$

ⓑ For a perfect dielectric $\frac{\sigma}{\omega\varepsilon} \to 0$

ⓒ For a good dielectric $\frac{\sigma}{\omega\varepsilon} \ll 1$

(8) Uniform Plane Wave in a Good Conductor

(a) Attenuation constant (α)

for good conductor $\frac{\sigma}{\omega\varepsilon} \gg 1$

W.K.T $\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2+1} - 1\right]\right\}^{1/2}$

Consider $\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2+1} - 1 \approx \sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1 = \frac{\sigma}{\omega\varepsilon} - 1 \approx \frac{\sigma}{\omega}$

Hence $\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\frac{\sigma}{\omega\varepsilon}\right\}^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}}$

(b) Phase constant (β)

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1 \approx \frac{\sigma}{\omega\varepsilon}$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

(c) Wave Velocity (V)

$$V = \frac{\omega}{\beta}$$

$$V = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

(d) Propagation constant (γ)

$$\gamma = \alpha + j\beta$$

$$\gamma = \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\gamma = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j)$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}(1+j)^2}$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}(1+j^2+2j)} = \sqrt{\frac{\omega\mu\sigma}{\cancel{2}}(\cancel{2}j)}$$

$$= \sqrt{\omega\mu\sigma j}$$

$$\boxed{\gamma = \sqrt{\omega\mu\sigma}\,\angle 45^\circ}$$

Uniform plane wave in a perfect Dielectric

(a) Attenuation constant (α)

For perfect dielectric the condition is $\frac{\sigma}{\omega\varepsilon} \to 0$

$$\left\{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right\} \approx \sqrt{1+0} - 1 = 0$$

$$\alpha = 0$$

(b) Phase constant (β)

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1 = \sqrt{1+0} + 1 = 2$$

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}(2)\right\}^{1/2}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$$

(c) Velocity of wave (v)

$$v = \omega/\beta = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

(d) Propagation Constant (γ)

$$\boxed{\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}}$$

(3) Uniform plane wave in good dielectric

(10)

For a good dielectric $\frac{\sigma}{\omega\varepsilon} << 1$

(a) Attenuation constant (α)

Consider $\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} = \left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2}$

We know by binomial theorem that when a is small

$(1+a)^n = 1+na$ i.e., $\left[1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right]^{1/2} = \left[1+\frac{1}{2}\frac{\sigma^2}{\omega^2\varepsilon^2}\right]$

$$\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1 \approx 1+\frac{\sigma^2}{2\omega^2\varepsilon^2} - 1 = \frac{\sigma^2}{2\omega^2\varepsilon^2}$$

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left(\frac{\sigma^2}{2\omega^2\varepsilon^2}\right)\right\}^{1/2} = \omega\sqrt{\mu\varepsilon}\left\{\frac{\sigma}{2\omega\varepsilon}\right\}$$

$$\boxed{\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}}$$

(b) Phase constant (β)

$$\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1 \approx 1+\frac{\sigma^2}{2\omega^2\varepsilon^2} + 1 = 2+\frac{\sigma^2}{2\omega^2\varepsilon^2}$$

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left(2+\frac{\sigma^2}{2\omega^2\varepsilon^2}\right)\right\}^{1/2} = \omega\sqrt{\mu\varepsilon}\left\{1+\frac{\sigma^2}{4\varepsilon^2\omega^2}\right\}^{1/2}$$

Again by Binomial theorem

$$\beta = \omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma^2}{8\varepsilon^2\omega^2}\right]$$

Since $\frac{\sigma}{\varepsilon\omega} << 1$ $\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$

(c) Velocity of wave (v)

$$V = \frac{\omega}{\beta} = \frac{\omega}{\omega \cdot \sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

(d) Propagation Constant (γ)

$$\gamma = \alpha + j\beta = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}$$

$$\boxed{\gamma = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}}$$

1.2

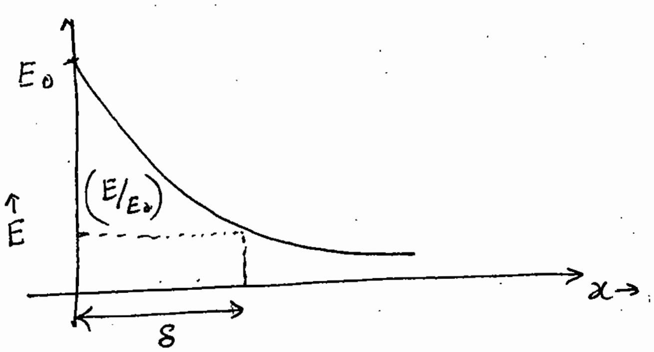
Skin Depth & Skin Effect

When an electromagnetic wave enters a conducting medium, its amplitude decreases exponentially, and becomes practically zero after penetrating a small distance. As a result, the current induced by the wave exists only near the surface of the conductor. This effect is known as skin effect.

If x is the distance travelled in the medium, and E is the amplitude, then the dependance can be represented as

$$E = E_0 e^{-\alpha x}$$

where, E_0 is the amplitude either at the time of incidence or at some point in the medium where x is taken as zero, α is the attenuation constant.



At the penetrating depth of $x = \delta$ let the value of x be

$$x = \frac{1}{\alpha} \qquad \text{i.e.,} \quad x = \delta = \frac{1}{\alpha}$$

$$e^{-\alpha x} = \frac{1}{e}$$

i.e., $$\boxed{E = \frac{1}{e} E_0}$$

(14) δ is called depth of penetration or skin depth.

Thus the skin depth is defined as the depth of conductor at which the amplitude of an incident wave drops to $(1/e)$ times its value of amplitude at the time of incidence.

But $\delta = 1/\alpha$

For good conductor $\alpha = \sqrt{\dfrac{\omega\mu\sigma}{2}}$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f\mu\sigma}}$$

For good conductors since σ is very high, δ will be very small. Further for a given conductor δ depends on inverse of the frequency f. Higher the frequency lesser will be the penetration. For copper $\sigma = 5.8 \times 10^7$ S/m

If $\mu = \mu_0$ $$\delta = \frac{0.066}{\sqrt{f}}$$

which gives values for δ as 1 cm at 50 Hz, 2 mm at 1 kHz and 0.066 mm at 1 MHz.

Poynting Theorem

"Poynting theorem states that the energy dissipated in a given volume, free of any sources is equal to the sum of the rate at which the decrease in electric and magnetic energies stored in the volume takes place and the rate at which energy inflow occurs through its surface."

→ This theorem may be considered as a statement of conservation of energy in electromagnetism.

→ Poynting vector represents the rate at which energy is transported per unit area at a point on the path of an electromagnetic wave, and is given by the cross product $\vec{E} \times \vec{H}$ at the same point.

We have the Maxwell's equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \longrightarrow (1)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \longrightarrow (2)$$

Consider $\nabla \cdot [\vec{E} \times \vec{H}]$, From vector identity we have

$$\nabla \cdot [\vec{A} \times \vec{B}] = B \cdot [\nabla \times \vec{A}] - \vec{A} \cdot [\nabla \times \vec{B}]$$

Similarly

$$\nabla \cdot [\vec{E} \times \vec{H}] = \vec{H} \cdot [\nabla \times \vec{E}] - \vec{E} \cdot [\nabla \times \vec{H}] \longrightarrow (3)$$

(16) From equations (1) & (2)

$$\nabla \cdot \left[\vec{E} \times \vec{H}\right] = \vec{H}\left[-\mu \frac{\partial \vec{H}}{\partial t}\right] - \vec{E}\cdot\left[\vec{J} + \varepsilon\frac{\partial \vec{E}}{\partial t}\right]$$

$$\nabla \cdot \left[\vec{E} \times \vec{H}\right] = -\mu\left[\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}\right] - \vec{E}\cdot\vec{J} - \varepsilon\left[\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}\right]$$

We know that

$$\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t} \quad \text{and} \quad \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}$$

$$\nabla \cdot \left[\vec{E} \times \vec{H}\right] = -\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t} - \vec{E}\cdot\vec{J}$$

Taking volume integral on both sides.

$$\int_V \nabla\cdot\left[\vec{E}\times\vec{H}\right] dv = -\int_V \frac{\partial}{\partial t}\left[\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right]dv - \int_V \vec{E}\cdot\vec{J}\, d$$

By Stoke's theorem

$$\int_V \nabla\cdot\left[\vec{E}\times\vec{H}\right] dv = \oint_S \left[\vec{E}\times\vec{H}\right] d\vec{s}$$

Substituting above expression in equation

$$\boxed{\int_V \vec{E}\vec{J}\, dv = -\int_V \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right) dv - \oint_S \left(\vec{E}\times\vec{H}\right) d\vec{s}} \rightarrow (4)$$

The above expression signifies the Poynting's theorem.

Proof for $\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t}$.

Consider $$\frac{\partial}{\partial t}\left(\vec{H}\cdot\vec{H}\right) = \vec{H}\cdot\frac{\partial \vec{H}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{H}}{\partial t}$$

$$\frac{\partial H^2}{\partial t} = 2\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} \Rightarrow \boxed{\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t}}$$

(17)

(a) Meaning of $\int \vec{E}\cdot\vec{J}\, dv$

We know that for a conductor of any area of cross section (say A) in which current I is flowing under a potential difference of E, the power dissipated/unit length $= EI$

Power dissipated by a conductor of length $l = EIl$

Power dissipated by a unit volume of the conductor

$$= \frac{EIl}{V} = \frac{EIl}{l \times A} = \frac{EI}{A} = EJ$$

$\therefore \int \vec{E}\cdot\vec{J}\, dv =$ the power dissipated by a unit volume of the conductor.

(b) Meaning of $-\int_V \left[\frac{\partial}{\partial t}\left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right)\right] dv$

$\frac{\mu H^2}{2}$ and $\frac{\varepsilon E^2}{2}$ are stored magnetic field and electric field energies respectively.

$-\int_V \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right) dv =$ the rate at which the stored field energy is decreasing in a volume v over which the integration is carried out.

18

(c) Meaning of $\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$ and the Poynting vector (Power = rate at which energy is supplied)

From equation (4)

→ Rate at which energy is dissipated in a volume V = [Rate of decrease of stored field energy in the volume V] − [$\oint_S [\vec{E} \times \vec{H}] \cdot d\vec{s}$]

→ As per law of conservation of energy, the last term in the above equation must also represent rate of energy change. In order to satisfy the logic, the second term must represent rate of energy input into the volume V.

thus $-\oint_S [\vec{E} \times \vec{H}] \cdot d\vec{s}$ rate of energy flow into the volume through its surface.

$\oint_S [\vec{E} \times \vec{H}] \cdot d\vec{s}$ is the rate of energy flow out of volume through its surface.

→ Thus vector product $[\vec{E} \times \vec{H}]$ represents rate of energy flow/unit area.

→ For an electromagnetic wave the value of this product at a particular point on the path of the wave, represents the rate at which energy is transported across the point.

→ The product $\vec{E} \times \vec{H}$ is another vector denoted by $\vec{P}$ directed perpendicular to the plane containing $\vec{E}$ & $\vec{H}$ vectors. i.e., $\vec{P}$ is called Poynting Vector $\boxed{\vec{P} = \vec{E} \times \vec{H}}$

Polarization of Electromagnetic Wave

→ The polarization is defined as time varying behaviour of the electric field intensity vector $\vec{E}$ at some fixed point in space along the direction of propagation.

→ Consider a uniform plane wave travelling in positive z-direction. Then the field vectors $|\vec{E}|$ and $\vec{H}$ lie in x-y plane. As EM wave travels in space both the fields undergo same variations with respect to time.

→ There are different types of polarization

1) Linear Polarization.
2) Elliptical Polarization
3) Circular Polarization.

→ Because of polarization there will be variations in magnitude and direction of the electric fields. These variations can be seen in direction of propagation of wave, hence in the plane perpendicular to the direction of propagation we get $\vec{E}$ along straight line, circle or ellipse.

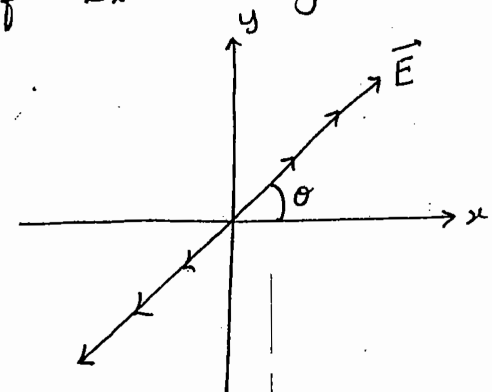
→ Consider $\vec{E}$ as resultant of $\vec{E}_x$ and $\vec{E}_y$.

1) Linear Polarization

→ If electric field $\vec{E}$ has only x component and y component of $\vec{E}$ is zero. Then looking from the direction of propagation. The wave is linearly polarized in x-direction.
Similary if only y component of E is present then wave is said to be linearly polarized in y-direction.

→ Let us assume that both the components of E are present. denoted by $\vec{E}_x$ and $\vec{E}_y$ and both the components are in phase with different amplitudes. If the amplitudes of $\vec{E}_x$ increases or decreases the amplitude of $\vec{E}_y$ also increases or decreases.

→ The electric field $\vec{E}$ is resultant of $\vec{E}_x$ and $\vec{E}_y$ and the direction of it depends on the relative magnitude of $\vec{E}_x$ and $\vec{E}_y$.



Thus the angle made by $\vec{E}$ with x-axis is given by

$$\theta = \tan^{-1} \frac{E_y}{E_x}$$

This angle is constant with respect to time.

→ Thus the resultant $\vec{E}$ is constantly oriented in a direction with respect to time, thus wave is said to be linearly polarized.

→ when both the amplitudes are same then we get polarization of $\vec{E}$ as linear polarization with constant angle of 45°.

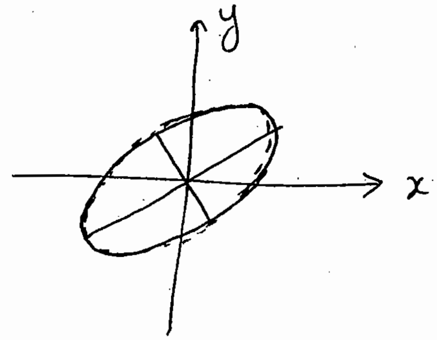
→ Thus when $\vec{E}_y$ and $\vec{E}_x$ components are in phase with either equal or unequal amplitudes for a uniform plane wave travelling in z-direction, the polarization is linear.

Elliptical Polarization

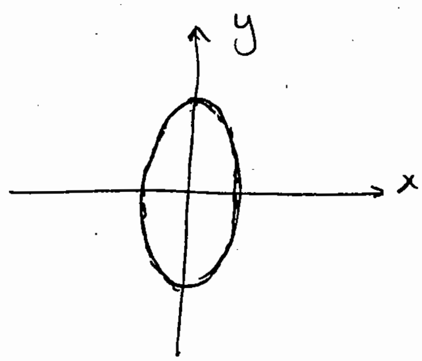
→ Consider that the electric field $\vec{E}$ has both components which are not having same amplitudes and are not in phase. As the wave propagates $\vec{E}_x$ and $\vec{E}_y$ will have maximum and minimum amplitudes at different instant of time.

→ If the locus of end points of $\vec{E}$ is traced it is observed that $\vec{E}$ moves elliptically. Then such a wave is said to be elliptically polarized.

→ When the amplitudes of $\vec{E}_x$ and $\vec{E}_y$ are different and the phase difference between two is other than 90° then the axes of the ellipse are inclined at an angle θ with coordinate axes.



→ When the amplitudes of $\vec{E}_x$ and $\vec{E}_y$ are unequal and phase difference between the two components be 90° exactly, then the axes of ellipse ~~are inclined~~ lie along the coordinate axes.



Circular Polarization

→ If we consider that $\vec{E}$ has two components $\vec{E_x}$ and $\vec{E_y}$ of equal amplitude but the phase difference between them is exactly 90°.

→ At any instant of time, when the amplitude of any one of the component is maximum then amplitude of remaining component becomes zero. Also when one component gradually increases or decreases, the other component decreases or increases respectively

→ Thus at any instant of time magnitude of $\vec{E}$ is constant But the direction of resultant vector $\vec{E}$ changes with respect to the time. The locus of projected points will form a circle with the centre on z-axis. Such wave is said to be circularly polarized.

→ If $\vec{E}$ rotates in clockwise direction such that the locus of all such points represents a circle, as per IEEE definitions this type of polarization is called left-circular polarization

→ If $\vec{E}$ rotates in anticlockwise direction, such that the locus of all such points represents a circle this type of polarization is called right circular polarization.

Conditions for the Polarization of a sinusoidal wave

Consider that the electric field $\vec{E}$ of a uniform plane wave travelling in z-direction is expressed as

$$\vec{E} = \vec{E_0}\, e^{j(\omega t - \beta z)}$$

As the wave propagates in z-direction, $\vec{E}$ must lie in a plane perpendicular to the direction of propagation i.e., x-y plane. Thus $\vec{E}$ is resultant of two components i.e, $\vec{E_x}$ and $\vec{E_y}$ along x-axis and y-axis respectively. Let E_1 and E_2 be the amplitudes of $\vec{E_x}$ & $\vec{E_y}$ respectively. If both variations are sinusoidal & if δ is the phase difference between the two components. The two components can be expressed in phasor form as.

$$\vec{E_x} = E_1\, e^{j(\omega t - \beta z)}$$
$$\vec{E_y} = E_2\, e^{j(\omega t - \beta z - \delta)} \longrightarrow (1)$$

$\vec{E}$ is resultant of $\vec{E_x}$ & $\vec{E_y}$

$$\vec{E} = \vec{E_x} + \vec{E_y}$$
$$\vec{E} = E_1\, e^{j(\omega t - \beta z)}\hat{a}_x + E_2\, e^{j(\omega t - \beta z - \delta)}\hat{a}_y \longrightarrow (2)$$

The above equation is true for all values of z.

Let $z=0$

$$\vec{E} = E_1\, e^{j(\omega t)}\hat{a}_x + E_2\, e^{j(\omega t - \delta)}\hat{a}_y \longrightarrow (3)$$

$$\vec{E} = E_1\left[\cos(\omega t) + j\sin(\omega t)\right]\hat{a}_x + E_2\left[\cos(\omega t-\delta) + j\sin(\omega t-\delta)\right]\hat{a}_y \longrightarrow (4)$$

$$\vec{E} = E_x\,\hat{a}_x + E_y\,\hat{a}_y$$

Equating above equations (RHS) with the real part

$$E_x\hat{a}_x + E_y\hat{a}_y = Re\left\{E_1\left(\cos\omega t + j\sin\omega t\right)\hat{a}_x + E_2\left[\cos(\omega t-\delta) + j\sin(\omega t-\delta)\right]\hat{a}_y\right\}$$

$$E_x\hat{a}_x + E_y\hat{a}_y = \left[E_1\cos\omega t\right]\hat{a}_x + \left[E_2\cos(\omega t-\delta)\right]\hat{a}_y \longrightarrow (5)$$

$$E_x = E_1\cos(\omega t) \longrightarrow (6)$$

$$E_y = E_2\cos(\omega t-\delta) \longrightarrow (7)$$

$$\left[\frac{E_x}{E_1}\right] = \cos(\omega t) \longrightarrow (8)$$

$$\left[\frac{E_x}{E_1}\right]^2 = \cos^2(\omega t) = 1-\sin^2(\omega t)$$

$$\sin^2(\omega t) = 1-\left[\frac{E_x}{E_1}\right]^2$$

$$\sin(\omega t) = \sqrt{1-\left(\frac{E_x}{E_1}\right)^2} \longrightarrow (9)$$

From equation (7)

$$\frac{E_y}{E_2} = \cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta) \longrightarrow (10)$$

Putting values of $\cos(\omega t)$ & $\sin(\omega t)$ in equation $\rightarrow$ (10).

$$\frac{E_y}{E_2} = \frac{E_x}{E_1}\cos(\delta) + \sqrt{1-\left(\frac{E_x}{E_1}\right)^2}\sin\delta \longrightarrow (11)$$

$$\left[\frac{E_y}{E_2} - \frac{E_x}{E_1}\cos\delta\right] = \sqrt{1-\left(\frac{E_x}{E_1}\right)^2}\,\sin\delta$$

Squaring both sides.

$$\left[\frac{E_y}{E_2}\right]^2 + \left[\frac{E_x}{E_1}\right]^2\cos^2\delta - \frac{2E_yE_x}{E_1E_2}\cos\delta = \sin^2\delta - \left(\frac{E_x}{E_1}\right)^2\sin^2\delta$$

Simplifying above expression.

$$\boxed{\left(\frac{E_x}{E_1}\right)^2\left(\cos^2\delta+\sin^2\delta\right) + \left(\frac{E_y}{E_2}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right)\cos\delta = \sin^2\delta} \longrightarrow (12)$$

Above equation is the equation for polarization of sinusoidal wave. By applying different conditions to above equation different types of polarization can be obtained.

$$\boxed{\left[\frac{E_x}{E_1}\right]^2 + \left[\frac{E_y}{E_2}\right]^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right)\cos\delta = \sin^2\delta} \longrightarrow (12)$$

Equation for polarization.

Condition 1: $\vec{E}_x$ & $\vec{E}_y$ are in phase $\delta = 0$

Equation (12) reduces to

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right) = 0$$

$$\left[\left(\frac{E_x}{E_1}\right) - \left(\frac{E_y}{E_2}\right)\right]^2 = 0$$

$$\boxed{E_x = \left(\frac{E_1}{E_2}\right)E_y} \longrightarrow (13)$$

For a given wave the amplitudes of E_x and E_y remain constant. i.e., E_1 & E_2 are constants.
Then the equation (13) similar to straight line passing through origin $y = mx$ then wave is said to be linearly polarized.

Condition 2 : $\vec{E_x}$ and $\vec{E_y}$ Components of unequal amplitudes with phase difference $\delta \neq 0$ Let us assume $\delta = \pi/2^c$

Applying this condition to equation (12)

$$\boxed{\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1}$$

This equation represents equation of ellipse. Hence wave is elliptically polarized.

Condition 3 : $\vec{E_x}$ and $\vec{E_y}$ components of equal amplitude with phase difference between two as $\delta = \pi/2^c$ -
the equation (12) reduces to

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1$$

Let $E_1 = E_2 = E_0$

$$\boxed{E_x^2 + E_y^2 = E_0^2}$$

This equation represents equation of circle similar to $x^2 + y^2 = a^2$. Hence wave is circularly polarized.