DATA 582: Bayesian Inference

Quiz 1 Block 6

INSTRUCTOR: Dr. Irene Vrbik **DURATION:** 80 minutes **DATE:**

This is a **closed**-book test. The only tools permitted are a writing utensil and a non-programmable calculator. Communication with anyone but the instructor is strictly prohibited. In order to receive full marks, neatly and clearly provide all of your answers directly on the quiz booklet. If you run out of space, more paper will be provided.

Beta-Binomial model

Let Y be a Binomial(n, θ) random variable and θ be the unknown Binomial proportion that can take any value between 0 and 1. The Beta-Binomial model has the following form:

$$\begin{aligned} \theta &\sim \text{Beta}(\alpha, \beta) && \textit{Prior} \\ Y \mid \theta &\sim \text{Binomial}(n, \theta) && \textit{Likelihood} \\ \theta \mid Y = y &\sim \text{Beta}(y + \alpha, n - y + \beta) && \textit{Posterior} \end{aligned}$$

Gamma-Poisson model

Let $\theta > 0$ be an unknown rate parameter and $Y = (Y_1, Y_2, \dots, Y_n)$ be an independent identically distributed Poisson(θ) sample. The Gamma-Poisson Bayesian model has the following form:

$$\begin{aligned} \theta &\sim \text{Gamma}(\alpha, \beta) && \textit{Prior} \\ Y_i \mid \theta &\sim \text{Poisson}(\theta) && \textit{Likelihood} \\ \theta \mid Y = y &\sim \text{Gamma}(\alpha + \sum_i y_i, \beta + n). && \textit{Posterior} \end{aligned}$$

Beta Distribution

The beta pdf, for $0 \leq x \leq 1$, and shape parameters $\alpha, \beta > 0$ is given by:

$$p(x \mid \alpha, \beta) = \frac{1}{\mathrm{B}(\alpha, \beta)} x^{\alpha - 1}(1 - x)^{\beta - 1} \tag{1}$$

where $\mathrm{B}(\alpha, \beta)$ is the Beta function $= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$ and $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$. A Beta random variable X has equal mean and variance equal to:

$$\mathrm{E}[X] = \frac{\alpha}{\alpha + \beta} \qquad \mathrm{Var}[X] = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

This exam has 4 pages including (NOT including cover page and formula sheets)

Gamma Distribution

The gamma pdf, for $x \geq 0$, shape parameter $\alpha > 0$ and rate parameter $\beta > 0$, is given by:

$$p(x \mid \alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \tag{2}$$

A Gamma random variable X has equal mean and variance equal to:

$$\mathrm{E}[X] = \frac{\alpha}{\beta} \qquad \mathrm{Var}[X] = \frac{\alpha}{\beta^2}$$

Poisson Distribution

The poisson pmf, for $x = 0, 1, 2, \ldots$, rate parameter $\lambda > 0$, is given by:

$$p(x \mid \lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!} \tag{3}$$

A Poisson random variable X has equal mean and variance equal to:

$$\mathrm{E}[X] = \lambda \qquad \mathrm{Var}[X] = \lambda$$

Binomial Distribution

The binomial pmf, for $x = 0, 1, 2, \ldots, n$, with parameters n and $0 < p < 10$, is given by:

$$p(x \mid n, p) = \binom{n}{x} p^{x} (1-p)^{n-x} \tag{4}$$

A Binomial random variable X has equal mean and variance equal to:

$$\mathrm{E}[X] = np \qquad \mathrm{Var}[X] = np(1-p)$$