CHALMERS
UNIVERSITY OF TECHNOLOGY

FFR105 Stochastic optimization algorithms

Stochastic optimization algorithms 2019 Home problems, set 1

Submitted By:
Dhurai Isaac Prabhahar
951228-5835
dhurai@student.chalmers.se

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Abstract

This home problem is the first part of the stochastic optimization algorithm course. The objective of this assignment is to implement simple classical and stochastic optimization algorithms in MATLAB in order to optimize the given function. Hence the main work was done in MATLAB whereas the analytical methods that are to be performed by hand where done on pen and paper. The main results from MATLAB and analytical methods by hand are documented in this final report in the order of the given questions.

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1 Penalty Method

The objective of this task is to use one of the classical optimization algorithm i.e the penalty method to find the minimum of the given function under constraint. Penalty method is used to solve a constrained optimization problem. In this method the constrained optimization problem is transformed to a unconstrained optimization problem and then it is solved using gradient descent method.

Given function:

$$f(x_1, x_2) = (x_1 - 1)^2 + 2(x_2 - 2)^2 \tag{1}$$

Constraint:

$$g(x_1, x_2) = x_1^2 + x_2^2 - 1 \leq 0 \tag{2}$$

1.1 Function definition

$$f_p(x; \mu) = f(x) + p(x; \mu) \tag{3}$$

where $p(x; \mu)$ is the penalty function and

$$p(x; \mu) = \mu(max(g(x), 0))^2 \tag{4}$$

$$f_p(x; \mu) = (x_1 - 1)^2 + 2(x_2 - 2)^2 + \mu * (max((x_1^2 + x_2^2 - 1), 0))^2 \tag{5}$$

1.2 Gradient

$$f_p(x; \mu) = \begin{cases} (x_1 - 1)^2 + 2(x_2 - 2)^2 + \mu(x_1^2 + x_2^2 - 1)^2, & \text{if } x_1^2 + x_2^2 - 1 \geq 0 \\ (x_1 - 1)^2 + 2(x_2 - 2)^2 & \text{otherwise} \end{cases} \tag{6}$$

When constrain is not fulfilled:

$$f_p(x; \mu) = (x_1 - 1)^2 + 2(x_2 - 2)^2 + \mu(x_1^2 + x_2^2 - 1)^2 \tag{7}$$

$$\nabla f_p(x : \mu) = \left(\frac{\partial f_p}{\partial x_1}, \frac{\partial f_p}{\partial x_2}, \frac{\partial f_p}{\partial \mu} \right)^T \tag{8}$$

$$\frac{\partial f_p}{\partial x_1} = (2x_1 - 2) + 4\mu(x_1^3 + x_1 x_2^2 - x_1) \tag{9}$$

$$\frac{\partial f_p}{\partial x_2} = (4x_2 - 8) + 4\mu(x_2^3 + x_2 x_1^2 - x_2) \tag{10}$$

$$\frac{\partial f_p}{\partial \mu} = max((x_1^2 + x_2^2 - 1)^2, 0) \tag{11}$$

When constrain is fulfilled:

$$f_p(x; \mu) = (x_1 - 1)^2 + 2(x_2 - 2)^2 \tag{12}$$

$$\nabla f_p(x : \mu) = \left(\frac{\partial f_p}{\partial x_1}, \frac{\partial f_p}{\partial x_2}, \frac{\partial f_p}{\partial \mu} \right)^T \tag{13}$$

$$\frac{\partial f_p}{\partial x_1} = (2x_1 - 2) \tag{14}$$

$$\frac{\partial f_p}{\partial x_2} = (4x_2 - 8) \tag{15}$$

$$\frac{\partial f_p}{\partial \mu} = 0 \tag{16}$$

1.3 Unconstrained Minimum

In order to get the starting points, gradients of the function (1) are differentiated with respect to x_1 and x_2 and then set to zero.

Starting point 1:

$$\frac{\partial f}{\partial x_1} = 2x_1 - 2 = 0 \tag{17}$$

$$x_1 = 1 \tag{18}$$

Starting point 2:

$$\frac{\partial f}{\partial x_2} = 4x_2 - 8 = 0 \tag{19}$$

$$x_2 = 2 \tag{20}$$

The stationary points (x_1^*, x_2^*) are (1,2).

1.4 Output Results

For this task the file RunPenaltyMethod.m was executed and the given parameters were chosen for the program.The final results are tabulated as shown,

μ	x_1^*	x_2^*
1	0.434	1.21
10	0.331	0.996
100	0.314	0.955
1000	0.312	0.951

From the above table, we can see that the x_1^* and x_2^* converges near $\mu = 100$ and $\mu = 1000$. Hence the results are reasonable.

2 Constrained Optimization

2.1 Analytical Method:

In this task, the objective is to find the global minimum of the given function using an Analytical method.

The given Function is

$$f(x_1, 2_2) = 4x_1^2 - x_1x_2 + 4x_2^2 - 6x_2 \tag{21}$$

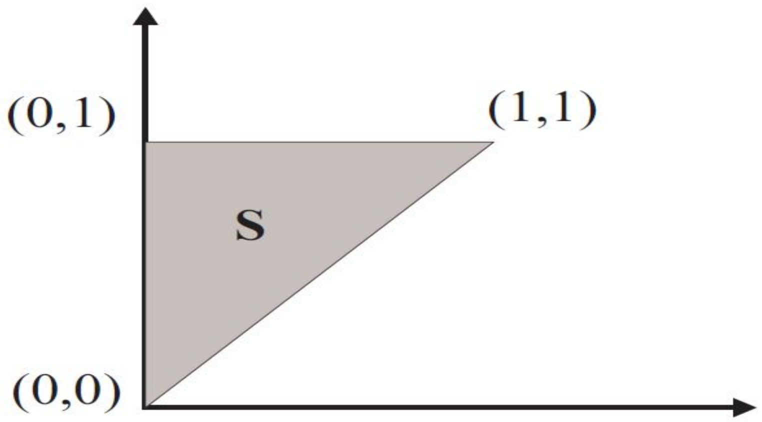

Figure 1: The set S used in this problem

Interior Point:

We first find the stationary points on the inside of the boundary S.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)^T \tag{22}$$

$$\frac{\partial f}{\partial x_1} = 8x_1 - x_2 = 0 \tag{23}$$

$$\frac{\partial f}{\partial x_2} = -x_1 + 8x_2 - 6 = 0 \tag{24}$$

Solving equation (23) and (24), we the stationary points (x^*, x^*)

$$P_1 = \left(\frac{2}{21}, \frac{16}{21}\right)$$

condition:

$$0 < x_1 = \frac{2}{21} < 1$$
$$0 < x_2 = \frac{16}{21} < 1$$

Hence the condition satisfies and point P_1 is in the region S

Points on the Boundary ∂S :

Now we check for the points in the boundary ∂S of S. First we will the check boundary of the line from (0,0) and (0,1). So $x_1 = 0$ and $0 < x_2 < 1$. For this condition we the function

$$f(0, x_2) = 4x_2^2 - 6x_2 \tag{25}$$

Taking derivative of equation (29) and setting it to zero we get.

$$\frac{\partial f}{\partial x_2} = 8x_2 - 6 = 0 \tag{26}$$

Solving the above equation we get $x_2 = \frac{3}{4}$

$$P_2 = (0, \frac{3}{4})$$

Similarly for the next line (0,1) and (1,1). Here $0 < x_1 < 1$ and $x_2 = 1$.

$$f(x_1, 1) = 4x_1^2 - x_1 - 2 \tag{27}$$

$$\frac{\partial f}{\partial x_1} = 8x_1 - 1 = 0 \tag{28}$$

Solving the above equation we get $x_2 = \frac{1}{8}$

$$P_3 = (\frac{1}{8}, 1)$$

For the last line from (1,1) and (0,0) its $x_1 = x_2$. The function is

$$f(x_1, x_2) = 4x_1^2 - x_1^2 + 4x_1^2 - 6x_1$$
$$= 7x_1^2 - 6x_1 \tag{29}$$

$$\frac{\partial f}{\partial x_1} = 14x_1 - 6 = 0 \tag{30}$$

Solving the above equation we get $x_1 = \frac{3}{7}$ and since $x_1 = x_2$ we get

$$P_4 = \left(\frac{3}{7}, \frac{3}{7}\right)$$

Endpoints:
At last we also consider the end points at the corners of the triangle and they are

$$P_5 = (0, 0)$$
$$P_6 = (0, 1)$$
$$P_7 = (1, 1)$$

Function Minimum:
To find the minimum of the seven different points, the corresponding x_1 and x_2 values were substituted into the given function. The values of x_1 and x_2 which gives the lowest value is the global minimum of the given function. The results are tabulated as follows.

Point	x_1, x_2	Function value
P_1	$\left(\frac{2}{16}, \frac{16}{21}\right)$	-2.285
P_2	$\left(0, \frac{3}{4}\right)$	-2.25
P_3	$\left(\frac{1}{8}, 1\right)$	-2.062
P_4	$\left(\frac{3}{7}, \frac{3}{7}\right)$	-1.286
P_5	(0,0)	0
P_6	(0,1)	-2
P_7	(1,1)	1

From the above table we can see that the global minimum is -2.285 which corresponds to the Points $P_1 = \left(\frac{2}{16}, \frac{16}{21}\right)$.

2.2 Lagrange multiplier method:

In this task the objective is to find the minimum $(x_1^*, x_2^*)^T$ of the given function subjected to the constraint using the Lagrange multiplier method.

The given function:

$$f(x_1, x_2) = 15 + 2x_1 + 3x_2 \tag{31}$$

Constrain function:

$$h(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 - 21 = 0 \tag{32}$$

Lagrange multiplier function is defined as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda h(x_1, x_2) \tag{33}$$

$$L(x_1, x_2, \lambda) = 15 + 2x_1 + 3x_2 + \lambda(x_1^2 + x_1x_2 + x_2^2 - 21 = 0) \tag{34}$$

Here λ is the Lagrange multiplier.

Taking the derivative of the lagrange multiplier function with respect to x_1, x_2, λ and then setting it to Zero we get,

$$\frac{\partial L}{\partial x_1} = 2 + 2\lambda x_1 + \lambda x_2 = 0 \tag{35}$$

$$\frac{\partial L}{\partial x_2} = 3 + \lambda x_1 + 2\lambda x_2 = 0 \tag{36}$$

$$\frac{\partial L}{\partial \lambda} = x_1^2 + x_1x_2 + x_2^2 - 21 = 0 \tag{37}$$

Solving equation (35), (36) and (44) for $x_1, x_2 and \lambda$ we get,

$$\lambda = \pm\frac{1}{3}$$

Hence there are two solution. For the first case i.e ($\lambda = \frac{1}{3}$) we get,

$$x_1 = -1$$
$$x_2 = -4$$

Substituting the values of $\lambda, x_1$and x_2 in equation (34) we get,

$$L(x_1, x_2, \lambda) = 1 \tag{38}$$

Now for the second case i.e for ($\lambda = -\frac{1}{3}$) we get,

$$x_1 = 1$$
$$x_2 = 4$$

Again, Substituting the new values of $\lambda, x_1$and x_2 in equation (34) we get,

$$L(x_1, x_2, \lambda) = 29 \tag{39}$$

Hence the minimum of the function is obtained when,

$$\begin{aligned} x_1 &= -1 \\ x_2 &= -4 \\ \lambda &= \frac{1}{3} \end{aligned} \tag{40}$$

The minimum (x_1^*, x_2^*) = (-1,-4) and its corresponding function value is 1.

3 Basic GA Program

3.1 Modified Genetic Algorithm

In this task, the main genetic algorithm program from the introductory programming problem was modified according to the modification requirement given in the Handout. But the main task is to get the global minimum (x_1^*, x_2^*) of the given function,

$$g(x_1, x_2) = (1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2))* \\ (30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)) \quad (41)$$

After running the modified genetic algorithm the global minimum is found to be (x_1^*, x_2^*) = (0.00021,-1.0003) ≈ (0,-1).

3.2 Computing the median

In this task, the objective is to compute the median for four different mutation probability rates. The results are as shown in the table.

Mutation rate	0.0	0.02	0.05	0.10
Median fitness value	0.3189	0.3323	0.3317	0.3289

From the above, we can see that the fitness value increase to 0.02 and then starts decreasing after that. The most highest fitness is achieved when the mutation rate is 0.02 which is the inverse of the number of genes(1/m). This is proved in Appendix B2.5(1)

3.3 Analytical proof

In order to proof that the points found in the task 3.2,$(x_1^*, x_2^*)^T$ is the stationary point of the given function, we need to prove,

$$\nabla g(x_1, x_2) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right)^T = 0 \quad (42)$$

The given function equation is,

$$g(x_1, x_2) = (1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2))* \\ (30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)) \quad (43)$$

In order to compute this long equation, we'll split this long equation and solve it using the product rule. So,

$$u = (1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)) \tag{44}$$

$$v = (30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)) \tag{45}$$

Now differentiating equation (44) and (45) with respest to x_1 and x_2 we get,

$$\begin{aligned}\frac{\partial u}{\partial x_1} = 2(x_1 + x_2 + 1)(3x_1^2 + 3x_2^2 - 14x_1 - 14x_2 + 6x_1x_2 + 19) + \\ 2(3x_1 + 3x_2 - 7)(x_1 + x_2 + 1)^2\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{\partial u}{\partial x_2} = 2(x_1 + x_2 + 1)(3x_1^2 + 3x_2^2 - 14x_1 - 14x_2 + 6x_1x_2 + 19) + \\ 2(3x_1 + 3x_2 - 7)(x_1 + x_2 + 1)^2\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{\partial v}{\partial x_1} = 4(2x_1 - 3x_2)(12x_1^2 + 27x_2^2 - 32x_1 + 48x_2 - 36x_1x_2 + 18) \\ -4(9x_2 - 6x_1 + 8)(2x_1 - 3x_2)^2\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial v}{\partial x_2} = 6(9x_2 - 6x_1 + 8)(2x_1 - 3x_2)^2 - 6(2x_1 - 3x_2) \\ (12x_1^2 + 27x_2^2 - 32x_1 + 48x_2 - 36x_1x_2 + 18)\end{aligned} \tag{49}$$

From the previous task the stationary points (x_1^*, x_2^*) were found to be (0,-1). Now substituting these points in equation (44),((45),(46),(47),(48) and (49) we get,

$$\begin{aligned}u &= 1 + 0 = 1 \\ v &= 30 + 9(-3) = 3\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= 2(0)(3 + 14 + 19) + 2(-3 - 7)(0) = 0 \\ \frac{\partial u}{\partial x_2} &= 2(0)(3 + 14 + 19) + 2(-3 - 7)(0) = 0 \\ \frac{\partial v}{\partial x_1} &= 4(3)(27 - 48 + 18) - 4(-9 + 8)(9) = 0 \\ \frac{\partial v}{\partial x_2} &= 6(-9 + 8)(9) - 6(3)(27 - 48 + 18) = 0\end{aligned}$$

According to the product rule of derivatives,

$$\frac{\partial g}{\partial x_1} = \frac{\partial u}{\partial x_1} v + u \frac{\partial v}{\partial x_1} \tag{50}$$

$$\frac{\partial g}{\partial x_2} = \frac{\partial u}{\partial x_2} v + u \frac{\partial v}{\partial x_2} \tag{51}$$

After substituting the respective values in the above equation we get,

$$\begin{aligned} \frac{\partial g}{\partial x_1} &= 0(3) + 1(0) = 0 \\ \frac{\partial g}{\partial x_1} &= 0(3) + 1(0) = 0 \end{aligned} \tag{52}$$

Hence,∇g(0,-1) = (0,0). Thus proved that (0,-1) are the stationary point.

References

[1] Wahde, M. (Mattias). 2008. *Biologically inspired optimization methods.*