Lecture 2

Intro to Generalized Linear Models

Review – maximum likelihood estimation

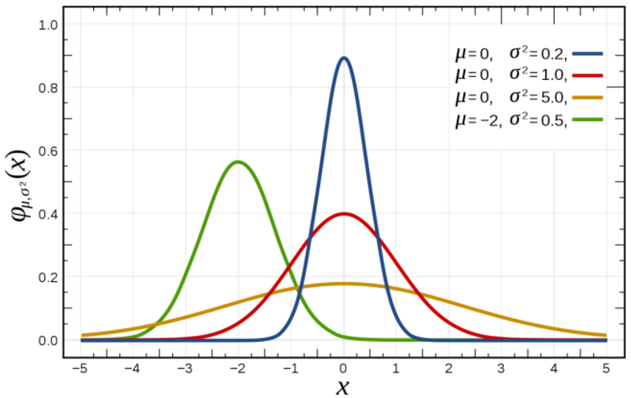
$$\log\big(L(\Theta)\big) = \sum_{i=1}^{n} f(y_i; \Theta)$$

We want to find some value that maximizes log-likelihood

We previously looked at the form for the Gaussian and Poisson distributions

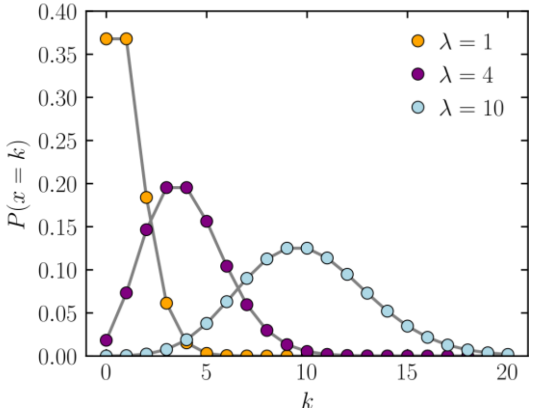
Common Probability Distributions

Gaussian



- Parameters are mean and variance
- Real values
- Most physical properties and measurement errors
- Easy to add together

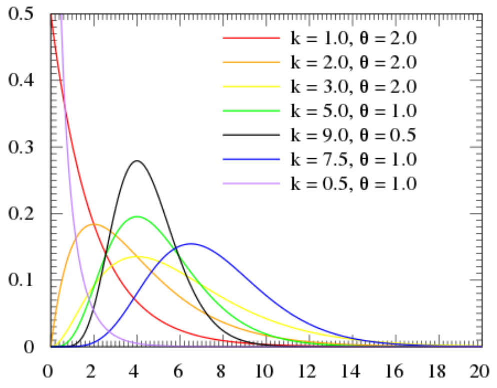
Poisson



- Parameter is the rate
- Discrete values
- Counts the number of events that occur in an interval given a **constant rate** and **independent events** – called a Poisson process

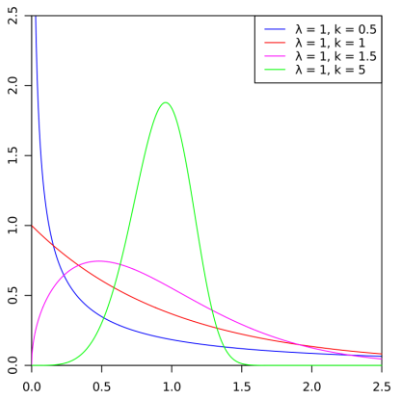
Common Probability Distributions

Gamma



- Parameters are shape (k) and scale (θ)
- Positive real values
- Measures time before k events occur in a Poisson process
- Case when k=1 is the **Exponential** distribution

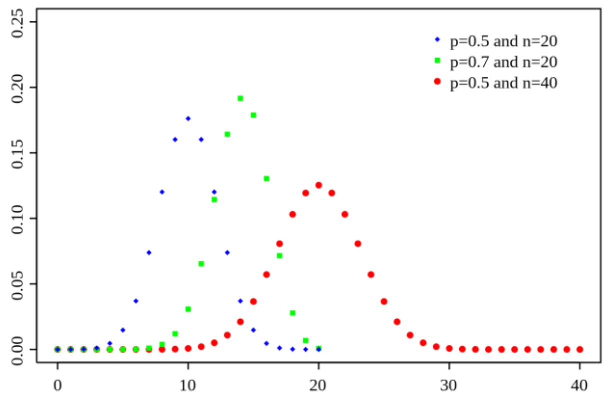
Weibull



- Parameters are scale (λ) and shape (k)
- Positive real values
- Measures a time-to-failure, when failure rate changes over time
  - K < 1 means failure rate decreases over time
  - K = 1 means failure rate is independent of time, this is also the **Exponential** distribution
  - K > 1 means failure rate increases over time

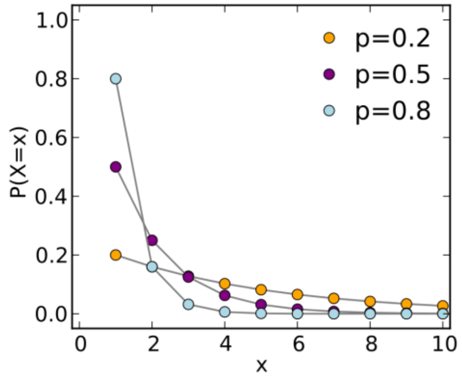
Common Probability Distributions

Binomial



- Parameters are number of trials (n) and success probability (p)
- Positive discrete values
- Measures the probability of getting k successes from a Bernoulli process with success probability p

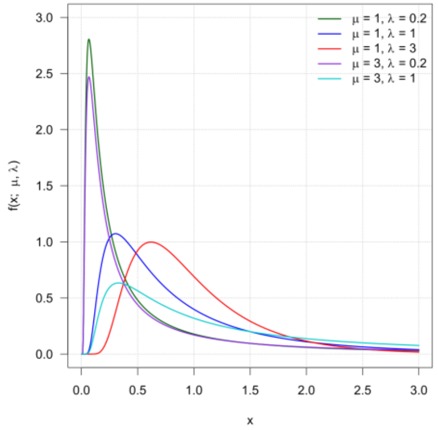
Geometric



- Parameter is success probability (p)
- Positive discrete values
- Measures the number of Bernoulli trials needed to get one success

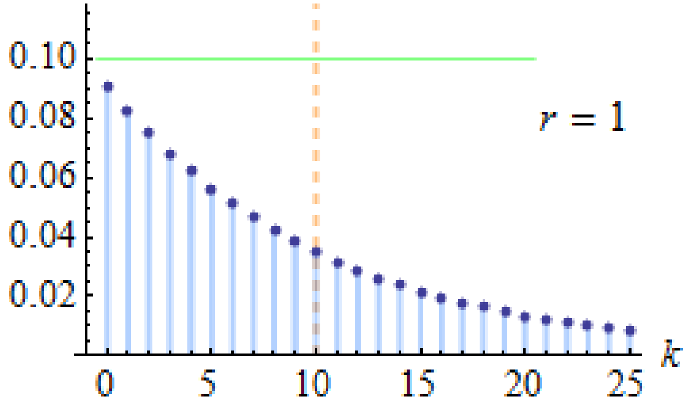
Common Probability Distributions

Inverse Gaussian



- Parameters are mean (μ) and shape (λ)
- Positive real values
- If the normal distribution is values in a random walk, the inverse Gaussian distribution is the number of steps taken to reach a given level

Negative Binomial



- Parameters are number of successes (r) and success probability (p)
- Measures the number of failures in a Bernoulli process before *r* successes

Comparing Probability Distributions

My cookie factory keeps having downtime. I want to understand this process better.

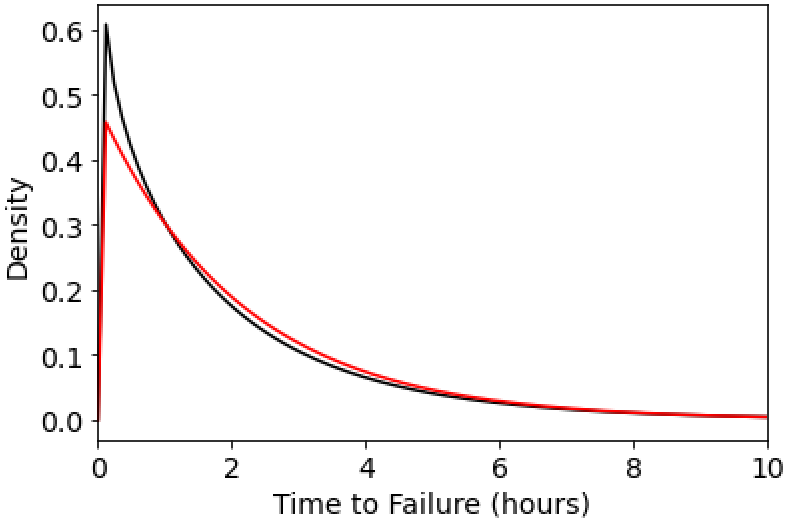
I want to know if the likelihood of a failure is dependent on the time since the last failure.

Start Time
8/24/2023 18:54
8/25/2023 11:11
8/25/2023 12:34
8/25/2023 12:53
8/25/2023 14:02
8/25/2023 14:46
8/27/2023 23:44
8/28/2023 0:01
8/28/2023 2:29
8/28/2023 4:28
8/28/2023 5:19
8/28/2023 5:30
8/28/2023 7:30
8/28/2023 8:34
8/28/2023 9:13
8/28/2023 10:22
8/28/2023 10:38
8/24/2023 18:54
8/25/2023 11:11
8/25/2023 12:34

from scipy.stats import weibull_min, expon

Comparing Probability Distributions

```
# ttf is a numpy array of time to failure
weibull_dist = weibull_min.fit(ttf)
exponential_dist = expon.fit(ttf)

expon_loglik = np.sum([np.log(expon(*exponential_dist).pdf(x)) for x in ttf])
weibull_loglik = np.sum([np.log(weibull_min(*weibull_dist).pdf(x)) for x in ttf])

print(expon_loglik, weibull_loglik)
>>> -1413.542   -1407.631
```




```
print(Weibull_dist)
>>> (0.921326, 0.061388, 1.911398)

# Shape (k) is 0.921
# Location is 0.061
# Scale (lambda) is 1.911
```

f(x-L) where L is location - python just moves it to the origin

Extending MLE – the Exponential Family

$$f(x|\theta) = e^{\frac{x*\theta - b(\theta)}{a(\phi)} + c(x,\theta)}$$

- θ is the **canonical parameter** of the distribution
- ϕ is the **diffusion parameter** of the distribution

Examples of the Exponential Family

$$f(x|\theta) = e^{\frac{x*\theta - b(\theta)}{a(\phi)} + c(x,\theta)}$$

For a normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x_j - \mu_0\right)^2}{2\sigma^2}\right)$$

- Canonical parameter (θ) is μ
- Diffusion parameter (ϕ) is σ
- $a(\phi) = \sigma^2$
- $b(\theta) = \frac{\mu^2}{2}$
- $c(x, \phi) = -\frac{x^2}{2\phi} - \log\sqrt{2\pi\phi}$

For a Poisson distribution:

$$f(x) = \frac{\lambda^k e^{-\lambda}}{k!}$$

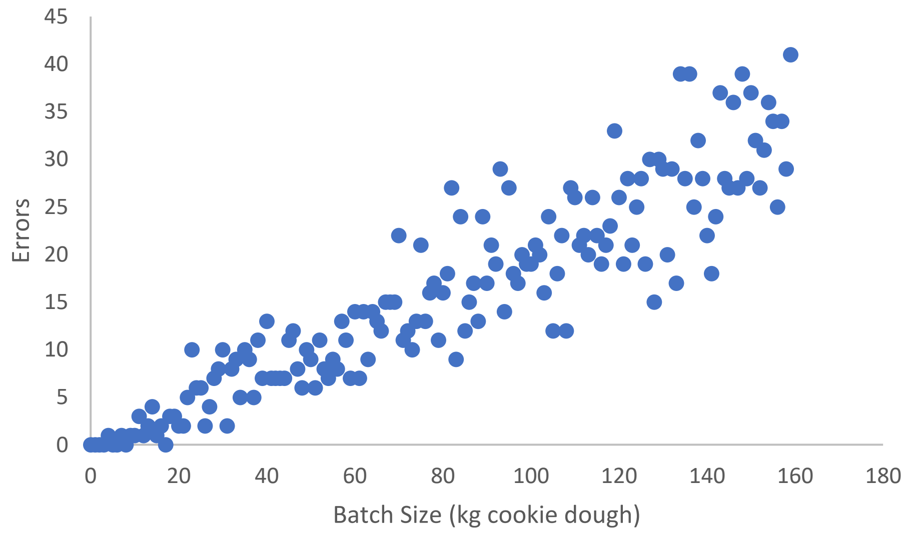
- Canonical parameter (θ) is $\log(\lambda)$
- Diffusion parameter (ϕ) is 1
- $a(\phi) = 1$
- $b(\theta) = \lambda = e^{\theta}$
- $c(x, \phi) = -\log\sqrt{x!}$

Motivating example

My cookie company is struggling with high error rates in cookie production.

I have measured the error rate from different cookie batch sizes.

How can I use this information to predict error rate in the future?



Predictive Modelling with Independent Variables

- Inclusion of **covariate** information allows us to reduce the number of parameters in the model, while providing some predictive power.

- At a minimum, we seek estimators with
  1. low bias.
  2. small variance (*i.e.* we seek efficiency).
  3. consistency: the estimator converges to the true parameter in probability when sample size goes to infinity.

Poisson Regression

- Since we are modelling a rate of discrete occurrences (sales), we can assume it follows a Poisson distribution

- Our response variable is dependent on an independent variable; we want to understand this relationship

Poisson Regression

- We can begin by fitting a **different** distribution to **each** data point

- Taking a distribution from the exponential family

$$f(\overset{y}{\cancel{x}}|\theta) = e^{\frac{\overset{y}{\cancel{x}}*\theta - b(\theta)}{a(\phi)} + c(\overset{y}{\cancel{x}},\theta)}$$

- Gives us the log-likelihood

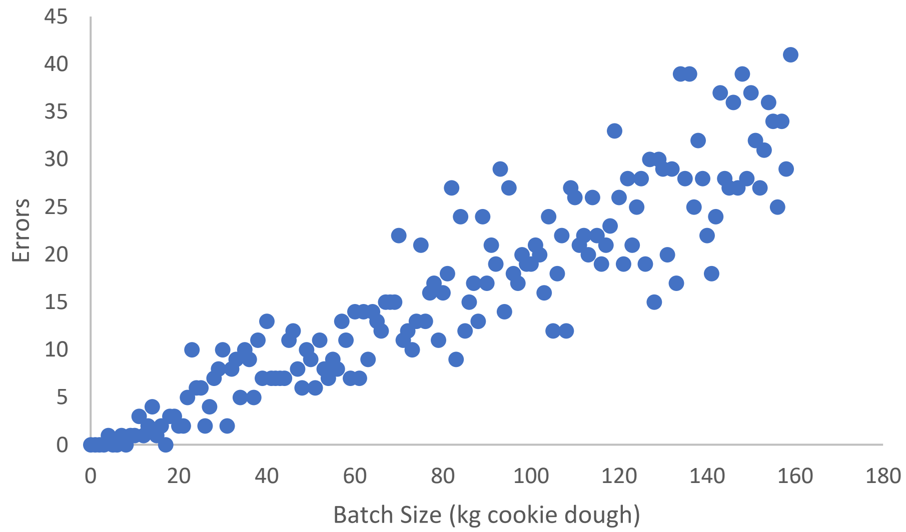
$$l(\theta) = {y\theta - b(\theta)}\Big/{a(\phi)} + c(y,\phi)$$

- Differentiating with respect to θ and finding the root gives us

$$l'(\theta) = \frac{y - b'(\theta)}{a(\phi)} \qquad\qquad \hat{\theta} = b'^{-1}(y)$$

Poisson Regression

- Find $\hat{\theta}$ for every data point
- This gives us a unique distribution for each data point
- Such a model is called a **saturated model**



Poisson Regression

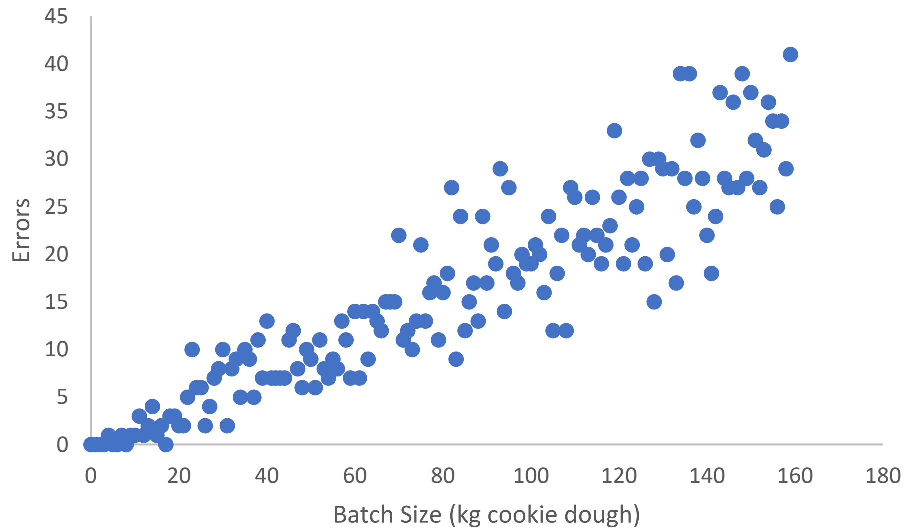
- Assume our data follows a Poisson distribution with a canonical parameter (θ) **linearly dependent** on our batch size (x)

$$l(\theta) = {y\theta - b(\theta)}\Big/{a(\phi)} + c(y, \phi)$$

theta = beta*x

$$l_i = y_i \beta x_i - b(\beta x_i) + c(y, \phi)$$

$$\frac{\partial l_i}{\partial \beta} = y_i x_i - b'(\underline{\beta x_i}) x_i$$



Poisson Regression

- Take the derivation with respect to β

b'(theta)

$$\boxed{\frac{\partial l_i}{\partial \beta} = y_i x_i - \overbrace{b'(\beta x_i)}^{b'(\theta)} x_i}$$

- Set the derivative to zero

$$0 = y_i x_i - e^{\beta x_i} x_i$$

think about link function from data-583 (Shao pings class)

- Rearrange the equation to solve

$$\underline{\hat{\beta} = \log(y_i) / x_i}$$

Poisson Regression

Instead of individual values of β, we want to come up with a single value that best represents all our data. We have some equations to help us:

sub

$\theta = \beta * x$	$\hat{\beta} = \log(y_i)\,/x_i$
$a(\phi) = 1$ $b(\theta) = \lambda = e^{\theta}$ $c(x, \phi) = -\log\sqrt{x!}$	sub $\rightarrow$ $l_i = y_i \beta x_i - b(\beta x_i) + c(y, \phi)$

Substituting values into the log-likelihood, we can derive the equation:

$$l(\vec{y}) = \sum_{i=1}^{n} y_i * \beta * x_i - e^{\beta * x_i} - \log(y_i!)$$

Finding $argmax\big(l(\vec{y})\big)$ is not trivial and is done numerically.

A full derivation and associated R code is available at https://statomics.github.io/SGA2019/assets/poissonIRWLS-implemented.html

Generalized Linear Model

- Generalized Linear Models (GLMs) predict a distribution in response to independent variables

- A GLM has three properties
  - A value y (dependent variable) is generated from a distribution
  - The mean of the distribution depends on some independent variables X
  - The link between X and the mean μ is called a **link function**

A closer look at link functions

- A **link function** is function that relates the independent variable to the mean of the distribution
- A **canonical link function** is derived from the distribution's density function
  - Link functions exist for every distribution in the exponential family
  - You do not have to use the canonical link function, but it's a good start
- When using a distribution with canonical parameter θ the link function has the form $\theta = b(\mu)$

Canonical Link Functions

Distribution	Uses	Canonical Link Function
Normal	Data with linear responses	μ this is the identity link
Poisson	Counts of occurrences within time/space	$\log(\mu)$ this is log link
Exponential	Time between events	$-\frac{1}{\mu}$
Gamma	Sum of exponential response variables	$-\frac{1}{\mu}$
Binomial	Count of 1s in a series of [0,1] trials	$\log\left(\frac{\mu}{n-\mu}\right)$

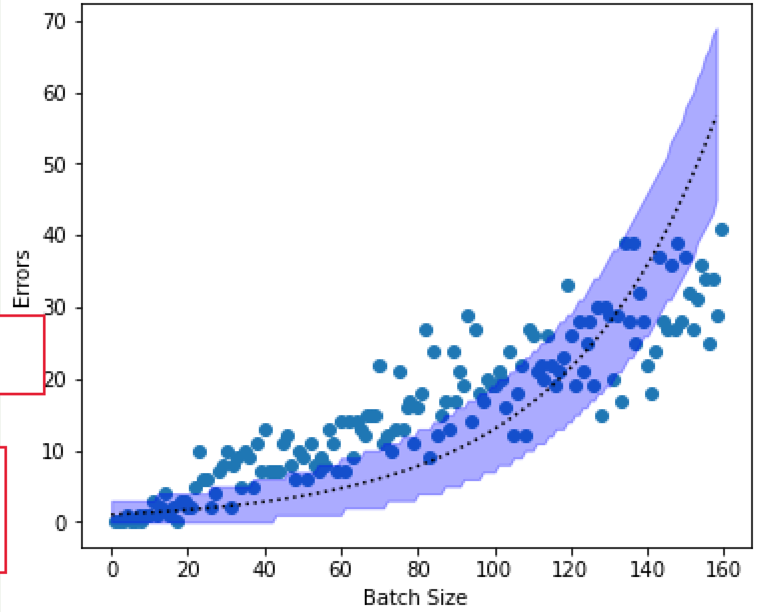
Completing the motivating example

```
import statsmodels.api as sm
import pandas as pd
from scipy.stats import poisson

df = pd.read_excel("lecture2_figures.xlsx“)
poisson_glm = sm.GLM(df['Errors'],
                      df['Batch Size’],
               family=sm.families.Poisson())
results = poisson_glm.fit()

predictions = results.predict()
low_bar = [poisson.ppf(0.05,x) for x in predictions]
high_bar = [poisson.ppf(0.95,x) for x in predictions]
```

dependent variable first (y)

independent variable second (x)

default uses the canonical link function but this is not the correct link function to use which is why the fit looks bad



Completing the motivating example

```
import statsmodels.api as sm
import pandas as pd
from statsmodels.genmod.families.links import Identity
from scipy.stats import poisson

df = pd.read_excel("lecture2_figures.xlsx“)
poisson_glm = sm.GLM(df['Errors'],
                     df['Batch Size’],
       family=sm.families.Poisson(link=Identity())
       )
results = poisson_glm.fit()

predictions = results.predict()
low_bar = [poisson.ppf(0.05,x) for x in predictions]
high_bar = [poisson.ppf(0.95,x) for x in predictions]
```