Topology

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1 What have I learnt?

Topology is one of the topics which have piqued my interest. I am always inclined towards learning new mathematical topics,especially wanted to study one of the topics of abstract algebra. I am currently using *"Topology without tears"* as a base reference for studying Topology.
I have completed 3 chapters from that book,namely

1. *topological spaces*
2. *The Euclidean Topology*
3. *Limit points*

and I am enjoying learning topology, of course there are portions where i get stuck, but then I reattempt it to improve my understanding. I will sumarise my learning below with some examples.

2 Summary

2.1 What is a Topology?

2.1.1 Definition: Topology

Let X be a non-empty set. A set T of subsets of X is said to be a **topology on *X*** if

(i) X and the empty set , ϕ , belong to T

(ii) the union of any finite or infinite number of sets in T belongs to T, and

(iii) the intersection of any two sets in T belongs to T.

The pair *(X,T)* is called a topological space.

Example: Let $X = \{a, b, c, d, e, f\}$ and
$T_1 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$.

As T_1 satisfies all conditions of Definition 2.1.1 , it is a topology on X

Example: Let $X = \{a, b, c, d, e\}$ and
$T_2 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, e\}, \{b, c, d\}\}$.

Then T_2 is not a topology on X as the union

$$\{c, d\} \cup \{a, c, e\} = \{a, c, d, e\}$$

doesn't belong to T_2

2.1.2 Definition: discrete space

Let X be any non empty set and let T be the collection of all subsets of X. Then T is called the **discrete topology** on the set X. The topological space (X, T) is called a **discrete space**.

2.1.3 Definition: indiscrete topology

Let X be any non empty set and $T = \{X, \phi\}$. Then T is called **indiscrete topology** and (X, T) is said to be an **indiscrete space**.

Propositon: If (X, T) is a topological space such that, for every $x \in X$,the singleton set $\{x\}$ is in T , then T is the discrete topology.

Proof:
every set is the union of all its singleton subsets; that is , if S be any subset of X, then

$$S = \bigcup_{x \in S} \{x\}$$

Since we are given that each $\{x\}$ is in T , and the above equation imply that $S \in T$. As S is an arbitrary subset of X,we have that T is the discrete topology.

2.1.4 Definition: Open sets

Let (X, T) be any Topological space. Then the members of T are said to be **open sets**.

2.1.5 Proposition:

If (X, T) is any topological space, then

(i) X and ϕ are open sets,

(ii) the union of any (finite or infinite) number of open sets is an open set and

(iii) the intersection of any finite number of open sets is an open set.

2.1.6 Definition: Closed sets

Let (X, T) be a topological space. A subset S of X is said to be a ***closed set*** in (X, T) if its complement in X , namely $X \backslash S$, is open in (X, T).

2.1.7 Definition: Clopen

A subset S of a topological space (X, T) is said to be ***clopen*** of ot os both open and closed in (X, T).

Some points to remember:

(i) In every topological space (X, T) both X and ϕ are clopen.

(ii) In a discrete space all subsets of X are clopen.

(iii) In an indiscrete space the only clopen subsets are X and ϕ

2.1.8 Definition: Cofinite topology

Let X be any non-empty set. A topology T on X is called the ***finite-closed topology*** or the ***cofinite topology*** if the closed subsets of X are X and all finite subsets of X; that is, the open sets are ϕ and all subsets of X which have finite complements.

2.2 The Euclidean Topology

2.2.1 Definition: The Euclidean topology on $\mathbb{R}$

A subset S of $\mathbb{R}$ is said to be open in the ***euclidean topology on*** $\mathbb{R}$ if it has the following property:
(*) For each $x \in S$, there exist a, b in $\mathbb{R}$, with $a < b$, such that $x \in (a, b) \subseteq S$

2.2.2 Definition: Basis

Let (X, T) be a topological space. A collection B of open subsets of X is said to be a ***basis*** for the topology T if every open set is a union of members of B. **Example:**
Let (X, T) be a discrete space and B the family of all singleton subsets of X; that is, $B = \{\{x\} : x \in X\}$,Then B is a basis for T. **Proposition:** Let X be a non-empty set and let $\mathbf{B}$ be a collection of subsets of X. Then $\mathbf{B}$ is a basis for a topology on X if and only if $\mathbf{B}$ has the following properties:

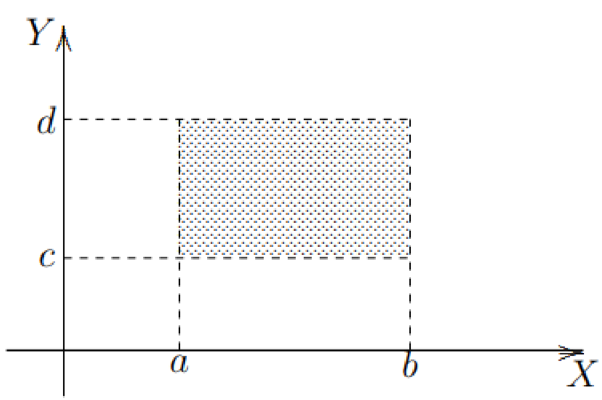
(a) $X = \bigcup\limits_{B \in \mathbf{B}} B$, and

(b) for any $B_1, B_2 \in \mathbf{B}$, the set $B_1 \cap B_2$ is a union of members of $\mathbf{B}$

Example: Let B be the collection of all "open rectangles"

$$\{\langle x, y\rangle : \langle x, y\rangle \in \mathbb{R}^2, a < x < b, c < y < d\}$$

in the plane which have each side parallel to the $X - orY - axis$.



Then B is a basis for a topology on the plane. This topology is called the euclidean topology.

Proposition: Let (X, T) be a topological space. A family $\mathbf{B}$ of open subsets of X is a basis for T if and only if for any point x belonging to any open set U, there is a $B \in \mathbf{B}$ such that $x \in B \subseteq U$ **Proposition:** Let $\mathbf{B}_1 and \mathbf{B}_2$ be bases for topologies T_1 and T_2, respectively, on a non-empty set X. Then $T_1 = T_2$ if and only if

(i) for each $B \in \mathbf{B}_1$ and eaxh $x \in B$, there exists a $B' \in \mathbf{B}_2$ such that $x \in B' \subseteq B$, and

(ii) for each $B \in \mathbf{B}_2$ and each $x \in B$, there exists a $B' \in \mathbf{B}_1$ such that $x \in \mathbf{B}' \subseteq B$.

2.3 Limit points:

2.3.1 Definition: Limit point

Let A be a subset of a topological space (X,T).A point $x \in X$ is said to be a ***limit point of*** A if every open set,U,containing x contains a point of A different from x.
Example: Consider the topological space (X,T) where the set $X = \{a,b,c,d,e\}$, the topology $T = \{X, \phi, \{a\}, \{c.d\}, \{a,c,d\}, \{b,c$
and $A = \{a,b,c\}$.Then $b, dande$ are limit point of A but a and c are not the limit points of A.
Proof:
The set $\{a\}$ is open and contains no other point of A. So a is not a limit point of A. The set$\{c,d\}$ is an open set containing c but no other point of A. So c is not a limit point of A.
Proposition: Let A be a subset of a topological space (X,T) and $A^{'}$ the set of all limit points of A. Then $A \cup A^{'}$ is a closed set.

2.3.2 Definition: Closure of$A$

Let A be a subset of a topological space (X,T). Then the set $A \cup A^{'}$ consisting of A and all its limit points is called the ***Closure of*** A and is denoted by $\bar{A}$.

2.3.3 Definition: dense in X

Let A be a subset of a topological space (X,T). Then A is said to be ***dense*** in X or ***everywhere dense*** in X if $\bar{A} = X$

2.4 Homeomorphism

2.4.1 Definition: Subspaces

Let Y be a non-empty subset of a topological space (X,T). The collection $T_Y = \{O \cap Y : O \in T\}$ of subsets of Y is a topology on Y called ***subspace topology*** (or the ***relative topology*** or the ***induced topology*** or the ***topology induced on*** Y ***by*** T) The topological space (Y, T_Y) is said to be a ***subspace*** of (X,T).
Example : Let $X = \{a,b,c,d,e,f\}$,$T = \{X, \phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e,f\}\}$, and $Y = \{b,c,e\}$. Then the subspace topology on Y is $T_Y = \{Y, \phi, \{c\}\}$.

2.4.2 Definition : Homeomorphism

Let (X,T) and (Y,T_1) be topological spaces. Then they are said to be ***homeomorphic*** if there exists a $f : X \longrightarrow Y$ which has the following properties:

(i) f is one-to-one

(ii) f is onto

(iii) for each $U \in T_1, f^{-1}(U) \in T$

(iv) for each $V \in T, f(V) \in T_1$

Further, the map f is said to be a homeomorphism between (X,T) and (Y,T_1). We rite $(X,T) \cong (Y,T_1)$
Example Let$(X,T), (Y,T_1)$ and (Z,T_2) be topological spaces. $(X,T) \cong (Y,T_1)$ and $(Y,T_1) \cong (Z,T_2)$, prove that (X,T_2).
Solution: As $(X,T) \cong (Y,T_1)$ and $(Y,T_1) \cong (Z,T_2)$, there exist homeomorphism $f : (X,T) \to (Y,T_1)$ and $g : (Y,T_1) \to (Z,T_2)$. The composite function $g \circ f : X \to Z$ is one one and onto and satisfies all conditions thus $(X,T) \cong (Z,T_2)$.
Proposition: Any topological space homeomorphic to a connected space is connected.

2.4.3 Definition: Interval

A subset S of $\mathbb{R}$ is said to be an ***interval*** if it has the following property: if $x \in S, z \in S$ and $y \in \mathbb{R}$ are such that $x < y < z$,then $y \in S$. **NOTE:**

(i) Each singleton set $\{x\}$ is an interval.

(ii) Every interval has one of the following forms: $\{a\}, [a, b], (a, b), [a, b), (a, b], (-\infty, a), (-\infty, a], (a, \infty), [a, \infty), (-\infty, \infty)$

(iii) Every interval is homeomorphic to $(0, 1), [0, 1], [0, 1), or 0$

Proposition : A subspace S of $\mathbb{R}$ is connected if and only if it is an interval.

2.5 Continuous Mappings

2.5.1 Lemma:

Let f be a function mapping $\mathbb{R}$ into itself. Then f is continuous if and only if for each $a \in \mathbb{R}$ and each open set U containing $f(a)$,there exists an open set V containing a such that $f(V) \subseteq U$.

2.5.2 Definition : Continuous Mapping

Let $(X, T)$and (Y, T_1) be topological spaces and f a function from X into Y. Then $f : (X, T) \to (Y, T_1)$ is said to be ***continuous mapping*** if for each $U \subset T_1, f^{-1}(U) \in T$
Example: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = c$, for c a constant and all $x \in \mathbb{R}$. Then let U be any open set in R. Clearly $f^{-1}(U) = \mathbb{R}$ if $c \in U$ and ϕ if $c \notin U$. In both cases,$f^{-1}(U)$ is open. So f is continuous.

2.5.3 Definition: Pathwise connected

A topological space (X, T) is said to be ***path-connected*** if for each pair of (distinct) points a and b of X there exists a continuous mapping $f : [0, 1] \to (X, T)$, such that $f(0) = a$ and $f(1) = b$. The mapping f is said to be a ***path joining a to b*** .

2.5.4 Theorem : Weierstrass Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) \neq f(b)$. Then for every number p between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = p$. **Proof:**
As $[a, b]$ is connected and f is continuous, $f([a, b])$ is connected. This implies that $f([a, b])$ is an interval. Now $f(a)$ and $f(b)$ are in $f([a, b])$. So if p is between $f(a)$ and $f(b)$,$p \in f([a, b])$, that is, $p = f(c)$, for some $c \in [a, b]$.

2.6 Metric spaces

2.6.1 Definition : Metric Space

Let X be a non-empty set and d a real-valued function defined on $X \times X$ such that for $a, b \in X$:

(i) $d(a, b) \geq 0$ and $d(a, b) = 0$ if and only if $a = b$;

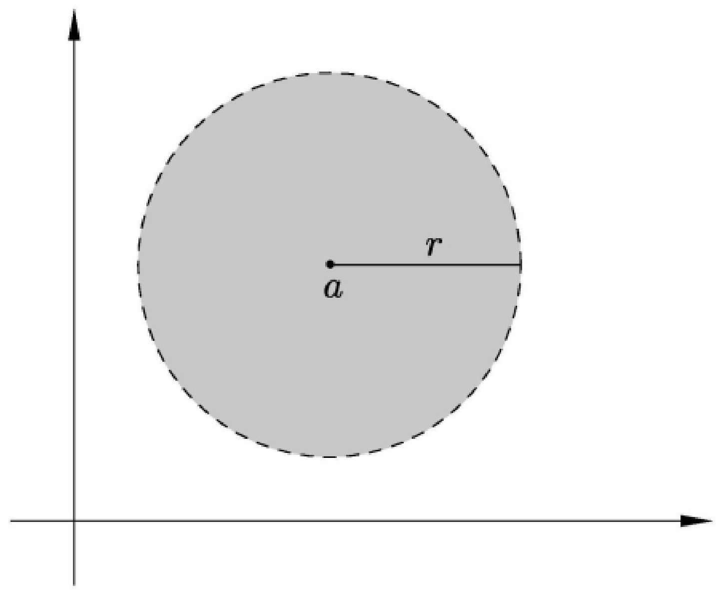
(ii) $d(a, b) = d(b, a)$ and

(iii) $d(a, c) \leq d(a, b) + d(b, c)$ for all a, b and c in X.

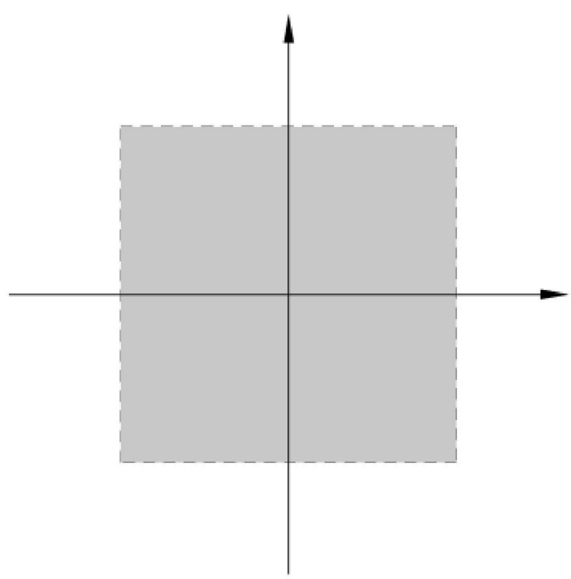
Then d is said to be a ***metric*** on X, (X, d) is called a ***metric space*** and $d(a, b)$ is referred to as the ***distance*** between a and b.

2.6.2 Definition : Open ball about a point

Let (X, T) be a metric space and r any positive real number. Then the ***open ball about*** $a \in X$ ***of radius*** r is the set $B_r(a) = \{x : x \in X \text{ and } d(a, x) < r\}$.
Example: In $\mathbb{R}^2$ with euclidean metric, $B_r(a)$ is the open disc with centre a and radius r.



Example : In $\mathbb{R}^2$ with the metric d^* given by $d^*(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) = max\{|a_1 - b_1|, |a_2 - b_2|\}$, the open ball $B_1(\langle 0, 0\rangle) looks like$



Lemma : Let (X, d) be a metric space and a and b points of X. further, let δ_1 and δ_2 be positive real numbers. If $c \in B_{\delta_1}(a) \cap B_{\delta_2}(b)$, then there exists a $\delta > 0$ such that $B_\delta(c) \subseteq B_{\delta_1}(a) \cap B_{\delta_2}(b)$.

2.6.3 Definition: Equivalent

Metrics on a set X are said to be ***equivalent*** if they induce the same topology on X.

2.6.4 Definition: Hausdorff space

A topological space (X, T) is said to be a ***Hausdorff space*** if for each pair of distinct points a and b in X, there exist open sets U and V such that $a \in U, b \in V$, and $U \cap V = \phi$.

2.6.5 Definition : Metrizability

A space (X, T) is said to be ***metrizable*** if there exists a metric d on the set X with the property that T is the topology induced by d.

2.6.6 Definition : Convergence of Sequences

Let (X, T) be a metric space and $x_1, \ldots, x_n, , \ldots$ a sequence of points in X.Then the sequence is said to ***converge to*** $x \in X$ if given any $\epsilon > 0$ there exists an integer n_0 such that for all $n \geq n_0, d(x, x_n) < \epsilon$. This is denoted by $x_n \longrightarrow x$.
The sequence $y_1, y_2, \ldots, y_n, \ldots$ of points in (X, d) is said to be ***convergent*** if there exists a point $y \in X$ such that $y_n \longrightarrow y$.

2.6.7 Definition : Cauchy sequence

A sequence $x_1, x_2, \ldots, x_n, \ldots$ of points in ametric space (X,d) is said to be a ***Cauchy sequence*** if given any real number $\epsilon > 0$, there exists a positive integer n_0, such that for all integers $m \geq n_0$ and $n \geq n_0$, $d(x_m, x_n) < \epsilon$.

2.6.8 Definition : Complete

A metric space (X, d) is said to be ***complete*** if every Cauchy sequnece in (X, d) converges to a point in (X, d).

2.6.9 Definition: subsequence

If $\{x_n\}$ is any sequence then the sequnece $x_{n_1}, x_{n_2}, \ldots$ is said to be a ***subsequence*** if $n_1 < n_2 < n_3 < \ldots$

2.6.10 Definition : Monotonic

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then it is said to be an ***increasing sequence*** if $x_n \leq x_{n+1}$, for all $n \in \mathbb{N}$. It is said to be a ***decreasing sequence*** if $x_n \geq x_{n+1}$, for all $n \in \mathbb{N}$. A sequence which is either increasing or decreasing is said to be ***monotonic***.

2.6.11 Definition: Peak point

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then $n_0 \in \mathbb{N}$ is said to be a ***peak point*** if $x_n \leq x_{n0}$, for every $n \geq n_0$.

This is what I have read till now