Randomized Block Designs

UBCO MDS — DATA 543

Introduction

- To eliminate as much of the natural variation as possible, we choose the experimental units to be as homogeneous as possible.
- On the other hand, we would like the conclusions of our experiment to be widely applicable.
- If feasible, we prefer a plan with blocking because we can better control confounding by holding an explanatory variate fixed within each block.
- This adds precision to our estimation of treatment effects, in somewhat similar ways as stratified sampling did in the context of parameter estimation

Introduction

- In this section, we look at the comparison of t treatments when we form blocks.
- We consider the simplest situation with b blocks where we apply all t treatments in a random order within each block. Note that this plan is balanced.
- The blocks are groups of t units selected to have similar values of one or more explanatory variates.
- Today we'll consider a *randomized block design* (RBD), wherein each treatment is repeated once in each block.

Example of blocks

If experimental units represent physical entities, blocking by similar physical characteristics often results in more homogeneous groups.

- For example, plots of land in agricultural experiments are usually blocked by proximity because plots in close proximity normally have similar soil characteristics.
- When experimental units are animals, the grouping of genetically similar animals, such as littermates, often reduces variability within groups.
- When experimental units are simply trials, or points in time where treatments will be applied, they are often blocked by time since many lurking variables may change over time and trials in close temporal proximity are more alike.

Randomized Block Design

- In a RBD a group of heterogeneous experimental units is used so that the conclusions can be more general.
- These heterogeneous experimental units are grouped into homogeneous sub-groups before they are randomly assigned to treatment factor levels.
- The act of grouping the experimental units together in homogeneous groups is called blocking.
- Randomly assigning treatment factor levels to experimental units within the smaller homogeneous subgroups of experimental units, or blocks, has the same effect as using only homogeneous units, yet it allows the conclusions to be generalized to the entire class of heterogeneous experimental units used in the study.

Randomized Block Design

- In a RBD with one treatment factor having t levels there will be b blocks (or subgroups of homogeneous experimental units) that each contain exactly t experimental units for a total of $t \times b$ experimental units.
- The t experimental units within each block are as similar as possible, and the groups of experimental units vary enough from block to block to allow general conclusions to be drawn.
- The randomization of experimental units to treatment factor levels is performed within each block.
- Variability among blocks represented the range of conditions over which conclusions would be generalized.

Comparing Two Treatments with Blocking

- Returning again to the amblyopia example from lecture 5, suppose we recruited patients from family doctors, and recruited 2 children from each of 10 family doctors.
- We could use a patient's doctor as a **block**.
- For each pair of children from a specific doctor, we randomize one to receive 6 hours of patching, the other 2.
- This could help remove any effect of a patient's doctor on the effect of treatment (e.g., perhaps a particularly attentive doctor will recommend exercises the patient can try in addition to the treatment).

Note that in this setup we do not have replicates within each block.

Randomized Block Design: the Model

The complete randomized block design is given by

$$Y_{ij} = \mu + \tau_i + \beta_j + R_{ij} \quad i = 1, 2, \ldots, t \quad j = 1, ..., b$$

- μ: overall average across treatments and blocks.
- τ_i: treatment effect
- β_j: block effect
- $R_{ij} \sim N(0, \sigma^2)$
- Constraints: $\sum_{i=1}^{t} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$

The word "complete" indicates that each block contains all the treatments.

Randomized Block Designs Model Notes

- the residual variance σ^2 is constant across treatments and blocks
- the model is additive, with no interaction between treatments and blocks.
- We find the estimates by minimizing

$$\sum_i \sum_j (y_{ij} - \mu - \tau_i - \beta_j)^2$$

subject to the constraints (using Lagrange multipliers).

Model Comparison with and without Blocking

Comparing treatments without blocking:

$$Y_{ij} = \mu + \tau_i + R_{ij} \quad R_{ij} \sim N(0, \sigma^2)$$

Comparing treatments with blocking:

$$Y_{ij} = \mu + \tau_i + \beta_j + R_{ij} \quad R_{ij} \sim N(0, \sigma^2)$$

Randomized Block Designs: Example

A company wants to replace some software

- Four possible replacements called A, B, C, D.

As with any large acquisition a selection team is formed and conducts a trial to compare the four products.

Randomized Block Designs: Example

- The team selects 6 different commonly performed tasks.
- They allocate 24 employees in 6 task groups.
- Within each group, one employee will use each product to complete the task.

The response of interest is the time taken to complete the task. The data are available as **RBDProducts.csv** in the Lab 4 folder on GitHub.

Randomized Block Designs

Let's rephrase this more succinctly:

- 4 treatments (products A, B, C, D).
- 6 blocks (the 6 tasks).
- 24 response measurements (time to complete task).

Q: Are there differences between the products?
Q: Which product is the best?

Note: we are not interested in differences between the blocks (the tasks).

Example: Data

task	brand	time	task	brand	time
1	A	6.5	4	A	13.4
1	B	10	4	B	12.9
1	C	5	4	C	12.1
1	D	6.8	4	D	15.6
2	A	14.1	5	A	6.6
2	B	14.5	5	B	10
2	C	14.4	5	C	6.8
2	D	15	5	D	8.8
3	A	9.9	6	A	8.2
3	B	13.2	6	B	7.4
3	C	10.5	6	C	6.9
3	D	12.2	6	D	11.4

Degrees of Freedom

For the variance estimate $\hat{\sigma}^2$ we need to think about degrees of freedom (df) again. The degrees of freedom equals the sample size minus the number of estimated parameters, plus the number of constraints.

$$\begin{aligned} \text{df} &= \text{sample size} \\ &\quad - \#\text{ estimated parameters} \\ &\quad + \#\text{ constraints} \end{aligned}$$

$$\text{df} = tb - (1 + t + b) + 2 = tb - t - b + 1 = (t-1)(b-1)$$

Variance Estimation

Our variance estimate is

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{(t-1)(b-1)}\sum_{i=1}^{t}\sum_{j=1}^{b}(y_{ij}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j)^2 \\ &= \frac{1}{(t-1)(b-1)}\sum_{i,j}(y_{ij}-\overline{y}_{i\cdot}-\overline{y}_{\cdot j}+\overline{y}_{\cdot\cdot})^2\end{aligned}$$

The residual variance estimator given by

$$\tilde{\sigma}^2 = \frac{1}{(t-1)(b-1)}\sum_{i,j}(Y_{ij}-\overline{Y}_{i\cdot}-\overline{Y}_{\cdot j}+\overline{Y}_{\cdot\cdot})^2$$

is unbiased for σ^2 regardless of any hypothesis about treatment effects.

Test Statistic

- Our path to a test statistic is similar to the CRD case.
- We want to derive a similar estimator that is unbiased for σ^2 under the assumption that $\tau_1 = \tau_2 = ... = \tau_t = 0$.

Recall for the CRD we had a **sum of squares decomposition**:

$$\sum_{i,j}(y_{ij} - \overline{y}_{..})^2 = r\sum_{i}(\overline{y}_{i.} - \overline{y}_{..})^2 + \sum_{i,j}(y_{ij} - \overline{y}_{i.})^2$$

or, in words (writing 'SS' for 'sum of squares'):

$$\text{Total SS} = \text{Treatment SS} + \text{Residual SS}$$

Can we do something similar here? Perhaps:

$$\text{Total SS} = \text{Treatment SS} + \textit{Block SS} + \text{Residual SS}$$

(Spoiler: yes!)

Sum of Squares (SS)

We can define sum of squares expressions for the treatments, blocks, residual, and total for the RBD:

SSTr Treatment SS: $b\sum_i(\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2$

SSB Block SS: $t\sum_j(\overline{y}_{\cdot j} - \overline{y}_{\cdot\cdot})^2$

SSR Residual SS: $\sum_{i,j}(y_{ij} - \overline{y}_{i\cdot} - \overline{y}_{\cdot j} + \overline{y}_{\cdot\cdot})^2$

SSTo Total SS: $\sum_{i,j}(y_{ij} - \overline{y}_{\cdot\cdot})^2$

where the residual SS is our variance estimate multiplied by the degrees of freedom.

SS Decomposition

We can show that, with these definitions, the sum of squares decomposition

SSTo = Treatment SS + *Block SS* + Residual SS

really does hold. Algebraically, $\sum_{i,j}(y_{ij} - \overline{y}_{..})^2 =$

$$b\sum_{i}(\overline{y}_{i.} - \overline{y}_{..})^2 + t\sum_{j}(\overline{y}_{.j} - \overline{y}_{..})^2 + \sum_{i,j}(y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..})^2$$

1^{st} Variance Estimator

Recall our residual variance estimator

$$\tilde{\sigma}^2 = \frac{1}{(t-1)(b-1)} \sum_{i,j} (Y_{ij} - \overline{Y}_{i\cdot} - \overline{Y}_{\cdot j} + \overline{Y}_{\cdot\cdot})^2$$

is unbiased for σ^2 regardless of any hypothesis about treatment effects, and so

$$\sum_{i,j} (Y_{ij} - \overline{Y}_{i\cdot} - \overline{Y}_{\cdot j} + \overline{Y}_{\cdot\cdot})^2 / \sigma^2 \sim \chi^2_{t-1,b-1}$$

$$\frac{\text{Residual SS}}{\sigma^2} \sim \chi^2_{t-1,b-1}$$

Treatment Estimator

For each treatment average we have

$$\overline{Y}_{i\cdot} \sim N(\mu + \tau_i, \sigma^2/b)$$

so then for any contrast

$$\theta = \sum_{i=1}^{t} a_i \tau_i \qquad \text{such that} \qquad \sum_{i=1}^{t} a_i = 0$$

$$\tilde{\theta} = \sum_{i=1}^{t} a_i \overline{Y}_{i\cdot} \sim N\left(\theta, \sigma^2 \frac{\sum_{i=1}^{t} a_i}{b}\right)$$

2^{nd} Variance Estimator

Following similar logic to the CRD case, we can show that if $\tau_1 = \tau_2 = ... = \tau_t = 0$

$$\overline{Y}_{i\cdot} \sim N(\mu, \sigma^2/b)$$

so

$$b\sum_i (\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot})^2/\sigma^2 \sim \chi^2_{t-1}$$

$$\frac{\text{Treatment SS}}{\sigma^2} \sim \chi^2_{t-1}$$

and this estimator is independent of the other variance estimator!

Randomized Block Designs

Side-note: If $X \sim \chi^2_m$ and $Y \sim \chi^2_n$ are independent, then $F = \frac{X/m}{Y/n} \sim F(m, n)$ and we have

$$\frac{\text{Residual SS}}{\sigma^2} \sim \chi^2_{(t-1)(b-1)} \quad \text{and} \quad \frac{\text{Treatment SS}}{\sigma^2} \sim \chi^2_{t-1}$$

and so

$$\frac{\text{Treatment SS}/\text{df(Treatment)}}{\text{Residual SS}/\text{df(Residual)}} = \frac{MS_T}{MS_R} \sim F_{t-1,(t-1)(b-1)}$$

Randomized Block Designs

We take the ratio of the mean square for treatment (MS_T) and mean square for residuals (MS_R) and compare it to the F-distribution with $t-1, (t-1)(b-1)$ degrees of freedom.

Under the null hypothesis (i.e., $\tau_1 = ...\tau_t$) this ratio should be approximately 1.

It is helpful to see the ANOVA table:

Source	Sum of squares	DF	Mean square	F_{obs}
Treatments	$b\sum_i(\overline{y}_{i\cdot}-\overline{y}_{\cdot\cdot})^2$	$t-1$	$\frac{b\sum_i(\overline{y}_{i\cdot}-\overline{y}_{\cdot\cdot})^2}{t-1}$	$\frac{MS_T}{MS_R}$
Blocks	$t\sum_j(\overline{y}_{\cdot j}-\overline{y}_{\cdot\cdot})^2$	$b-1$	$\frac{t\sum_j(\overline{y}_{\cdot j}-\overline{y}_{\cdot\cdot})^2}{b-1}$	
Residual	$\sum_{i,j}(y_{ij}-\overline{y}_{i\cdot}-\overline{y}_{\cdot j}+\overline{y}_{\cdot\cdot})^2$	$(t-1)(b-1)$	$\frac{\sum_{i,j}(y_{ij}-\overline{y}_{i\cdot}-\overline{y}_{\cdot j}+\overline{y}_{\cdot\cdot})^2}{(t-1)(b-1)}$	
Total	$\sum_{i,j}(y_{ij}-\overline{y}_{\cdot\cdot})^2$	$tb-1$		

- For 'by-hand' calculations it will be easier to compute the residual SS by calculating the total SS and subtracting the treatment and block SS.

Source	Sum of squares	DF	Mean square	F_{obs}
Treatments	SSTr	$t-1$	$\text{MSTr} = \dfrac{SSTr}{t-1}$	$\dfrac{\text{MSTr}}{\text{MSR}}$
Blocks	SSB	$b-1$	$\text{MSB} = \dfrac{SSB}{b-1}$	
Residual	SSR	$(t-1)(b-1)$	$\text{MSR} = \dfrac{SSR}{(t-1)(b-1)}$	
Total	SSTo	$tb-1$		

Example Summary Statistics

Back to the company trying to choose a new product.

	brand				
task	**A**	**B**	**C**	**D**	**average**
1	6.5	10	5	6.8	7.08
2	14.1	14.5	14.4	15	14.5
3	9.9	13.2	10.5	12.2	11.45
4	13.4	12.9	12.1	15.6	13.5
5	6.6	10	6.8	8.8	8.05
6	8.2	7.4	6.9	11.4	8.48
average	9.78	11.33	9.28	11.63	10.51
variance	9.78	11.33	9.28	11.63	

Example Summary Statistics

- The sample variance of the response times is 10.37
- The sample variance of the treatment averages is 1.32
- The sample variance of the task averages is 9.55

Example: Computing the ANOVA Table

- $SSTr = b\sum_i(\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2 = 1.32 \times (4-1) \times 6 = 23.8$
- $SSB = t\sum_j(\overline{y}_{\cdot j} - \overline{y}_{\cdot\cdot})^2 = 9.55 \times (6-1) \times 4 = 190.9$
- $SSTo = \sum_{i,j}(y_{ij} - \overline{y}_{\cdot\cdot})^2 = 23 \times 10.37 = 238.6$

We can then use the total SS to calculate the residual SS because $SSTo = SSTr + SSB + SSR$:

- $SSR = SSTo - SSTr - SSB = 23.82$

Example: Computing the ANOVA Table

We get our mean squares by dividing our sum of squares by the corresponding degree of freedom:

- $MS_T = \dfrac{SST}{df_t} = \dfrac{SST}{t-1} = \dfrac{SST}{4} = 7.945,$
- $MS_B = \dfrac{SSB}{df_b} = \dfrac{SSB}{b-1} = \dfrac{SSB}{5} = 38.189,$
- $MS_R = \dfrac{SSR}{df_R} = \dfrac{SSR}{(t-1)(b-1)} = \dfrac{SSR}{15} = 1.588.$

Randomized Block Designs: Example

- We compute $F_{obs} = \dfrac{MS_T}{MS_R} = \dfrac{7.945}{1.588} = 5.00$
- and compare to an $F_{t-1,(t-1)(b-1)} = F_{3,15}$
- using R we find that the p-value is

```
1 - pf(Fobs, df.t, df.r)
```

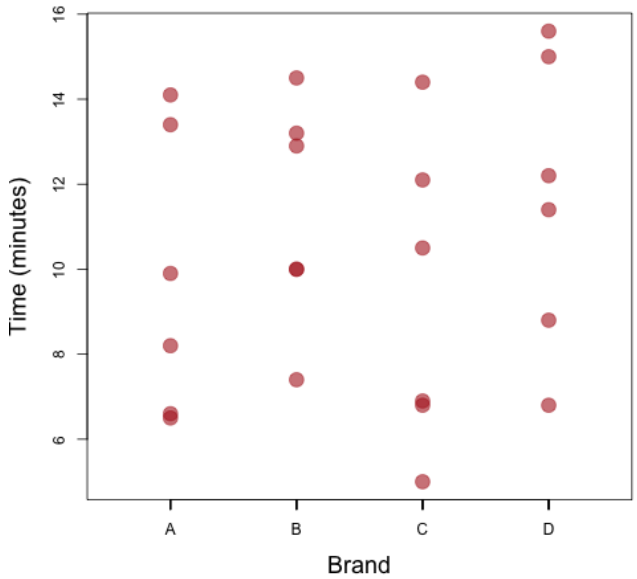
0.01334361

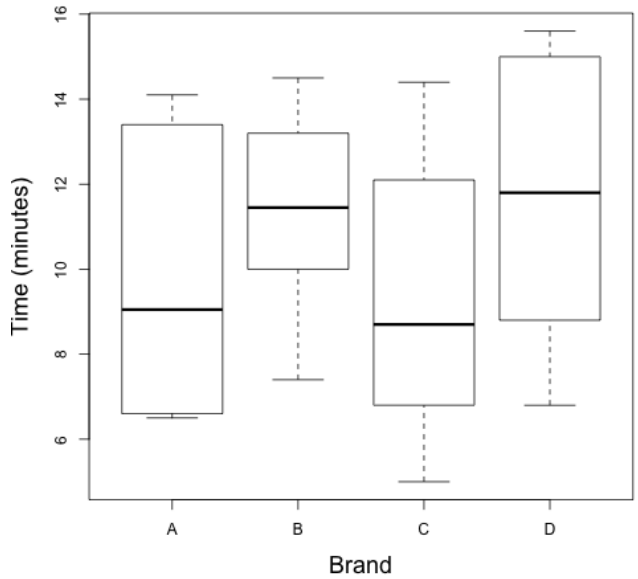
Example: ANOVA Table

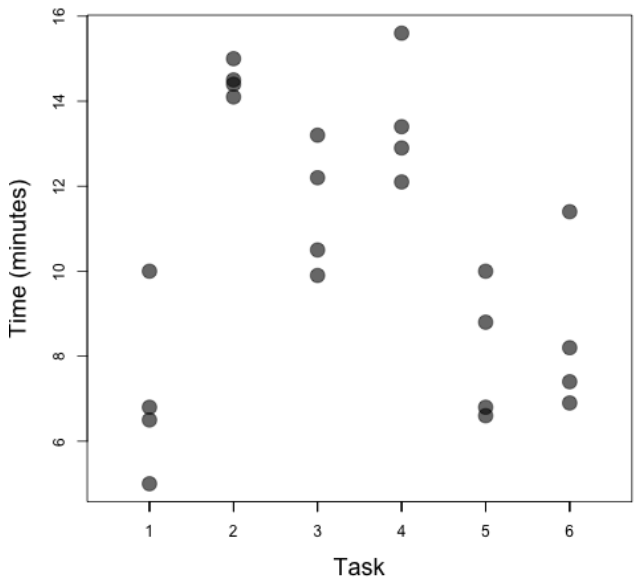
Source	Sum of squares	DF	Mean square	F_{obs}	p-value
Treatments	23.83	3	7.94	5.003	0.0133
Blocks	190.94	5	38.19		
Residual	23.82	15	1.59		
Total	238.6	23			

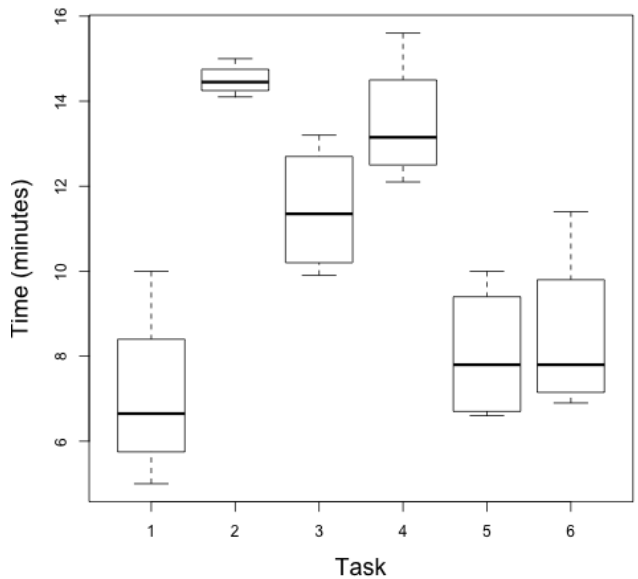
Therefore there is evidence against the null hypothesis that all four products represent populations with the same mean task completion time!

Randomized Block Designs: Example Plots



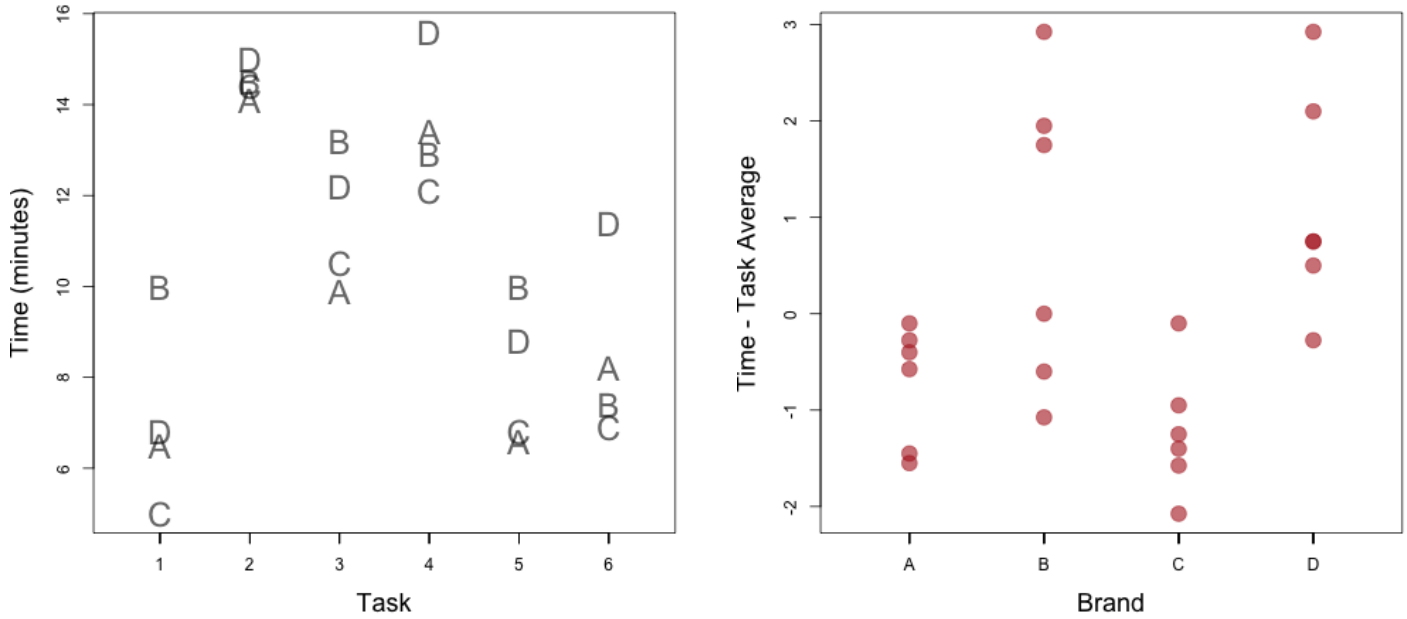






It is difficult to see the brand effects because of the task effects.

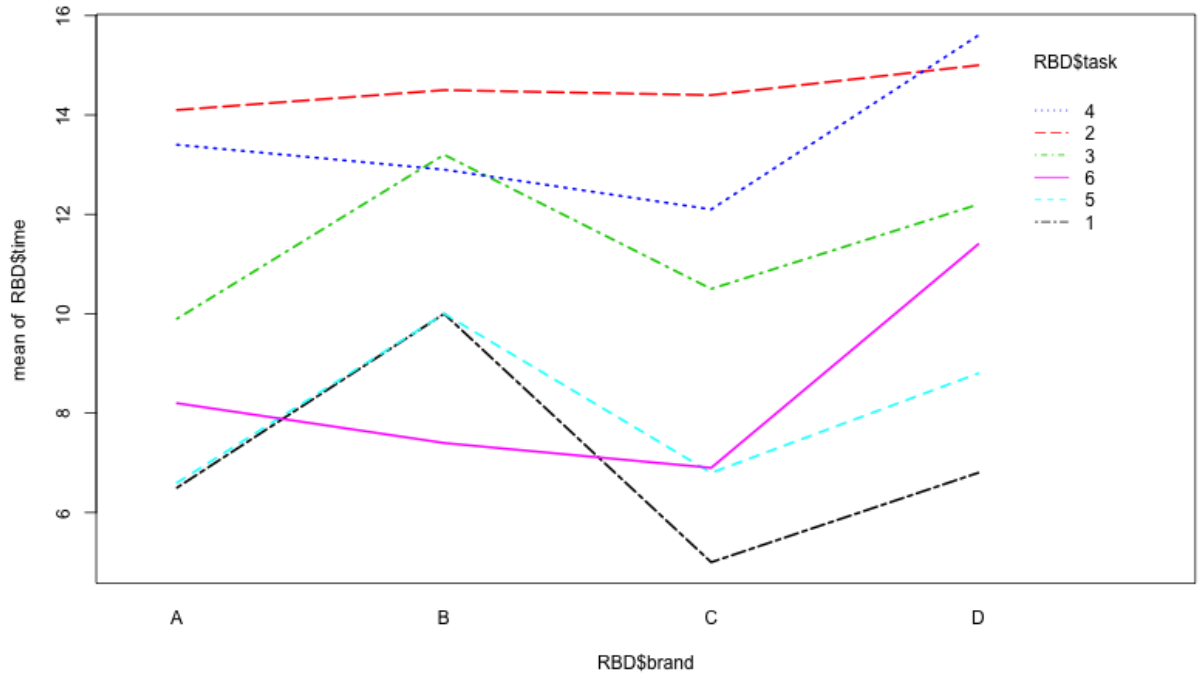
Randomized Block Designs: Example Plots



Removing the block average helps identify the treatment effect

Interaction Plots

If we want to plot both factors, we can use a so-called interaction plot.The following plots response as a function of treatment (x-axis) for each block labelled with different colour and line type



Randomized Block Designs: Contrast

To test hypotheses or build confidence intervals for $\theta = \sum_i a_i \tau_i$ where $\sum_i a_i = 0$ we use

$$\widetilde{\theta} = \sum_i a_i \overline{Y}_{i\cdot} \sim N\left(\theta, \sigma^2 \frac{\sum_i a_i^2}{b}\right)$$

$$\text{and} \quad \frac{(t-1)(b-1)\widetilde{\sigma}_r^2}{\sigma^2} \sim \chi^2_{(t-1)(b-1)}$$

to obtain

$$\frac{\left(\widetilde{\theta} - \theta\right)}{\sqrt{\widetilde{\sigma}_R^2 \frac{\sum_i a_i^2}{b}}} \sim t_{(t-1)(b-1)}$$

Contrast Example

Is the average time from A & C equal to average time from B & D.

- $\theta = (\tau_1 + \tau_3)/2 - (\tau_2 + \tau_4)/2$
- $\widehat{\theta} = (9.78 + 9.28)/2 - (11.33 + 11.63)/2 = 1.95$
- From the ANOVA table, $MS_R = 1.59$
- $SE\left(\widehat{\theta}\right) = \sqrt{\widehat{\sigma}_R^2 \frac{\sum_i a_i^2}{b}} = 0.515$
- For a 95% CI the required c from a t_{15} is 2.131.
- Hence, a 95% CI is

$$\widehat{\theta} \pm c \times SE\left(\widehat{\theta}\right) = [0.855, 3.045]$$

Why is blocking (potentially) helpful?

Intuition: by deliberately separating our treatments out across different blocks, we remove the risk of randomly assigning the same treatment to units in the same block.

- e.g., if our blocks were patients with very severe and mildly severe symptoms, how would it look if treatment A was assigned to all the very severe patients?

Important: blocking makes it easier to detect a treatment effect *if there truly is one to detect in the first place*.]

- If there is genuinely no treatment effect, blocking should not artificially cause one to appear!

Why block? Reduces the Residual SS

In the CRD case, we have

$$\text{Total SS} = \text{Treatment SS} + \text{Residual SS}$$

Very informally, if the treatment SS is (much) bigger than the residual SS, we have evidence of a treatment effect. In the RBD case, we have

$$\text{Total SS} = \text{Treatment SS} + \text{Block SS} + \text{Residual SS}$$

- The residual SS from the CRD gets split into the block and residual SS in the RBD case.

The treatment SS stays the same, but the residual SS is smaller.

Example Revisited RBD vs CRD

When we ignore the blocking variable, an ANOVA table assuming completely randomized design (CRD) is

Source	SS	DF	Mean square	F_{obs}	p-value
Treatments	23.83	3	7.94	0.74	0.541
Residual	214.76	20	10.738		
Total	238.6	23			

An ANOVA table assuming randomized block design (CRD) is

Source	SS	DF	Mean square	F_{obs}	p-value
Treatments	23.83	3	7.94	5.003	0.0133
Blocks	190.94	5	38.19		
Residual	23.82	15	1.59		
Total	238.6	23			

Ignoring the blocking variable

- When we ignore the blocking variable, we no longer get a significant result!
- In addition the mean square for error has increased from 1.59 to 10.738!
- Hence, all of the variability due to blocks is now in the error term ($190.94 + 23.82 = 214.76$).
- This is why we sometimes call the RBD a noise-reducing design technique.
- Note that the treatment SS is the same for this analysis as the blocked one!

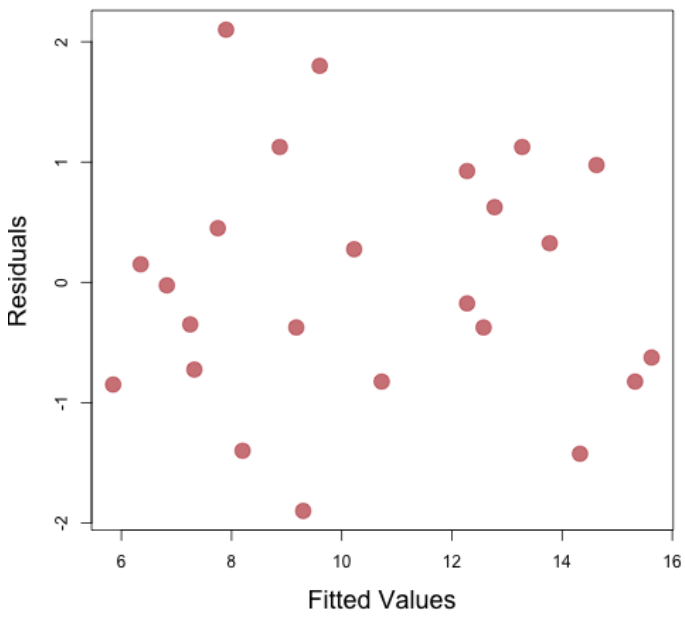
Checking Assumptions

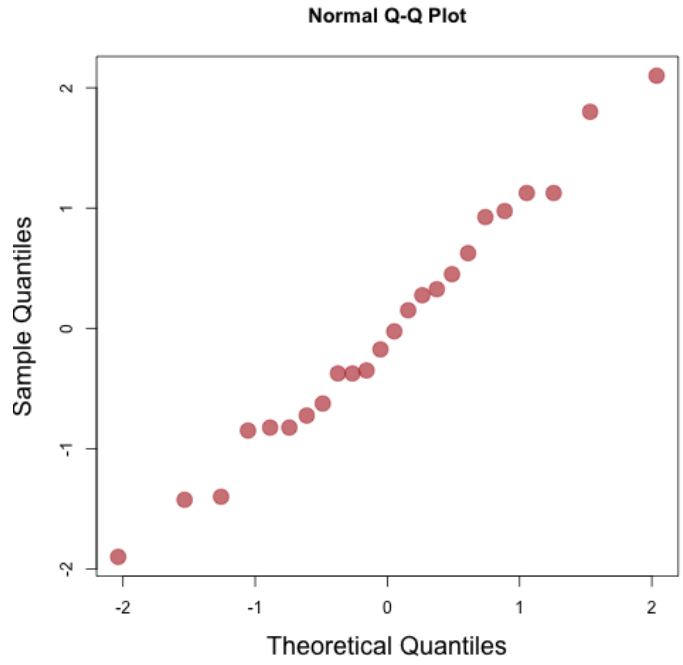
Assumptions are similar to the CRD case, but be careful!

1. Independence can be checked by looking at the study design.
    - Recall: experimental units (between blocks) are independent, and treatments are assigned at random (within blocks).

2. Normality of response and constant variance σ^2.
    - We can check the residuals.

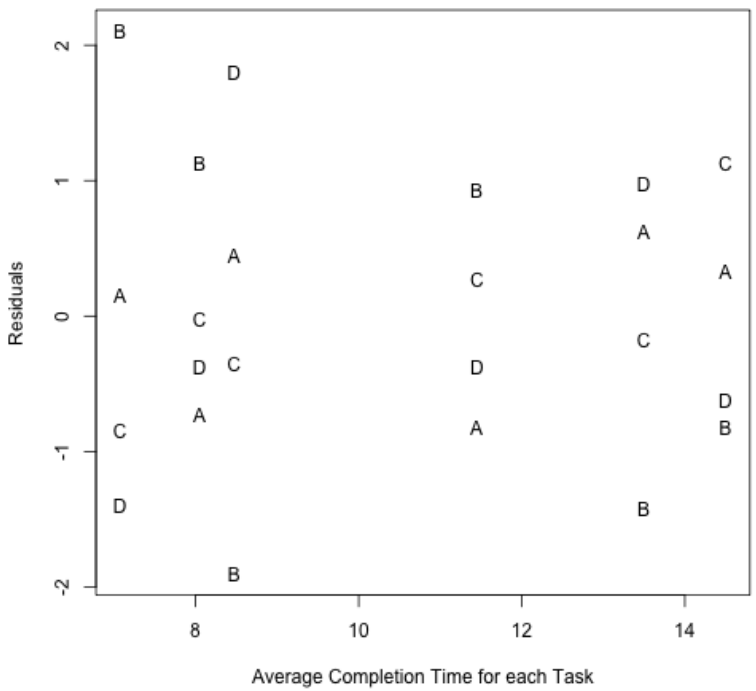
Randomized Block Designs: Assumptions





Checking Assumptions

3. Treatment and block effects are additive.
   - This means that the difference on average between any two products is the same for each task.
   - We might be concerned that the difference would be larger for more difficult tasks.



We can inspect this by plotting the residual for each product/task combination, and see if the residuals increase for more difficult (i.e., slower) tasks.

Randomized Block Designs: Assumptions

How we might phrase our opinion of these assumptions:

- Based on the study design, we assume independence between blocks as the 24 employees were randomized to each task (block).
- Based on residual plots the normality and additive assumptions seem reasonable.
- Based on summary statistics/boxplots the constant variance assumption also seems reasonable.
- We do not know whether treatments (products) were assigned at random within blocks, so it's possible that employees chose for themselves in a non-random manner.
    - We would need more information about this stage of the study to assess this assumption.

Incomplete Block Designs

This blocking analysis is the simplest form of blocking. There are many variations, such as:

- *Incomplete block designs* can occur when we have many treatments and every block does not receive all the treatments e.g. suppose we only had 12 employees available to us.
- With 2 blocking variables we might use a *Latin Square* e.g. suppose we only had 4 employees available to us and 4 tasks

Examples

Let's consider 2 scenarios: suppose we have 12 employees available to us and 6 tasks (left table) or 4 tasks (right table).

Task	Products
1	A B
2	A C
3	A D
4	B C
5	B D
6	C D

Task	Products
1	A B C
2	A B D
3	A C D
4	B C D

Incomplete Block Designs

Suppose we only had four tasks then we have

Task	Products
1	A B C
2	A B D
3	A C D
4	B C D

This is an example of a balanced incomplete block design.

Latin Square Design

Another important example is the **Latin Square** design. Suppose we have 4 employees where each one will use each product to carry out each task.

	1	2	3	4
Will	*A*	*B*	*C*	*D*
Xavier	*B*	*C*	*D*	*A*
Yan	*C*	*D*	*A*	*B*
Zoey	*D*	*A*	*B*	*C*

This setup creates two blocks - employee and task number - with each product (A-D) being used once for each task and once by each employee. The signature of a latin square design is that a given treatment of interest appears only once in a given row and a given column.