Tutorial on Lie groups and Lie algebra

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March 1, 2016

The purpose of this tutorial is to introduce the concept of Lie groups and Lie algebra in a very basic manner. **Lie groups**

As the word "Lie group" has group in it, Lie group displays the property of a group, i.e., the group has an identity element; every element of the group has an inverse and given two group elements and performing the group operation on the elements, gives an element of the group itself. Mathematically,

For a Lie group U,

1. There exists an element $e \in U$, such that e is the identity element of the group.
2. For any $x \in U$, there exists an element $x^{-1} \in U$, such that $xx^{-1} = e$.
3. For any two elements $x, y \in U$, $xy \in U$.

For being a Lie group, there is one extra property needed for the group operator and the inverse operation, which is, it should be differentiable.

A very simple example of a Lie group is a *Circle Group*. The elements of a circle group are the complex numbers with unit magnitude, i.e., a unit circle in the complex plane. The group operator of this group is the multiplication of the complex numbers.
The identity element of the circle group is $(1, 0)$. Representing every element of the circle in terms of magnitude and angle ($(1)e^{\iota\theta}$, where θ is in degrees), then every element has the inverse represented by $(1)e^{\iota(360-\theta)}$. The group operation here is multiplication, which is a differentiable operator. Hence, the circle group is a Lie group.

Another example is the General Linear group ($GL(n)$). The General Linear group is the group of $n \times n$ invertible matrices and the group operation is multiplication. The identity element of the group is the $n \times n$ identity matrix and all the group elements are invertible and matrix multiplication is also a differentiable operation. Hence, it is a Lie group.

The subgroups of $GL(n)$ such as the group of all orthogonal matrices($O(n)$), group of matrices with determinant $+1$ ($SL(n)$) etc. are also Lie groups, as all the subgroups of a Lie group is also a Lie group.

Lie Algebra

Lie algebra is defined as a vector space on some field F along with an operation $[,] : F \times F \to F$. The operation is called a Lie bracket and it should have the following properties:

1. **Bi-linearity:**

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$and\ [z, ax + by] = a[z, x] + b[z, y],$$
$$where\ a, b \in F\ and\ x, y, z\ in\ the\ vector\ space.$$

2. $$[x, x] = 0\ For\ all\ x\ in\ the\ vector\ space.$$

3. **Jacobi identity:**

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0\ where\ x, y, z\ in\ the\ vector\ space.$$

Lie algebra was introduced to study *infinitesimal transformation*. Lie Algebra is closely related to Lie groups. Every finite dimensional Lie algebra can be associated with a Lie group. The association can be described using a map from either Lie group to Lie algebra or from the Lie algebra to Lie group. The exponential map is defined as a map from Lie algebra to Lie group. The exponential map is said to be "de-linearizing" the Lie algebra. For understanding this, let's take an example.
The example of the vector field R with addition and multiplication operations

and the Lie bracket defined as

$$[x, y] = xy - yx = 0,\ x, y \in R$$

As multiplication is commutative, the Lie bracket in this case is always 0.

The exponential map can be defined as,

$$x \in R \Rightarrow z = Exp(x) = e^{\iota\theta} = cos\theta + \iota sin\theta$$

As $|e^{\iota\theta}| = 1$, the range of the exponential map is the unit circle in the complex plane, which is the circle group.

In the above example, the addition operation in the Lie algebra leads to the multiplication operation of the Lie group.

$$\theta_1 + \theta_2 \Rightarrow e^{\iota(\theta_1+\theta_2)} = e^{\iota\theta_1} e^{\iota\theta_2}$$

Here, addition operation of the Lie algebra is a linear operation, whereas taking the exponential map leads to the multiplication operation of the Lie group, which is not a linear operation. Hence, in some sense, taking the exponential map "de-linearizes" the Lie algebra.

Another example of a Lie algebra is *gl(n)*, which on taking the exponential map leads to the General Linear group. Elements of *gl(n)* are all $n \times n$ matrices and the Lie bracket is defined as,

$$[A, B] = AB - BA$$

The exponential maps for the Lie group of $GL(n)$ and all its subgroups is the normal matrix exponential, which is defined as follows:

$$e^A = \sum_{k \geq 0} \frac{A^k}{k!} = I_n + \sum_{k \geq 1} \frac{A^k}{k!}$$

Some more examples of Lie groups, Lie algebra and the exponential maps between the two are as follows: All the subgroups of $GL(n)$ are also smooth manifolds. First, let us understand what a smooth manifold is. An n-dimensional manifold is a set of mappings from subsets of an open set (where neighborhood of every point in every direction is not empty) to R^n, where the neighborhood of every point locally acts as a vector space. Note that, a manifold is not a vector space. Different subsets of the open set can be mapped through different functions. These different functions are known as surface patches. Suppose, there are two surface patches of a manifold $\psi_1 : U \rightarrow R^n$ *and* $\psi_2 : V \rightarrow R^n$, such that $U \cap V \neq \phi$. Then, there can be a mapping for the elements in the intersection, such that it is $f = \phi_2^{-1}\phi_1 : \ R^n \rightarrow R^n$. This mapping f is called a Transition map. If, the transition map is smooth, which means partial derivatives of the transition map of all orders exists and all the first order partial derivatives are linearly independent of each other (*which means that all the directions in which change is happening are independent of each other*), then it is called a smooth manifold.

As the Lie groups which are smooth manifolds, the properties of both Lie groups and manifolds can be combined together to get a better understanding along with applications of the subject.For example, smooth manifolds have the properties of curvature, geodesics etc, which can be studied easily using the property of the Lie groups. Taking the Lie group as the manifold, Lie algebra is the tangent plane at identity and exponential map is the map from tangent plane to points on the manifold.