```
## Warning:  package 'MPV' was built under R version
4.3.1

## Warning:  package 'KernSmooth' was built under R
version 4.3.1

## Warning:  package 'randomForest' was built under R
version 4.3.1
```

DATA 581

Modeling and Simulation II

Lecture 8: Hidden Markov Model

Outline

- Introduction
- Hidden Markov Model
- A Simple Example
- Dynamic Programming - How to get from New York to LA in a hurry
- Dynamic Programming ; Viterbi Algorithm
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- Application to Windspeed Data

```
set.seed(222696)   # use this to reproduce output
```

Introduction

- **A Hidden Markov Models (HMMs) is a statistical model in which the system being modelled is assumed to be markov process with unobserved hidden states.**
- **It used to describe the revolution of observable events that depend on factors that are not directly visible and are hidden from view.**
- **They allow us to predict a sequence of unknown (hidden) variables from a set of observed variables.**
- **A simple example of an HMM is predicting the weather, Rainy/ Sunny (hidden variable) based on your friend's mood; Grumpy/Happy (observed values).**

Introduction

- The idea of HMM allow us compute the joint probability of a set of hidden states given a set of observed states.

- Once we know the joint probability of a sequence of hidden states, we determine the best possible sequence i.e. the sequence with the highest probability. (any method?)

- **Note:** The ratio of hidden states to observed states is not necessarily 1 is to 1. key idea is that one or more observations allow us to make an inference about a sequence of hidden states.

General Approach

We need three types of information to compute the joint probability of a sequence of hidden states.

1. **Transition probability; the probability of transitioning to a new state conditioned on a present state.**

2. **Emission probability; the probability of transitioning to an observed state conditioned on a hidden state.**

3. **Initial state information; the initial probability of transitioning to a hidden state. This is also known as the prior probability.**

Note: **Usually, the term *"states"* are used to refer to the hidden states and *"observations"* are used to refer to the observed states.**

General Approach

There are three types of problems in HMMs;

1. **Evaluation:** **Given the HMM model, m, and observation sequence O. Caluculate the probability that model m has generated sequence O.**

2. **Decoding:** **Given the HMM model, m, and observation sequence O. Caluculate the most likely sequence of hidden states $S - i$ that generated sequence O.**

3. **Learning:** **Given some training observation sequences O and general structure of HMM (visible and hidden states); Determine HMM parameters that best fit the training data.**

A Simple Hidden Markov Model

Suppose we observe data on a discrete random variable

$$Y : 1, 0, 0, 0, 1, 1, 0.$$

We can model this as Bernoulli data, but we suspect there is some hidden dependence.

So we choose to model it as

$$Y_j = B(0.25 + 0.5X_j)$$

where X_j is a Bernoulli random variable which is part of an underlying (hidden) Markov chain.

Note: The *emission probabilities* are assumed:
$P(Y_j = 1|X_j = 1) = .75$ **and** $P(Y_j = 1|X_j = 0) = .25$**.**

A Simple Hidden Markov Model

Since X_j is part of a Markov chain, we need a transition matrix.

Since X_j is Bernoulli and can take only two possible values, there are two hidden states, the transition matrix is 2×2.

For example,

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}.$$

Together, the emission probabilities and the transition matrix make up a Hidden Markov Model (HMM).

Usually, we do not know the emission probabilities P_E or the transition probabilities P.

These are usually estimated by maximizing the likelihood function:

$$L(P_E, P) = P(Y_1, Y_2, \ldots, Y_n).$$

Because the Y_j's are not independent, we need to be careful how to evaluate the likelihood.

A Simple Hidden Markov Model

What we can calculate easily:

$$P(Y_1, Y_2, \ldots, Y_n | X_1, X_2, \ldots, X_n) = \Pi_{j=1}^{n} (.25 + .5X_j)^{Y_j} (.75 - .5X_j)^{1-Y_j}.$$

$$P(X_1, \ldots, X_n) = P(X_1)P(X_2|X_1) \cdots P(X_n|X_{n-1})$$

Taking the product of these, we can calculate:

$$P(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$$

We then have to sum over all combinations of the Xs to get the likelihood, a big job.

Dynamic Programming

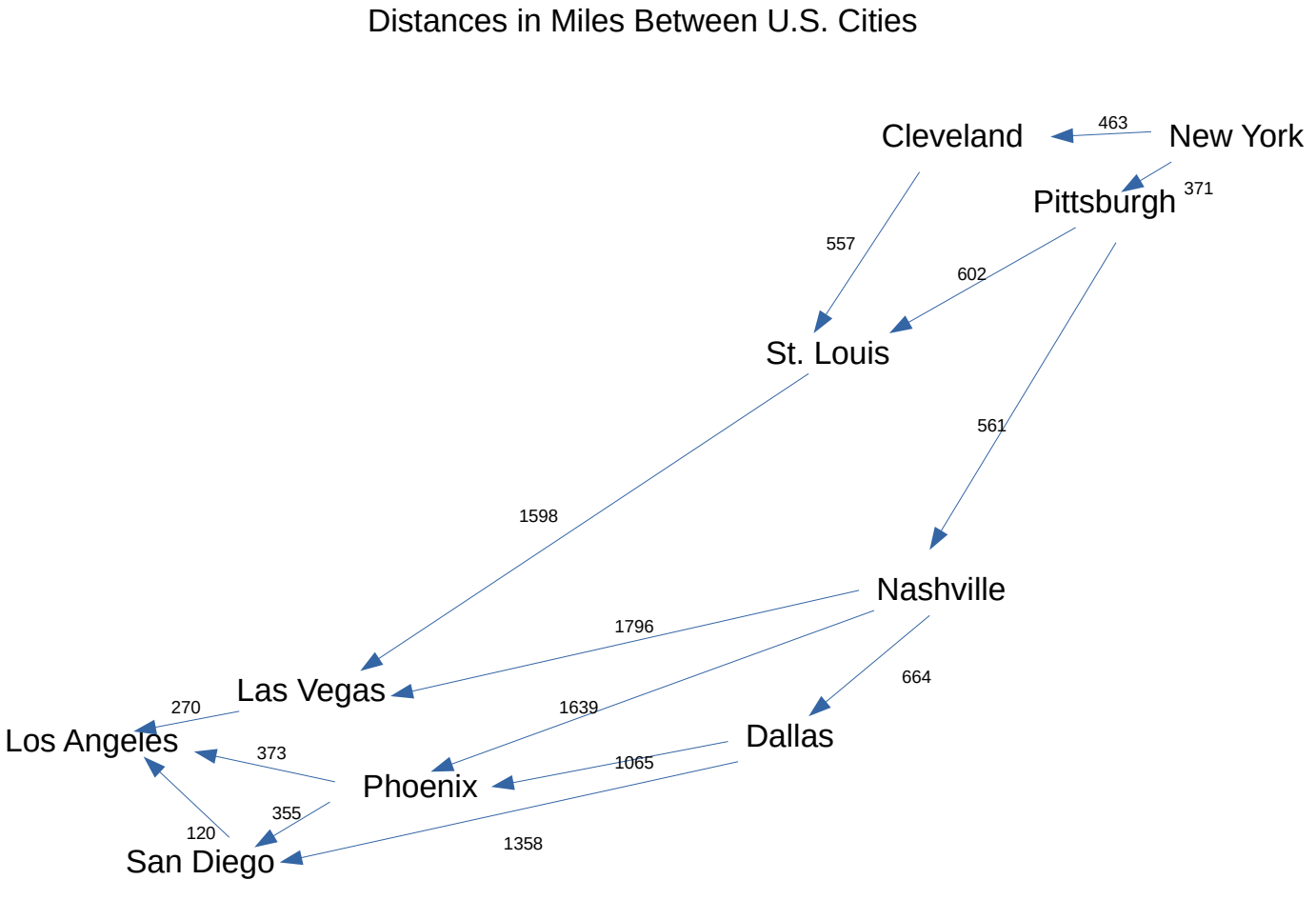
- We can calculate these probabilities quite efficiently using dynamic programming, and relying on the key property of the Markov chain.

- Dynamic programming is an optimization procedure due to Richard Bellman (early 1960's) which efficiently finds the minimum distance route through a network.

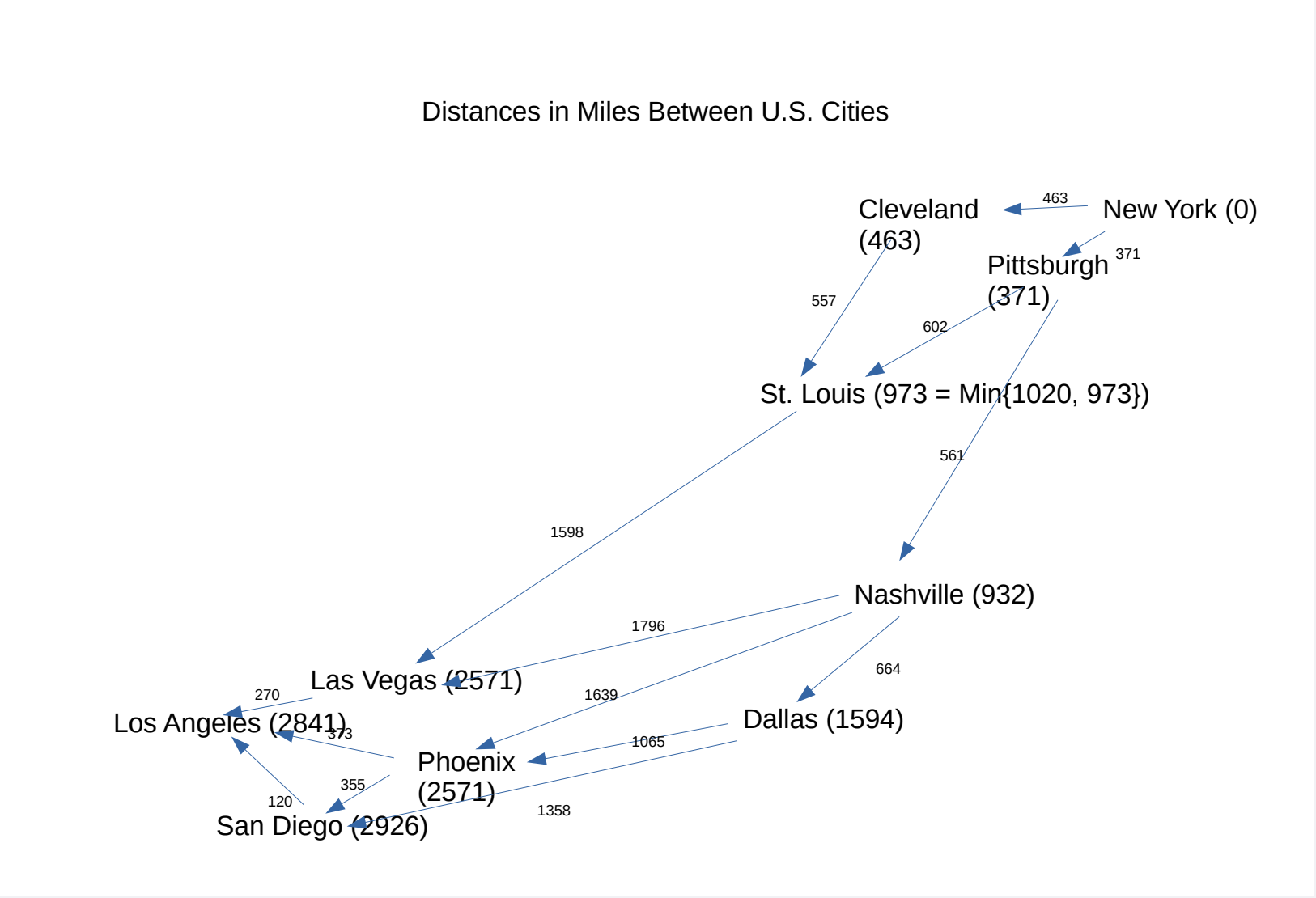
- Consider the problem of driving from New York to Los Angeles. There are many possible routes.

- Dynamic programming allows us to break the bigger network problem into a set of smaller network problems which can be solved recursively.

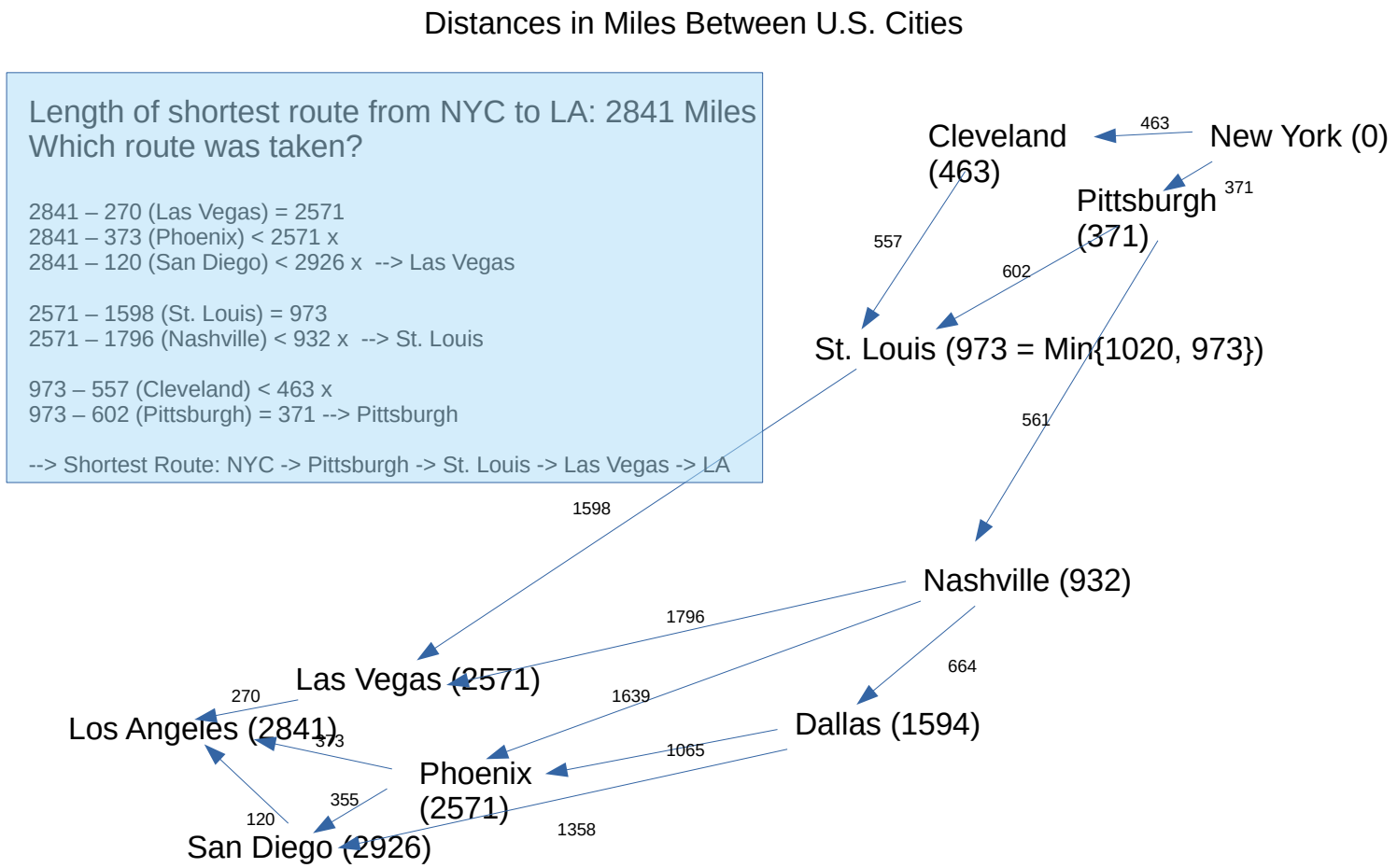
Find Shortest Route from New York to Los Angeles



Find Shortest Route from New York to Los Angeles



Find Shortest Route from New York to Los Angeles



Viterbi Algorithm

The Viterbi algorithm employs dynamic programming to find the most likely sequence of hidden Markov chain states which could give rise to the observed data (Veterbi path).

Using the Viterbi path, we can calculate a kind of maximum likelihood estimate for the emission probability matrix and the transition matrix.

The probability distributions of hidden states is not always known. In this case, we use Expectation Maximization (EM) models in order to determine hidden state distributions. A popular algorithm is the Baum-Welch algorithm

Built-in Softwares

- The *depmixS4* package fits hidden Markov models on mixed categorical and continuous (time series) data.

- A simpler-to-use package is *HMM*, which can be used to fit and simulate discrete-time discrete-state Hidden Markov Models.

Built-in Software - the *HMM* package

We illustrate the use of the software for the Hidden Markov Model for which the transition matrix for the hidden 2 state Markov chain X is

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

and the emission matrix for the probability distributions of the observed values Y, given X is

$$E = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

The state space of the Markov chain is $S = \{0, 1\}$.

The first row of the emission corresponds to the distribution of Y, given $X = 0$ and the second row corresponds to $X = 1$.

We assume that the observed data are $Y = 1, 0, 0, 0, 1, 1, 0$.

Built-in Software - the *HMM* package

We load the package, enter the information about the transition and emission matrices as well as the observed data.

```
library(HMM)

## Warning:  package 'HMM' was built under R version
4.3.0
```

```
hmm <- initHMM(c("0","1"), c("0","1"),
         transProbs=matrix(c(1, 2, 2, 1)/3,2),
         emissionProbs=matrix(c(3, 1, 1, 3)/4, 2))
observations <- as.character(c(1, 0, 0, 0, 1, 1, 0))
```

Using the `viterbi` function, we can calculate the Viterbi path, the most likely X values:

```
viterbi <- viterbi(hmm, observations)
print(viterbi)

## [1] "1" "0" "1" "0" "1" "1" "0"
```

Training (Estimating) the HMM from Y Observations

Based on the preceding Viterbi sequence, we could get estimates of the transition matrix elements by calculating the proportion of the various transitions.

In the example, the Markov chain appears to have visited state 0 two times, not including the last value.

The Markov chain entered state 1 both times, so we would estimate the P_{01} to be 1 and P_{00} as 0. Similarly, we would estimate P_{10} as $3/4$ and P_{11} as $1/4$.

We can also estimate the emission probabilities by estimating the distribution of Y for each value of X: $P(Y = 0|X = 0) = 3/3$ and $P(Y = 1|X = 0) = 0/3$. $P(Y = 0|X = 1) = 1/4$ and $P(Y = 1|X = 1) = 3/4$.

Training (Estimating) the HMM from Y Observations

1. **Begin with an initial guess as to the transition matrix and emission matrix.**

2. **Apply the Viterbi algorithm to the Y observations to obtain the most probable set of X observations, using the most recent estimates of the transition and emission matrices.**

3. **Use the Viterbi sequence of X's to estimate the transition matrix probabilities and the emission matrix probabilities.**

4. **Return to step 2, unless the transition and emission matrices have converged to within a given tolerance.**

The function `viterbiTraining` is an implementation of this algorithm.

An Example Using Simulated Data

We simulate X's from a two state Markov chain, and use these to generate Y observations which take values 0, 1 and 2, according to different distributions, depending on the corresponding value of X.

The transition matrix for the hidden Markov chain X:

$$P = \begin{bmatrix} 0.1 & 0.9 \\ 0.2 & 0.8 \end{bmatrix}$$

The emission matrix for the probability distributions of the observed values Y, given X:

$$E = \begin{bmatrix} 0.75 & 0.2 & 0.05 \\ 0.05 & 0.4 & 0.55 \end{bmatrix}$$

The first row corresponds to the distribution of Y, given $X = 0$ and the second row corresponds to $X = 1$.

An Example Using Simulated Data

```
P <- matrix(c(.1, .9, .2, .8), nrow=2, byrow=TRUE)
E <- matrix(c(.75, .2, 0.05, .05, 0.4, .55), nrow=2, byrow=TRUE)
current.state <- 1; n <- 500
X <- numeric(n); Y <- numeric(n)
for (i in 1:n) {
    current.state <- sample(0:1, size = 1, prob = P[current.state+1,])
    X[i] <- current.state  #hidden states
    Y[i] <- sample(0:2, size = 1, prob = E[current.state+1,])
    #Y observed data
}
```

First few simulated observations:

```
Y[1:5]

## [1] 1 0 1 2 1
```

An Example Using Simulated Data

Fitting the HMM to the data, using an arbitrary starting guess for the emission matrix and the transition matrix.

```
hmm <- initHMM(c("0","1"), c("0","1", "2"),
        transProbs=matrix(c(.5, .5, .5, .5),2),
        emissionProbs=matrix(c(.5, .4, .1, .8, .05, .15), 2))
observations <- as.character(Y)
simDataFit <- viterbiTraining(hmm, observations)
```

An Example Using Simulated Data

The output from the algorithm includes the estimated transition matrix and emission matrix:

```
print(simDataFit$hmm$transProbs)   # transition matrix estimate
```

```
##     to
## from         0         1
##    0 0.1276596 0.8723404
##    1 0.2024691 0.7975309
```

```
print(simDataFit$hmm$emissionProbs)   # emission matrix estimate
```

```
##       symbols
## states 0         1         2
##      0 1 0.0000000 0.0000000
##      1 0 0.4408867 0.5591133
```

The estimated matrices are not terribly far from the truth, but there is still considerable error. Note that the sample size is fairly large, and we are fitting a very simple model.

Application to the Windspeed Data

Data preparation:

```
source("wind.R")
ws <- wind$h12  # Winnipeg noon hour windspeed (kmh)
wsCut <- cut(ws, c(-1e-10, 15, 22, 35, 45, 80))
levels(wsCut) # see what the cut function did

## [1] "(-1e-10,15]" "(15,22]"     "(22,35]"     "(35,45]"
## [5] "(45,80]"

levels(wsCut) <- c("N", "L", "M", "H", "E")
  # Nil, Low, Moderate, High, Extreme
WindStates <- factor(wsCut, ordered = TRUE)
```

Application to the Windspeed Data

1. We set up an HMM object with our initial guess for the Wind Speed HMM:

```
WindHMM <- initHMM(c("0","1","2"), levels(WindStates),
     transProbs=matrix(c(2, 1, 1, 1, 2, 2, 3, 3, 3)/6, 3),
     emissionProbs=matrix(c(4, 2, 0, 3, 2, 1, 2, 2, 2, 1,
   2, 3, 0, 2, 4)/10, nrow=3))
```

Application to the Windspeed Data

Let's view the transition matrix and emissions probabilities for the initial guess:

```
print(WindHMM$transProbs)
```

```
##      to
## from     0     1   2
##    0 0.333 0.167 0.5
##    1 0.167 0.333 0.5
##    2 0.167 0.333 0.5
```

```
print(WindHMM$emissionProbs)
```

```
##        symbols
## states   N   L   M   H   E
##      0 0.4 0.3 0.2 0.1 0.0
##      1 0.2 0.2 0.2 0.2 0.2
##      2 0.0 0.1 0.2 0.3 0.4
```

Interpretation of the initial guess emissions probability matrix P_E:

$$P_E[i, j] = P(Y = j | X = i)$$

where X is the current state of the hidden Markov chain, and Y is the current observation.

e.g. $P_E[2, 3] = 0.2$ so the probability of Moderate wind when in Markov Chain state 1 is 0.2.

Application to the Windspeed Data

2. We can fit a HMM model by calling `viterbiTraining()`, it gets an inital object and observed values.

The algorithm fits the Hidden Markov Model by finding transition probabilities and emission probabilities that maximize the likelihood.

```
WindHMMFit <- viterbiTraining(WindHMM, WindStates)
```

We can view the transition matrix and emissions probabilities for the fitted model:

```
print(WindHMMFit$hmm$transProbs)
```

```
##      to
## from      0      1      2
##    0 0.8047 0.0000 0.1953
##    1 0.0000 0.0000 1.0000
##    2 0.2020 0.2424 0.5556
```

```
print(WindHMMFit$hmm$emissionProbs)
```

```
##        symbols
## states      N      L       M      H      E
##      0 0.5560 0.3739 0.07007 0.0000 0.0000
##      1 0.5623 0.4377 0.00000 0.0000 0.0000
##      2 0.0000 0.0000 0.79976 0.1557 0.0445
```

Application to the Windspeed Data

Now we can also simulate from the fitted model using `simHMM()`.

```
WindSim <- simHMM(WindHMMFit$hmm, length(WindStates))
```

We have simulated the same number of observations as in the original data set.

Application to the Windspeed Data

We have use the `rle()` (Run Length Encoding) function which computes the lengths and values of runs of equal values in a vector.

Compare run lengths of the various states in observed data:

```
table(rle(as.character(WindStates)))
```

```
##         values
## lengths    E    H    L    M    N
##       1  133  409 1017 1028 1031
##       2    7   49  217  410  307
##       3    1    6   37  175  115
##       4    0    0   13   74   39
##       5    0    0    4   28   22
##       6    0    0    3   14    8
##       7    0    0    0    2    6
##       8    0    0    1    3    4
##       9    0    0    0    1    2
```

Application to the Windspeed Data

Compare run lengths of the various states from the simulated data.

```
table(rle(WindSim$observation))
```

```
##         values
## lengths    E    H    L    M    N
##       1  148  437 1031  943 1107
##       2    4   51  176  404  291
##       3    0    4   58  178  107
##       4    0    0   15   87   36
##       5    0    0    2   40   19
##       6    0    0    2    9    5
##       7    0    0    0    8    3
##       8    0    0    0    3    2
##       9    0    0    0    0    2
##       12   0    0    0    1    1
```

Summary

1. **Hidden Markov Models (HMM) provide a way to model data that have complex dependencies in a structured way: a Markov chain for the hidden information, together with a matrix of probability distributions for the observed data.**

2. **The Viterbi algorithm provides a basic way to train the data, based on finding the most likely path through the hidden Markov states.**

3. **Hidden Markov Models are an important tool in diverse areas such as climate science, genomics, and speech processing.**