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QUESTION:44

A continuous real-valued signal $x(t)$ has finite positive energy and $x(t) = 0, \forall\ t < 0$. From the list given below, select ALL the signals whose continuous-time Fourier transform is purely imaginary.

1) $x(t) + x(-t)$
2) $x(t) - x(-t)$
3) $j(x(t) + x(-t))$
4) $j(x(t) - x(-t))$

(GATE IN 2023)

Solution: $X(\omega)=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\ dt$

Parameter	Description
$x(t)$	Continuous real valued signal
t	time
ω	angular velocity of the signal
$X(\omega)$	Fourier Transfom of x(t)
$X(-\omega)$	Fourier Transform of x(-t)

TABLE I
VARIABLES AND THEIR DESCRIPTIONS

A fourier transform of a signal is purely imaginary if the signal is real and odd signal and imaginary even. Let us consideer a odd signal $x(t)$, i.e $x(-t)=-x(t)$

$$\mathcal{F}\{x(t)\} = X(\omega) \tag{1}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\ dt \tag{2}$$

$$= \int_{-\infty}^{0} x(t)\, e^{-j\omega t}\ dt + \int_{0}^{\infty} x(t)\, e^{-j\omega t}\ dt \tag{3}$$

$$= \int_{0}^{\infty} x(-t)\, e^{j\omega t}\ dt + \int_{0}^{\infty} x(t)\, e^{-j\omega t}\ dt \tag{4}$$

$$= -\int_{0}^{\infty} x(t)\, e^{j\omega t}\ dt + \int_{0}^{\infty} x(t)\, e^{-j\omega t}\ dt \tag{5}$$

$$= 2\int_{0}^{\infty} jx(t)\, sin(\omega t)\ dt \tag{6}$$

$$\tag{7}$$

$\therefore$ Fourier transform of an real and odd signal is purely imaginary.
Let us consider an imaginary even signal $jx(t)$
i.e $jx(-t)=jx(t)$

$$\mathcal{F}\{x(t)\} = X(\omega) \tag{8}$$

$$X(\omega) = \int_{-\infty}^{\infty} jx(t)\, e^{-j\omega t}\ dt \tag{9}$$

$$= \int_{-\infty}^{0} jx(t)\, e^{-j\omega t}\ dt + \int_{0}^{\infty} jx(t)\, e^{-j\omega t}\ dt \tag{10}$$

$$= -\int_{0}^{\infty} jx(-t)\, e^{j\omega t}\ dt + \int_{0}^{\infty} jx(t)\, e^{-j\omega t}\ dt \tag{11}$$

$$= -\int_{0}^{\infty} jx(t)\, e^{j\omega t}\ dt + \int_{0}^{\infty} jx(t)\, e^{-j\omega t}\ dt \tag{12}$$

$$= -2\int_{0}^{\infty} jx(t)\, sin(\omega t)\ dt \tag{13}$$

$$\tag{14}$$

$\therefore$ Fourier transform of an imaginary even signal is purely imaginary.
1)$x(t) + x(-t)$
Let,

$$f(t) = x(t) + x(-t) \tag{15}$$

$$f(-t) = x(-t) + x(t) \tag{16}$$

$$f(t) = f(-t) \tag{17}$$

$\therefore$ $x(t) + x(-t)$ is even siganl
$\therefore$ $x(t) + x(-t)$ is not Purely imaginary.
2) $x(t) - x(-t)$
Let,

$$f(t) = x(t) - x(-t) \tag{18}$$

$$f(-t) = x(-t) - x(t) \tag{19}$$

$$f(-t) = -f(t) \tag{20}$$

$\therefore$ $x(t) - x(-t)$ is odd siganl.
$\therefore$ $x(t) - x(-t)$ is purely imaginary.
3) $j(x(t) + x(-t))$ Let,

$$f(t) = j(x(t) + x(-t)) \tag{21}$$
$$f(-t) = j(x(-t) + x(t)) \tag{22}$$
$$f(t) = f(-t) \tag{23}$$

$\because$ $f(t)$ is imaginary and even function.
$j(x(t) + x(-t))$ is Purely imaginary.
4) $j(x(t) - x(-t))$ Let,

$$f(t) = j(x(t) - x(-t)) \tag{24}$$
$$f(-t) = j(x(-t) - x(t)) \tag{25}$$
$$f(t) = -f(-t) \tag{26}$$

$\because$ $f(t)$ is imaginary and odd function.
$j(x(t) + x(-t))$ is not Purely imaginary.