Path Finding for N-Link Robotic Arm

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***Abstract*—The report discusses about modelling of an n-link robot using transformation matrices; finding the forward and inverse kinematics equations and solutions along with the best possible path for going to the given end effector point. The approach uses concepts from differential geometry and Lie theory to model the n-link robot and to calculate the best possible path which minimizes the total rotation needed to achieve the final configuration.**

Include at least 5 keywords or phrases.

I. INTRODUCTION

Nowadays, robots are very useful in industry as they are capable of performing many different kinds of tasks with much more precision then a human can and that too without any safety requirements along with comfort elements. But, to make a robot work automatically and even to make a manual robot requires a lot of effort and resources. The intelligent robots or the robots which are able to function without manual control needs to be programmed for the specific tasks. Robotic arm is one of many types of robotic elements, required to perform task similar to that of a human hand. This report describes different ways to find the path from initial to end position.

A robotic arm is a mechanical arm, usually programmable giving similar functions as that of a human arm, which can be the system as whole or part of a more complex robotic system. The links of such a manipulator are connected by joints which allows the arm either rotational motion or a translational displacement. The links of such a manipulator can be considered to form a kinematic chain and the end of the last chain is called the end effector. A robotic arm with n links is called as n-link robotic arm and has $(n-1)$ joints.

Modelling the configuration of n-link robotic arm in terms of matrices requires the transformation of the robotic arm to be modelled first. As above, every link of the n-link robotic arm may have translational, rotational motion and also reflections. The combination of translational and rotational motions are known as rigid body motions. Rotation of a vector in 3-dimensional Euclidean space can be represented by a matrix known as rotation matrix. All the rotation matrices form a group. Rotation matrices are orthogonal matrices with determinant $+1$. The group of the 3×3 orthogonal matrices with determinant $+1$ are called $SO(3)$, *Special Orthogonal Group* of order 3. Every rigid motion can be represented as a linear transformation of homogeneous coordinates of any point in $\mathbb{R}^3$. For example, a rigid body motion consisting of a rotation matrix R and translation vector T can be represented by a 4×4 matrix:

$$\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

The group of the matrices of rigid body motion are called $SE(3)$, *Special Euclidean Group*.

In the project, we have first defined a model to represent the n-link robotic arm. Then found the forward and inverse kinematics of a 2-link robot. The forward kinematics of a robot is, finding the location of end-effector of the arm given the quantity of rotation of every link. Inverse kinematics of a robotic arm is finding the rotations of each link given the position of end effector. The other objective of the project was to find the path for the robotic arm to travel. The problem statement is given the location of the end effector find the path to travel from the initial location to the end effector. Two approaches to this problem is discussed and applied in this project. The first approach is to first finding the inverse kinematics solution for the given end effector and then finding the shortest path to travel from the initial position to the end-effector. The second approach is to directly find the path using gradient descent.

In this report, first modelling of a 2-link robot is given, followed by the forward kinematics of the 2-link robot. For calculation of the best possible path for a robot for going from one point to another requires basics of differential geometry and Lie theory. Thus, the next section explains these basics. Then, the application of these concepts in finding the inverse kinematics of a 2-link robotic arm and then the best possible path for a n-link robot is given in the section after that.

II. MODELLING OF A 2-LINK ROBOTIC ARM

Assume the lengths of the links of the robotic arm are l_1 and l_2 respectively. Every link of the the robot will have three degrees of freedom, in which it can rotate. Suppose the initial configuration of the robotic arm is as given in the figure below:

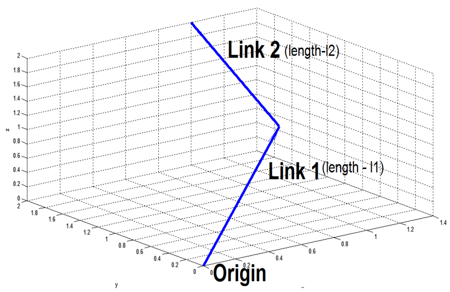

Fig. 1. Initial configuration for a 2-link robotic arm

Here, both the links can rotate individually. But, if the first link rotates, the second link will also rotate because it is joined to the first link. As, the rotation of the first link translates the starting position of the second link and rotates the second link as well, the second transformation is a combination of translation and rotation, which can be represented by an $SE(3)$ matrix. But, as the translation part of the second link is completely dependent on the rotation matrix of the first link, the whole configuration can be modelled using cross product of $SO(3)$ with itself, i.e. $SO(3) \times SO(3)$. Why this model cannot be further reduced to just $SE(3)$ is explained in a later section.

Now, suppose $(R_1, R_2) \in SO(3) \times SO(3)$ represents the initial position of the 2-link robotic arm. R_1 *and* R_2 represents the rotation matrices which rotate link-1 and link-2 from $(l_1, 0, 0)$ and $(l_2, 0, 0)$ to the initial positions of the links respectively.

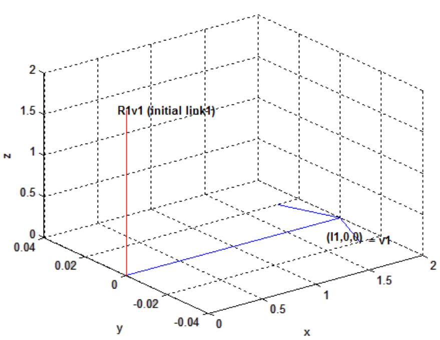

Fig. 2. Obtaining Initial configuration for link-1 of 2-link robotic arm

III. Forward Kinematics of a 2-link robotic arm

Now, let us see the effect of rotations of link-1 and link-2 to the final positions of both the links and the arm. Suppose $(R_1, R_2) \in SO(3) \times SO(3)$ represents the initial position of the 2-link robotic arm.

A. Effect of rotating Link-2

If link-2 is rotated by the rotation matrix $R_{\theta 2}$, then link-1 will remain as it is and thus, the starting position of link-2 will also remain the same. And thus,

$$The\ rotation\ matrix\ for\ Link-1: \ R_1$$

$$The\ rotation\ matrix\ for\ Link-2: \ R_{\theta 2} \cdot R_2$$

$$The\ translation\ for\ Link-2: \ R_1 \cdot v_1$$

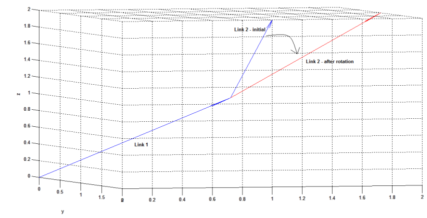

Fig. 3. Effect of rotating link 2

B. Effect of rotating Link-1

If link-1 is rotated, then the initial translation of link-2 will also change. Suppose that the rotation matrix for link-1 is $R_{\theta 1}$, then

$$The\ rotation\ matrix\ for\ Link-1: \ R_{\theta 1} \cdot R_1$$

$$The\ translation\ for\ Link-2: \ R_1 \theta 1 \cdot R_1 \cdot v_1$$

As, both the links are joined, link-2 will also rotate according to the rotation matrix $R_{\theta 1}$. Thus,

$$The\ rotation\ matrix\ for\ Link-2: \ R_{\theta 1} \cdot R_2$$

Studying the effects of rotating one link to all the other links of the robotic arm is known as forward kinematics.

As seen above, the rotation of link 1 affects the position and rotation of link 2. Thus, for an n-link robot rotation of link i will affect the position and rotation of link $i + 1$.

To understand the path problem, it is needed to understand basics of Lie theory and differential geometry. The following section will explain those concepts.

IV. Lie groups, Lie algebra and differential geometry

A. Lie groups

Lie groups are objects that lie in the intersection of group theory and differential geometry; Lie groups are groups which are continuous and differentiable. Lie group displays the property of a group, i.e., the group has an identity element; every element of the group has an inverse and given two group elements and performing the group operation on the elements, gives an element of the group itself. Mathematically,

Other properties of a Lie group are:

1) Group operation $(x, y) \Rightarrow xy \ and \ x \Rightarrow x^{-1}$, are differentiable.
2) The group elements should represent a differentiable manifold.

Example 1: Circle group

The elements of a circle group are the complex numbers with unit magnitude, i.e., a unit circle in the complex plane. The group operator of this group is the multiplication of the complex numbers.
Mathematically,

$$C = \{x \in C / |x| = 1\}$$

The identity element of the circle group is 1. Representing every element of the circle in terms of magnitude and angle ($(1)e^{i\theta}$, where θ is in degrees), then every element has the inverse represented by $(1)e^{i(360-\theta)}$. The group operation here is multiplication, which is differentiable. A circle is also a 1-dimensional manifold and can parametrized using the angle θ. Hence, the circle group is a Lie group.

Example 2: General Linear Group ($GL(n, R)$)

The General Linear group is the group of $n \times n$ invertible matrices and the group operation is multiplication. The identity element of the group is the $n \times n$ identity matrix and all the group elements are invertible and matrix multiplication is also differentiable and $GL(n)$ is an n^2-dimensional manifold. Hence, it is a Lie group.

The subgroups of $GL(n)$ such as the group of all orthogonal matrices($O(n)$), group of matrices with determinant +1 ($SL(n)$) etc. are also Lie groups.

B. Differential Geometry: Manifolds and some basics

A manifold is a topological space, which is locally Euclidean, which means that around every point on the manifold, there is a neighborhood which is topologically the same as a unit ball in R^n. For example, the surface of Earth at every point looks flat (which resembles a plane, which is a two-dimensional Euclidean space.), but if you keep walking in one direction on the surface, you will eventually end up at the same point, which indicates that the global structure of Earth may resemble to that of a sphere.

Manifold(Definition): *Manifold is a topological space such that every point has a non empty neighborhood which is homeomorphic to an open set in* $\mathbb{R}^n$.
A function is said to be homeomorphic when it is bijective, continuous and its inverse is also a continuous function. For example, graph of $y = x^{2/3}$ $(cusp)$, given in Fig. 4 is a 1-dimensional topological manifold. It is locally Euclidean, because it is homeomorphic to R via the function $f : (x, x^{2/3}) \mapsto x$. The sphere is a 2-dimensional manifold embedded in $\mathbb{R}^3$.

Fig. 4. Cusp as a 1-dimensional manifold

A homeomorphic map, which maps open sets to subsets of $\mathbb{R}^n$ of manifold is called a *coordinate chart* and the coordinates of the points for a particular map are called *local coordinates*. There can be more than one homeomorphic maps required to cover all the points of the manifold. In the set of coordinate charts, there can be some functions/surface maps whose domains are not disjoint. For this kind of coordinate charts, the intersection of the domains can be transformed from one coordinate chart representation to that of another. Let's take an example.

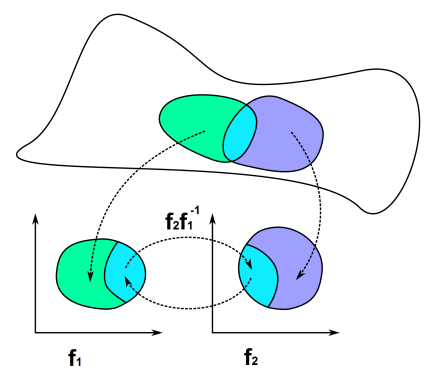

Fig. 5. Transition map

In the figure above[1], $f_1 \ and \ f_2$ are two coordinate charts and there is a subset of the ranges which is common between them. A map $f_2 \bullet f_1^{-1}$ can be defined which will change the coordinates from the first coordinate chart to that of the second coordinate chart. This map is called the transition map.
Regular Manifold (Definition): *If the transition map is smooth,which means partial derivatives of the transition map of all orders exists and all the first order partial derivatives*

[1]https://upload.wikimedia.org/wikipedia/commons/thumb/0/06/Two_coordinate_charts_on_a_manifold.svg/2000px-Two_coordinate_charts_on_a_manifold.svg.png

are linearly independent of each other (which means that all the directions in which change is happening are independent of each other), then the manifold is said to be a regular surface/manifold.[2]

Now, we will define the concepts from differential geometry which will be useful to work with Lie groups.

Tangent Space(Definition): *Every point of the manifold lies on many curves that are on the manifold and for each of these curves, there will be a tangent for that point of the manifold. The set of all the tangents at a point is called the tangent space and has algebraic properties of a vector space.*

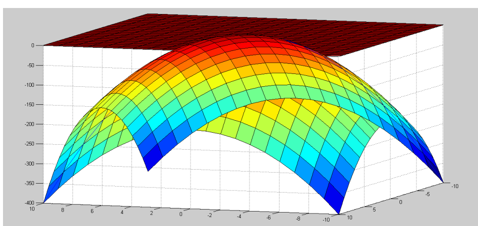

Fig. 6. Tangent plane at the origin for the paraboloid $z = -2x^2 - 2y^2$

Geodesics:
Geodesics are natural generalizations of straight lines on curved manifolds. Straight lines have two characteristics:

1) A straight line is the shortest path between two points on the line.
2) The straight line has no curvature at any point.

Thus, a geodesic is a curve with zero acceleration or whose curvature with respect to the manifold is zero at any point. i.e., for a curve γ_v to be a geodesic, $\bigtriangledown_{\gamma_v'} \gamma_v' = 0$. For example, great circles are geodesics of a sphere as they have zero curvature with respect to the surface of the sphere.

[2]For a regular surface, the need of the linear independence of the first order partial derivatives arise due to the fact that if the change in one direction is completely dependent on the change in other direction, then the degrees of freedom will decrease by one and will result in $(n-1)$-dimensional manifold instead of n-dimensional manifold as claimed.

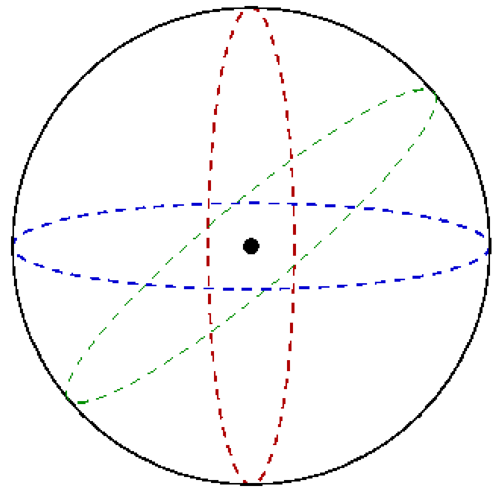

Fig. 7. Great circle - geodesic of a sphere

Exponential and log map:
The range of the exponential map at any point say P, is the set of points obtained from travelling from point P on every specific curve for unit time.
Exponential map(Definition): $exp_x : U \subset T_xX \Rightarrow X$ *is a homeomorphism, where* U *is a subset of Tangent space of the manifold,* $exp_p(v) = \gamma_v(1)$*, where,* γ_v *is the curve which is solution to the differential equation* $\bigtriangledown_{\gamma_v'} \gamma_v' = 0$ *with initial conditions* $\gamma_v'(0) = v$ *and* $\gamma_v = x$. The concept of exponential map will be used later.

Exponential map defines the map from a subset of tangent space to that of points on the manifold, whereas the log map is the inverse function of the exponential map.

If the smooth manifold is a Lie group, then the tangent space of that manifold also has an algebraic structure, which is studied using the concept of Lie algebra.

C. Lie Algebra

Lie algebra is defined as a vector space V on some field F along with an operation $[.,.] : V \times V \rightarrow F$. The operation is called a *Lie bracket* and it has the following properties:

1) **Bi-linearity:**

$$[ax+by, z] = a[x, z]+b[y, z]\ ,\ [z, ax+by] = a[z, x]+b[z, y],$$

$$\forall a, b \in F \ \ \forall x, y, z \in V$$

2)

$$[x, x] = 0 \ \ \forall x \in V$$

3) **Jacobi identity:**

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \ \ \forall x, y, z \in V$$

As Lie groups are differentiable manifolds, Lie algebra can be expressed as the tangent plane at identity element and the exponential map as the map from that tangent plant to the points on the manifold or the group elements. Since, we are interested in the modelling of n-link robots in $\mathbb{R}^3$,which requires the rigid body rotations and translations, we will focus on Lie group of $SO(3)$.

Examples of Lie Algebra
Example 1: Circle group and its Lie algebra

The tangent space at a point on the circle is just the tangent line for that point on the circle. Following is the image[3] of tangent line at point p of the circle.

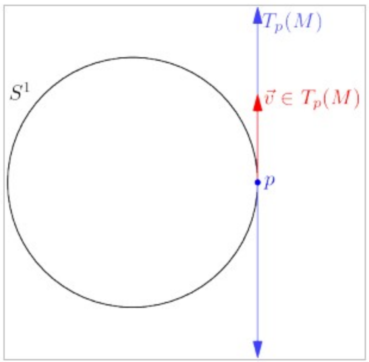

Fig. 8. Tangent line at point P

Now, all the points can be represented as line of real numbers rotated by some angle. Thus, the tangent space at identity for a circle can be represented by the set of real numbers. Hence, the vector space R is the Lie algebra of the circle group.

The vector space R with addition and multiplication operations and the Lie bracket defined as

$$[x, y] = xy - yx = 0, \ x, y \in R$$

As multiplication is commutative, the Lie bracket in this case is always 0.

The exponential map can be defined as,

$$x \in R \Rightarrow z = Exp(x) = e^{\iota\theta} = cos\theta + \iota sin\theta$$

As $\left|e^{\iota\theta}\right| = 1$, the range of the exponential map is the unit circle in the complex plane, which is the circle group.

In the above example, the addition operation in the Lie algebra leads to the multiplication operation of the Lie group.

$$\theta_1 + \theta_2 \Rightarrow e^{\iota(\theta_1+\theta_2)} = e^{\iota\theta_1}e^{\iota\theta_2}$$

Here, addition operation of the Lie algebra is a linear operation, whereas taking the exponential map leads to the multiplication operation of the Lie group, which is not a linear operation. Hence, in some sense, taking the exponential map "de-linearizes" the Lie algebra.

[3]https://usamo.wordpress.com/tag/tangent-space/

Example 2 : Lie Algebra for SO(3)

$SO(3)$ is the group of all 3×3 orthogonal matrices with determinant $+1$.

Suppose a curve $\gamma(t) \in SO(3), \ where \forall t \in (-\epsilon, \epsilon)$, passing through a point $R \in SO(3)$ such that $\gamma(0) = R$. The tangent vector of the curve $\gamma(t)$ at the point R will be $\dot{\gamma}(0)$.

As the curve is in $SO(3)$, for all points of the curve,

$$\gamma(t)^T\gamma(t) = Id \quad \gamma(t)^T\gamma(t) = Id$$

Now, differentiating on both the sides,

$$\begin{array}{ll}
\dot{\gamma}(t)^T\gamma(t) + \gamma(t)^T\dot{\gamma}(t) = 0 & \dot{\gamma}(t)\gamma(t)^T + \gamma(t)\dot{\gamma}(t)^T = 0 \\
\Rightarrow \dot{\gamma}(t)^T\gamma(t) = -\gamma(t)^T\dot{\gamma}(t) & \dot{\gamma}(t)\gamma(t)^T = -\gamma(t)\dot{\gamma}(t)^T \\
\Rightarrow (\gamma(t)^T\dot{\gamma}(t))^T = -\gamma(t)^T\dot{\gamma}(t) & \gamma(t)\dot{\gamma}(t)^{T^T} = -\gamma(t)\dot{\gamma}(t)^T \\
Now, \ Suppose & \\
A_1 = \gamma(t)^T\dot{\gamma}(t) & A_2 = \gamma(t)\dot{\gamma}(t)^T \\
A_1^T = -A_1 & A_2^T = -A_2
\end{array}$$

Hence, A_1 and A_2 both are skew symmetric matrices.

$$\begin{array}{ll}
A_1 & = \gamma(t)^T\dot{\gamma}(t) \\
\Rightarrow \dot{\gamma}(t) & = \gamma(t)A_1 \\
\Rightarrow \dot{\gamma}(0) & = \gamma(0)A_1 \\
\Rightarrow \dot{\gamma}(0) & = RA_1 \\
\Rightarrow R^{-1}\dot{\gamma}(0) & = A_1
\end{array}$$

$R^{-1}\dot{\gamma}(0)$ belongs to the tangent space at identity and hence to the Lie algebra. As the matrix A_1 is a skew symmetric matrix, the Lie algebra for the $SO(3)$ will be the group of skew symmetric matrices, which is $so(3)$.

The formula given below for this particular exponential map is knows as the *Rodrigue's Formula*, which is the formulae expressing a rotation of axis specified by the vector (a,b,c) and angle θ. And has applications in robotics, kinematics and motion interpolation.

For a 3×3 skew symmetric matrix

$$A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

The exponential map : *so*$(3) \rightarrow$ SO(3) is given by,

$$e^A = cos\theta I_3 + \frac{sin\theta}{\theta}A + \frac{1 - cos\theta}{\theta^2}B$$

V. Inverse kinematics of 2-link Robotic Arm

The problem of inverse kinematics is finding the positions of the links, given the required end effector position. For example, in a 2-link robot, suppose the end effector position needed is $(2, 1, 2)$ and the actual positions of link 1 is $(1, 1, 1,)$ and that of link 2 is $(1, 2, 2)$. Following is the figure of the given situation. Let the lengths of Link-1 and Link-2 be l_1 and l_2 respectively.

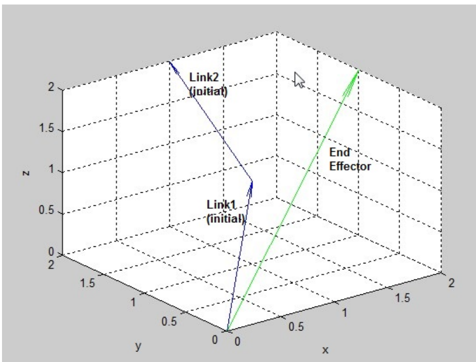

Fig. 9. Inverse Kinematics Problem

One solution can be produced by the following steps:

1) Find the end positions of the first link to get to the end effector position as the end position of link-2 is specified by the end effector position.
2) If more than one solution exists, find the configuration of the links which minimizes the total rotation.
3) Find the geodesic path (minimum distance path) for attaining that configuration given the initial configurations.

Following are the explanations for each step for a 2-link robotic arm:

Step 1: Find the end positions for both links

Here, the end positions of 2 links will make a triangle with the origin as shown in the following figure.

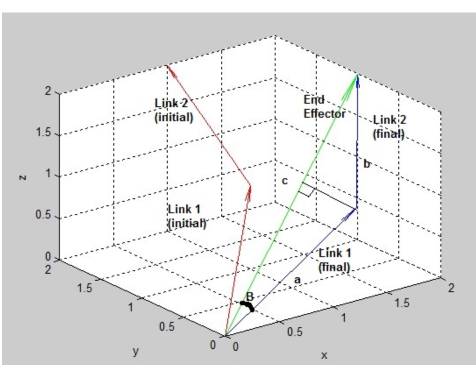

Fig. 10. Triangle formed by the final positions of the links

Here, there are infinite number of triangles possible in $\mathbb{R}^3$ for the given sides. The planes in which the triangle can lie should pass through the end effector line. Now, taking the end position of link-1 for all possible triangles, will make a circle as given in the following figure. Let a, b, c be the lengths of link-1, Link-2 and the end effector respectively.

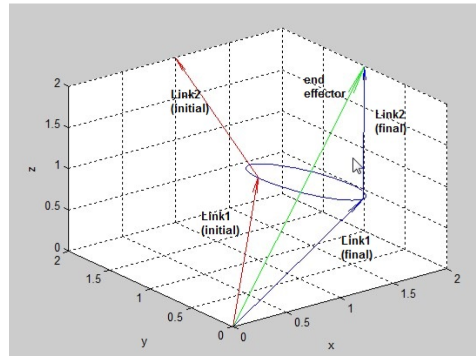

Fig. 11. Circle formed by the solutions of the triangle

Here, the radius of the circle will be, $r = ainB$, the center of the circle is the projection point of final position of link-1 on the end-effector and B is the angle formed between the end effector line and link-1 position vector. The equation of the given circle is as follows:

$$P(t) = rcos(t)\vec{u} + rsin(t)\vec{n} \times \vec{u} + \vec{c}$$

where, $\vec{n}$ is the normal vector to the plane in which the circle lies; $\vec{u}$ is any vector from the center of the circle to a point on the cicle and $\vec{c}$ is the vector to the center of the circle.

The value of angle B can be found using the cosine formulae as all the three sides of the triangle are known.

$$cosB = \frac{a^2 + c^2 - b^2}{2ac}$$

Step 2 & 3: Find the configuration which requires minimum rotation and find the geodesic

Now, for every point on the circle, corresponding values of the end positions of the two links can be found, which are the point on the circle and the end effector position for first and second links respectively. From the positions of both the links, it is easy to find the $SO(3)$ elements of both the links.

For, each point on the circle there will be 2 corresponding rotation matrices, which are independent of each other. Now, suppose the rotation matrices describing the initial position of the robotic arm are $R_{12} = (R_1, R_2)$ and the pair of rotation matrices corresponding to any point on the circle (one of final position of robotic arm) are $R_{12f} = (R_{1f}(t), R_{2f}(t))$. Now, the task is to find the point on the circle such that the distance between (R_1, R_2) and $(R_{1f}(t), R_{2f}(t))$ is minimum for the manifold $SO(3) \times SO(3)$. The approach taken for this task was by taking sample points on the circle and finding the geodesic

distance between the rotation matrices of sampled points to that of the rotation matrices for initial configuration of the robotic arm. The geodesic distance between two points on the manifold is the same as the value of $log_{R_{12}} R_{12f}$ (value of tangent vector of R_{12f} from R_{12}).
Following is the results of the given process done on the given example.

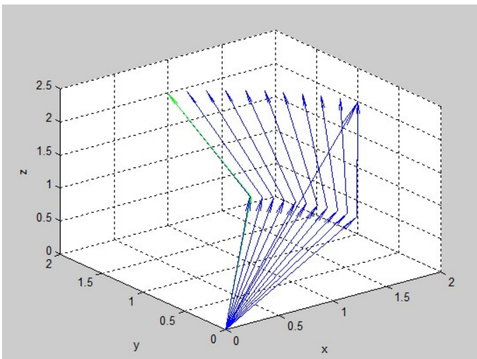

Fig. 12. Path from initial to final position

VI. General Approach for Inverse Kinematics and path finding for n-link robotic arm

As we have seen that for 2-link robot the starting position of the second link is completely dependent on the rotation matrix of the first link, the same is true for the case of n-link robot, i.e. the starting position of i^{th} link is dependent on the rotation matrices from 1^{st} to $(i-1)^{th}$ link. Hence, an n-link robotic arm can be modelled using direct product of n groups of $SO(3)$.

Here, for inverse kinematics of a n-link robotic arm the approach of gradient descent is used. Gradient descent is a first order local optimization algorithm which uses gradient of a function in order to find a local minimum. Gradient of a function gives the direction of maximum increase for any function, and hence the exact opposite direction from the direction given by gradient will be the direction of maximum decrease. The direction of maximum increase for a function $f(x)$ is given by the gradient of the function, $\bigtriangledown_x f(x)$ and thus, the direction of maximum decrease is given by $-\bigtriangledown_x f(x)$. Hence, the point in the direction of maximum decrease near the given point x_k is $x_{k+1} = x_k - \bigtriangledown_x f(x) \Delta t$.

In the given problem, if the final position of the end-effector is $\vec{pos}$; the rotation matrices for each link in the final configuration are R_{if} for i^{th}-link and $\vec{l_i} = (l_i, 0, 0)$, where l_i is the length of link i, then

$$\vec{pos} = \sum_{i=1}^{n} R_{if} \vec{l_i}$$

The different points on the path of the n-link robot can be written as a function of rotation matrices of all the links as follows:

$$\vec{f}(R_1, R_2, \ldots, R_n) = \sum_{i=1}^{n} R_i \vec{l_i}$$

where R_i is the i^{th} rotation matrix.

To compute the path we try to minimize the distance between current and desired end effector. This distance is a function of rotation matrices $R_1, R_2, \ldots, R_n$.

So, the cost function here is,

$$c(R_1, R_2, \ldots, R_n)$$

$$=< \vec{f}(R_1, R_2, \ldots, R_n) - \vec{pos}, \vec{f}(R_1, R_2, \ldots, R_n) - \vec{pos} >$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} l_i^T R - i^T R_j l_j + \sum_{i=1}^{n} l_i^T l_i - 2(\sum_{i=1}^{n} l_i^T R_i^T) pos$$

Now, for finding the gradient descent, we need to find the partial differentiation of the cost function c_t with respect to R_i.

$$\frac{\partial c(R_1, R_2, \ldots, R_n)}{\partial R_x} = 2 \sum_{i=1_{i \neq x}}^{n} (R_i l_i) l_x^T + 2 pos l_x^T \quad (1)$$

As the partial differentiation is in the tangent space of the Rotation matrix R_x, first we need to find the corresponding matrix in the tangent space of identity. This can be done by first left multiplying the partial differentiation by the inverse of the rotation matrix R_x and then taking the projection on the tangent space at identity.

$$\begin{aligned} A_x &= R_x^{-1} \frac{\partial c_t}{\partial R_x} \\ A_{xp} &= \frac{A_x - A_x^T}{2} \end{aligned} \quad (2)$$

Now, for the path, we need to find the rotation matrix corresponding to the skew-symmetric matrix A_{xp}, which is in the tangent plane of identity. This can be done using the exponential map. But as we need to find the matrix corresponding to a little change in the opposite direction given by A_{xp}, we will find the rotation matrix corresponding to $\Delta t(-A_{xp})$ and as the new rotation matrix corresponding to the new position of the link it has to be tranformed from identity to the given rotation matrix.

$$R_x^{new} = R_x^{old} exp((-A_{xp} \Delta t)) \quad (3)$$

The new rotation matrix for each link will be found till c_t reaches some minimum distance (it is very near to the final position of the end effector.)

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