Data Structures and Algorithms

UBCO Master of Data Science – DATA 532

Recap

We looked at the idea of Dynamic Programming

We specifically looked at two approaches

- Forward Approach
- Backward Approach

We solved in detail an example to understand the working of Dynamic Programming

Today…

Introduction to Discrete Optimization

- Applications of Discrete Optimization
- Challenges with Discrete Optimization problems
- Mathematical Formulation

Solving discrete optimization problems

- Linear Programming

What is a discrete optimization problem ?

It is a problem with

Optimization : there is a set of solutions. We look for the solution that maximize or minimize one criterion.

For example: *find the minimum of* $f(x) = (x - 0.9)^2$

- The answer is 0, when x=0.9

What is a discrete optimization problem ?

Discreteness : Refers to optimization where the variables are discrete (as opposed to continuous).

Some decisions must be taken as a whole :

- e.g. left or right, yes or no, 1, 2, 3, 4…

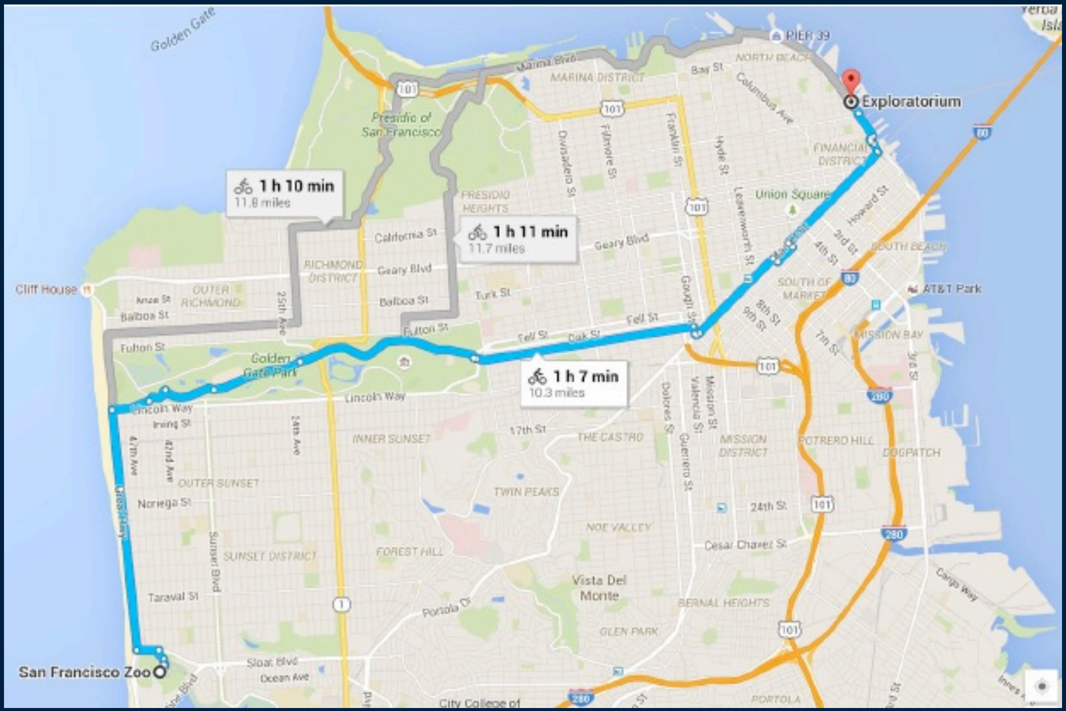
Very common in practice !

For example: *find the minimum of* $f(x) = (x - 0.9)^2$, where x is an integer

- The answer is 0.01, when x=1

Some important discrete problems…

The shortest path



A very popular discrete problem….

The traveling salesman problem (TSP)

Probably the most well-known discrete problem. . .

With many applications : Vehicle routing, VLSI design,. . .

Given n cities, in which order should one visit them in order to minimize the total distance ?

Example : Visit 23 EU cities with minimal traveling distance

Optimal solution

Why is that so theoretically complicated ?

After all, there is a finite number of solutions

In particular n! possible permutations

Imagine we can check 10^{12} possibilities per second
That is already a pretty amazing machine. . .

10 ! 0 sec
20 ! 28 days
30 ! 8400 billion years
40 ! 5 quadrillions times the age of the Earth. . .

A first success-story : the traveling salesman problem

Applications of the TSP

- Truck routing
- Arranging school buses routes (the very first application)
- Scheduling of a machine to drill holes in a circuit board
- Delivery of meals to homebound people
- Genome sequencing
- Cities are local strings, and the cost is the measure of likelihood that a sequence
- follows another
- Link points through fiber optic connections in order to minimize the total distance and ensure that any failure leaves the whole system operational
- A robot that needs to explore its environment

Another example of application

Exam scheduling

- Since 2015-2016, the exam schedule of the faculty is computed using discrete optimization algorithms
- 300 versions of courses, 1000 students, 600 different courses
- Make sure that no student has ever two exams on the same day !
- Try to maximize the rest days between two exams for a student
- Take care of professor constraints
- Compute the schedule in 10 hours of computing time - probably impossible to do with a customized algorithm

Example problem

Find the length-8 string of letters (e.g. aaaaaaaa, abcdefgh, etc.) that contains the most English words in it.

- The problem is discrete we might solve it by brute force search
- Alphabet has 26 lower-case letters -> 26^8 possibilities -> 208827064576

- It's not computationally feasible because there are too many options and it would be slow

- Need of smarter algorithms.
  - E.g. breadth-first search to find the shortest distance in a graph.

- *Formulating* the problem: converting a conceptual idea into an optimization problem (mathematical programming)

Specifying an optimization problem

We will focus on formulating problems using

mathematical optimization formulations.

That means that we want to define variables, mathematical constraints (inequalities and equalities essentially) and an objective function to maximize or minimize.

$$
\begin{aligned}
&\min c(x) \\
&s.t.\, f(x) \le b \\
&\quad g(x) = 0 \\
&\quad x \in X
\end{aligned}
$$

Why mathematical programming?

Typical programming approach

- Analyze the problem
- Write an algorithm to solve the problem using while, if, then, else,. . .
- Proof the correctness of the algorithm
- Analyze the complexity of the algorithm

Pros and cons

- Works well for well-posed problems.
- Works well for tractable (solvable by polynomial-time algorithms) problems.
- Needs to be done from scratch if there is a slight change in the problem or additional constraints

Why mathematical programming?

The mathematical programming approach

- Analyze the problem
- Write a static mathematical model
- Rely on meta-algorithms that work on all models that are correctly written

Pros and Cons

- Sometimes difficult to formulate the problems in the right way (no if, then, else)
- Works also for hard problems
- Two different models of the same problem may not be as good as the other
- Very flexible to add complicating constraints.

Scheduling nurses example

Constrains:

- A working day in a hospital is subdivided in 12 periods of two hours.
- A nurse works 8 hours a day and is entitled to a break after having worked for 4 hours.
- The personnel requirements change from period to period.

Objective function:

- Determine the minimum number of nurses required to cover all requirements

Linear Programming

Linear Programming (LP)

A model consisting of linear relationships representing a firm's objective and resource constraints

LP is a mathematical modeling technique used to determine a level of operational activity in order to achieve an objective, subject to restrictions called constraints

LP Model Formulation

Decision variables

- mathematical symbols representing levels of activity of an operation

Objective function

- a linear relationship reflecting the objective of an operation
- most frequent objective of business firms is to *maximize profit*
- most frequent objective of individual operational units (such as a production or packaging department) is to *minimize cost*

Constraint

- a linear relationship representing a restriction on decision making

LP Model Formulation (cont.)

Max/min $\mathbf{z = c_1x_1 + c_2x_2 + \ldots + c_nx_n}$

subject to:

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n\ (\leq, =, \geq)\ b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n\ (\leq, =, \geq)\ b_2 \\
\vdots \\
a_{m1}x1 + a_{m2}x_2 + \ldots + a_{mn}x_n\ (\leq, =, \geq)\ b_m
\end{cases}
$$

x_j = decision variables

b_i = constraint levels

c_j = objective function coefficients

a_{ij} = constraint coefficients

LP Model: Example

RESOURCE REQUIREMENTS			
PRODUCT	*Labor (hr/unit)*	*Clay (lb/unit)*	*Revenue ($/unit)*
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day

Decision variables

x_1 = number of bowls to produce

x_2 = number of mugs to produce

LP Formulation: Example

Maximize $Z = 40\,x_1 + 50\,x_2$

Subject to

$$
\begin{aligned}
x_1 + 2x_2 &\le 40 \text{ hr} && \text{(labor constraint)} \\
4x_1 + 3x_2 &\le 120 \text{ lb} && \text{(clay constraint)} \\
x_1 , x_2 &\ge 0
\end{aligned}
$$

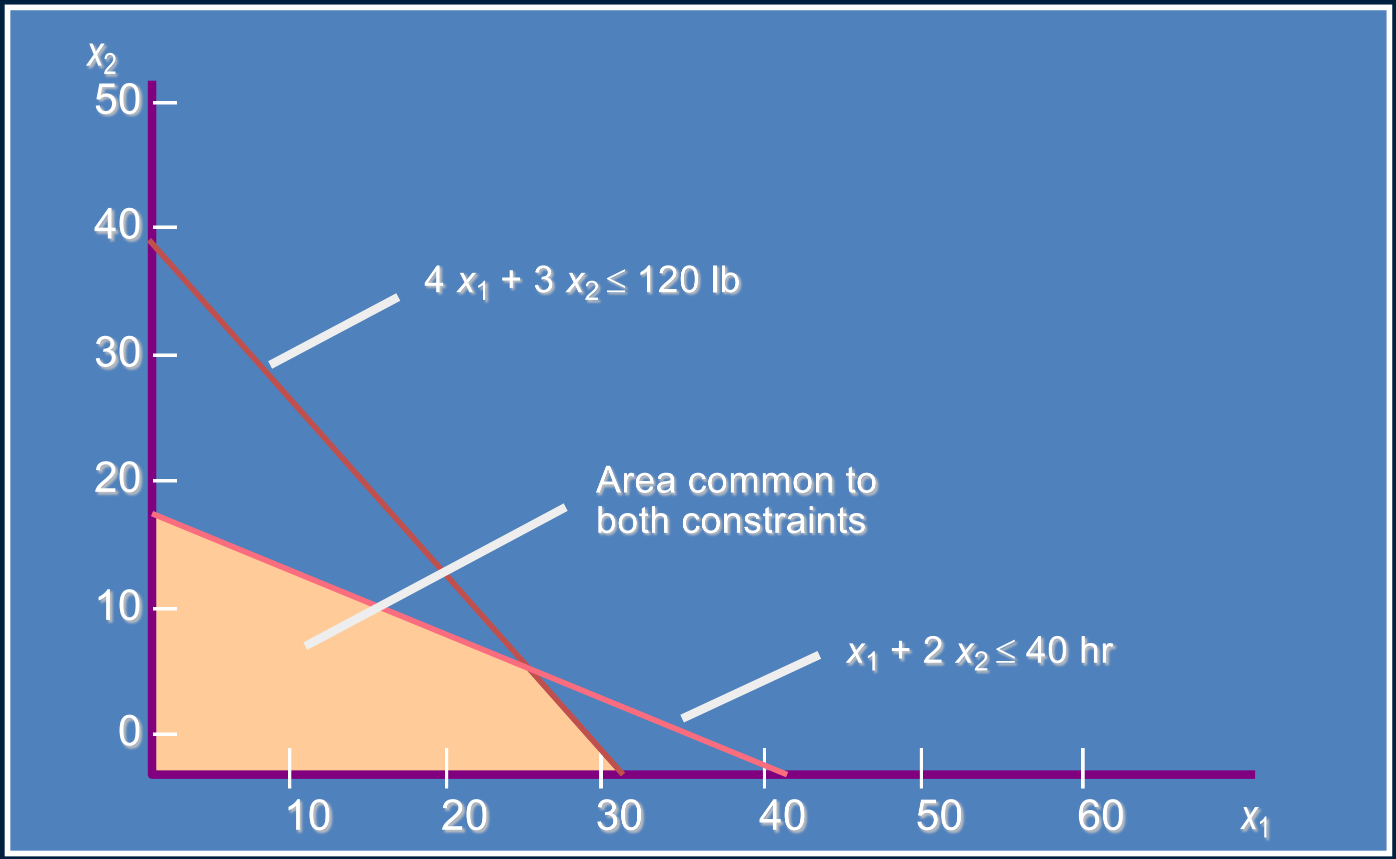
Solution is $x_1 = 24$ bowls $x_2 = 8$ mugs

Revenue = $1,360

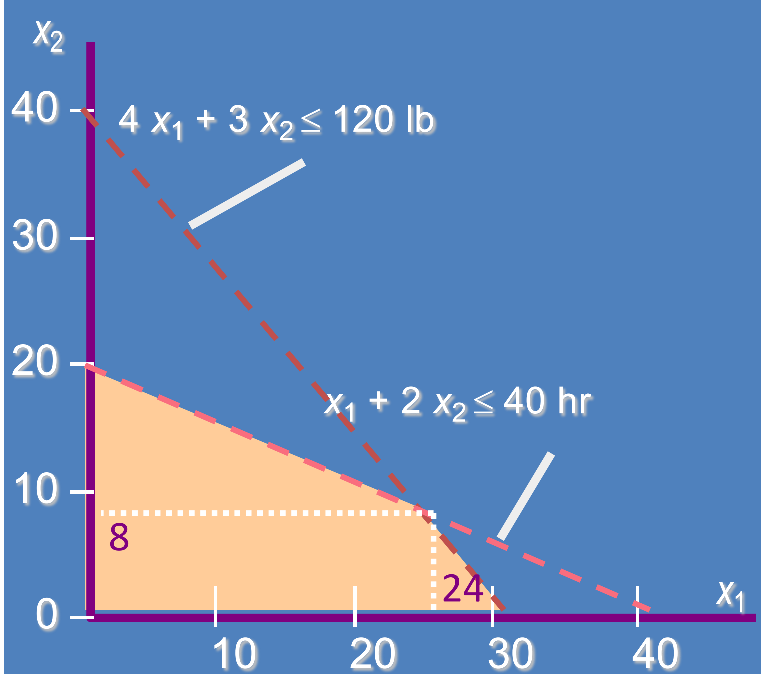
Graphical Solution Method

1. Plot model constraint on a set of coordinates in a plane
2. Identify the feasible solution space on the graph where all constraints are satisfied simultaneously
3. Plot objective function to find the point on boundary of this space that maximizes (or minimizes) value of objective function

Graphical Solution: Example



Computing Optimal Values



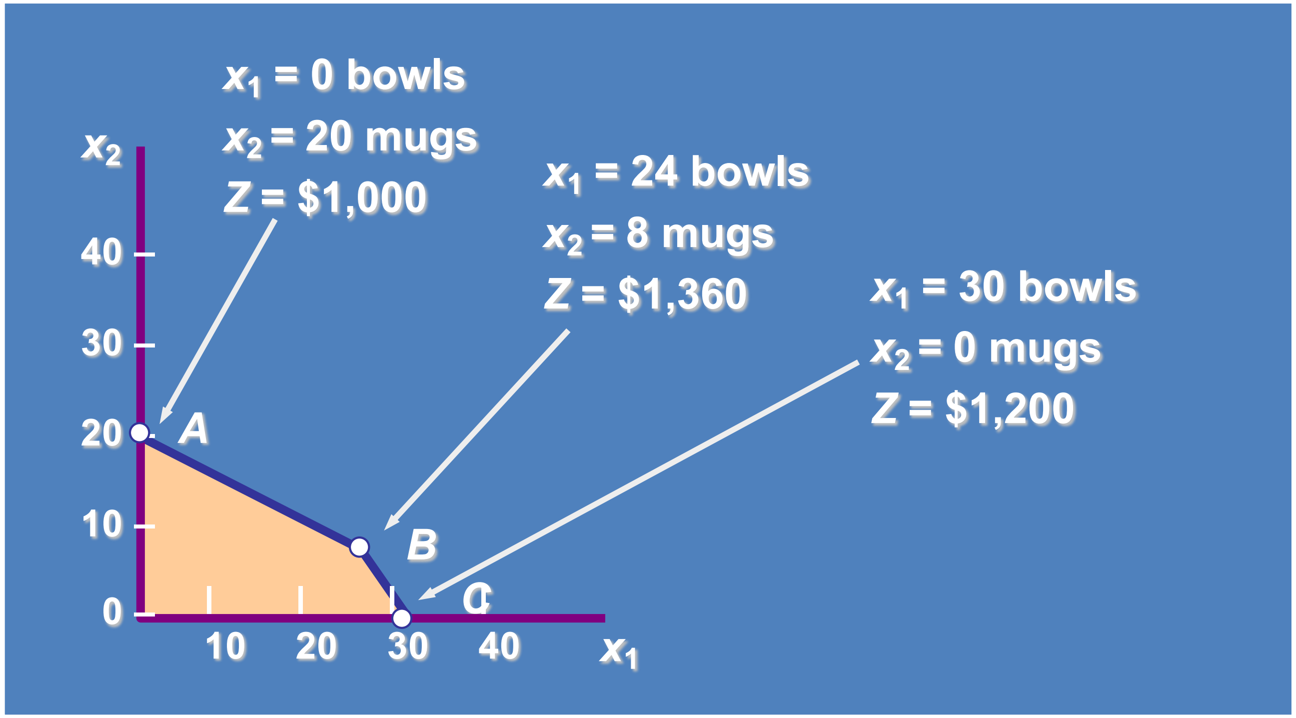
$$x_1 + 2x_2 = 40$$
$$x_1 = 40 - 2x_2$$

$$4x_1 + 3x_2 = 120$$
$$x_2 = 8$$

$$x_1 + 2(8) = 40$$
$$x_1 = 24$$

$Z = \$40(24) + \$50(8) = \$1,360$

Extreme Corner Points



Minimization Problem

CHEMICAL CONTRIBUTION

Brand	*Nitrogen (lb/bag)*	*Phosphate (lb/bag)*
Gro-plus	2	4
Crop-fast	4	3

Minimize $Z = \$6x_1 + \$3x_2$

subject to

$$2x_1 + 4x_2 \geq 16 \text{ lb of nitrogen}$$
$$4x_1 + 3x_2 \geq 24 \text{ lb of phosphate}$$
$$x_1, x_2 \geq 0$$

Summary

We looked at the concept of Discrete Optimization

We looked at various applications

We saw one of the forms used in Discrete Optimization (Linear Programming)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