The University of British Columbia

Data Science 581 Modelling and Simulation II

Lab Assignment 3 Solutions

1. The file *text.txt* contains some technical writing with punctuation stripped away, so it is just a sequence of words. You can read in the file with the `readLines()` function:

```
MyText <- readLines("text.txt")
```

2. R has some functions for handling character strings, including `strsplit` which splits strings at given characters. The following separates strings at spaces, resulting in lists of single words:

```
MyTextStrings <- strsplit(MyText, " ")
```

We will find it easier to work with a single vector of the words, so we use the `unlist()` function to remove the list structure, resulting in a plain vector of strings (none of which contain blank spaces):

```
MyTextStrings <- unlist(MyTextStrings)
```

Finally, to do some Markov chain analysis of the text strings, we might be interested in the lengths of each word. We can count the number characters in each string using the `nchar()` function:

```
lettercounts <- nchar(MyTextStrings)
```

To see the lengths of the first few words in the document, try

```
lettercounts[1:12]

##  [1]  3  4  2 13  2  2  7  3 10  2  3  5
```

Later, we will set up a Markov chain which models the sequence of word lengths. States for this Markov chain will be the word lengths, so the state space will be all possible word lengths.

3. Use the `table()` function to determine the number of words exceeding 12 characters, for example, and to see if there are any blank spaces remaining (and there are).

```
table(lettercounts)

## lettercounts
##   0   1   2   3   4   5   6   7   8   9  10  11  12  13  14
##  24  24 147 149 117  76  67  58  61  45  29  32  11   5   7
```

The following code will remove the remaining blank spaces:

```
lettercounts <- lettercounts[lettercounts > 0]
```

4. We will use the following truncation to reduce the size of the state space:

   ```
   lettercountsT <- lettercounts
   lettercountsT[lettercounts > 11] <- 12
   ```

   This means that the state '12' actually contains 13 and 14 as well as 12. Thus, our state space is now $\{1, 2, 3, \ldots, 12\}$. We only need to estimate 144 elements of our transition matrix instead of 196, so we gain some accuracy by giving up this degree of precision.

5. Construct the transition matrix as follows:

   ```
   P <- matrix(0, nrow=12, ncol=12)
   for (i in 2:length(lettercountsT)) {
     P[lettercountsT[i-1], lettercountsT[i]] <- P[lettercountsT[i-1],lettercountsT[i]] + 1
   }
   P <- P/as.numeric(table(lettercountsT[-length(lettercountsT)]))
   length(table(lettercountsT[-length(lettercountsT)]))

   ## [1] 12
   ```

   Note that we are estimating the entries of the transition matrix by calculating the proportion of the time that each type of transition occurs. In calculating these proportions, we are using the `table()` function to determine the numbers of times that we are in each state, and since we don't know what comes after the final observation, we do not include that in our count.

6. Find the mean and standard deviation of the word lengths in the original data set as well as the proportion of the time that the difference in subsequent word lengths exceeds 7, i.e.

   ```
   mean(diff(lettercountsT) > 7)

   ## [1] 0.05199516
   ```

   The mean and standard deviation of the (truncated) word lengths in the original data set are:

   ```
   mean(lettercountsT)

   ## [1] 5.111111

   sd(lettercountsT)

   ## [1] 2.905563
   ```

   We could also consider the mean and standard deviation of the raw word lengths:

```
mean(lettercounts)

## [1] 5.134058

sd(lettercounts)

## [1] 2.966178
```

Note that the truncation has had little effect on these quantities.

These are examples of statistics that could be used to make comparisons with other samples of text, e.g. in cases where one might be checking for forgeries.

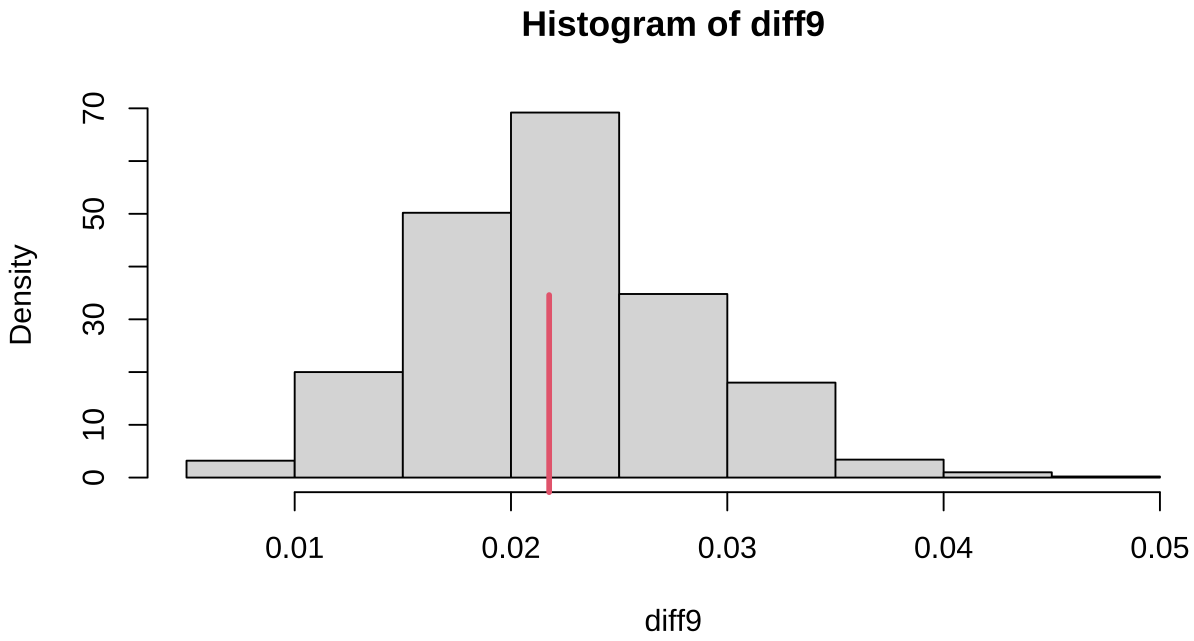
7. To check another text to see if its authorship would be different from the given text, we might run a simulation as follows:

```
wordlengthAVGs <- numeric(1000)
wordlengthSDs <- numeric(1000)
diff9 <- numeric(1000)
for (i in 1:1000) {
    Ntransitions <- length(lettercounts) # number of words
    wordlength <- numeric(Ntransitions)#initializing the Markov chain
    current.state <- lettercountsT[1] # initial wordlength
    for (j in 1:Ntransitions) {
        current.state <- sample(1:12,
           size = 1, prob = P[current.state, ])
        wordlength[j] <- current.state
    }
wordlengthAVGs[i] <- mean(wordlength)
wordlengthSDs[i] <- sd(wordlength)
diff9[i] <- mean(abs(diff(wordlength)) > 9)
}
```

In the above simulation, we have simulated 1000 realizations of the fitted Markov chain model. In each case, we have calculated the mean and standard deviation of the wordlength, as well as the proportion of time the absolute value of the subsequent difference in wordlength changes by more than 9 characters, a fairly extreme type of statistic.

The following is a histogram of the extreme changes, together with a rug plot indicating the location of the observed data:

```
hist(diff9, freq=FALSE)
rug(mean(abs(diff(lettercountsT)) > 9), col=2, lwd=3, ticksize=.5)
```

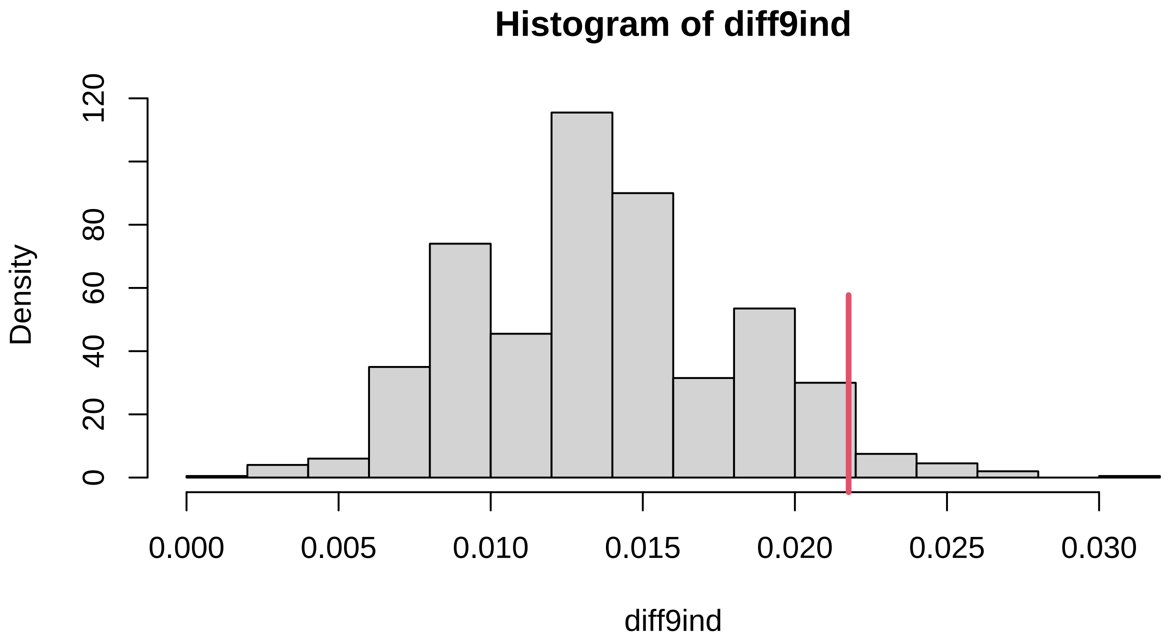


Note how the statistic lies in an area of high probability density. This is an indicator that our model is fitting well, and we would be suspicious of any text where the extreme statistic was larger than .04 or less than .01.

But was it necessary to use a Markov chain model? What if we just sample from the marginal distribution of the observed states as below (assuming that the changes in wordlength are independent from word to word)?

```
diff9ind <- numeric(1000)
for (i in 1:1000) {
    indepcounts <- sample(1:12, size = length(lettercounts),
          replace=TRUE, prob=table(lettercountsT)/
         length(lettercounts))
    diff9ind[i] <- mean(abs(diff(indepcounts)) > 9)
}
```

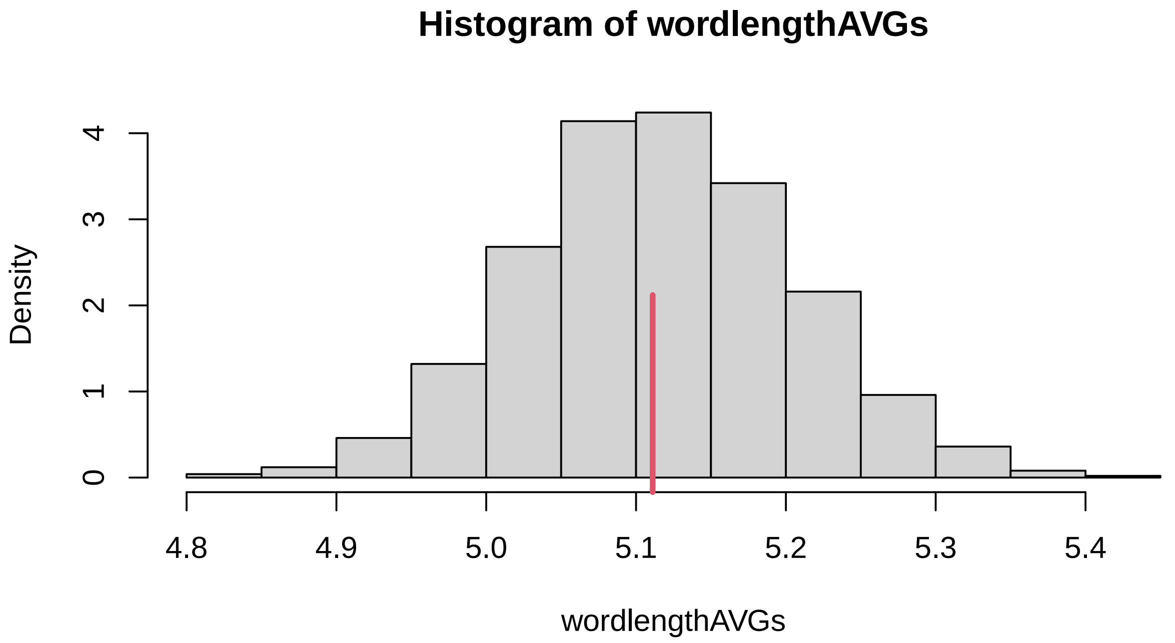
```
hist(diff9ind, freq=FALSE)
rug(mean(abs(diff(lettercountsT)) > 9), col=2, lwd=3, ticksize=.5)
```



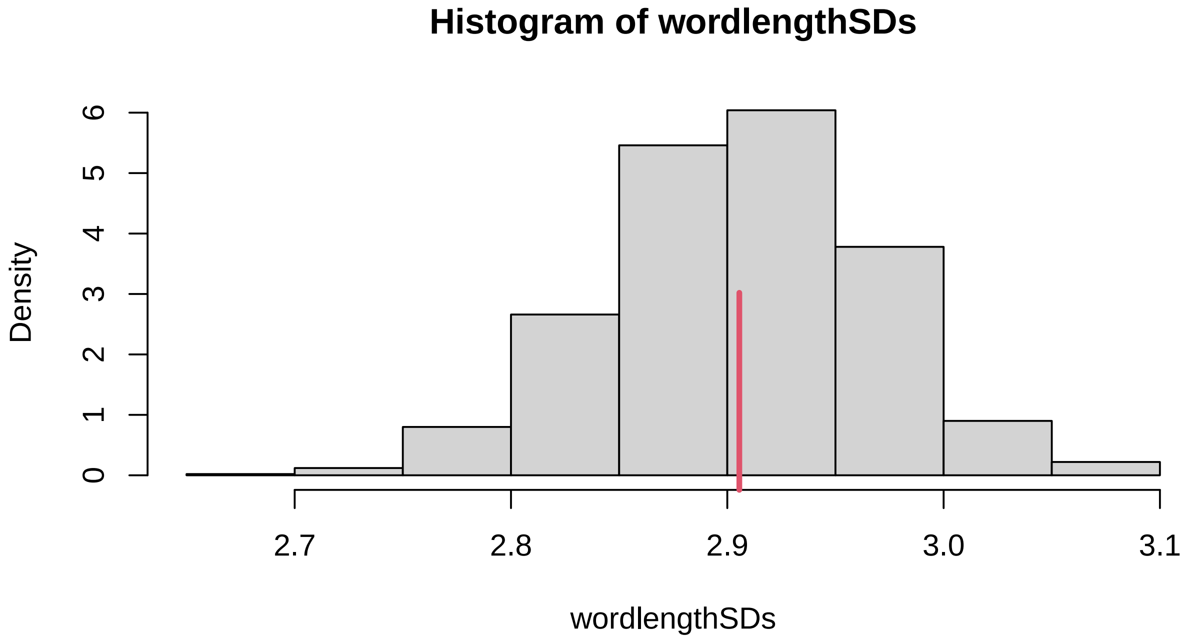
Note how the observed value of the extreme statistic is no longer in an area of high probability density. We would judge the actual author as a forger. We would also fail to detect forgery in a work where the extreme statistic is less than .01 – a further indication of inaccuracy.

8. Repeat the above graphical analyses in the cases of the mean and the standard deviation. What rule would you use to identify a forgery on the basis of the mean wordlength? How about on the basis of the standard deviation of the wordlength?

```
hist(wordlengthAVGs, freq=FALSE)
rug(mean(lettercountsT), col=2, lwd=3, ticksize=.5)
```



```
hist(wordlengthSDs, freq=FALSE)
rug(sd(lettercountsT), col=2, lwd=3, ticksize=.5)
```



If the average word length is used, we expect the authentic author to have a mean between 4.9 and 5.5. An average outside this interval could indicate a different author. The standard deviation is expected to be between 2.7 and 3.1.

In practice, a forgery detection method based on all 3 statistics (and perhaps more), simultaneously, would be most effective than the univariate approach outlined in this assignment.

9. **Problem from Lecture 3.** At the beginning of each day, a batch of containers arrives at a stockyard having capacity to store 6 containers. The batch size has the discrete probability distribution $\{q_0 = .4, q_1 = 0.3, q_2 = 0.2, q_3 = 0.1\}$. If the stockyard does not have sufficient space to store the whole batch, the batch as a whole is taken elsewhere. Each day, as long as there are containers in the stockyard, exactly one container is removed from the stockyard.

   (a) Find the transition matrix for the Markov chain $\{X_1, X_2, \ldots\}$, where $X_t =$ the number of containers in the stockyard at the beginning of the tth day.

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 & 0 & 0 \\ 0 & 0.4 & 0.3 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.5 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 0.7 & 0.3 \end{bmatrix}$$

   To see this, it is important to keep in mind that X_t gives the container count at the very beginning of the tth day. To get the first row of the matrix, there are no containers at the beginning of the tth day. If 0 or 1 container arrive that day (an event with probability 0.7*), the day would end with no containers, because any container present would be taken away. If 2 containers arrive (with probability 0.2), the day would end with 1 container in the yard, and if 3 containers arrive, the day would end with 2 containers. It is impossible for more containers to arrive, so the remaining entries of the first row are 0. Similar reasoning is used to fill in the rest of the matrix, noting that it is impossible to start the day with more than 5 containers, since 1 container is taken away each day.*

   (b) Find the long run distribution for this Markov chain.

```
P <- matrix(c(0.7, .2, .1, 0, 0, 0, 0.4,
  0.3, 0.2, 0.1, 0, 0,
  0, 0.4, 0.3, 0.2, 0.1, 0, 0, 0, 0.4,
  0.3, 0.2, 0.1, 0, 0, 0, 0.5, 0.3, 0.2,
  0, 0, 0, 0, 0.7, 0.3), nrow=6, byrow = TRUE)
A <- t(P) - diag(rep(1, 6)) # P^T - I
A <- rbind(A, rep(1,6))
RHS <- c(rep(0,6), 1)
options(digits=4)
pi <- qr.solve(A, RHS)
pi

## [1] 0.23273 0.17455 0.18909 0.18545 0.14909 0.06909
```

   (c) Suppose a profit of \$100 is realized for each container that spends a night at the stockyard. Calculate the long-run average weekly profit.

$$\text{Profit } = 100 \times X$$

   where X is the number of containers in the yard at the beginning of a day.

$$E[X] = \sum_{i=0}^{5} i\pi_i$$

```
sum(pi*(0:5))

## [1] 2.051
```

so

$$E[100X] = 205.10$$

(d) Write R code to simulate this Markov chain, and run a simulation of 100000 transitions, starting in state 1 (i.e. where there is 1 container in the yard).

```
Ntransitions <- 100000 # number of transitions
ncontainers <- numeric(Ntransitions)#initializing
current.state <- 1 # initial stock
for (t in 1:Ntransitions) {
   current.state <- sample(1:6,
        size = 1, prob = P[current.state, ])
   ncontainers[t] <- current.state
}
pi <- table(ncontainers)/Ntransitions # this tabulates the
   # distribution of the number of containers
pi

## ncontainers
##       1       2       3       4       5       6
## 0.23717 0.17534 0.18860 0.18197 0.14824 0.06868
```

10. Consider the Markov chain $X_0, X_1, X_2, \ldots,$ with transition matrix and state space $\{1, 2, 3\}$ and

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.25 & 0.75 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(a) Find the probability that $X_2 = 1$, given that $X_0 = 2$.
5/16. Obtain this by computing the $(2, 1)$ *element of* $\mathbf{P}^2$.

(b) Do states 1 and 3 communicate? Explain briefly.
The MC can go from 1 to 3 in one step with probability .5 so 1 leads to 3. It can go from 3 to 2 in one step with probability 1 and then to 1 directly w.p. .25, so 3 leads to 1. Therefore, 1 and 3 communicate.

(c) Use the fact that

$$\mathbf{P}^5 = \begin{bmatrix} 0.2891 & 0.5703 & 0.1406 \\ 0.2842 & 0.5732 & 0.1426 \\ 0.2852 & 0.5664 & 0.1484 \end{bmatrix}.$$

to find the probability that $X_5 = 3$, given that $X_0 = 1$.
$P(X_5 = 3|X_0 = 1) = \mathbf{P}_{13}^{(5)} = 0.2852$

(d) Is $\mathbf{P}$ a regular matrix?
Yes, since $P^k > 0$ *for some* k *(e.g.* $k = 5$*)*

(e) Find the stationary distribution for the given Markov chain.
$= [2/7 \;\; 4/7 \;\; 1/7]$. *This can be obtained by solving* $= \mathbf{P}$ *subject to the condition that the elements of sum to 1.*