Principal Component Analysis

UBCO MDS — DATA 572

Motivation

- Changing gears to dimensionality reduction, and in particular an unsupervised method for doing so.
- We will eventually bring this back around to supervised learning in the next lecture
- Note: dimensionality reduction is NOT (necessarily) the same as variable selection/feature reduction/etc. This will hopefully become clear while we progress...

Motivation

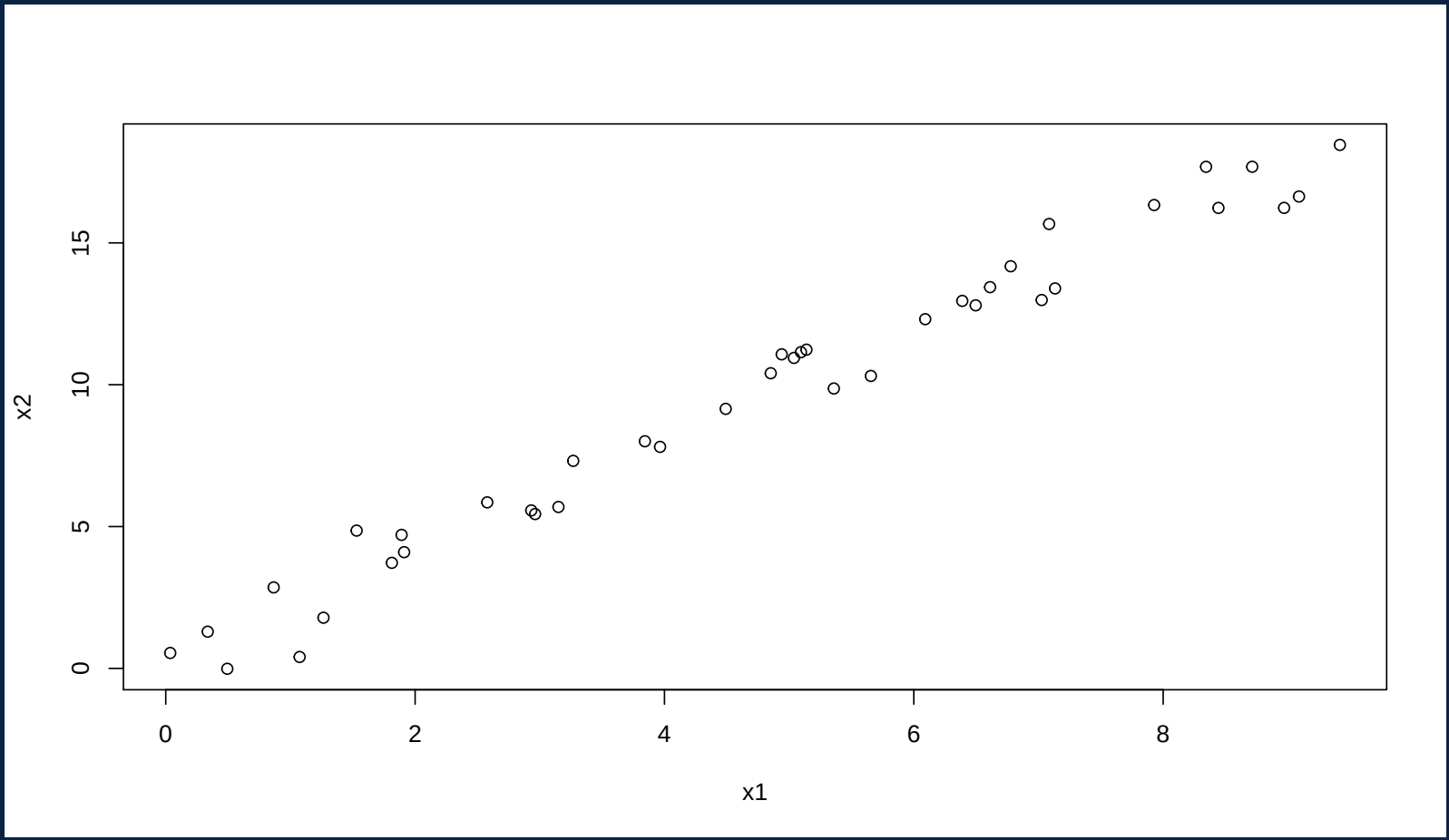
- Where we'll go with this on the application side...
- The heptathlon is a track and field competition with several (seven, specifically) running, throwing, and jumping events.
- The scoring system is...complex (we will outline it later). Can we use this particular form of dimensionality reduction to devise a 'simpler' scoring system?
- But first, let's get technical...

Basic Idea

- We have p predictors $X_1, X_2, \ldots, X_p$
- We will seek p 'new' variables, say $Z_1, Z_2, \ldots, Z_p$ that
  1. are linear combinations of $X_1, X_2, \ldots, X_p$
  2. are uncorrelated (that is, $\text{Cor}(Z_j, Z_k) = 0$ for all $j \neq k$)
  3. provide the bulk of the variation (aka, information) in $X_1, X_2, \ldots, X_p$ within the first few Z_j's

Simple Bivariate Example

- Suppose we have the following data



Simple Bivariate Example

- We can note

```
> cov(cbind(x1, x2))
          x1        x2
x1  7.616567 14.78788
x2 14.787881 29.46383
```

```
> cor(cbind(x1, x2))
          x1        x2
x1 1.0000000 0.9871468
x2 0.9871468 1.0000000
```

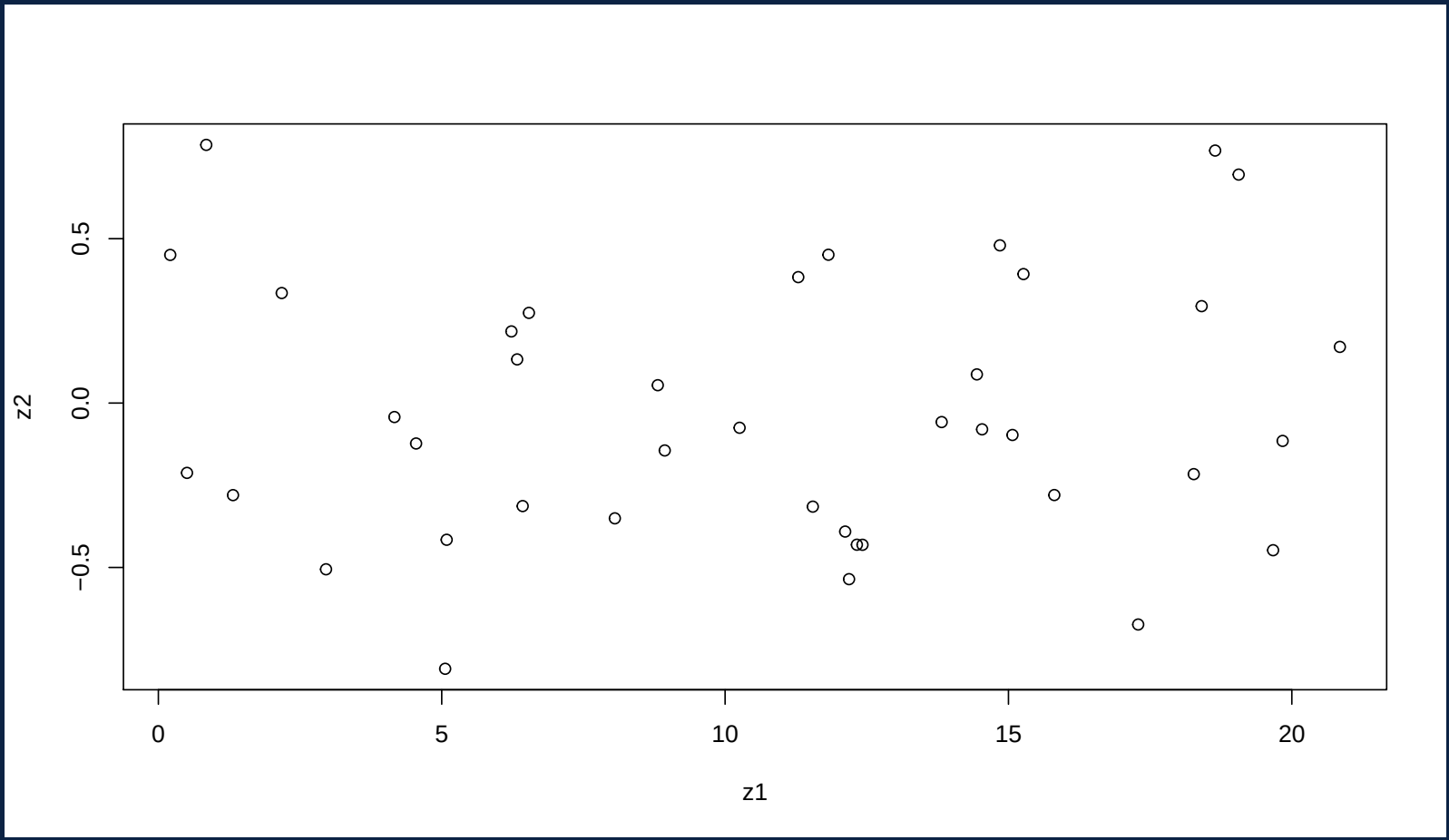
Simple Bivariate Example

- Now, suppose we create two new variables as linear combos of X_1 and X_2, namely...
  - $Z_1 = .45X_1 + .90X_2$
  - $Z_2 = .90X_1 - .45X_2$

- Note that with our current toolbox, this would seem to be a fairly random choice of coefficients for the linear combos...but let's see what the transformed data looks like...

Simple Bivariate Example

- Scatterplot of Z_1 and Z_2



Simple Bivariate Example

- And further note

```
> cov(cbind(z1, z2))
           z1         z2
z1 37.3862401 0.1354968
z2  0.1354968 0.1576611
```

```
> cor(cbind(z1, z2))
           z1          z2
z1 1.00000000 0.05580986
z2 0.05580986 1.00000000
```

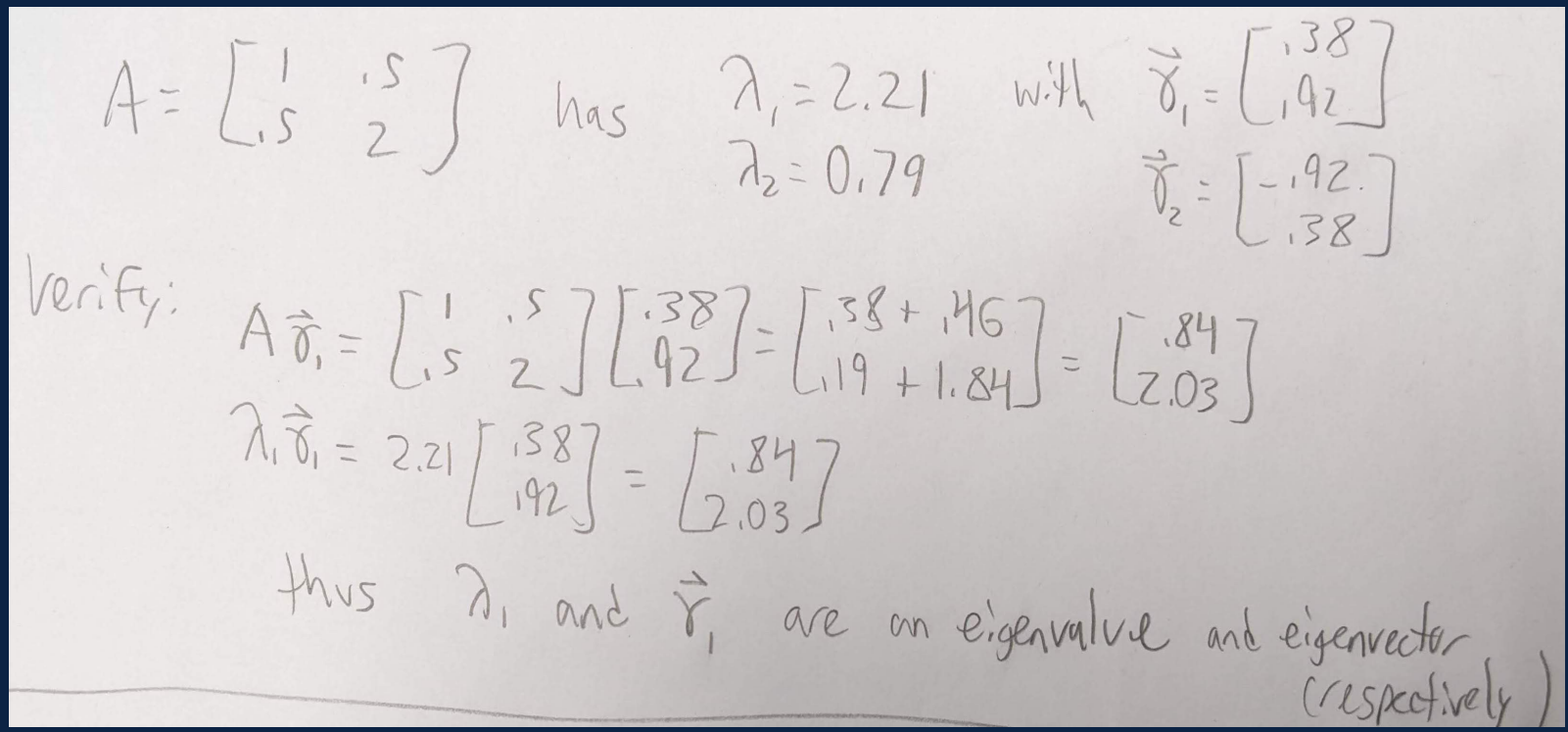
Some Linear Algebra

- A square $p \times p$ matrix A is said to have an eigenvalue λ with corresponding eigenvector $\gamma \neq \vec{0}$ if

$$A\gamma = \lambda\gamma$$

- If A is symmetric, then A has p eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ and p corresponding eigenvectors $\gamma_1, \gamma_2, \ldots, \gamma_p$

Some Linear Algebra



Some Linear Algebra

- If A is $p \times p$ symmetric with eigenvalues, then we can write

$$\mathrm{A} = \mathrm{P}\Lambda\mathrm{P}^T$$

where all matrices are $p \times p$.

- Further, note that

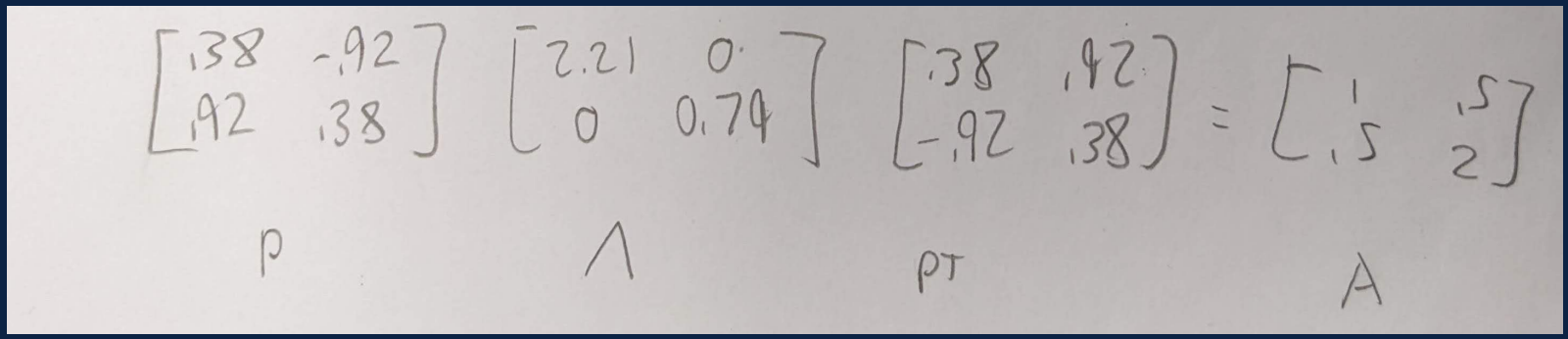
$$\mathrm{P} = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_p]$$

- and Λ is a diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ along the diag.

Some Linear Algebra

- Also, $PP^T = P^TP = I_p$, AKA the columns of P are orthonormal
- AKA, $\gamma_j^T \gamma_k = 0$ for all $j \neq k$ and $\gamma_j^T \gamma_j = 1$

Some Linear Algebra



Some Linear Algebra

- A symmetric $p \times p$ matrix A is positive semi-definite (psd) if

$$\vec{c}^T \mathrm{A} \vec{c} \geq 0 \quad \forall \quad \vec{c}$$

- If A is psd, then $\lambda_i \geq 0$ for all i.
- Note that covariance matrices are psd, and therefore have p non-negative eigenvalues.

Principal Components

- Recall from beginning of these slides...
- We will seek p 'new' variables, say $Z_1, Z_2, \ldots, Z_p$ that
  1. are linear combinations of $X_1, X_2, \ldots, X_p$
  2. are uncorrelated (that is, $\text{Cor}(Z_j, Z_k) = 0$ for all $j \neq k$)
  3. provide the bulk of the variation (aka, information) in $X_1, X_2, \ldots, X_p$ within the first few Z_j's

Principal Components

- Suppose covariance matrix Σ has eigenvalues ordered such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$ with corresponding eigenvectors $\gamma_1, \gamma_2, \ldots, \gamma_p$.
- It can be shown that γ_1 (aka, the eigenvector corresponding to the largest eigenvalue of Σ) provides coefficients such that $\text{Var}(\gamma_1^T X)$ is maximized subject to the constraint $\gamma_1^T \gamma_1 = 1$
- And furthermore, γ_2 maximizes $\text{Var}(\gamma_2^T X)$ subject to $\gamma_2^T \gamma_2 = 1$ AND $\gamma_2^T \gamma_1 = 0$
- Annnnnd so on for the remaining eigenvectors...

Principal Components

- In summary, the eigendecomposition of Σ provides the solution for our desired properties for principal components. AKA, we can define $Z_j = \gamma_j X$.
- So the eigenvectors provide the coefficients for the linear combo, but the eigenvalues are interesting too!
- Note that the diagonal of Σ contains the variance of each variable. Summing that up, $\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_p^2$, provides a measure of 'total variance'
- It can be shown through matrix properties (namely trace) that $\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_p^2 = \lambda_1 + \lambda_2 + \cdots + \lambda_p$.
- So the total variance of X still exists in PX. AKA, there is no information loss in principal components (at least, to this point of our discussion...)

Principal Components and Geometry

- A couple of geometric asides
- Not only is there no information loss, it is also true that distance between observations in the original data are preserved in the PCA-transformed space.
- Angles between vectors are also preserved.
- In fact, PCA is simply an orthogonal rotation about the origin.

Principal Components

- Furthermore, we can easily figure out the "proportion of variance explained" by any one component via $\frac{\lambda_j}{\sum_{j=1}^{p} \lambda_j} = \frac{\text{Var} Z_j}{\text{"total variance"}}$
- Brings us to an interesting point...
- Suppose once we get to the k^{th} principal component, we see the percent of variance explained as quite small, say 0.001.
- Can we then toss out that principal component? Along with the remaining principal components (which by definition will have smaller λ)?

Principal Components

- NOTE: THIS is where the dimensionality reduction occurs in PCA
- Since we transform p variables (X) into p variables (Z), it is only when we toss out principal components that we reduce the dimensionality of the data.
- It is also the only point at which we experience a loss of information from the original data.
- But also note: even if we only keep one principal component (transforming from p-variate to univariate data) we don't actually remove any of our original measurements. All p original variables are needed to calculate $Z_1 = \gamma_{11}X_1 + \gamma_{12}X_2 + \cdots + \gamma_{1p}X_p$.

PCA and Scaling

- BIG BIG NOTE: PCA is NOT scale invariant
- As we'll see in an example, this has huge implications...notably, any variable with large variance (relative to the rest) will dominate the first principal component.
- In most cases, this is undesirable. Most commonly, you will need/want to scale your data to have mean 0, variance 1 (almost certainly when your measures are on vastly different scales).
- This amounts to performing an eigendecomposition on the correlation matrix rather than the covariance matrix.

PCA: How many components?

- So how do we choose how many principal components to keep?
- There are several common options, we'll discuss three:
  1. Cumulative proportion/percent of variance
     - Keep number of components such that, say, 90% (or 95%, or 80%, etc) of the variance from original data is retained
  2. Kaiser criterion
     - Keep all $\lambda_j \geq \bar{\lambda}$ where $\bar{\lambda} = \frac{\sum_{j=1}^{p} \lambda_j}{p}$. Note this is further simplified if the data is scaled (mean 0, variance 1) since $\bar{\lambda} = \frac{p}{p} = 1$.
  3. Scree plot
     - Plot the (monotonically decreasing) eigenvalues, look for an 'elbow', or plateauing

Phew...

- And finally, an example...

PCA on Heptathlon Data

```
> matrix(rownames(heptathlon)[1:6], ncol=1)
     [,1]
[1,] "Joyner-Kersee (USA)"
[2,] "John (GDR)"
[3,] "Behmer (GDR)"
[4,] "Sablovskaite (URS)"
[5,] "Choubenkova (URS)"
[6,] "Schulz (GDR)"
> print(heptathlon[1:6,], row.names=FALSE)
 hurdles highjump  shot run200m longjump javelin run800m score
   12.69     1.86 15.80   22.56     7.27   45.66  128.51  7291
   12.85     1.80 16.23   23.65     6.71   42.56  126.12  6897
   13.20     1.83 14.20   23.10     6.68   44.54  124.20  6858
   13.61     1.80 15.23   23.92     6.25   42.78  132.24  6540
   13.51     1.74 14.76   23.93     6.32   47.46  127.90  6540
   13.75     1.83 13.50   24.65     6.33   42.82  125.79  6411
```

Heptathlon Scoring[1]

- Some notes on heptathlon scoring — it's not simple .
- The heptathlon scoring system was devised by Dr. Karl Ulbrich, a Viennese mathematician.
- There is designated "standard" performance (for example, approximately 1.82 m for the high jump) scores 1000 points.
- Each event also has a minimum recordable performance level (e.g. 0.75 m for the high jump), corresponding to zero points.
- Then...

[1] https://en.wikipedia.org/wiki/Heptathlon

Heptathlon Scoring[1]

Event	a	b	c
200 metres	4.99087	42.5	1.81
800 metres	0.11193	254	1.88
100 metres hurdles	9.23076	26.7	1.835
High jump	1.84523	75.0	1.348
Long jump	0.188807	210	1.41
Shot put	56.0211	1.50	1.05
Javelin throw	15.9803	3.80	1.04

- Running events (200m, 800m, 100m hurdles)

$$P = a(b - T)^c$$

- Jumping events (high, long)

$$P = a(M - b)^c$$

- Throwing events (shotput, javelin)

$$P = a(D - b)^c$$

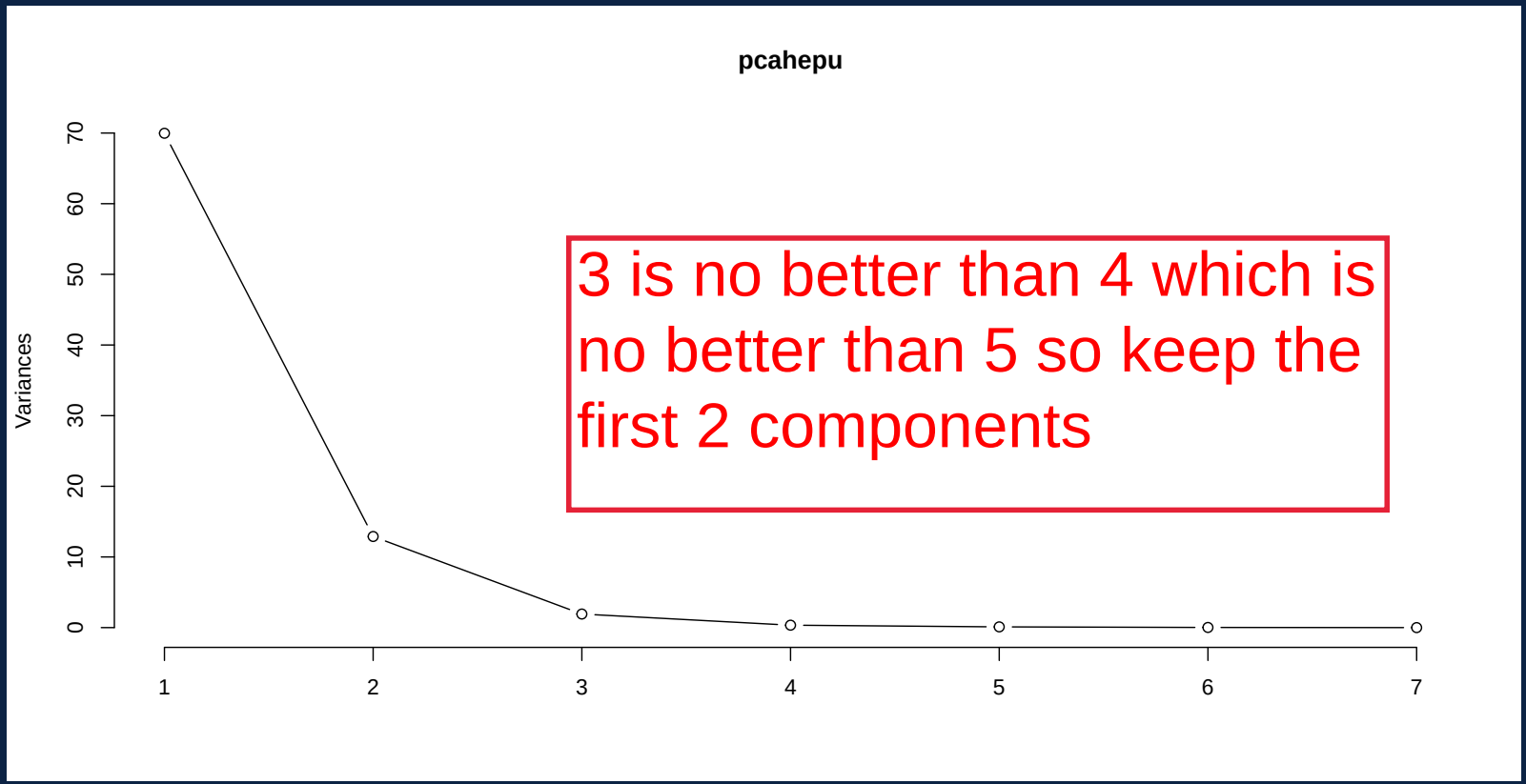
[1] https://en.wikipedia.org/wiki/Heptathlon

Scoring in General

- As a general concept, fairly combining scores from several sporting disciplines seems tricky
- But in effect, we want to find a scoring system that best separates the participants
- In more statistical lingo, we want to find a single variable (made of the original measures) which will provide the bulk of the variation present in the data
- In other words, PCA can suggest a different (simpler?) scoring system! We remove the score variable and work with the remaining...

PCA on Unscaled Measures

```
> pcahepu <- prcomp(heptathlon[,-8])
> plot(pcahepu, type="lines")
```



PCA on Unscaled Measures

- "rotation" are the eigenvectors, aka coefficients of the linear combo, aka component "loadings"

```
> pcahepu$rotation[,1:3]
                 PC1           PC2         PC3
hurdles    0.069508692 -0.0094891417  0.22180829
highjump -0.005569781  0.0005647147 -0.01451405
shot      -0.077906090  0.1359282330 -0.88374045
run200m    0.072967545 -0.1012004268  0.31005700
longjump -0.040369299  0.0148845034 -0.18494319
javelin    0.006685584  0.9852954510  0.16021268
run800m    0.990994208  0.0127652701 -0.11655815
```

- What do you notice?

PCA on Unscaled Measures

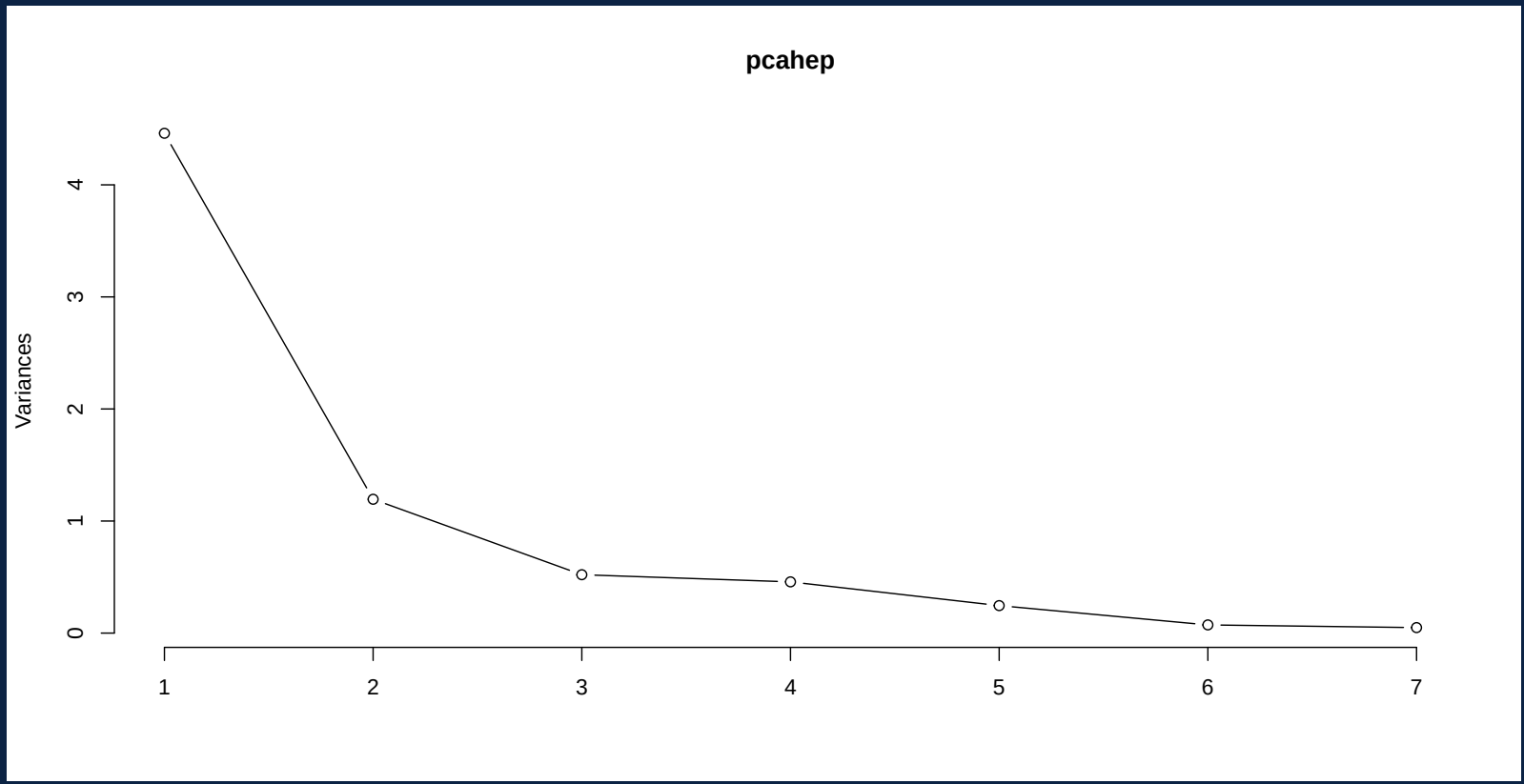
```
> round(pcahepu$rotation[,1:3], 2)
           PC1   PC2   PC3
hurdles   0.07 -0.01  0.22
highjump -0.01  0.00 -0.01
shot     -0.08  0.14 -0.88
run200m   0.07 -0.10  0.31
longjump -0.04  0.01 -0.18
javelin   0.01  0.99  0.16
run800m   0.99  0.01 -0.12
```

- What do you notice?

this is a tell-tale sign you didn't standardize because each component is dominated by 1 predictor (PC1 dominated by run800m)

PCA on Scaled Measures

```
> pcahep <- prcomp(heptathlon[,-8], scale.=TRUE)
> plot(pcahep, type="lines")
```



PCA on Scaled Measures

```
> round(pcahep$rotation[,1:3], 2)
           PC1   PC2   PC3
hurdles   0.45 -0.16 -0.05
highjump -0.38  0.25  0.37
shot     -0.36 -0.29 -0.68
run200m   0.41  0.26  0.08
longjump -0.46  0.06 -0.14
javelin  -0.08 -0.84  0.47
run800m   0.37 -0.22 -0.40
```

- What do you notice?

in PC1 positives are time based where if you take longer its worse, and negatives are measured in distance where more distance is better.

reason why PC2 doesn't hold this pattern is because the most amount of information is contained in the first principle component

PCA on Scaled Measures

```
> summary(pcahep)
Importance of components:
                          PC1    PC2     PC3     PC4     PC5     PC6    PC
Standard deviation     2.1119 1.0928 0.72181 0.67614 0.49524 0.27010 0.221
Proportion of Variance 0.6372 0.1706 0.07443 0.06531 0.03504 0.01042 0.007
Cumulative Proportion  0.6372 0.8078 0.88223 0.94754 0.98258 0.99300 1.000
```

- Scree plot suggests probably 2, most criterion would probably look at 2, 3, or 4

PCA on Scaled Measures

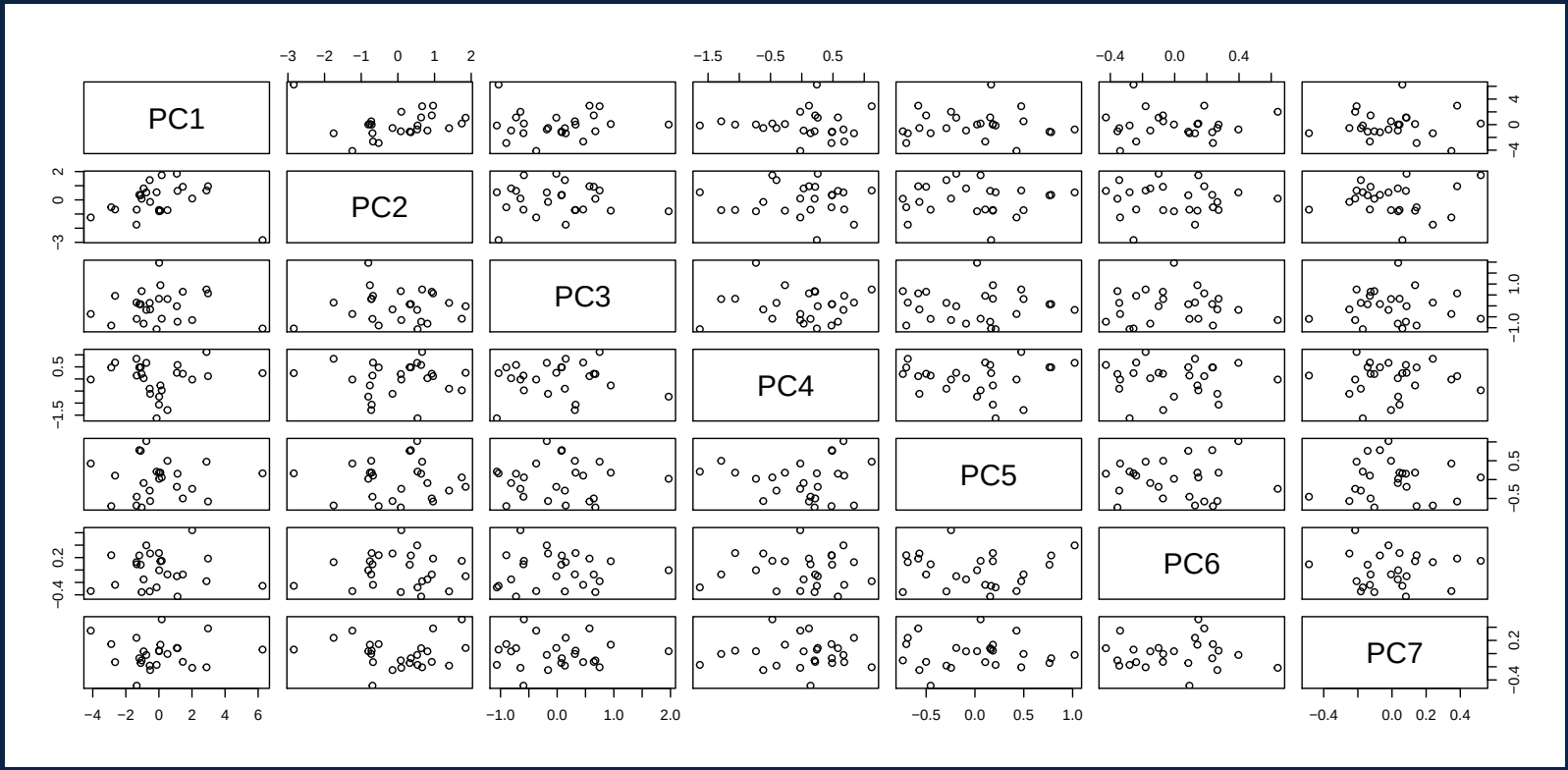
- Also contained in the pca object as "x" are what's commonly referred to as 'scores'. AKA, the transformed observations!

```
> head(pcahep$x)
                          PC1         PC2        PC3         PC4        PC
Joyner-Kersee (USA) -4.121448 -1.24240435 -0.3699131 -0.02300174  0.426006
John (GDR)          -2.882186 -0.52372600 -0.8974147  0.47545176 -0.703065
Behmer (GDR)        -2.649634 -0.67876243  0.4591767  0.67962860  0.105525
Sablovskaite (URS)  -1.343351 -0.69228324 -0.5952704  0.14067052 -0.453928
Choubenkova (URS)   -1.359026 -1.75316563  0.1507013  0.83595001 -0.687194
Schulz (GDR)        -1.043847  0.07940725  0.6745305  0.20557253 -0.737933
```

PCA on Scaled Measures

correlation of 0 does not mean there is no relationship, which is why PC1 and PC2 seem to have some sort of linear relationship

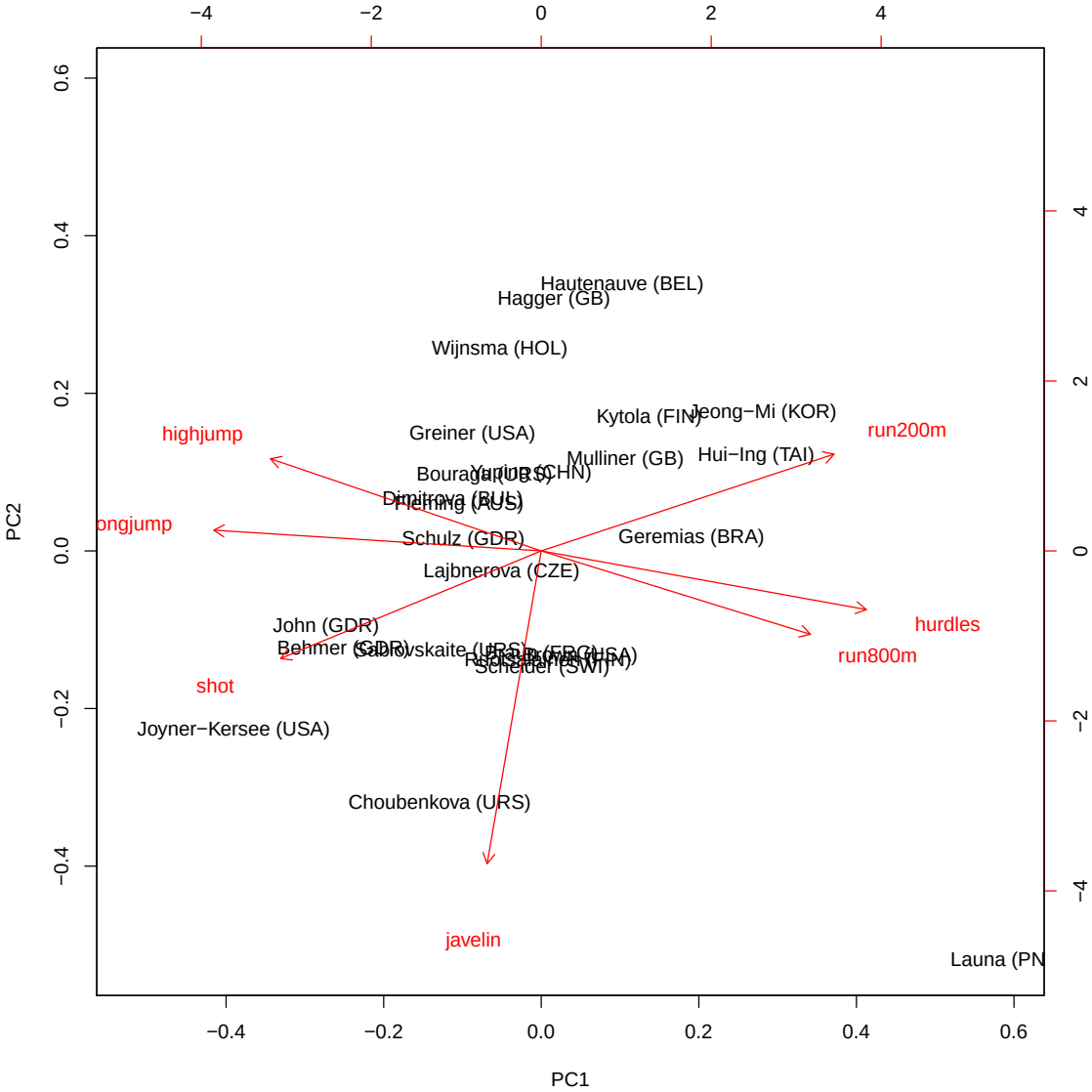
- Which we could visualize by a pairs plot, say



- Or perhaps better in a bivariate form...

PCA on Scaled Measures

```
> biplot(pcahep)
```



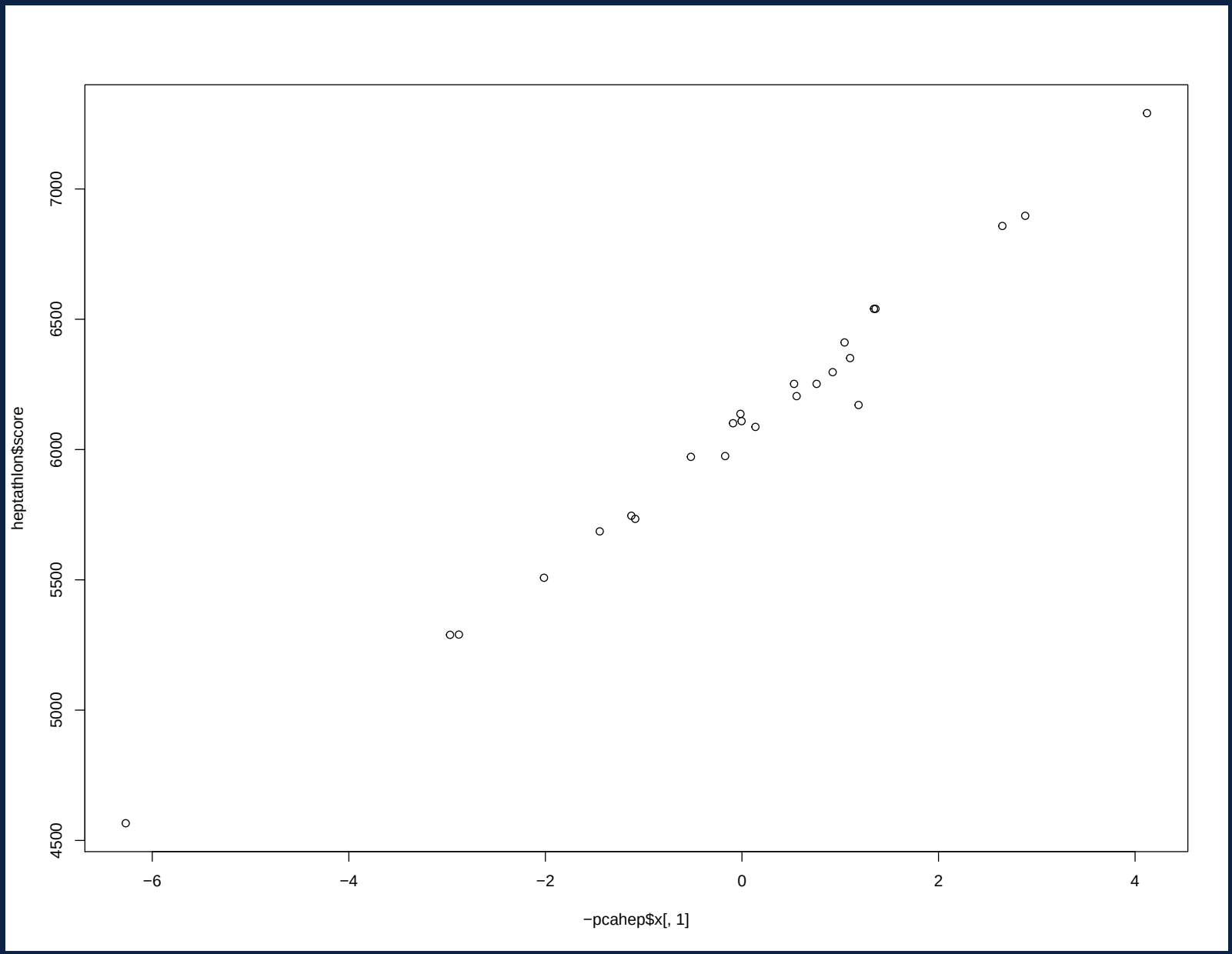
Back to the task...

- How does the first variable function as a scoring system?
- Because of the sign, minimizing would be the goal. For PCA, the sign of the entire component is arbitrary. Opposite signs within a component are meaningful, however.
- Thus, we can multiply an entire component by -1 without changing the underlying mathematics.

PCA on Scaled Measures

```
> round(-pcahep$rotation[,1], 2)
 hurdles highjump     shot  run200m longjump  javelin  run800m
   -0.45     0.38     0.36    -0.41     0.46     0.08    -0.37
> print(cbind(-sort(pcahep$x[,1]), rownames(heptathlon), heptathlon$score))
                     [,1]                  [,2]                    [,3]
Joyner-Kersee (USA) "4.12144762636023"    "Joyner-Kersee (USA)" "7291"
John (GDR)          "2.88218593484013"    "John (GDR)"          "6897"
Behmer (GDR)        "2.649633766599126"   "Behmer (GDR)"        "6858"
Choubenkova (URS)   "1.359025695554282"   "Sablovskaite (URS)"  "6540"
Sablovskaite (URS)  "1.343351209677757"   "Choubenkova (URS)"   "6540"
Dimitrova (BUL)     "1.186453832100095"   "Schulz (GDR)"        "6411"
Fleming (AUS)       "1.100385638571154"   "Fleming (AUS)"       "6351"
Schulz (GDR)        "1.043847470921169"   "Greiner (USA)"       "6297"
Greiner (USA)       "0.923173638862005"   "Lajbnerova (CZE)"    "6252"
Bouraga (URS)       "0.759819023916292"   "Bouraga (URS)"       "6252"
Wijnsma (HOL)       "0.556268302151919"   "Wijnsma (HOL)"       "6205"
Lajbnerova (CZE)    "0.530250688783237"   "Dimitrova (BUL)"     "6171"
Yuping (CHN)        "0.137225439803270"   "Scheider (SWI)"      "6137"
Braun (FRG)         "-0.003774222556983" "Braun (FRG)"          "6109"
```

PCA on Scaled Measures



PCA on Scaled Measures

Scheider (SWI)	"-0.015461226409337"	"Ruotsalainen (FIN)"	"6101"
Ruotsalainen (FIN)	"-0.0907477089383147"	"Yuping (CHN)"	"6087"
Hagger (GB)	"-0.1711286514492 38"	"Hagger (GB)"	"5975"
Brown (USA)	"-0.519252645741 11"	"Brown (USA)"	"5972"
Hautenauve (BEL)	"-1.085697646190 83"	"Mulliner (GB)"	"5746"
Mulliner (GB)	"-1.125481832771 36"	"Hautenauve (BEL)"	"5734"
Kytola (FIN)	"-1.447055499152 66"	"Kytola (FIN)"	"5686"
Geremias (BRA)	"-2.014029620424 39"	"Geremias (BRA)"	"5508"
Hui-Ing (TAI)	"-2.880298635278 55"	"Hui-Ing (TAI)"	"5290"
Jeong-Mi (KOR)	"-2.970118606982 08"	"Jeong-Mi (KOR)"	"5289"
Launa (PNG)	"-6.270021971628 09"	"Launa (PNG)"	"4566"

PCA on Scaled Measures

```
> head(heptathlon)
                    hurdles highjump  shot run200m longjump javelin run800
Joyner-Kersee (USA)   12.69     1.86 15.80   22.56     7.27   45.66  128.5
John (GDR)            12.85     1.80 16.23   23.65     6.71   42.56  126.1
Behmer (GDR)          13.20     1.83 14.20   23.10     6.68   44.54  124.2
Sablovskaite (URS)    13.61     1.80 15.23   23.92     6.25   42.78  132.2
Choubenkova (URS)     13.51     1.74 14.76   23.93     6.32   47.46  127.9
Schulz (GDR)          13.75     1.83 13.50   24.65     6.33   42.82  125.7
> tail(heptathlon)
                  hurdles highjump  shot run200m longjump javelin run800m s
Hautenauve (BEL)    14.04     1.77 11.81   25.61     5.99   35.68  133.90
Kytola (FIN)        14.31     1.77 11.66   25.69     5.75   39.48  133.35
Geremias (BRA)      14.23     1.71 12.95   25.50     5.50   39.64  144.02
Hui-Ing (TAI)       14.85     1.68 10.00   25.23     5.47   39.14  137.30
Jeong-Mi (KOR)      14.53     1.71 10.83   26.61     5.50   39.26  139.17
Launa (PNG)         16.42     1.50 11.78   26.16     4.88   46.38  163.43
```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