# Partial Fractions

## Introduction

I have a polynomial $f(x)$ and I want to write $1/f(x)$ as a sum of functions of the form $A/(x+B)$, if possible. Partial fractions, they call it.

$$\frac{1}{x^3-6x^2+11x-6} = \frac{1/2}{x-1} + \frac{-1}{x-2} + \frac{1/2}{x-3} \tag{1}$$

It's not always possible to be able to decompose $1/f(x)$ like that but suppose I have a polynomial $f(x)$ with all roots distinct. Then I can always write:

$$\frac{1}{f(x)} = \sum_{\substack{\alpha\in\mathbb{C}\\ f(\alpha)=0}} \frac{1/f'(\alpha)}{x-\alpha} \tag{2}$$

I won't have to worry about $f'(\alpha)=0$ since it would imply $\alpha$ is a repeated root. The equality (2) can be seen by multiplying $f(x)$ on both sides and the RHS is now a polynomial of degree 1 less than $f(x)$ but it equals the LHS on all the roots of $f$, so it's equal to the LHS always.

But what do I do with $f(x)$ who have repeated roots? If $f$ has a root $\alpha$ which is repeated once, I can't decompose $1/f$ into partial fractions without a term of the form $(Ax+B)/(x-\alpha)^2$. Similarly I'll need to keep terms of the form $p(x)/(x-\alpha)^k$ for more frequently repeated roots.

$$\frac{1}{x^6-10x^5+40x^4-82x^3+91x^2-52x+12} = \frac{(-17x^2+24x-11)/8}{(x-1)^3} + \frac{2x-5}{(x-2)^2} + \frac{1/8}{x-3} \tag{3}$$

An alternate representation would be to write $\frac{Ax+B}{(x-\alpha)^2}$ as $\frac{A}{(x-\alpha)} + \frac{B'}{(x-\alpha)^2}$ but I choose to not consider it for now because it will end up being a straightforward inter-conversion anyway. This is because (as will be evident shortly) I will write the numerator $(Ax+B)$ as $(A(x-\alpha)+B')$ instead from the beginning. Similarly, $p(x-\alpha)$ for higher degree polynomials $p(x)$.

Let $f(x)$ be a polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ where the multiplicity of the roots are (respectively) $m_1, m_2, \ldots, m_n$ so that the degree of $f$ is $m_1+m_2+\cdots+m_n$. We want to write:

$$\frac{1}{f(x)} = \sum_{i=1}^{n} \frac{p_i(x)}{(x-\alpha_i)^{m_i}} \tag{4}$$

where $p_i(x)$ is a polynomial of degree $m_i-1$. However I will be writing $p_i(x-\alpha_i)$ instead of $p_i(x)$ since it doesn't change the underlying meaning but we happen to get convenient expressions later on.

$$\frac{1}{f(x)} = \sum_{i=1}^{n} \frac{p_i(x-\alpha_i)}{(x-\alpha_i)^{m_i}} \tag{5}$$

The task now is to find $p_i(x)$

Rewrite (5) as:

$$-1 + \sum_{i=1}^{n} p_i(x-\alpha_i)\frac{f(x)}{(x-\alpha)^{m_i}} = 0 \tag{6}$$

Note that $f(x)/(x-\alpha_i)^{m_i}$ is also just a polynomial since $\alpha$ is a root with multiplicity $m_i$. Let $g_i(x-\alpha_i) = f(x)/(x-\alpha_i)^{m_i}$. Again, defining this way instead of $g_i(x)$ for no other reason than it being easier to follow later on.

$$-1 + \sum_{i=1}^{n} p_i(x-\alpha_i)g_i(x-\alpha_i) = 0 \tag{7}$$

Going on a similar line of thought as for the case of distinct roots, we see that the LHS is a polynomial of degree 1 less than the degree of $f$. If we can make the roots of LHS be the same as the roots of $f$, then we know that it must be zero always. But here we need to ensure that, for example, we make LHS have the root $\alpha_i$ with a multiplicity $m_i$. In other words, LHS is divisible by $(x-\alpha_i)^{m_i}$ for all $i \in 1, 2, \ldots, n$. But by definition, $g_j(x-\alpha_j)$ is divisible by $(x-\alpha_i)^{m_i}$ for all $j \neq i$. So we need to have the following:

$$(x-\alpha_i)^{m_i} \text{ divides } p_i(x-\alpha_i)g_i(x-\alpha_i) - 1 \qquad \forall i \in \{1, 2, \ldots, n\} \tag{8}$$

or equivalently,

$$x^{m_i} \text{ divides } p_i(x)g_i(x) - 1 \qquad \forall i \in \{1, 2, \ldots, n\} \tag{9}$$

Now this is an independent question for each $i$, where we are 'given' $g_i(x)$ and we have to 'find' a $p_i(x)$ of degree $m_i - 1$. The coefficients of $g_i(x)$ are 'known'. We have to set the first $m_i$ coefficients of $p_i(x)g_i(x)$ to 0 (or 1) and the $m_i$ coefficients of $p_i(x)$ are our unknowns. This forms a system of linear equations that we know how to solve. But it would be nice to be able to get some final expression too. For a general polynomial $P(x)$, let's denote the coefficient of $x^k$ in $P(x)$ as $P[k]$.

$$\sum_{j=0}^{k} p_i[j] \cdot g_i[k-j] = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \text{ and } k < m_i \end{cases} \tag{10}$$

We rewrite this set of linear equations as follows:

$$\text{Let } G_i = \begin{bmatrix} g_i[0] & 0 & 0 & 0 & \ldots & 0 \\ g_i[1] & g_i[0] & 0 & 0 & \ldots & 0 \\ g_i[2] & g_i[1] & g_i[0] & 0 & \ldots & 0 \\ g_i[3] & g_i[2] & g_i[1] & g_i[0] & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_i[m_i-1] & g_i[m_i-2] & g_i[m_i-3] & g_i[m_i-4] & \ldots & g_i[0] \end{bmatrix} \tag{11}$$

$$\text{Let } P_i = \begin{bmatrix} p_i[0] & p_i[1] & p_i[2] & p_i[3] & \ldots & p_i[m_I - 1] \end{bmatrix}^\top \tag{12}$$

then

$$G_i P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^\top \tag{13}$$

Let the matrix $G_c$ be such that each element of $G_c$ is the cofactor of the corresponding element from $G$ divided by $\det(G)$. This just means $G^{-1} = G_c^\top$. We also have $det(G) = (g_i[0])^{m_i}$. Rewrite (13) as:

$$P_i^\top = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} G_c \tag{14}$$

So the coefficients of $p_i(x)$ (what we require) is just the first row of $G_c$ (or in other words the cofactors of the first row of $G$ divided by $\det(G)$). There is a alternative way to represent $p_i(x)$ from this information.

The determinant of a matrix is the sum of element-cofactor products of (say) the first row. If we replace the first row of $G$ with powers of $x$ i.e. $(1, x, x^2, \ldots, x^{m_i-1})$ and take the determinant of that matrix, we get $p_i(x)$ since the coefficient of a power of $x$ is the cofactor of that same element in $G$'s first row as we just saw. We will of-course have to divide by $(g_i[0]^{m_i})$ after taking the determinant. Before all this however, we still need to be able to write $g_i[k]$ in terms of $f$ so we can form the matrix $G_i$. We shall do that now

We denote by $f^k(x)$ the $k$-th derivative of $f(x)$ where $f^0(x) = f(x)$. From the definition of $g_i$ it will follow that

$$g_i[k] = \frac{f^{k+m_i}(\alpha_i)}{(k+m_i)!} \tag{15}$$

To see why, note that coefficient of $x^k$ in $g_i(x)$ is the same as the coefficient of $x^{k+m_i}$ in $f(x+\alpha_i)$. And that in general $P[k] = P^k(0)/k!$. To introduce some notation, let $F_k(\alpha) := \frac{f^k(\alpha)}{k!}$. So $g_i[k] = F_{k+m_i}(\alpha_i)$

Thus we have:

$$p_i(x-\alpha_i) = \frac{1}{(F_{m_i}(\alpha_i))^{m_i}} \begin{vmatrix} 1 & (x-\alpha_i) & (x-\alpha_i)^2 & \ldots & (x-\alpha_i)^{m_i-1} \\ F_{m_i+1}(\alpha_i) & F_{m_i}(\alpha_i) & 0 & \ldots & 0 \\ F_{m_i+2}(\alpha_i) & F_{m_i+1}(\alpha_i) & F_{m_i}(\alpha_i) & \ldots & 0 \\ F_{m_i+3}(\alpha_i) & F_{m_i+2}(\alpha_i) & F_{m_i+1}(\alpha_i) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{2m_i-1}(\alpha_i) & F_{2m_i-2}(\alpha_i) & F_{2m_i-3}(\alpha_i) & \ldots & F_{m_i}(\alpha_i, m_i) \end{vmatrix} \tag{16}$$

For the sake of closure, we can now say that:

$$\frac{1}{f(x)} = \sum_{i=1}^{n} \frac{\det(\Delta_i)}{(F_{m_i}(\alpha_i)(x-\alpha_i))^{m_i}} \tag{17}$$

where $\Delta_i$ is an $m_i \times m_i$ matrix defined by:

$$[\Delta_i]_{jk} := \begin{cases} (x-\alpha_i)^{k-1} & j = 1 \\ F_{m_i+k-j}(\alpha_i) & j > 1 \end{cases}$$

and

$$F_k(\alpha) := \frac{f^k(\alpha)}{k!}$$

(row and column indices $j, k$ start at 1)

The zeros are automatically taken care of in this definition too. Also, when $m_i = 1$, $\Delta_i = [1]$ and $F_1(\alpha_i) = f'(\alpha_i)$ so this is consistent with the distinct roots case we started with.

## 'Partial fractions' for non-polynomials

Though this section is quite hand-wavy, most results do happen to be correct and well-known.

To decompose something into partial fractions as in (2) we just need to know the roots of $f(x)$ and the derivative of $f$ at those roots. Of course, $f(x)$ needs to be a polynomial too but what if I decide to humour the formulae? What if, say, $f(x) = \sin x$ ?

Think of $\sin x$ as a polynomial with roots $n\pi$ for all integers $n$. We may try to compare $\sin x$ to a more polynomial form of $x(x^2 - \pi^2)(x^2 - 4\pi^2)\cdots$. This doesn't look right because we probably need to multiply by some appropriate constant so that, say $\lim_{x\to 0}(\sin x/x) = 0$. So maybe we compare $\sin x$ to a 'polynomial' that looks like: $x\prod_{k=1}^{\infty}\left(1 - \left(\frac{x}{k\pi}\right)^2\right)$. This does come close to $\sin x$ which you can see by trying to graph a finite product. This is actually known as Weierstrass factorization. So what if we pretend this is $\sin x$ to see how $1/\sin x$ will break into partial fractions according to (2)? We get

$$\frac{1}{\sin x} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x - k\pi} \tag{18}$$

This funny looking equation is actually true, though more commonly written in the following equivalent format:

$$\frac{x\csc x - 1}{2x^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2\pi^2} \tag{19}$$

But there is an important issue with our line of reasoning so far. Other than the lack of rigour, of course. We seem to imply that any function $f(x)$ who equals zero only at $x = n\pi$ and has the derivative as $f'(n\pi) = (-1)^n$ will be able to take the place of $\sin x$ in (18). But clearly there are functions other than $\sin x$ that are zero at $n\pi$ with slope $(-1)^n$. You can construct any such example but let's take a simple one: $\sin x - a\sin^2 x$ for some $a \in (0, 1)$

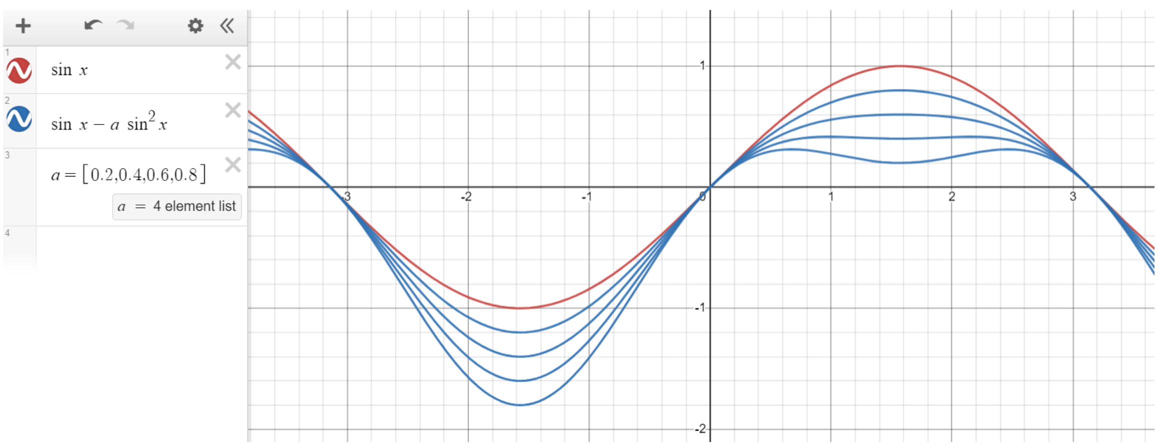


Figure 1: $(\sin x)$ vs $(\sin x - a\sin^2 x)$

Let $f_a(x) = \sin x - a\sin^2 x$. If we are going to pretend $f_a(x)$ and $f(x)$ are polynomials, then we need to be including their non-real roots too. If we define $\sin x$ for $x \in \mathbb{C}$ as $\frac{e^{ix} - e^{-ix}}{2i}$ then $f(x) = \sin x$ still has only real roots but $f_a(x)$ has some additional non real roots (from solving $\sin x = 1/a$) that we need to have corresponding terms for in the partial fraction decomposition. In other words, when we say that there are no roots other than $n\pi$, that includes non-real roots.

If (18) is correct, it seems to also imply that the exact shape of $\sin x$ is just the 'right' shape for starting at $x = 0$ with a slope of 1 and finishing at $x = \pi$ with a slope of -1, for example. If we try to do this by drawing any other shape there will apparently be some underlying non-real roots associated with it.

To test our claim about taking into account the non-real roots of $f_a(x)$ too in the partial fraction decomposition, we calculate that these are of the form

$$x_k = \alpha_k \mp i\beta \qquad\qquad \forall k \in \mathbb{Z} \tag{20}$$

where

$$\alpha_k = \frac{\pi}{2} + 2k\pi \qquad\qquad \beta = \ln\left(\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}\right) \qquad\qquad f_a'(x_k) = \mp i\sqrt{\frac{1}{a^2} - 1} \tag{21}$$

So the additional terms due to non real roots will be:

$$\frac{1}{\left(i\sqrt{\frac{1}{a^2}-1}\right)(x-(\alpha_k+i\beta))} + \frac{1}{\left(-i\sqrt{\frac{1}{a^2}-1}\right)(x-(\alpha_k-i\beta))} = \frac{\frac{2\beta a}{\sqrt{1-a^2}}}{x^2 - 2\alpha_k x + \alpha_k^2 + \beta^2} \tag{22}$$

And therefore we want to claim that:

$$\frac{1}{\sin x - a\sin^2 x} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x - k\pi} + \sum_{k=-\infty}^{\infty} \frac{\frac{2\beta a}{\sqrt{1-a^2}}}{x^2 - 2\alpha_k x + \alpha_k^2 + \beta^2} \tag{23}$$

I can't 'prove' this equality but approximating finite sums on any graphing calculator can show us that they are close enough and so (probably) equal. As a simpler example: (evident from subtracting $\csc x$ from both sides) this is also equivalent to saying:

$$\frac{1}{1 - a\sin x} = \sum_{k=-\infty}^{\infty} \frac{\frac{2\beta}{\sqrt{1-a^2}}}{x^2 - 2\alpha_k x + \alpha_k^2 + \beta^2} \tag{24}$$

So, if it works for $\sin x$ then it probably works for some other simple functions too. If $f(x) = \tan x$ then $f$ is zero at only $n\pi$ (including possible non-real roots) for $n \in \mathbb{Z}$ and $f'(n\pi) = 1$. So

$$\frac{1}{\tan x} = \sum_{k=-\infty}^{\infty} \frac{1}{x - k\pi} \tag{25}$$

which is also true and so is the case of $f(x) = \cos x$ and $f(x) = \cot x$ etc. Apparently the discontinuity at $x = k\pi + \pi/2$ when treating $f(x) = \tan x$ as a polynomial does not break (25). The wikipedia example of $\csc^2 x$ suggests that the equations for the repeated roots case also hold for 'non-polynomials'. I claim, without proof, that (2) and (17) are awesome.

## Polynomial Interpolation and the Vandermonde matrix

Suppose I know that a function $H(x)$ is supposed to take values $\beta_i$ at $x = \alpha_i$ for $i \in \{1, 2, \ldots, n\}$ where the $\alpha_i$ are distinct. If I want to interpolate $H$ from these points assuming that, say, $H(x)$ is a polynomial of degree $n-1$; then $H(x)$ can be 'uniquely' expressed as:

$$H(x) = \sum_{i=1}^{n} \frac{f(x)}{f'(\alpha_i)(x - \alpha_i)}\beta_i \tag{26}$$

Where $f(x) := \prod_{i=1}^{n}(x - \alpha_i)$ for ease of notation (and for relating to the previous formulaes). As before, $f(x)/(x - \alpha_i)$ is just a shorthand for $\prod_{k=1,k\neq i}^{n}(x - \alpha_k)$. The justification for this is also the same as before, $H(x)-$RHS is 0 at $n$ distinct values (the $\alpha_i$) but is a $n-1$ degree polynomial (at most). So it is zero. These are actually called the Lagrange Interpolating Polynomials.

As a sidenote, we can again extend the same problem for the case of repeated roots. It would be a very convoluted, made-up problem statement but then what isn't? Suppose I know that a function $H(x)$ takes values $\beta_i$ at $x = \alpha_i$ with a multiplicity $m_i$ for $i \in \{1, 2, \ldots, n\}$ where the $\alpha_i$ are distinct. This means that as a function, $H$ has the property that

$$\lim_{x \to \alpha_i} \frac{H(x) - \beta_i}{(x - \alpha_i)^k} = 0 \qquad \forall k \text{ such that } 0 \le k < m_i, \forall i \tag{27}$$

If I was to interpolate $H$ from this information assuming that, say, $H(x)$ is a polynomial of appropriate degree; then $H(x)$ can be expressed as:

$$H(x) = \sum_{i=1}^{n} \frac{p_i(x - \alpha_i) f(x)}{(x - \alpha_i)^{m_i}} \beta_i \tag{28}$$

Where $f(x) := \prod_{i=1}^{n} (x - \alpha_i)^{m_i}$ and $f(x)/(x - \alpha_i)^{m_i}$ would be a shorthand as before. $p_i(x)$ is a polynomial of degree $m_i - 1$ as described in the previous section. The justification for this is also the same as before, $H(x)-$RHS is divisible by $(x - \alpha_i)^{m_i}$ for all $i$ (write in terms of $p(x)g(x) - 1$ to see how). But it's of degree 1 less than $\sum m_i$, so it is zero.

Now a polynomial of degree $n - 1$ is described completely by its $n$ coefficients. But by (26) we can see that its also completely described by the values it returns at some $n$ inputs $\alpha_i$. So we have 2 different ways to express the coefficients, given the outputs $\beta_i$ returned at some $\alpha_i$. We can look at the coefficients in (26) or we can solve for the coefficients from a system of linear equations. Let's equate the two. Let the coefficient of $x^k$ in $H(x)$ from (26) be $c_k$ where $k \in \{0, 1, \ldots, n - 1\}$. The following relation exists between $\alpha_i, \beta_i, c_k$:

$$\text{let } V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{bmatrix} \tag{29}$$

(btw $V$ stands for Vandermonde)

$$V \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = V^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \tag{31}$$

Notice how in the RHS of (31) we have a linear combination of the columns of $V^{-1}$, with multipliers $\beta_i$ similar to how in (26) we have a summation with multipliers $\beta_i$. From this we can conclude that the $i$-th column of $V^{-1}$ should ideally just be the coefficients of the following polynomial:

$$\frac{f(x)}{f'(\alpha_i)(x - \alpha_i)} = \prod_{k=1, k \neq i}^{n} \frac{x - \alpha_k}{\alpha_i - \alpha_k} \tag{32}$$

Let $C$ be the cofactor matrix of $V$ (each element is the cofactor of the corresponding element in $V$, and therefore $V^{-1} = C^{\top}/\det(V)$). The $k$-th element of the $i$-th column of $V^{-1}$, which is $\frac{C^{\top}_{k,i}}{\det(V)}$, should be the coefficient of $x^{k-1}$ of $\prod_{k=1,k\neq i}^{n} \frac{x-\alpha_k}{\alpha_i-\alpha_k}$. An insightful way to write this would be:

$$\sum_{k=1}^{n} \frac{x^{k-1} C_{i,k}}{\det(V)} = \prod_{k=1,k\neq i}^{n} \frac{x-\alpha_k}{\alpha_i-\alpha_k} \tag{33}$$

$$\sum_{k=1}^{n} x^{k-1} C_{i,k} = \frac{\det(V)}{\prod_{k=1,k\neq i}^{n}(\alpha_i-\alpha_k)} \prod_{k=1,k\neq i}^{n} (x-\alpha_k) \tag{34}$$

Think about the explicit expression for a vandermonde determinant, and notice that the product $\prod_{k=1,k\neq i}^{n}(\alpha_i-\alpha_k)$ contains all the terms in $\det(V)$ that are not in $C_{i,n}$ (which is another smaller vandermonde determinant). Even the $(\pm)$ signs work out just right so that we can write

$$\frac{\det(V)}{\prod_{k=1,k\neq i}^{n}(\alpha_i-\alpha_k)} = C_{i,n} \tag{35}$$

Now we could have just inferred this directly from (34) by looking at the coefficient of $x^{n-1}$ but that is the very equation we are trying to verify independently. Atleast, I am. It now becomes:

$$\sum_{k=1}^{n} x^{k-1} C_{i,k} = C_{i,n} \prod_{k=1,k\neq i}^{n} (x-\alpha_k) \tag{36}$$

(36) does not contain the variable $\alpha_i$ anywhere but that variable is independent from all the other $\alpha_j$ $(j \neq i)$. So it doesn't matter if I replace $x$ with $\alpha_i$. Think about what that means. I'm not substituting $x = \alpha_i$ in the traditional sense of evaluating a polynomial/function. (36) is supposed to be an algebraic identity, true always because it is the same algebraic expression on each side. Just like $x^2 + ax = xa + x^2$ is an identity whose validity is not changed if I substitute $x$ with $b$. Why do I want to do this? Because the LHS becomes $\sum_{k=1}^{n} \alpha_i^{k-1} C_{i,k}$ and we know this to be $\det(V)$ by definition of the determinant. And the supposed identity we wanted to verify has now just reduced to (35) which we have already proved.

This made for a very fancy proof that gives the impression of being elegant but it was never clear what underlying property of vandermonde cofactors $C_{i,k}$ was actually being used to equate the two expressions.

$$C_{i,k} = C_{i,n} \left( \text{coeff. of } x^{k-1} \text{ in } \prod_{j=1,j\neq i}^{n} (x-\alpha_j) \right) \tag{37}$$

$$(-1)^{i+k} C_{i,k} = (-1)^{i+n} C_{i,n} \left( \sum_{\substack{S\subseteq U \\ |S|=n-k}} \prod_{j\in S} \alpha_j \right) \qquad U = \{1,2,\ldots,n\} - \{i\} \text{ and } k \neq n \tag{38}$$

Where the multiplied term on RHS; probably familiar as Vieta's formula for polynomial coefficients; is kinda hard to write in a compact notation, as I found out just now. (38) is an identity in the $n-1$ variables $a_k$ for $k \in \{1,2,3,\ldots,n\} - \{i\}$. Once we explicitly write the cofactors $C_{i,k}, C_{i,n}$ as determinants themselves we see that this is an identity concerning the $n-1 \times n-1$ vandermonde matrix and tells us what would happen to its determinant if we remove its $k$-th column and add a new column at the right end continuing the pattern of powers. For the sake of brevity (that's a word, right?) we will state this property for an $n \times n$ vandermonde matrix instead of $n-1 \times n-1$. So, in short, as far as the underlying identity involving matrix determinants is concerned, it is equivalently:

(for new variables $x_i$ with $i \in \{1, 2, \ldots, n\}$)

$$\text{let } V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} & x_1^k & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} & x_2^k & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{k-1} & x_n^k & \cdots & x_n^{n-1} \end{bmatrix}$$

$$\text{let } V_k = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-2} & x_1^k & \cdots & x_1^{n} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-2} & x_2^k & \cdots & x_2^{n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{k-2} & x_n^k & \cdots & x_n^{n} \end{bmatrix} \qquad 1 \leq k \leq n$$

$$\det(V_k) = \det(V)\left(\sum_{\substack{S \subseteq U \\ |S| = n-k+1}} \prod_{j \in S} x_j\right) \qquad U = \{1, 2, \ldots, n\}$$

Here $V$ is a standard Vandermonde matrix and $V_k$ is formed by removing the $k$-th column from $V$ and adding a new column at the right end for the next power ($n$).
We also know that $\det(V) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$ so this becomes an explicit expression for $\det(V_k)$.

As an example, this is what the identity in question implies for $n = 4$:

$$\begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}(a + b + c + d)$$

$$\begin{vmatrix} 1 & a & a^3 & a^4 \\ 1 & b & b^3 & b^4 \\ 1 & c & c^3 & c^4 \\ 1 & d & d^3 & d^4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}(ab + bc + cd + da + ac + bd)$$

$$\begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}(abc + bcd + cda + dab)$$

$$\begin{vmatrix} a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \\ d & d^2 & d^3 & d^4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}(abcd)$$

The $n = 4$ case is not particularly groundbreaking, it just gives context for a result that is true for general $n$